THE CENTURY'S PROGRESS IN APPLIED MATHEMATICS.

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THE honor of election to the presidency of the AMERICAN MATHEMATICAL SOCIETY carries with it the difficult duty of preparing an address, which may be at once interesting and instructive to a majority of the membership, and which may indicate at the same time the lines along which progress may be expected in one or more branches of our favorite science. In partial recognition of the honor you have conferred upon me it has seemed that I could do no better than to consider with you some of the principal advances that have been made in mathematical science during the past century. But here at the outset one must needs feel sharply restricted by the limitations of his knowledge and by the wide extent of the domain to be surveyed. Especially must this be the case with one who belongs to no school of mathematicians unless it be the 'old school' of inadequate opportunities and desultory training. On account of these conditions, I have found it essential to accept the ordinary division of the science into pure and applied mathematics and to confine my attention in this address wholly to applied mathematics. Here again, however, it is necessary to impose restrictions, for the domain thus divided is still far too large to be reviewed adequately in the brief interval allotted to the present occasion. I have therefore limited my considerations chiefly to those branches of applied mathematics which were already recognized as such at the beginning of the present century. The most important of these branches appear to be analytical mechanics, geodesy, dynamical astronomy, spherical or observational astronomy, the theory of elasticity, and hydromechanics. This rather arbitrary subdivision may be made to include several important branches not enumerated, while it must exclude others of equal or greater importance. Thus the theory of heat diffusion which led Fourier to the wonderful analysis which bears his name may be alluded to under physical geodesy ; the theories of sound and light may be regarded as applications merely of the theories of elasticity and hydromechanics ; while the theories of electricity, magnetism, and thermodynamics, which are the peculiar and perhaps most important developments of the present century, must be excluded almost altogether.

Another difficulty which besets one who would speak of the progress in question is that arising from the technicalities of the subjects to be discussed. Beautiful and important as these subjects are when arrayed in their mathematical dress, and thrilling as they truly are when rehearsed with appropriate terminology in the quiet of one's study, it must be confessed that they are on the whole rather uninviting for the purposes of semi-popular exposition. In order to meet this difficulty it seems best to relegate technicalities which demand expression in symbols to foot notes and, while freely using technical terminology, to translate it into the vernacular whenever essential. Thus it is hoped to avoid the dullness of undue condensation on the one hand and the superficiality of mere literary description on the other.

The end of the last century marks one of the most important epochs in the history of mathematical science. This time, one hundred years ago, the master work of Lagrange (1736–1813), the Mécanique Analytique, had been published about eleven years. The first two volumes of the Mécanique Céleste of Laplace (1749–1827), undoubtedly the greatest systematic treatise ever published, had just been issued. Fourier (1768–1830), whose mathematical theory of heat was destined to play a wonderful rôle in pure and applied mathematics, was a soldier statesman in Egypt, where with Napoleon he stood before the pyramids while the centuries looked down upon them.* Gauss (1777–1855), who with Lagrange and Cauchy (1789–1857) must be ranked among the founders of modern pure mathematics, was a promising but little known student whose Disquisitiones Arithmeticae and other papers were soon to win him the directorship of the observatory at Göttingen. Poisson (1781–1840), to whom we owe in large part the beginnings of mathematical physics, had just started on his brilliant career as a student and professor in the École Polytechnique. Bessel (1784–1846), whose theories of observational astronomy and geodesy were destined soon to assume a prominence which they still hold, was an accountant in a trading house at Bremen. Dynamical astronomy, the favorite science of

*The bombastic words of Bonaparte, "Songez que du haut de ces pyramides quarante siècles vous contemplent," may be excused, perhaps, in view of the fact that Fourier, Monge, and Berthollet were present on the occasion.

the day was under the dominating genius of Laplace, with no one to dispute his preëminence, and with only Lagrange and Poisson as friendly competitors in the same field. Rational mechanics as we now know it, was soon to be simplified and systematized by Poinsot (1777–1859), Poisson, Möbius (1790–1868), and Coriolis (1792–1843), who were all at this time under twenty-five years of age. The undulatory theory of light, in which Young (1773–1829), Fresnel (1788–1827), Arago (1786–1853), and Green (1793–1841) were to be the most conspicuous early figures, was just beginning to be considered as an alternative to the emission theory of Newton. The theory of elasticity, or the theory of stress and strain as it is now called, was about to be reduced to the definiteness of formulas at the hands of Navier (1785–1836), Poisson, Cauchy, and Lamé (1795–1870). Planetary and sidereal astronomy, to which so much of talent, time, and treasure have since been devoted, was soon to receive the fruitful impetus imparted to it by the German school of Gauss, Bessel, Encke (1791–1865), and Hansen (1795–1874).

