is even (2^{α_0}) *then the subgroup must involve operators of order* 4 *and* $\alpha_0 > 3$. Since any number of these factors may be non-abelian, there cannot be an upper limit to the number of non-abelian groups which may be conformal with one abelian group. This fact may be seen in many other ways.

STANFORD UNIVERSITY,
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THE INFINITESIMAL GENERATORS OF CERTAIN PARAMETER GROUPS.

BY DR. S. E. SLOCUM.

(Read before the American Mathematical Society, October 26, 1901.)

BY means of the r independent infinitesimal transformations

$$X_j \equiv \sum_1^n{}_k\, \xi_{jk}(x_1, \cdots, x_n) \frac{\partial}{\partial x_k} \qquad (j = 1, 2, \cdots, r)$$

we may construct a family of transformations

$$(1) \qquad x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

with r essential parameters $a_1, \cdots, a_r$, where $f_i(x, a)$ is defined in the neighborhood of the identical transformation by the series

$$f_i(x, a) \equiv x_i + \sum_1^r{}_j\, a_j X_j x_i + \frac{1}{2\,!} \sum_1^r{}_j \sum_1^r{}_k X_j X_k x_i + \cdots$$
$$(i = 1, 2, \cdots, n).$$

The transformations defined by these equations for assigned values of the a's may be denoted by T_a. Let the differential operators X_j $(j = 1, 2, \cdots, r)$ satisfy Lie's criterion, that is, let

$$X_j X_k - X_k X_j \equiv \sum_1^r{}_s\, c_{jks} X_s \quad (j, k = 1, 2, \cdots, r).$$

Then by Lie's chief theorem, the family of transformations T_a, defined by equations (1), forms a group G.* Conse-

* Continuierliche Gruppen, pp. 390–391.

quently the transformation obtained by the successive application to the manifold $x_1, \cdots, x_n$, in the order named, of the transformations denoted by T_a and T_b respectively, with parameters a and b, will be a transformation of the group G, say T_c; that is to say, we shall have

$$T_b T_a = T_c,$$

where

$$c_k = \varphi_k(a_1, \cdots, a_r, b_1, \cdots, b_r) \quad (k = 1, 2, \cdots, r).$$

If this system of equations is written in the form

$$a_k' = \varphi_k(a_1, \cdots, a_r, \alpha_1, \cdots, \alpha_r) \quad (k = 1, 2, \cdots, r),$$

it can be shown that they define an r-parameter group in the variables a and a', with continuous parameters $\alpha_1, \cdots, \alpha_r$, and also that each transformation of the group is generated by an infinitesimal transformation of the group. The group thus defined is termed the *parameter group* of the given group G.*

On pages 97–103 of the *Proceedings of the American Academy of Arts and Sciences*, volume 36, I have shown that the symbols of the infinitesimal transformations which generate the parameter groups are the same for all groups of the same structure; and in the same pages I have also given a method by which these symbols may be obtained from the structural constants belonging to any given structure. The following is a résumé of the method given in that paper.

Let α denote the differential operator $\alpha = \sum_1^r{}_k a_k X_k$. Then equations (1) may be written in the symbolic form

$$x_i' = e^\alpha x_i \qquad (i = 1, 2, \cdots, n),$$

where e^α denotes the operator

$$e^\alpha f = f + \alpha f + \frac{\alpha^2}{2\,!} f + \frac{\alpha^3}{3\,!} f + \cdots$$

and $\alpha^{m+1} f = \alpha(\alpha^m f)$. By making the parameters in equations (1) infinitesimal we obtain an infinitesimal transformation of the family, that is to say, a transformation infinitely near the identical transformation. Let δt denote an infinitesimal, and let γ denote the operator $\gamma = \sum_1^r{}_k c_k X_k$, the

