Two general defects seem to be patent, one in presentation, one in range. It is the author's aim to emphasize the force and directness of the analysis by having the results appear after a minimum of intermediate work, but this work has frequently been so abbreviated that the reading is at times extremely difficult. Again, the author was naturally ambitious to have his method apply to as large a range of subjects as possible, but its applicability seems but questionably successful when applied to projective properties.

For the differential analysis of curves and surfaces the work of Cesàro is certainly a powerful instrument and can be used with profit by every student of the subject.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
December, 1902.

GAUSS'S COLLECTED WORKS.

Carl Friedrich Gauss Werke. Achter Band. Herausgegeben von der königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B. G. Teubner in Leipzig, 1900. 4to., 458 pp.

WITH the death of Ernst Schering, the venerable editor of the first six volumes of Gauss's collected works, it became incumbent on younger hands to continue the labor of studying, selecting and preparing for the press everything of interest that still remained in the great mass of manuscript that Gauss left behind. Professor Klein, with customary energy and dispatch, has made the necessary arrangements to bring Schering's labor to a prompt termination.

Four new volumes are planned. Professor Brendel, of Göttingen, is editor-in-chief, having the editorial supervision of the entire undertaking. He has also, in particular, the preparation of volume VII. This is to contain the final edition of the Theoria motus and Gauss's voluminous work on the theory of perturbation of the smaller planets, the theory of the moon, etc.

Volume VIII is the volume under review and will be spoken of later.

The contents of volume IX fall into two parts. One of these is to be devoted to mathematical physics, and is in charge of Professor Wiechert, director of the magnetic observatory in

Göttingen. The other part in charge of Professors Börsch and Krüger of the geodetic Centralinstitut in Potsdam, is to contain matter relating to geodesy.

Volume X will contain extracts from Gauss's very extensive scientific correspondence, also biographical notes. Finally a supplementary volume is planned to give a complete index of the preceding ten volumes.

In this connection it is interesting to learn that the movement inaugurated by Professor Klein to collect letters written by or to Gauss, copies of his lectures, scattered manuscripts, note books, in short anything and everything which will help posterity to understand and appreciate Gauss's genius, has been eminently successful. The so-called Gauss rooms in the observatory at Göttingen now serve to house Gauss's papers, and memorabilia relating to him. These, when finally arranged, will without doubt be made accessible to the public and form a fresh inducement, if indeed any were needed, for the student of mathematics to visit Göttingen.

We turn now to the consideration of volume VIII, the first of the new series to appear. Its contents may be regarded as addenda to the first three volumes of Schering's edition, and to the fourth, excluding geodesy.

The new material falls under the following heads : Arithmetic and algebra, pp. 3–32 ; Analysis and function theory, pp. 36–117 ; all under the charge of Professor Fricke, of Braunschweig. Numerical calculation, pp. 121–130 ; Calculus of probabilities, pp. 133–156, in care of Professors Börsch and Krüger, of Potsdam. Foundations of geometry, pp. 159–268 ; Geometria situs, pp. 271–286 ; Problems and theorems in elementary geometry, pp. 289–300 ; Application of complex numbers to geometry, pp. 303–362 ; Theory of surfaces, pp. 365–449, all under the care of Professor Stäckel, in Kiel, with the exception of a single paper on developable surfaces, three pages in length, by Professor Weingarten, of Charlottenburg.

As the reader may know, the first three volumes of Schering's edition, devoted chiefly to the theory of numbers and algebra, the elliptic and hypergeometric functions, contained a very large amount of material taken from Gauss's posthumous papers. It is not astonishing, then, that a second gleaning does not bring to light much new matter on these topics. Interesting however are the following. First we note two short fragments on cubic and biquadratic residues. In the two epoch-making memoirs on biquadratic residues published in 1828 and 1832 Gauss took the momentous step of introducing imaginary numbers into the theory of numbers. In two places in these memoirs Gauss states that he began to study the higher laws of reciprocity in 1805 and was soon led to introduce the roots of unity into his speculations as the true basis of this theory. The above notes have now an especial interest to the devotees of the sublime science, as Gauss often styled the higher arithmetic, because they belong to this early epoch, 1805–1808. Let us observe in passing that the ideas contained in the memoir of 1832, properly generalized by Kummer, Dedekind and Kronecker, have led to the modern theory of algebraic numbers.