The advances that have been made during the past century in analytical mechanics must be measured from the elevated standard attained by Lagrange in his Mécanique Analytique. To work any improvement over this, to simplify its demonstrations, or to elaborate its details, was a task fit only for the keenest intellects. Lagrange had, as he supposed, reduced mechanics to pure mathematics. Geometrical reasonings and diagrammatic illustrations were triumphantly banished from this science and replaced by the systematic and unerring processes of algebra. "Ceux qui aiment l'Analyse," he says, "verront avec plaisir la Mécanique en devinir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine." The mathematical world has not only accepted Lagrange's estimate of his work, but has gone further, and considers his achievement one of the most brilliant and important in the whole range of mathematical science. "The mechanics of Lagrange," as Mach has well said, "is a stupendous contribution to the economy of thought."*

Nevertheless, improvements were essential, and they came in due time. As we can now see without much difficulty, Lagrange and most of his contemporaries in their eagerness to put mechanics on a sound analytical basis overlooked to

* The Science of Mechanics, by Dr. Ernst Mach. Translated from the German by Thomas J. McCormack. Chicago, Open Court Publishing Co. 1893.

a serious extent its more important physical basis. The prevailing mathematical opinion was that a science is finished as soon as it is expressed in equations. One of the first to protest against this view was Poinsot, though the preëminent importance of the physical aspect of mechanics did not come to be adequately appreciated until the latter half of the present century. The animating idea of Poinsot was that in the study of mechanics one should be able to form a clear mental picture of the phenomena considered; and that it does not suffice to put the data and hypotheses into the hopper of our mathematical mill and then to trust blindly to its perfection in grinding the grist. In elaborating this idea he produced two of the most important elementary treatises on mechanics of the century. These are his Éléments de Statique published in 1804, and his Théorie Nouvelle de la Rotation des Corps published in 1834.* In the former work he developed the beautiful and fruitful theory of couples and their composition, and the conditions of equilibrium, as they are now commonly expressed in elementary books. In the latter work he took up the more recondite question of rendering a clear account of the motion of a rigid body. This problem had been treated already by the illustrious Euler, d'Alembert, Lagrange, and Laplace, and it seemed little short of temerity to hope for any improvement. But Poinsot entertained that hope and his efforts proved surprisingly successful. His little volume of about one hundred and fifty pages is still one of the finest models of mathematical and mechanical exposition; and his repeated warning, "gardons-nous de croire qu'une science soit faite quand on l'a réduite à des formules analytiques," has been fully justified. He gave us what may be called the descriptive geometry of the kinetics of a rotating rigid body, the "image sensible de cette rotation"; he clarified the theory of moments of inertia and principal axes; he made plain the meaning of what we now call the conservation of energy and the conservation of moment of momentum of systems which are started off impulsively; and he surpassed Laplace himself in expounding the theory of the invariable plane.

Another elementary work of prime importance in the progress of mechanics was Poisson's Traité de Mécanique.

* Outlined in a communication to the Paris Academy in 1834. In the introduction to the edition of 1852 he says, "Voici une des questions qui m'ont le plus souvent occupé, et, si l'on me permet de parler ainsi, une des choses que j'ai le plus désiré de savoir en dynamique. "Tout le monde se fait une idée claire du mouvement d'un point, * * * Mais, s'il s'agit du mouvement d'un corps de grandeur sensible et de figure quelconque, il faut convenir qu'on ne s'en fait qu'une idée très-obscure."

Poisson belonged to the Lagrangian school of analysts, but he was so profoundly devoted to mathematical physics that almost all his mathematical work was suggested by and directed towards practical applications. His facility and lucidity in exposition rendered all his works easy and attractive reading, and his treatise on mechanics is still one of the most instructive books on that subject. He was one of the first to call attention to the value of the principle of homo geneity in mechanics,* a principle which, as expanded in Fourier's theory of dimensions,† has proved of the greatest utility in the latter half of the century. The influence of Poisson's work in mechanics proper, very widely extended, of course, by his memoirs in all departments of mathematical physics, is seen along nearly every line of progress since the beginning of the century.