* Transformationsgruppen, vol. 1, pp. 401 *et seq.*

c's being arbitrary parameters. Then the transformation $e^{\alpha+\delta t\gamma}$ is infinitely near the transformation e^{α}. Consequently the transformation obtained by the successive application to the manifold $x_1, \cdots, x_n$, in the order named, of the transformation $e^{-\alpha}$, inverse to e^{α}, and the transformation $e^{\alpha+\delta t\gamma}$ is an infinitesimal transformation. If we denote its parameters by $\delta t b_1, \cdots, \delta t b_r$ and let $\beta = \sum_1^r {}_k b_k X_k$, we have

$$(2) \qquad e^{-\alpha} e^{\alpha+\delta t\gamma} = e^{\delta t\beta},$$

whence

$$1+\delta t\left\{\gamma - \frac{1}{2!}(\alpha, \gamma) + \frac{1}{3!}[\alpha, (\alpha, \gamma)] - \frac{1}{4!}\{\alpha, [\alpha, (\alpha, \gamma)]\} + \cdots\right\}$$
$$+ \cdots\cdots = 1 + \delta t\beta + \cdots,$$

where (α, γ) denotes the alternant $\alpha\gamma - \gamma\alpha$. Equating coefficients of δt,

$$(3) \quad \beta = \gamma - \frac{1}{2!}(\alpha, \gamma) + \frac{1}{3!}[\alpha, (\alpha, \gamma)] - \frac{1}{4!}\{\alpha, [\alpha, (\alpha, \gamma)]\} + \cdots,$$

whence

$$(4) \qquad b_k = \sum_1^r {}_j P_{kj} c_j \qquad (j = 1, 2, \cdots, r),$$

the P's being power series in $a_1, \cdots, a_r$, convergent for all finite values of $a_1, \cdots, a_r$.

Let Δ denote the determinant of the P's. Then if $\Delta \neq 0$ the a's and b's may be taken arbitrarily and the c's determined by means of equations (4), in which case

$$(5) \qquad c_j = \sum_1^r {}_k \frac{Q_{jk}}{\Delta} b_k \qquad (j = 1, 2, \cdots, r)$$

where Q_{jk} denotes the first minor of Δ relative to P_{kj}. Let the transformation $e^{\alpha+\delta t\gamma}$ be denoted by e^{α_1}, where

$$\alpha_1 = \sum_1^r {}_k a_k^{(1)} X_k,$$

the $a_k^{(1)}$ $(k = 1, 2, \cdots, r)$ being arbitrary parameters. Then

$$(6) \qquad e^{\alpha_1} = e^{\alpha+\delta t\gamma},$$

whence

$$(7) \qquad a_k^{(1)} = a_k + \delta t c_k \qquad (k = 1, 2, \cdots, r).$$

This system of equations defines the infinitesimal transformations which generate the parameter group ; but these are also defined by the equations

$$(8)\qquad a_k^{(1)} = a_k + \sum_1^r{}_j \xi_{kj}(a) b_j \delta t \quad (k = 1, 2, \cdots, r).$$

Consequently, from equations (7) and (8),

$$(9)\qquad c_k = \sum_1^r{}_j \xi_{kj}(a) b_j = \sum_1^r{}_j \frac{Q_{kj}}{\Delta} b_j.$$

Therefore if $A_1, \cdots, A_r$ denote the symbols of the infinitesimal transformations which generate the parameter group, we have

$$A_j = \sum_1^r{}_k \xi_{jk}(a) \frac{\partial}{\partial a_k} = \sum_1^r{}_k \frac{Q_{jk}}{\Delta} \frac{\partial}{\partial a_k} \qquad (j = 1, 2, \cdots, r).$$

Since the form of Δ and of the Q's depends only on the structural constants, the symbols $A_j (j = 1, 2, \cdots, r)$ will be the same for all groups of the same structure.

To illustrate what precedes, consider the three-parameter structure

$$(X_1, X_2) \equiv 0, \ (X_1, X_3) \equiv X_1, \ (X_2, X_3) \equiv \beta X_2, \ (\beta \neq 0, 1).$$

Equation (3) gives

$$\begin{aligned} b_1X_1 + b_2X_2 + b_3X_3 &= c_1X_1 + c_2X_2 + c_3X_3 \\ &- \frac{1}{2!}\left\{(a_1c_3 - a_3c_1)X_1 + \beta(a_2c_3 - a_3c_2)X_2\right\} \\ &- \frac{1}{3!}\left\{a_3(a_1c_3 - a_3c_1)X_1 + a_3\beta^2(a_2c_3 - a_3c_2)X_2\right\} \\ &- \frac{1}{4!}\left\{a_3^2(a_1c_3 - a_3c_1)X_1 + a_3^2\beta^3(a_2c_3 - a_3c_2)X_2\right\} \\ &- \frac{1}{5!}\left\{a_3^3(a_1c_3 - a_3c_1)X_1 + a_3^3\beta^4(a_2c_3 - a_3c_2)X_2\right\} \\ &- \cdots \cdots, \end{aligned}$$

whence

$$(10)\qquad \begin{aligned} b_1 &= \frac{c_1}{a_3}(e^{a_3} - 1) - \frac{a_1c_3}{a_3^2}(e^{a_3} - a_3 - 1), \\ b_2 &= \frac{c_2}{a_3\beta}(e^{a_3\beta} - 1) - \frac{a_2c_3}{a_3^2\beta}(e^{a_3\beta} - a_3\beta - 1), \\ b_3 &= c_3. \end{aligned}$$