In the section devoted to Analysis and theory of functions, three papers attract our attention. One is the celebrated letter of 1811 to Bessel. Here Gauss discusses summarily the nature of a function and an integral taken between complex limits. In the first place, he advocates warmly the introduction of the complex variable into analysis. He urges:* "Es ist hier nicht von praktischem Nutzen die Rede, sondern die Analysis ist mir eine selbständige Wissenschaft die durch Zurücksetzung jener fingirten Grössen ausserordentlich an Schönheit und Rundung verlieren und alle Augenblick Wahrheiten, die sonst allgemein gelten, höchst lästige Beschränkungen beizufügen genötigt sein würde."

After stating Cauchy's theorem that $\int f(x)dx$ around a closed contour containing no singular points is zero he remarks†: "Dies ist ein sehr schöner Lehrsatz dessen eben nicht schweren Beweis ich bei einer schicklichen Gelegenheit geben werde. Er hängt mit schönen andern Wahrheiten, die Entwickelungen in Reihen betreffend, zusammen."

The latter part of this quotation suggests at once far-reaching consequences: Does Gauss have in mind the developments of Taylor, Laurent and Fourier? In this letter Gauss also emphasizes the importance of those functions which we call to-day one valued integral transcendental functions. In place of the function Soldner had introduced, viz., $\int(\log x)^{-1}dx$, he suggests $\int(e^x - 1)x^{-1}dx$.

* P. 90. We take this occasion to remark that all references without indication of the title refer to Gauss's Works, Vol. VIII.

† P. 91.

The second paper, which Fricke dates about 1800, relates to the inversion of the elliptic integral

$$u = \int \frac{dx}{\sqrt{(1 - x^2)(1 - \mu^2 x^2)}}.$$

We find to our surprise that the development given by Gauss is nothing more nor less than Weierstrass's celebrated expression of x as the quotient of two integral transcendental functions, viz., $Al_1(u)$, $Al_0(u)$.

The third interesting feature in the section are three fragments on elliptic modular functions. If the mathematical world was filled with astonishment in 1827–28 when it became gradually known that Gauss had had in his possession for thirty years a good part of Abel's and Jacobi's results, we to-day are hardly less astonished to learn that Gauss certainly knew some of the modern theory of the elliptic modular functions. In fact, on page 104 we find for example a modular figure with the angles $\pi/4$, $\pi/4$, $\pi/4$, complete with its *orthogonal circle.* It is worthy of note that on page 478 of volume III appears another figure very familiar to students of the modular functions, viz., the fundamental domain of the modulus $-ik^2$. But as this volume was published in 1876, while Fuchs's and Dedekind's papers, which gave birth to the modular functions, did not appear till the following year, the figure last mentioned could not mean much to Schering, otherwise he would have searched the manuscripts of Gauss more carefully for other fragments of this nature.

The section devoted to the calculus of probabilities contains some interesting letters of Gauss. The theory of least squares, so important in the adjustment of observations, was discovered independently by Legendre and Gauss. The first published account of it is found in Legendre's Nouvelles méthodes pour la détermination des orbites des comètes, which appeared in 1806. In the Theoria motus, published in 1809, Gauss claimed to have discovered it in 1795. A controversy arose, very painful to the friends of Gauss. For example, Legendre in a letter* to Jacobi dated 1827, speaking of Gauss's claims of priority in the elliptic functions, breaks out: "Comment se fait-il que M. Gauss ait osé vous faire dire que la plupart de vos théorèmes lui etait connus, et qu'il en avait fait la decouverte dès 1808? * * * Cet

*Jacobi, Werke, Vol. I, p. 398.

excès d'impudence n'est pas croyable de la part d'un homme qui a assez de mérite personnel pour n'avoir pas besoin de s'approprier les découvertes des autres * * * Mais c'est le même homme qui en 1801 voulut s'attribuer la découverte de la loi de reciprocité publiée en 1785 et qui voulut s'emparer en 1809 de la méthode des moindre carrés publiée en 1805. D'autres exemples se trouveraient en d'autres lieux, mais un homme d'honneur doit se garder de les imiter."

