NOTE ON THE PROJECTIONS OF THE ABSOLUTE ACCELERATION IN RELATIVE MOTION.

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INASMUCH as the usual methods of forming the projections of the absolute acceleration on the moving axes rest to a certain extent on geometric considerations it has seemed worth while to treat the question from a purely analytic point of view and derive these expressions by purely algebraic processes. This method possesses the advantage of being immediately extensible to any number of dimensions, and in the latter part of this note I shall give the expressions for n dimensions.

Connecting the moving axes Ox, Oy, Oz with the fixed axes by the following schema

	x	y	z
X	a	b	c
Y	a'	b'	c'
Z	a''	b''	c'',

I shall rapidly recall the following equations :

(1) $$X = Sax, \quad Y = Sa'x, \quad Z = Sa''x,$$

where S denotes summation with respect to a, b, c and x, y, z.

(2)
$$
\begin{cases}
\dfrac{dX}{dt} = Sa\,\dfrac{dx}{dt} + Sx\,\dfrac{da}{dt}, \\[2ex]
\dfrac{dY}{dt} = Sa'\,\dfrac{dx}{dt} + Sx\,\dfrac{da'}{dt}, \\[2ex]
\dfrac{dZ}{dt} = Sa''\,\dfrac{dx}{dt} + Sx\,\dfrac{da''}{dt};
\end{cases}
$$

$$(3) \quad \begin{cases} \dfrac{d^2X}{dt^2} = Sa\dfrac{d^2x}{dt^2} + Sx\dfrac{d^2a}{dt^2} + 2S\dfrac{dx}{dt}\dfrac{da}{dt}, \\[3mm] \dfrac{d^2Y}{dt^2} = Sa'\dfrac{d^2x}{dt^2} + Sx\dfrac{d^2a'}{dt^2} + 2S\dfrac{dx}{dt}\dfrac{da'}{dt}, \\[3mm] \dfrac{d^2Z}{dt^2} = Sa''\dfrac{d^2x}{dt^2} + Sx\dfrac{d^2a''}{dt^2} + 2S\dfrac{dx}{dt}\dfrac{da''}{dt}; \end{cases}$$

$$(4) \quad \begin{cases} p = \sum c\dfrac{db}{dt} = -\sum b\dfrac{dc}{dt}, \\[3mm] q = \sum a\dfrac{dc}{dt} = -\sum c\dfrac{da}{dt}, \\[3mm] r = \sum b\dfrac{da}{dt} = -\sum a\dfrac{db}{dt}. \end{cases}$$

The projections of the absolute velocity on the moving axes are then

$$(4) \quad \begin{cases} V_x = \sum a\dfrac{dX}{dt} = \dfrac{dx}{dt} + qz - ry, \\[3mm] V_y = \sum b\dfrac{dX}{dt} = \dfrac{dy}{dt} + rx - pz, \\[3mm] V_z = \sum c\dfrac{dX}{dt} = \dfrac{dz}{dt} + py - qx. \end{cases}$$

The direction cosines satisfy the following equations

$$(5) \quad \frac{da}{dt} = br - cq, \quad \frac{db}{dt} = cp - ar, \quad \frac{dc}{dt} = aq - bp,$$

with two similar sets in a', b', c', and a'', b'', c'', which I shall denote by $(5)'$ and $(5)''$.

The projections of the absolute acceleration on the moving axes are

$$(6) \quad \begin{cases} J_x = \sum a\dfrac{d^2X}{dt^2} = \sum a\,Sa\dfrac{d^2x}{dt^2} + \sum a\,Sx\dfrac{d^2a}{dt^2} + 2\sum aS\dfrac{dx}{dt}\dfrac{da}{dt}, \\[3mm] J_y = \sum b\dfrac{d^2X}{dt^2} = \cdots, \\[3mm] J_z = \sum c\dfrac{d^2X}{dt^2} = \cdots. \end{cases}$$

Calculating $\sum aSa \dfrac{d^2x}{dt^2}$ and remembering the relations between the direction cosines we have at once

(7) $$\sum aSa \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2}.$$

Again

$$\sum aSx \frac{d^2a}{dt^2} = x \sum a \frac{d^2a}{dt^2} + y \sum a \frac{d^2b}{dt^2} + z \sum a \frac{d^2c}{dt^2}.$$

Differentiating (5) (5)′ and (5)″ and calculating

$$\sum a \frac{d^2a}{dt^2}, \quad \sum a \frac{d^2b}{dt^2}, \quad \sum a \frac{d^2c}{dt^2},$$

we have

(8) $$\begin{cases} \sum a \dfrac{d^2a}{dt^2} = -(q^2 + r^2), \\[2mm] \sum a \dfrac{d^2b}{dt^2} = -\dfrac{dr}{dt} + pq, \\[2mm] \sum a \dfrac{d^2c}{dt^2} = \dfrac{dq}{dt} + pr. \end{cases}$$

Again

(9) $$\sum aS \frac{dx}{dt} \frac{da}{dt} = \frac{dx}{dt} \sum a \frac{da}{dt} + \frac{dy}{dt} \sum a \frac{db}{dt} + \frac{dz}{dt} \sum a \frac{dc}{dt}$$

$$= -r \frac{dy}{dt} + q \frac{dz}{dt}.$$

Hence

(10) $$J_x = \frac{d^2x}{dt^2} - (q^2 + r^2)x + \left(pq - \frac{dr}{dt}\right)y + \left(pr + \frac{dq}{dt}\right)z$$

$$- 12\left(r \frac{dy}{dt} - q \frac{dz}{dt}\right).^*$$

Now (10) may be written

(11) $$J_x = \frac{d}{dt}\left(\frac{dx}{dt} + qz - ry\right) + q\left(\frac{dz}{dt} + py - qx\right)$$

$$- r\left(\frac{dy}{dt} + rx - pz\right).$$

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* This formula appears to have been first given by J. Liouville, *Jour. de Math. pures et appl.*, series 2, vol. 3 (1858), pp. 10–11.