Consequently,

$$\varDelta = \begin{vmatrix} \dfrac{e^{a_3}-1}{a_3} & 0 & -\dfrac{a_1}{a_3^2}(e^{a_3}-a_3-1) \\ 0 & \dfrac{e^{a_3\beta}-1}{a_3\beta} & -\dfrac{a_2}{a_3^2\beta}(e^{a_3\beta}-a_3\beta-1) \\ 0 & 0 & 1 \end{vmatrix}$$

and equations (10) give

$$c_1 = \frac{a_3}{e^{a_3}-1}\left\{ b_1 + \frac{a_1}{a_3^2}(e^{a_3}-a_3-1)b_3 \right\} \equiv \sum_1^r {}_j\, \xi_{1j}(a)b_j,$$

$$c_2 = \frac{a_3\beta}{e^{a_3\beta}-1}\left\{ b_2 + \frac{a_2}{a_3^2\beta}(e^{a_3\beta}-a_3\beta-1)b_3 \right\} \equiv \sum_1^r {}_j\, \xi_{2j}(a)b_j,$$

$$c_3 = b_3 \equiv \sum_1^r {}_j\, \xi_{3j}(a)b_j.$$

Therefore the symbols of the infinitesimal transformations which generate the parameter group corresponding to the above structure are

$$A_1 \equiv \frac{a_3}{e^{a_3}-1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_3\beta}{e^{a_3\beta}-1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3}-a_3-1)}{a_3(e^{a_3}-1)}\frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3\beta}-a_3\beta-1)}{a_3(e^{a_3\beta}-1)}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

In the following table are enumerated all possible types of structure of two-, three- and four-parameter complex groups as given by Lie,* and under each structure are given the symbols of the infinitesimal transformations which generate the parameter group corresponding to that structure, obtained by the method explained above.

Groups With Two Parameters.

Type I. $(X_1, X_2) \equiv X_1.$

The symbols of the infinitesimal transformations which generate the parameter group corresponding to this structure are

$$A_1 \equiv \frac{a_2}{e^{a_2}-1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_1(e^{a_2}-a_2-1)}{a_2(e^{a_2}-1)}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2}.$$

* Continuierliche Gruppen, pp. 565, 571, 574–589 ; Transformationsgruppen, vol. 3, pp. 713, 716, 723–730.

Type II. $(X_1, X_2) \equiv 0.$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2}.$$

Groups With Three Parameters.

Type I. $(X_1, X_2) \equiv X_1, \quad (X_1, X_3) \equiv 2X_2, \quad (X_2, X_3) \equiv X_3,$

$$A_1 \equiv \left\{ -\frac{a_2}{2} + \frac{\varphi(e^{\phi} - e^{-\phi})}{2(e^{\phi} + e^{-\phi} - 2)} + \frac{a_1 a_3}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_1} + \left\{ -a_3 + \frac{a_2 a_3}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_2} + \frac{a_3^2}{\varphi^2}\psi \frac{\partial}{\partial a_3},$$

$$A_2 \equiv \left\{ \frac{a_1}{2} - \frac{a_1 a_2}{2\varphi^2}\psi \right\} \frac{\partial}{\partial a_1} + \left\{ \frac{a_2^2}{\varphi^2} - \frac{2a_1 a_3 (e^{\phi} - e^{-\phi})}{\varphi(e^{\phi} + e^{-\phi} - 2)} \right\} \frac{\partial}{\partial a_2} + \left\{ -\frac{a_3}{2} - \frac{a_2 a_3}{2\varphi^2}\psi \right\} \frac{\partial}{\partial a_3},$$

$$A_3 \equiv \frac{a_1^2}{\varphi^2}\psi \frac{\partial}{\partial a_1} + \left\{ a_1 + \frac{a_1 a_2}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_2} + \left\{ \frac{a_2}{2} + \frac{\varphi(e^{\phi} - e^{-\phi})}{2(e^{\phi} + e^{-\phi} - 2)} + \frac{a_1 a_3}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_3},$$

where

$$\varphi \equiv \sqrt{a_2^2 - 4a_1 a_3}, \qquad \psi \equiv \frac{e^{\phi}(\varphi - 2) - e^{-\phi}(\varphi + 2) + 4}{e^{\phi} + e^{-\phi} - 2}.$$