Such direct accusations were of course not published, but they were circulated in private. Enough was said in print and in private to cause Gauss's friend Schumacher to write him (1831) as follows: * "Da Sie die Resultate Ihrer Rechnung geben, so scheint es mir, ist es leicht zu zeigen dass diese durch die Methode der kleinsten Quadrate abgeleitet sind. Zach lebt zudem noch, und hat gewiss Ihren Brief aufgehoben. Finden Sie es nicht der Mühe werth endlich die Sache einmal, selbst gegen die mir vor allen widerlichen *höflichen* Zweifel der Franzosen unwidersprechlich abzumachen"? To this Gauss replies: † "Ich glaube Ihnen schon einmal geschrieben zu haben, dass ich auf keinen Fall diese Stelle worin die Methode zum erstenmale öffentlich angedeutet ist releviren werde, auch nicht wünsche, dass einer meiner Freunde mit meiner Zustimmung es thue. Diess hiesse *anerkennen als bedürfe* meine Anzeige (Theoria Motus Corporum Coelestium) dass ich seit 1794 diese Methode vielfach gebraucht habe, einer Rechtfertigung, und dazu werde ich mich nie verstehen. Als Olbers attestirte dass ich ihm 1803 die ganze Methode mitgetheilt habe, war diess zwar gut gemeint: hätte er mich aber vorher gefragt, so würde ich es hautement gemissbilligt haben."

The grandeur and inflexibility of Gauss's nature is finely illustrated by this passage. In a later letter ‡ (1840) to Schumacher we find Gauss's own estimate of the importance of the discovery of least squares. He writes: "Sie wissen dass ich selbst auf das von mir seit 1794 gebrauchte Verfahren * * * niemals grossen Werth gelegt habe. Verstehen Sie mich recht; nicht in Beziehung auf den grossen Nutzen den sie leistet, der ist klar genug, aber *danach* taxire *ich* die Dinge nicht. Sondern deshalb oder in so fern legte ich nicht viel Werth darauf, als vom ersten Anfang an der Gedanke mir so

* P. 137.
† P. 138.
‡ P. 141.

natürlich, so äusserst nahe liegend schien dass ich nicht im Geringsten zweifelte, viele Personen, die mit Zahlenrechnung zu verkehren gehabt, müssten von selbst auf einen solchen Kunstgriff gekommen sein und ihn gebraucht haben ohne deswegen es der Mühe werth zu halten viel Aufhebens von einer so natürlichen Sache zu machen."

The precocious genius of Gauss will stand forth when we recall that he was but seventeen years old when he discovered his method of least squares.

Before passing on to the next section we must call attention to a curious mishap which has befallen the editors. Thirty of the hundred pages given to the section of analysis and theory of functions are occupied by a memoir entitled: "De integratione formulae differentialis $(1 + n \cos \phi)^{\nu} d\phi$." The manuscript of this memoir was discovered in 1893 and published by Schering with explanatory notes in the *Göttinger Nachrichten.* Fricke in a note * remarks: "Es fehlt in der Urschrift der Name des Verfassers und eben so auch eine Angabe über Ort, und Zeit der Abfassung. In dessen beseitigte der Vergleich sonstiger Manuscripte von Gauss mit der vorliegenden Handschrift jeden Zweifel welcher sonst vielleicht an Gauss' Autorschaft bestehen könnte." Fricke places it in the year 1795 or before, *i. e.*, before or at the beginning of Gauss's student days in Göttingen. It turns out now, however, that the memoir was not written by Gauss at all, but by a certain Thibaut, who presented it to the Academy of Göttingen in the year 1799.†

We come now to the most interesting section of the book, the Foundations of Geometry, or the Non-Euclidean Geometry. Here as in the case of the elliptic functions Gauss had explored far into a territory a quarter of a century or more in advance of his contemporaries and for years was in possession of epoch-making results. But he never published anything on this subject. In letters to a few of his intimate friends he would occasionally explain some of his views or indicate in a fragmentary manner some of his results. Also on several occasions he reviewed in the *Göttinger Anzeigen* works touching on the theory of parallels. All that is known of Gauss's views relating to the non-euclidean geometry has been collected and put in this section.