Of other works which paved the way to the present advanced state of mechanical science, it may suffice to mention the Cours de Mécanique ‡ of Poncelet (1788–1867), the Traité de Mécanique des Corps Solides et de l'Effet des Machines § of Coriolis, and the Lehrbuch der Statik ‖ of Möbius. To the two former we owe the fixation of ideas and terminology concerning the doctrine of mechanical work, while the suggestive treatise of Möbius foreshadowed a new type of mechanical concepts since cultivated by Hamilton, Grassmann, and others under the general designation of vector analysis.

Following close after the development of the elementary ideas whose history we have sketched came the important mathematical improvements in the Lagrangian analysis due to Hamilton ¶ (Sir W. R., 1805–1865). With these additions of Hamilton, amplified and clarified by the labors of Jacobi, Poisson, and others,** analytical mechanics may be said to have reached its present degree of perfection so far as mathematical methods are concerned. By these methods every mechanical question may be stated in either of three characteristic though interconvertible ways, namely: by the

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* See Article 23, Tome I., Traité de Mécanique, 2nd ed., Paris, 1833.

† Théorie Analytique de la Chaleur, Paris, 1822.

‡ Metz, 1826.

§ Paris, 1829.

‖ Leipzig, 1837.

¶ "On a general method in dynamics." *Philosophical Transactions*, 1834–35.

** For an account of these additions and a complete list of papers bearing on the subject (up to 1857), one should consult the admirable report of Cayley on "Recent progress in dynamics," published in the Report of the British Association for the Advancement of Science for 1857.

equations of d'Alembert, by the equations of Lagrange, and by the equation of Hamilton. Each way has special advantages for particular applications, and together they may be said to condense into the narrow space of a few printed lines the net results of more than twenty centuries of effort in the formulation of the phenomena of matter and motion.

Such was the state of mechanical science when the great physical discovery of the century, the law of conservation of energy, was made. To give adequate expression to this law it was only necessary to recur to the Mécanique Analytique, for herein Lagrange had prepared almost all of the needful machinery. So well, indeed, were the ideas and methods of Lagrange adapted to this purpose that they have not only furnished the points of departure for many of the most important discoveries * of the present half century, but they have also supplied the criteria by means of which mechanical phenomena in general are most easily and effectively defined and interpreted.

Of the special branches of analytical mechanics which have undergone noteworthy development during this century, by far the most important is that known as the theory of the potential function. This function first appeared in mathematical analysis in a memoir of Lagrange in 1777 † as the expression of the perturbative function, or force function. It next appeared in 1782 ‡ in a memoir by Laplace. In this memoir Laplace's equation § appears for the first time, being here expressed in polar coördinates. In 1787 || the same equation appears in the more usual form as expressed by rectangular coördinates.

Strange as it now seems when viewed by the light of this end of the century, nearly thirty years elapsed before Laplace's equation was generalized. Laplace had found only half of the truth, namely, that which applies to points external to the attracting masses.¶ Poisson discovered the

*Especially those in the theories of electricity, magnetism, and thermodynamics.

† *Nouveaux Mémoires de l'Académie des Sciences et Belles Lettres de Berlin.* See also remarks of Heine, Handbuch der Kugelfunctionen, Band II., p. 342.

‡ *Paris Mémoires* for 1782, published in 1785.

§ $$\Delta^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

$\Delta^2 V$ is called the Laplacian of V.

|| *Paris Mémoires* for 1787, published in 1789.

¶ That is, Laplace's equation is $\Delta^2 V = 0$, while Poisson's is $\Delta^2 V + 4\pi k \rho = 0$; V being the potential and ρ the density at the point (x, y, z), and k being the gravitation constant.

other half in 1813.* Thus the honors attached to the introduction of this remarkable theorem are divided between them, and we now speak of the equation of Laplace and the equation of Poisson, though the equation of Poisson includes that of Laplace.

Next came the splendid contributions of George Green under the modest title of "An essay on the application of mathematical analysis to the theories of electricity and magnetism."† It is in this essay that the term 'potential function' first occurs. Herein also his remarkable theorem in pure mathematics, since universally known as Green's theorem, and probably the most important instrument of investigation in the whole range of mathematical physics, made its appearance.