Type II. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv \beta X_2,$ $(\beta \neq 0, 1).$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_3 \beta}{e^{a_3 \beta} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(a^{a_3\beta} - a_3\beta - 1)}{a_3(e^{a_3\beta} - 1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

Type III. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv X_2.$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

Type IV. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv X_1 + X_2.$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1},$$

$$A_2 \equiv -\frac{a_3[e^{a_3}(a_3 - 1) + 1]}{(e^{a_3} - 1)^2} \frac{\partial}{\partial a_1} + \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \left\{ \frac{a_1(e^{a_3} - a_3 - 1) + (a_2 - e^{a_2})[e^{a_3}(1 - a_3) - 1]}{a_3(e^{a_3} - 1)} \right\} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

Type V. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv 0.$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3}.$$

Type VI. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1.$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv -\frac{a_3}{2} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2}{2} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3}.$$

Type VII. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv 0.$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{\partial}{\partial a_3}.$$

Groups With Four Parameters.

A. *Without three-parameter involution group.*

Type I.

$$(X_1, X_2) \equiv X_1, \quad (X_1, X_3) \equiv 2X_2, \quad (X_2, X_3) \equiv X_3,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0.$$

The symbols of the infinitesimal transformations which generate the parameter group corresponding to this structure are the same as those given under *Type I* of three-parameter structures with the addition of the symbol

$$A_4 \equiv \frac{\partial}{\partial a_4}.$$

Type II.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1,$$

$$(X_1, X_4) \equiv \beta X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv (\beta - 1)X_3,$$

$$(\beta \neq 1).$$

$$A_1 \equiv \frac{a_4\beta}{e^{a_4\beta} - 1}\frac{\partial}{\partial a_1},$$

$$A_2 \equiv \frac{a_3(\beta e^{a_4} - e^{a_4\beta} - \beta + 1)}{(e^{a_4\beta} - 1)(e^{a_4} - 1)(\beta - 1)}\frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2(1 - e^{a_4\beta} - \beta e^{-a_4})}{(e^{a_4\beta} - 1)(e^{a_4(\beta-1)} - 1)}\frac{\partial}{\partial a_1} + \frac{a_4(\beta - 1)}{e^{a_4(\beta-1)} - 1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{1}{a_4(e^{a_4\beta} - 1)}\{a_1(e^{a_4\beta} - a_4\beta - 1)$$

$$+ a_2a_3\beta[(1 - \beta)e^{a_4(\beta-1)} + (\beta - 2)e^{a_4\beta} + e^{a_4}]\}\frac{\partial}{\partial a_1}$$

$$+ \frac{a_2(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_2}$$

$$+ \frac{a_3(e^{a_4(\beta-1)} - a_4(\beta - 1) - 1)}{a_4(e^{a_4(\beta-1)} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type III.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1,$$

$$(X_1, X_4) \equiv 2X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv 2X_2 + X_3.$$

$$A_1 \equiv \frac{2a_4}{e^{2a_4} - 1}\frac{\partial}{\partial a_1},$$

$$A_2 \equiv -\frac{a_3(e^{2a_4} - 2e^{a_4} + 1)}{(e^{2a_4} - 1)(e^{a_4} - 1)}\frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \left\{\frac{4a_3[(a_4-1)e^{a_4}+1]+a_2(e^{2a_4}-2e^{a_4}+1)}{(e^{2a_4}-1)(e^{a_4}-1)}\right\}\frac{\partial}{\partial a_1} - \frac{2a_4(a_4e^{a_4}-e^{a_4}+1)}{(e^{a_4}-1)^2}\frac{\partial}{\partial a_2} + \frac{a_4}{e^{a_4}-1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \{e^{a_4}[(2-a_4)(e^{2a_4}+1)+2(a_4-2e^{a_4})]\}\frac{\partial}{\partial a_1} + \frac{2a_3(a_4e^{a_4}-e^{a_4}+1)}{e^{a_4}-1}\frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type IV.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1,$$
$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv 0.$$