* P. 64.

† F. Klein *Göttinger Nachrichten* (*Geschäftliche Mitth.*), 1902, p. 12.

Gauss's reason for not publishing his results will be made clear from the following extracts. In a letter * to his student friend, W. Bolyai (1799), he writes: "Mach' doch ja Deine Arbeit bald bekannt; gewiss wirst Du dafür den Dank nicht zwar des grossen Publicums (worunter auch mancher gehört, der für einen geschickten Mathematiker gehalten wird) einernten, denn ich überzeuge mich immer mehr dass die Zahl wahrer Geometer äusserst gering ist, und die meisten die Schwierigkeiten bei solchen Arbeiten weder beurtheilen noch selbst einmal sie verstehen können — aber gewiss den Dank aller derer, deren Urtheil Dir allein wirklich schätzbar sein kann."

In 1818 Gerling writes † Gauss that he intends to prepare a new edition of Lorenz's Reine Mathematik and wishes to know Gauss's opinion "wie es mit der Parallelentheorie wohl am besten zu halten ist." Gauss replies: ‡ "Ich freue mich, dass Sie den Muth haben sich so auszudrücken, als wenn Sie die Möglichkeit, dass unsere Parallelentheorie, mithin unsere ganze Geometrie, falsch wäre, anerkennten. Aber die Wespen, deren Nest Sie aufstören werden Ihnen um den Kopf fliegen."

In a letter to Bessel (1829) Gauss writes: § "Auch hier habe ich manches noch weiter consolidirt und meine Überzeugung dass wir die Geometrie nicht vollständig a priori begründen können, ist, wo möglich, noch fester geworden. Inzwischen werde ich wohl noch lange nicht dazu kommen meine *sehr ausgedehnten* Untersuchungen darüber zur öffentlichen Bekanntmachung auszuarbeiten, und vielleicht wird dies auch bei meinen Lebzeiten nie geschehen, da ich das Geschrei der Böotier scheue, wenn ich meine Ansicht *ganz* aussprechen wollte."

The wasps and the Bœotians that Gauss here refers to are probably not so much the mathematicians as the philosophers. As will be recalled, Kant in his Kritik der reinen Vernunft (1781) had endeavored to show that our notions of space were *essentially* à priori, *i. e.*, independent of all experience. In fact his demonstration of the possibility of synthetic judgments à priori, one of the corner stones of his philosophy,|| depends largely on this view of space.

* P. 159.
† P. 178.
‡ P. 179.
§ P. 200.
|| "Auf der Auflösung dieser Aufgabe oder einem genugthuenden Beweise, dass die Möglichkeit die sie erklärt zu wissen verlangt in der That gar nicht stattfinden, beruht nun das Stehen und Fallen der Metaphysik." Kritik der reinen Vernunft, ed. B. Erdmann (1884), p. 41.

Gauss's opinion regarding the à priori character of our notions of space is perhaps seen best from the following extract of a letter to Bessel (1830). He writes :* "Wahre Freude hat mir die Leichtigkeit gemacht, mit der Sie in meine Ansichten über die Geometrie eingegangen sind, zumal da so wenige offenen Sinn dafür haben. Nach meiner innigsten Überzeugung hat die Raumlehre in unserem Wissen a priori eine ganz andere Stellung wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Notwendigkeit (also auch von ihrer absoluten Wahrheit) ab die der letzern eigen ist; wir müssen in Demuth zugeben dass, wenn die Zahl *bloss* unsers Geistes Product ist, der Raum auch ausser unserm Geiste eine Realität hat der wir à priori ihre Gesetze nicht vollständig vorschreiben können."