We are all now able to understand, in a general way at least, the importance of Green's work, and the progress made since the publication of his essay in 1828. But to fully appreciate his work and subsequent progress one needs to know the outlook for the mathematico-physical sciences as it appeared to Green at this time and to realize his refined sensitiveness in promulgating his discoveries.

"It must certainly be regarded as a pleasing prospect to analysts," he says in his preface, "that at a time when astronomy, from the state of perfection to which it has attained, leaves little room for further applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it." * * * "Should the present essay tend in any way to facilitate the application of analysis to one of the most interesting of the physical sciences, the author will deem himself amply repaid for any labor he may have bestowed upon it; and it is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means as other indispensable avocations which offer but few opportunities of mental improvement, afforded." Where in the history of science have we a finer instance of that sort of modesty which springs from a knowledge of things?

The completion of the potential theory, so far as it depends on the Newtonian law of the inverse square of the distance, must be credited to Gauss, though a host of writers

* Poisson's equation was derived in a paper published in *Nouveau Bulletin* * * * *Société Philomatique*, Paris, Dec., 1813.
† Nottingham, 1828.

has since contributed many valuable additions in the way of details. Early in the century Gauss had begun the study of the absorbing problems of the day, namely, problems of attractions and repulsions. The prevailing notion of mathematical physicists seems to have been that all mechanical phenomena may be attributed to attractions and repulsions between the ultimate particles of matter and the ultimate particles of 'fluids' associated with matter. The difficulties of action at a distance, without the aid of an intervening medium, happily, did not trouble them at that time; for who shall say that their labors would have been more fruitful if they had stopped to remove these difficulties? Gauss's first memoir in this field relates to the attractions of homogeneous ellipsoidal masses,* and dates from 1813. It was in this memoir that he published a number of the elegant theorems † which are now found in the elementary books on the theory of the potential function. In 1829 he published his theory of fluid figures in equilibrium, ‡ and in 1832 there followed one of the most important papers of the century on the intensity of terrestrial magnetic force expressed in what we now call absolute units.§

Six years later he published his wonderful theory of the earth's magnetism || and applied it to all existing observational data. This theory is a splendid application of the potential theory, and his entire investigation is one of the most beautiful and useful contributions to mathematical physics of the century. Well was he qualified, therefore, to complete the theory of the Newtonian potential function in the collection of theorems published in his memoir ¶ of

* "Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum," 1813. See Gauss's Werke, Band V., Göttingen, 1877.

† Especially the theorem giving the values of the surface integral

$$\int \frac{\cos(s, n)}{s^2} dS,$$

where dS is an element of any closed surface, s the distance from dS to any fixed point, and n indicates the normal to the surface at dS. This gave the key to the very important theorem of the surface integral of the normal acceleration, or

$$\int \frac{\partial V}{\partial n} dS.$$

‡ "Principia generalia theoriæ figuræ fluidorum in statu æquilibrii," 1829. Werke, Band V.

§ "Intensitas vis magneticae terrestris ad mensuram absolutam revocata." Werke, Band V.

|| Allgemeine Theorie des Erdmagnetismus. Werke, Band V.

¶ "Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte." Werke, Band V.

1840. This is still the fundamental memoir on the subject of which it treats, and must be regarded as one of the most perfect models of mathematical exposition. In respect to clearness and elegance, indeed, the works of Gauss are unsurpassed. "In his hands," as Todhunter has said,* "Latin and German rival French itself for clearness and precision." "Alles gestaltet sich neu unter seinen Händen," was the tribute † of Bessel; and the lapse of two generations has served only to increase admiration for the genius and industry which made Gauss one of the most conspicuous figures in the science of the nineteenth century.

The importance of the theory of the potential function, when considered in its historical aspects, is found to consist not so much in the rich harvest of results it has afforded in the field of gravitation, as in its direct bearing on the developments of other branches of mathematical physics. For the points of view and the analytical methods of the Newtonian function have been adapted and extended with brilliant success to the interpretation of almost all kinds of mechanical phenomena. Thus it has come about that we have now to deal with many kinds of potential, as logarithmic potential, velocity potential, displacement potential, electric potential, magnetic potential, thermodynamic potential, etc.; each of which bears a more or less close mathematical analogy to the Newtonian function.