$$A_1 \equiv \frac{a_4}{e^{a_4}-1}\frac{\partial}{\partial a_1},$$

$$A_2 \equiv -\frac{a_3\left[e^{a_4}(a_4-1)+\frac{a_4^2}{2}+1\right]}{(e^{a_4}-1)^2}\frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4}-1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \left\{\frac{a_1(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)} + \frac{a_2a_3}{e^{a_4}-1}\left[e^{a_4}(a_4-3)\left(e^{a_4}+\frac{a_4}{2}\right) + \frac{a_4+2}{2}\left(4e^{a_4}-1\right)\right]\right\}\frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_4}.$$

Type V.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_2,$$
$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0.$$

$$A_1 \equiv \frac{a_4}{e^{a_4}-1}\frac{\partial}{\partial a_1}, \quad A_2 \equiv \frac{a_3}{e^{a_3}-1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2(e^{a_3}-a_3-1)}{a_3(e^{a_3}-1)}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_4}.$$

B. With three-parameter involution group.

Type I.

$$(X_1,\, X_2) \equiv (X_2,\, X_3) \equiv (X_3,\, X_1) \equiv 0,$$

$$(X_1,\, X_4) \equiv \alpha X_1, \quad (X_2,\, X_4) \equiv \beta X_2, \quad (X_3,\, X_4) \equiv \gamma X_3,$$

$$(\alpha \neq \beta \neq \gamma).$$

$$A_1 \equiv \frac{a_4 \alpha}{e^{a_4\alpha} - 1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4 \beta}{e^{a_4\beta} - 1}\frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{a_4 \gamma}{e^{a_4\gamma} - 1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4\alpha} - a_4\alpha - 1)}{a_4(e^{a_4\alpha} - 1)}\frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4\beta} - a_4\beta - 1)}{a_4(e^{a_4\beta} - 1)}\frac{\partial}{\partial a_2}$$

$$+ \frac{a_3(e^{a_4\gamma} - a_4\gamma - 1)}{a_4(e^{a_4\gamma} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type II.

$$(X_1,\, X_2) \equiv (X_2,\, X_3) \equiv (X_3,\, X_1) \equiv 0,$$

$$(X_1,\, X_4) \equiv \alpha X_1, \quad (X_2,\, X_4) \equiv \beta X_2, \quad (X_3,\, X_4) \equiv X_2 + \beta X_3,$$

$$(\alpha \neq \beta).$$

$$A_1 \equiv \frac{a_4 \alpha}{e^{a_4\alpha} - 1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4 \beta}{e^{a_4\beta} - 1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_4[e^{a_4\beta}(a_4\beta - 1) + 1]}{(e^{a_4\beta} - 1)^2}\frac{\partial}{\partial a_2} + \frac{a_4\beta}{e^{a_4\beta} - 1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4\alpha} - a_4\alpha - 1)}{a_4(e^{a_4\alpha} - 1)}\frac{\partial}{\partial a_1}$$

$$+ \frac{1}{a_4\beta(e^{a_4\beta} - 1)}\Big\{a_3[e^{a_4\beta}(a_4\beta - 1) + 1](2e^{a_4\beta} - a_4\beta - 2)$$

$$+ a_2\beta(e^{a_4\beta} - a_4\beta - 1)\Big\}\frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4\beta} - a_4\beta - 1)}{a_4(e^{a_4\beta} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type III.

$$(X_1,\, X_2) \equiv (X_2,\, X_3) \equiv (X_3,\, X_1) \equiv 0,$$

$$(X_1,\, X_4) \equiv X_1, \quad (X_2,\, X_4) \equiv X_1 + X_2, \quad (X_3,\, X_4) \equiv X_2 + X_3.$$

$$A_1 \equiv \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_1},$$

$$A_2 \equiv \frac{a_4[e^{a_4}(a_4 - 1) + 1]}{(e^{a_4} - 1)^2}\frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_4^2 e^{a_4}}{(e^{a_4}-1)^3}\left[e^{a_4} - \frac{a_4}{2}(e^{a_4}+1) - 1\right]\frac{\partial}{\partial a_1}$$