His scorn for the philosophers is manifested in a letter to Taurinus (1824); he writes,† speaking of some of the peculiarities of the non-euclidean geometry: "Aber mir deucht, wir wissen, trotz der nichtssagenden Wort-Weisheit der Metaphysiker eigentlich zu wenig oder gar nichts über das wahre Wesen des Raumes, als dass wir etwas uns unnatürlich vorkommendes mit *Absolut Unmöglich* verwechseln dürfen."

Again in a letter to Schumacher (1846), speaking of the notion of right and left, he writes : ‡ "Welche Geltung diese Sache in der Metaphysik hat, und dass ich darin die schlagende Widerlegung von Kants Einbildung finde, der Raum sei *bloss* die Form unserer äussern Anschauung habe ich succinct in den Göttingischen Gelehrten Anzeigen 1831, S. 635 ausgesprochen."

We have expressed our astonishment at Gauss's reluctance to prepare for publication the results of his speculations. We now wish to cite a couple of passages which exhibit another trait of his character, the utter lack of envy he displays when his results have been discovered by others and the carelessness he constantly manifests in establishing his rights of priority.

In a letter (1804) to W. Bolyai,§ speaking of his and Bolyai's investigations in non-euclidean geometry, he writes: "Ich habe zwar noch immer die Hoffnung dass jene Klippen einst und noch vor meinem Ende, eine Durchfart erlauben werden. Indess habe ich jetzt so manche andere Beschäftigungen vor

* P. 201.
† P. 187.
‡ P. 247.
§ P. 160.

der Hand dass ich gegenwärtig daran nicht denken kann, und glaube mir es soll mich herzlich freuen wenn Du mir zuvorkommst, und es Dir gelingt alle Hindernisse zu übersteigen. Ich würde dann mit der innigsten Freude alles thun, um Dein Verdienst gelten zu machen und ins Licht zu stellen so viel in meinen Kräften steht."

Again on the same subject in a letter to Gerling (1819) he writes: * "Die Notiz von Herrn Professor Schweikart hat mir ungemein viel Vergnügen gemacht und ich bitte ihm darüber von mir recht viel Schönes zu sagen. Es ist mir fast alles aus der Seele geschrieben."

Finally, the following extract † from a letter to W. Bolyai (1830): "Jetzt Einiges über die Arbeit Deines Sohnes. Wenn ich damit anfange '*dass ich solche nicht loben darf*': so wirst Du wohl einen Augenblick stutzen: aber ich kann nicht anders; sie loben hiesse mich selbst loben: denn der ganze Inhalt, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen er geführt ist, kommen fast durchgehends mit meinen eigenen, zum Theile seit 30–35 Jahren angestellten Meditationen überein. In der That bin ich dadurch auf das Aeusserste überrascht. Mein Vorsatz war von meiner eigenen Arbeit, von der übrigens bis jetzt wenig zu Papier gebracht war, bei meinen Lebzeiten gar nichts bekannt werden zu lassen. Die meisten Menschen haben gar nicht den rechten Sinn für das, worauf es dabei ankommt, und ich habe nur wenige Menschen gefunden, die das, was ich ihnen mittheilte, mit besonderm Interesse aufnahmen. Um das zu können, muss man erst recht lebendig gefühlt haben, was eigentlich fehlt, und darüber sind die meisten Menschen ganz unklar. Dagegen war meine Absicht, mit der Zeit alles so zu Papier zu bringen, dass es wenigstens mit mir dereinst nicht unterginge. Sehr bin ich also überrascht, dass diese Bemühung mir nun erspart werden kann, und höchst erfreulich ist es mir dass gerade der Sohn meines alten Freundes es ist der mir auf eine so merkwürdige Art zuvorgekommen ist."

Before leaving this section we must mention two fragments which give rise to bold surmises. We recall that the advance that Riemann made in his Habilitationsschrift (1854) "Über die Hypothesen welche der Geometrie zu Grunde liegen" ‡ was

* P. 181.
† P. 220.
‡ Riemann's Werke, p. 254 seq.

due in part to his introduction of differential geometry into his investigations, whereas the Bolyais, Lobachevsky and Gauss (certainly in his early work) employed the synthetic methods of Euclid.