In the closing paragraph of his Exposition du Système du Monde, Laplace refers to the immense progress made in astronomy since the geocentric theory was displaced by the heliocentric theory of the solar system. This progress is specially remarkable when we consider that it depended on the discovery, so humiliating to man, of the relatively insignificant dimensions and inconspicuous rôle of our planet. But we agree with Laplace that "Les resultats sublimes auxquels cette découverte l'a conduit sont bien propres à le consoler du rang qu'elle assigne à la Terre, en lui montrant sa propre grandeur dans l'extrême petitesse de la base qui lui a servi pour mesurer les cieux." All astronomy is based on a knowledge of the size, the shape and the mechanical properties of the earth; and it is not surprising, therefore, that a large share of the mathematical investigations of the century should have been directed to the science of geodesy. Founded in the middle of the last century by Clairaut ‡ and

* History of the Theories of Attraction and Figure of the Earth, Vol. II., p. 235.

† In a letter to Olbers, 1818.

‡ Clairaut's work, Théorie de la Figure de la Terre, Paris, 1743, was the pioneer work in physical geodesy.

his contemporaries; recast by Laplace and Legendre* (1752–1833) in the early part of this century; systematized and extended to a remarkable degree by the German geodesists, led especially by the incomparable Bessel,† this science has now come to occupy the leading position in point of perfection of methods and precision of results. So great, in fact, has been the growth of this science during the century that recent writers have found it desirable to subdivide the subject into two parts, called mathematical geodesy and physical geodesy, respectively, though both parts are nothing if not mathematical.‡

In a former address I have considered somewhat in detail certain of the more salient mathematical problems which have arisen in the study of the earth; § and the present review may hence be restricted to a rapid résumé of the less salient, but perhaps more recondite problems, and to the briefest mention of problems already discussed.

Adopting the convenient nomenclature of geologists, we may consider the earth as made up of four parts, namely: the atmosphere; the hydrosphere, or oceans; the lithosphere, or crust; and the nucleus. Beginning with the first of these we are at once struck by the fact that much greater progress has been made during the century in the investigation of the kinetic phenomena of the atmosphere than in the study of what may be called its static properties. Evidently, of course, the phenomena of meteorology are essentially kinetic, but it would seem that the questions of pressure, temperature, and mass distribution of the atmos phere ought to be determined with a close approximation from purely statical considerations. This appears to have been the view of Laplace, who was the first to bring adequate knowledge to bear on such questions. He investigated the terrestrial atmosphere as one might investigate the gaseous envelope of an unilluminated planet.|| He

* The name of Legendre is famous in geodesy by reason of his beautiful theorem which makes the solution of a geodetic triangle almost as easy as the solution of a plane triangle.

† Bessel's contributions to astronomy and geodesy are collected in Abhandlungen von F. W. Bessel, herausgegeben von Rudolf Engelmann, in drei Bänden, Leipzig, Wilhelm Engelmann, 1875.

‡ See, for example, Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie von Dr. F. R. Helmert, Leipzig, B. G. Teubner, Teil I., 1880; Teil II., 1884.

§ On the Mathematical Theories of the Earth. Vice-presidential address before the Section of Astronomy and Mathematics of the American Association for the Advancement of Science, 1889. *Proceedings of A. A. A. S.* for 1889.

¶ Mécanique Céleste, Livre III., Chap. VII., and Livre X, Chapts. I.–IV.

reached the conclusion that the atmosphere is limited by a lenticular shaped surface of revolution whose polar and equatorial diameters are about 4.4 and 6.6 times the diameter of the earth respectively, and whose volume is about 155 times that of the rest of the earth.* If this conclusion be true our atmosphere should reach out to a distance of about 26,000 miles at the equator and to a distance of about 17,000 miles at the poles. It does not appear, however, that Laplace attempted to assign the distribution of pressure and density, and hence total mass, of the atmosphere within this envelope; and I am not aware that any subsequent investigator has published a satisfactory solution of this apparently simple problem.†

On the other hand, the general character of the circulation of the atmosphere and the meteorological consequences

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* Laplace's equation to a meridian section of this envelope is

$$x^{-1} - x_0^{-1} + \tfrac{1}{2}ax^2\cos^2\phi = 0,$$

where $x = r/a$, r being the radius vector measured from the center of the earth and a the mean radius of the earth; α is the ratio of centrifugal to gravitational acceleration at the equator of the earth; φ is the geocentric latitude, and x_0 is the value of x for $\phi = \pi/2$.