Hence we have for the projections the ordinary expressions

$$(11) \quad \begin{cases} J_x = \dfrac{dV_x}{dt} + q\,V_z - r\,V_y, \\[2ex] J_y = \dfrac{dV_y}{dt} + r\,V_x - p\,V_z, \\[2ex] J_z = \dfrac{dV_z}{dt} + p\,V_y - q\,V_x. \end{cases}$$

To extend this method to n dimensions I shall connect the moving axes $Ox_1,\ Ox_2\ \cdots,\ Ox_n$ with the fixed axes OX_α, $OX_\beta,\ \cdots,\ OX_\nu$ by the following table :

	x_1	x_2	$\cdots$	x_n
X_α	a_1	a_2	$\cdots$	a_n
X_β	β_1	β_2	$\cdots$	β_n
$\vdots$	$\vdots$	$\vdots$	$\vdots\,\vdots$	$\vdots$
X_ν	ν_1	ν_2	$\cdots$	ν_n

Then

$$(12) \quad X_\lambda = \sum_{i=1}^{n} \lambda_i x_i,$$

$$(13) \quad \frac{dX_\lambda}{dt} = \sum_{i=1}^{n} \lambda_i \frac{dx_i}{dt} + \sum_{i=1}^{n} x_i \frac{d\lambda_i}{dt},$$

$$(14) \quad \frac{d^2 X_\lambda}{dt^2} = \sum_{i=1}^{n} \lambda_i \frac{d^2 x_i}{dt^2} + \sum_{i=1}^{n} x_i \frac{d^2 \lambda_i}{dt^2} + 2\sum_{1=i}^{n} \frac{dx_i}{dt}\,\frac{d\lambda_i}{dt},$$

$$(15) \quad V_{x_\tau} = Sa_\tau \frac{dX_\alpha}{dt} = Sa_\tau \sum_{i=1}^{n} a_i \frac{dx_i}{dt} + Sa_\tau \sum_{i=1}^{n} x_i \frac{da_i}{dt}$$

$$= \sum_{i=1}^{n} Sa_\tau a_i \frac{dx_i}{dt} + \sum_{i=1}^{n} Sa_\tau \frac{da_i}{dt} x_i$$

$$= \frac{dx_\tau}{dt} + \sum_{i=1}^{n} x_i Sa_\tau \frac{da_i}{dt} = \frac{dx_\tau}{dt} + \sum_{i=1}^{n} x_i p_{i\tau},$$

where

$$(16) \qquad\qquad p_{i\tau} \equiv Sa_\tau \frac{da_i}{dt} = -Sa_i \frac{da_\tau}{dt},$$

$$(17) \quad J_{x_\tau} = Sa_\tau \frac{d^2 X_a}{dt^2} = Sa_\tau \sum_{i=1}^n a_i \frac{d^2 x_i}{dt^2} + Sa_\tau \sum_{i=1}^n x_i \frac{d^2 a_i}{dt^2}$$

$$+ 2Sa_\tau \sum_{i=1}^n \frac{dx_i}{dt} \frac{da_i}{dt}$$

$$= \sum_{i=1}^n \frac{d^2 x_i}{dt^2} Sa_\tau a_i + \sum_{i=1}^n x_i Sa_\tau \frac{d^2 a_i}{dt^2} + 2 \sum_{i=1}^n \frac{dx_i}{dt} Sa_\tau \frac{da_i}{dt}.$$

The differential equation satisfied by the direction cosine λ_i is found by expressing the fact that the point at unit distance on the axis OX_λ has a velocity zero along the moving axes. Hence

$$(18) \qquad\qquad \frac{d\lambda_i}{dt} + \sum_{j=1}^n \lambda_j p_{ji} = 0.$$

From (18)

$$(19) \qquad\qquad \frac{d^2\lambda_i}{dt^2} = \sum_{j=1}^n \lambda_j \frac{dp_{ij}}{dt} + \sum_{j=1}^n \frac{d\lambda_j}{dt} p_{ij}.$$

Whence

$$(20) \quad Sa_\tau \frac{d^2 a_i}{dt^2} = \sum_{j=1}^n \frac{dp_{ij}}{dt} Sa_\tau a_j + \sum_{j=1}^n p_{ij} Sa_\tau \frac{da_i}{dt} = \frac{dp_{i\tau}}{dt} + \sum_{j=1}^n p_{ij} p_{j\tau}.$$

Hence

$$(21) \quad J_{x_\tau} = \frac{d^2 x_\tau}{dt^2} + \sum_{i=1}^n x_i \left(\frac{dp_{i\tau}}{dt} + \sum_{j=1}^n p_{ij} p_{j\tau} \right) + 2 \sum_{i=1}^n \frac{dx_i}{dt} p_{i\tau}$$

$$= \frac{d}{dt} \left(\frac{dx_\tau}{dt} + \sum_{i=1}^n x_i p_{i\tau} \right) + \sum_{i=1}^n \left(\frac{dx_i}{dt} + \sum_{j=1}^n x_j p_{ji} \right) p_{i\tau}$$

$$= \frac{d}{dt} V_{x\tau} + \sum_{i=1}^n V_{x\tau} p_{i\tau} \qquad (\tau = 1, 2 \cdots, n).$$

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