Now in one of the above mentioned fragments Gauss treats this problem: Assuming that the geometry of Euclid is valid for infinitely small triangles, determine the relations between the sides and the angles of finite triangles. This leads to a differential equation whose constant of integration Gauss represents by k, which is precisely the letter he employs in his Disquisitiones generales circa superficies curvas to represent the measure of curvature. In the second fragment we have a deduction of the equations of the pseudosphere which Gauss calls "das Gegenstück der Kugel." There is in these two fragments no direct evidence that Gauss saw that his non-euclidean geometry found in the pseudosphere a geometric substratum, nor that he saw the intimate relation between the non-euclidean geometry and his theory of developable surfaces; but the supposition lies very near and is not improbable.

We pass over the intermediate sections in order to employ the remaining space at our disposal to touch on a topic of great interest to the English-speaking race, viz., Gauss's relation to linear associative algebra in general and to quaternions in particular.

We are all familiar with Hamilton's efforts, extending over a period of fifteen years, to discover a system of numbers in three units which would bear the same relation to space as the ordinary complex numbers $x + iy$ do to the plane, and which at the same time obey the formal laws of ordinary algebra. Finally in 1843 he discovered the quaternions. Meantime, Gauss had considered the same problem. In his report in the *Göttinger Anzeigen* (1831) of his memoirs on biquadratic residues which we have already referred to, Gauss after discussing the nature of the complex numbers $x + iy$ in a masterly manner of closes his remarks with the following oracular statemen:* "Wir haben geglaubt, den Freunden der Mathematik durch diese kurze Darstellung der Hauptmomente einer neuen Theorie der sogenannten imaginären Grössen einen Dienst zu erweisen. Hat man diesen Gegenstand bisher aus einem falschen Gesichtspunkt betrachtet, und eine geheimnissvolle Dunkelheit dabei gefunden, so ist diess grössentheils den wenig

* Volume II, pp. 177.

schicklichen Benennungen zuzuschreiben. * * * Der Verfasser hat sich vorbehalten den Gegenstand welcher in der vorliegenden Abhandlung eigentlich nur gelegentlich berührt ist, künftig vollständiger zu bearbeiten, wo dann auch die Frage warum die Relationen zwischen Dingen die eine Mannigfaltigkeit von mehr als zwei Dimensionen darbieten, nicht noch andere in der allgemein Arithmetik zulässige Arten von Grössen liefern können, ihre Beantwortung finden wird."

From this passage we see that Gauss already in 1831 had recognized the non-existence of a number system such as Hamilton so long vainly sought to find. The question at once suggests itself: If Gauss was able to state a theorem of such fundamental importance in the theory of higher complex number systems, how far did his researches reach, what was their nature? The new volume of the Werke, while it still leaves us entirely in the dark on this subject, brings however one great surprise. In a fragment * entitled Transformationen des Raumes we find the multiplication table of quaternions. This note is dated by Stäckel, 1819. That the multiplication is not commutative is noted page 360 and the relation between quaternions and spherical triangles is noted pages 360–361. To prevent a possible disappointment to any one referring to these passages we observe that Gauss does not employ the usual quaternion notation $a + bi + cj + dk$ but the earlier notation of Hamilton (a, b, c, d). The introduction of certain symbols for the units (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) is entirely unnecessary and is made merely for convenience. From a theoretical standpoint the subject even wins in clearness by leaving them out at the start.