The problem of the statical properties of the atmosphere may be stated in three equations, namely:

$$\Delta^2 V + 4\pi k\rho - 2\omega^2 = 0, \qquad dp = \rho dV, \qquad p = f(\rho, \tau).$$

In these V is the potential at any point of the atmosphere, p, ρ, τ being the pressure, density and temperature at the same point; k is the gravitation constant; and ω is the angular velocity of the earth. The above equation of Laplace neglects the mass of the atmosphere in comparison with the mass of the rest of the earth. An essential difficulty of the problem lies in the unknown form of the function $f(\rho, \tau)$.

† I have sought a solution with a view especially to determining the mass of the atmosphère. A class of solutions satisfying the mechanical conditions of the following assumptions has been worked out. Thus, assuming $p = c\rho^m$, which includes the adiabatic relation, $p = c\rho^{1.41}$, and the famous Laplacian relation, $\partial p/\partial\rho = 2c\rho$; and the law of Charles and Gay-Lussac, $p = C\rho\tau$; there results

$$\frac{p}{p_0} = \left(\frac{Q}{Q_0}\right)^{\frac{m}{m-1}}, \qquad \frac{\rho}{\rho_0} = \left(\frac{Q}{Q_0}\right)^{\frac{1}{m-1}}, \qquad \frac{\tau}{\tau_0} = \frac{Q}{Q_0},$$

where $$Q = x^{-1} - x_0^{-1} + \tfrac{1}{2}ax^2\cos^2\phi$$

defined above; Q_0 is the value of Q for $x = 1$ and $\phi = \pi/2$; and p_0, ρ_0, τ_0 are the values of p, ρ, τ at the same point ($x = 1$, $\phi = \pi/2$).

Using the adiabatic law the above formula for ρ leads to a mass for the atmosphere of about 1/1200th of the entire mass of the earth. But since the adiabatic law gives too low a pressure, density, and temperature gradient, this can only be regarded as an upper limit to the mass of the atmosphere. A lower limit of about 1/1000000th of the earth's mass is found by assuming that the mass of the atmosphere is equal to a mass of mercury or water which would give an equivalent pressure at the earth's surface.

thereof, have been brought within the domain of mathematical research if they have not yet been wholly reduced to quantitative precision. The pioneer in this work was a fellow-countryman, William Ferrel,* (1817–1891), who, like Green, came near being lost to science through the obscurity of his early environment. It is a curious though lamentable circumstance, illustrating at once the peculiar shyness of Ferrel and the proverbial popular indifference to discoveries which cannot be patented, that a man who had mastered the Principia and the Mécanique Céleste and who had laid the foundations of our theory of the circulation of the atmosphere, should have found no better medium for the publication of his researches than the semi-popular columns of a journal devoted to medicine and surgery. But such was the medium through which Ferrel's "Essay on the winds and the currents of the ocean"† appeared in 1856. Since that time notable progress has been made at the hands of Ferrel, Helmholtz (1821–1894), Oberbeck, Bezold, and others;‡ so that we may entertain the hope that the apparently erratic phenomena of the weather will presently yield to mathematical expression just as the similar phenomena of oceanic tides and of terrestrial magnetism have already yielded to the powers of harmonic analysis.

When we pass from the atmosphere to the hydrosphere, several questions concerning the nature and properties of their common surface, or what is usually called the sea surface immediately demand attention. The most important of these are what may be distinguished as the static and the kinetic phenomena of the sea surface. Since tidal oscillations belong more properly to hydrokinetics, we may here confine attention to the static phenomena.

Starting from the datum plane fixed by Laplace, the most important contribution to the theory of physical geodesy since his time is found in the remarkable memoir of Sir George Stokes "On the variation of gravity at the surface of the earth."§ Adopting the hypothesis of original fluidity,

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* For a biography and autobiographical sketch of Ferrel, and a list of his publications, see Biographical Memoirs of the National Academy of Sciences, Vol. III., pp. 265–309. Washington, 1895.