$$- \frac{a_4[e^{a_4}(a_4-1)+1]}{(e^{a_4}-1)^2}\frac{\partial}{\partial a_2} + \frac{a_4}{e^{a_4}-1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{1}{a_4(e^{a_4}-1)}\left\{a_1(e^{a_4}-a_4-1)\right.$$

$$+ a_2(a_4 e^{a_4} - e^{a_4} + 1)(2e^{a_4} - a_4 - 2)$$

$$+ \frac{a_3}{e^{a_4}-1}\left[(a_4^2 e^{a_4} - 2a_4 e^{a_4} + 2e^{a_4} - 2)\frac{2e^{a_4}-a_4-2}{2}\right.$$

$$\left.\left. + \frac{a_4(a_4 e^{a_4} - e^{a_4} + 1)^2}{e^{a_4}-1}\right]\right\}\frac{\partial}{\partial a_1}$$

$$+ \frac{1}{a_4(e^{a_4}-1)}\left\{a_2(e^{a_4}-a_4-1) + a_3 a_4(a_4 e^{a_4} - e^{a_4} + 1)\right\}\frac{\partial}{\partial a_2}$$

$$+ \frac{a_3(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type IIIa.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv X_1, \quad (X_3, X_4) \equiv X_2.$$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv -\frac{a_4}{2}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_4^2}{12}\frac{\partial}{\partial a_1} - \frac{a_4}{2}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \tfrac{1}{2}\left(a_2 - \frac{a_3 a_4}{6}\right)\frac{\partial}{\partial a_1} + \frac{a_3}{2}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_4}.$$

Type IV.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv \alpha X_1, \quad (X_2, X_4) \equiv \alpha X_2, \quad (X_3, X_4) \equiv \gamma X_3,$$

$$(\alpha \neq \gamma).$$

$$A_1 \equiv \frac{a_4\alpha}{e^{a_4\alpha}-1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4\alpha}{e^{a_4\alpha}-1}\frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{a_4\gamma}{e^{a_4\gamma}-1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4 a} - a_4 a - 1)}{a_4(e^{a_4 a} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4 a} - a_4 a - 1)}{a_4(e^{a_4 a} - 1)} \frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4 \gamma} - a_4 \gamma - 1)}{a_4(e^{a_4 \gamma} - 1)} \frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type V.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv X_2 + X_3.$$

$$A_1 \equiv \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv -\frac{a_4[e^{a_4}(a_4 - 1) + 1]}{(e^{a_4} - 1)^2} \frac{\partial}{\partial a_2} + \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)} \frac{\partial}{\partial a_1} + \frac{1}{a_4(e^{a_4} - 1)} \{a_2(e^{a_4} - a_4 - 1) + a_3 a_4 (a_4 e^{a_4} - e^{a_4} + 1)\} \frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)} \frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type Va.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv X_2.$$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2},$$

$$A_3 \equiv -\frac{a_4}{2} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}, \qquad A_4 \equiv \frac{a_3}{2} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_4}.$$

Type VI.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv X_3.$$

$$A_1 \equiv \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)} \frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)} \frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type VIa.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0.$$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{\partial}{\partial a_3}, \qquad A_4 \equiv \frac{\partial}{\partial a_4}.$$

UNIVERSITY OF CINCINNATI,
October, 1901.

SHORTER NOTICES.

Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variablen. Von Dr. LUDWIG SCHLESINGER, ordentlichem Professor an der Universität zu Klausenburg. Leipzig, Göschen, 1900. Pp. viii + 310.

THIS little volume, which forms part of the "Sammlung Schubert" (cf. the BULLETIN for January, 1901, p. 192), gives, we believe, the best introduction which has yet appeared to that important side of the theory of ordinary differential equations in which the points of view are those of the theory of functions of a complex variable. Thus the discussion of the nature of singular points holds a central position in the treatment given. The author has been particularly successful in his choice of topics. He has on the one hand restricted himself to the simpler parts of the subject, more than half the volume being devoted to linear differential equations of the second order, and the remainder to the case of a single equation of the first order. By doing this he has succeeded in avoiding long analytical developments which only confuse a beginner without really teaching him anything. On the other hand the author has treated these simple cases in such a way as to bring out clearly a large number of the most important points of view of the modern theory of differential equations. Some of Dr. Schlesinger's own investigations, to mention only one point, on the Laplacian and Eulerian transformations are here set forth in particularly attractive form, although, of course, only for very special differential equations. The reader can turn to a large treatise for further information on these or