Apropos of quaternions we cannot resist the temptation to cite a passage from a letter † to Schumacher (1843) relating to Möbius's Barycentric Calculus. What Gauss here says of the barycentric calculus can be applied directly to quaternions and the vector analysis, a subject dear to many English and American mathematicians. After mentioning the prejudice he first had toward Möbius's work and his doubt "ob es der Mühe werth sei, eine recht artig ausgesonnene Rechnungsweise sich anzueignen, wenn man durch dieselbe nichts leisten könne was sich nicht eben so leicht ohne sie leisten lasse," follows this remarkable passage which Hamilton and Gibbs would

* P. 358.
† P. 297.

doubtless subscribe to most cordially: "Ueberhaupt verhält es sich mit allen solchen neuen Calculs so, dass man durch sie nichts leisten kann, was nicht ohne sie zu leisten wäre; der Vortheil ist aber der, dass, wenn ein solcher Calcul dem innersten Wesen vielfach vorkommender Bedürfnisse correspondirt, jeder, der sich ihn ganz angeeignet hat, auch ohne die gleichsam unbewussten Inspirationen des Genies die niemand erzwingen kann, die dahin gehörigen Aufgaben lösen, ja selbst in so verwickelten Fällen gleichsam mechanisch lösen kann, wo ohne eine solche Hülfe auch das Genie ohnmächtig wird. * * * Es werden durch solche Conceptionen unzählige Aufgaben, die sonst vereinzelt stehen, und jedesmal neue Efforts (kleinere oder grössere) des Erfindungsgeistes erfordern, gleichsam zu einem organischen Reiche."

In closing, we wish to praise the care the editors have taken in preparing the contents of the present volume for publication. As far as possible the dates of the various papers have been ascertained. Obscure passages have been explained and errors either corrected or pointed out. There are instances where the editors have been too sparing with their commentary, but they have doubtless preferred to err on the side of brevity rather than diffuseness.

Only one oversight has been noticed by the reviewer. In the extract printed on page 247, Gauss makes reference to his "schlagende Widerlegung" of Kant's theory of space which appeared in the *Göttinger Anzeigen* for 1831. This is certainly a matter of some importance and a reference should have been given where readers who have not the files of the *Anzeigen* at hand — and their number is relatively small — can find it in the Werke. The passage Gauss has in mind is to be found in volume 2, page 177 and footnote of the Werke. In this connection we wish to express the hope that further volumes will give *exact* references to the places where each particular paper was first published. This is only partially done in the volumes which have so far appeared. Indeed it seems desirable that this deficiency in the early volumes be made good in the supplementary volume.

In regard to misprints, we have observed but one, viz., page 160, last line: omit one *herzlich*.

To sum up, we may say that the volume under review takes its place worthily besides the six preceding volumes which enjoyed the loving attention of one who sacrificed a large part of his scientific activity in erecting a lasting monument to the memory of Germany's princeps mathematicorum.

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ANALYTIC PROJECTIVE GEOMETRY.

Lehrbuch der analytischen Geometrie in homogenen Koordinaten. Von WILHELM KILLING. 2 Teile, 8vo. Paderborn, F. Schöningh. Teil 1: *Die ebene Geometrie,* 1900. xiii + 220 pp. Teil 2: *Die Geometrie des Raumes,* 1901. viii + 361 pp.

THE first and perhaps the most noticeable feature of Professor Killing's text-book on the analytic geometry of homogeneous coördinates is the elaborately methodical arrangement. The line of march has been laid out with extreme care. Notice, for example, the titles and page numbers of the first few sections of the two volumes:

In the plane, Vol. 1.	In space, Vol. 2.
§ 1. Theory of cross ratio, p. 1.	§ 1. Division of dihedral angles, p. 1.
§ 2. The coördinate triangle, p. 8.	§ 2. The coördinate tetrahedron, p. 7.
§ 3. The straight line, p. 13.	§ 3. The straight line and plane, p. 13.
§ 4. The perpendiculars dropped upon a straight line from the vertices of the coördinate triangle, p. 19.	§ 4. The perpendiculars dropped upon a plane from the vertices of the coördinate tetrahedron, p. 21.
§ 5. The most general trimetric coördinates, p. 25.	§ 5. The most general tetrahedral coördinates, p. 29.
§ 6. The ratios of the coördinates as cross ratios, p. 37.	§ 6. The elements at infinity, p. 38.
§ 8. The elements at infinity, p. 48.	§ 7. The quotients of the coördinates as cross ratios, p. 44.
Total number of pages, 49.	Total number of pages, 52.