† In *Nashville Journal of Medicine and Surgery*, Oct. and Nov., 1856.

‡ Some of the most important papers and memoirs on this subject, collected and translated by Professor Cleveland Abbe, have been published by the Smithsonian Institution under the title "The Mechanics of the Atmosphere." Smithsonian Miscellaneous Collections, No. 843, Washington, 1891.

§ Read April, 1849; Mathematical and Physical Papers by G. G. Stokes, Cambridge University Press, 1883, Vol. II.

or the more general hypothesis of a symmetrical arrangement of the strata of the earth, with increasing density towards the center, Laplace had shown that the acceleration of gravity in passing from the equator to the poles should increase as the square of the sine of the latitude.* This conclusion agreed well with the facts of observation; and Laplace rested content in the opinion that his hypothesis was verified. But Stokes showed that the law of variation of the acceleration of gravity at the surface of the sea is wholly determined by that surface, regardless of the mode of distribution of the earth's mass. This, as we now see, of course, is a direct result of the theory of the potential function; for the sea surface is an equipotential surface, and since it is observed to be closely spheroidal, the formula of Laplace follows independently of all hypothesis save that of the law of gravitation. But while Laplace's formula and the arguments by which he reached it throw no light on the distribution of the earth's mass, a slight extension of his methods gives a formula which shows that any considerable differences between the equatorial moments of inertia of the earth would produce a variation in the acceleration of gravity dependent on the longitude of the place of observation.† Thus it is possible by means of pendulum observations alone to reach the conclusion that the mass of the earth is very nearly symmetrically distributed with respect to its equator and with respect to its axis of revolution.

A question of great interest with which the acceleration of gravity at the sea surface is closely connected is that of the earth's mass as a whole. About two years ago I published a short paper which gives the product of the mean density of the earth and the gravitation constant in terms of the coefficients in Laplace's formula and the dimensions of the earth.‡ It was shown that this product can be

*Laplace's formula is $g = \alpha + \beta \sin^2 \phi$, where α is the value of g at the equator, β is a constant, and ϕ is the latitude of the place.

†See Helmert, Geodäsie, Band II., p. 74. The expression for the acceleration is

$$g = \alpha + \beta \sin^2 \phi + \gamma \cos^2 \phi \cos 2\lambda,$$

where α, β, γ are constants, and ϕ, λ are latitude and longitude respectively; and the constant γ involves the difference of the equatorial moments of inertia as a factor.

‡ *The Astronomical Journal*, No. 424. This product is expressed thus:

$$k\rho = \frac{2\pi}{T^2} + \frac{3(\alpha s_1 + \beta s_2)}{4\pi a \sqrt{1-e^2}};$$

wherein k is the gravitation constant, ρ is the mean density of the earth, T is the number of mean solar seconds in a sidereal day, α and β are the first two constants in the formula $g = \alpha + \beta \sin^2 \phi + \gamma \cos^2 \phi \cos 2\lambda$ for easily computed from existing data to five significant figures with an uncertainty of only one or two units in the last figure, thus making it possible to obtain the mass of the earth to a like degree of precision if the constant of gravitation can be equally well determined. In a subsequent communication to this Society it was explained that the product in question is equal to 3π divided by the square of the periodic time of an infinitesimal satellite which would pass around the earth, just grazing the equator, if there were no atmosphere to impede its progress. The periodic time of such a satellite would be 1 hour, 24 minutes, 20.9 seconds. Attention is called to this subject with the hope that some mathematician may point out another possible relation between the gravitational constant and the mean density of the earth which can be accurately observed, or that some physicist may show how the gravitational constant can be measured directly with a precision extending to five significant figures.

The lithosphere is the special province of the geologist, and we may hence pass on to the nucleus, or chief part of the mass of the earth. Much time and attention have been devoted to the study of the important but intricate problems which the geometers of the early part of the century left to their successors. But while the obscurities and vagaries of our predecessors have been cleared away, it must be confessed that our improved mathematical apparatus has not brought us very far ahead of the positions of Laplace and Fourier as regards the constitution and properties of the nucleus. With respect to the law of distribution of density in the nucleus it may be said that, although Laplace's law * is probably not exact, it is yet quite as nearly correct as our observed information requires.†

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the acceleration of gravity at the sea surface in latitude φ and longitude λ; a is the half major axis and e is the eccentricity of the earth's spheroid; and

$$s_1 = \left\{1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}, \qquad s_2 = \frac{1-e^2}{4e^2}\left\{\frac{2e}{1-e^2} + \log\frac{1-e}{1+e}\right\}.$$

The resulting numerical value is

$$k\rho = 36797 \times 10^{-11}/(\text{second})^2.$$

* The Laplacian distribution of pressure, density, and potential in the earth are defined essentially (neglecting the effect of rotation) by the following three equations:

$$\Delta^2 V + 4\pi k\rho = 0, \qquad dp = \rho dV, \qquad \frac{\partial p}{\partial \rho} = c\rho;$$

where p, ρ, V are the pressure, density, and potential at any point of the mass, k is the gravitation constant, and c is a constant securing the equality of the members of the last equation.

† With regard to what constitutes an adequate theory in any case, see

Another question of widely general, and of peculiar mathematical interest, is the problem first attacked by Fourier, of the distribution and consequent effects of the earth's internal heat. The most interesting phase of this question is that which relates to the time that has elapsed since the crust of the earth became stable and sufficiently cool to support animal life. It is now nearly forty years since Lord Kelvin* startled geologists especially by telling them that Fourier's theory of heat conduction forbids anything like such long intervals of time as they were in the habit of assigning to the aggregate of paleontological phenomena. On several occasions since then Kelvin has restated his arguments with a cogency that has silenced most geologists if it has not convinced most mathematicians. Quite recently, however, the question has become somewhat less one-sided, since geologists and paleontologists are beginning to defend their position,† while that of Kelvin is being attacked from the mathematical side.‡ My own views on this subject were expressed somewhat at length ten years ago, in the address already referred to, and it seems unnecessary here to go into the matter any further than to reaffirm my conviction that the geologists have adduced the weightier arguments. Beautiful as the Fourier analysis is, and absorbingly interesting as its application to the problem of a cooling sphere § is, it does not seem to me to afford anything like so

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an instructive paper by Dr. G. Johnstone Stoney on "The kinetic theory of gas, regarded as illustrating nature," *Proceedings Royal Dublin Society*, Vol. VIII. (N. S.), Part IV., No. 45.

* In a memoir "On the secular cooling of the earth," *Trans. Royal Society of Edinburgh*, 1862. Republished in Kelvin and Tait's Treatise on Natural Philosophy, appendix D. Kelvin's latest paper on this subject is entitled "The age of the earth as an abode fitted for life," and is published in *Philosophical Magazine*, January, 1899; also in *Science*, May 12, 1899.

† See Professor T. C. Chamberlain's paper, "Lord Kelvin's address on the age of the earth as an abode fitted for life," *Science*, June 30, 1899; also Sir Archibald Geikie's presidential address to Geological Section of the British Association for the Advancement of Science, Dover meeting, 1899.

‡ Notably by Professor John Perry. See *Nature*, January 3, and April 18, 1895.

§ I have recast this problem of Fourier in two papers published in the *Annals of Mathematics*, Vol. III., pp. 75–88 and pp. 129–144. The solution there given is the only one, so far as I am aware, which lends itself to computation for all values of the time in the history of cooling. A point of much mathematical interest on which this solution depends is the equivalence of the two following series:

$$ru = \frac{2r_0u_0}{\pi}\sum_{n=1}^{n=\infty}\left(\frac{-1}{n}\right)^{n+1}e^{-a^2(n\pi/r_0)^2t}\sin n\pi\frac{r}{r_0},$$

definite a measure of the age of the earth as the visible processes and effects of stratification to which the geologists appeal. In short, the only definite results which Fourier's analysis appears to me to have contributed to knowledge concerning the cooling of our planet are the two following, namely: first, that the process of cooling goes on so slowly that nothing less than a million years is a suitable time unit for measuring the historical succession of thermal events; and secondly, that this process of cooling goes on substantially as if the earth possessed neither oceans nor atmosphere.

It was the well-founded boast of Laplace in the early part of the century that astronomy is the most perfect of the sciences;* and expert contemporary opinion, as we have seen in the case of no less a personage than Green, agreed that the Mécanique Céleste left little room for further advances. Indeed, it would appear that the completeness and the brilliancy of the developments of celestial dynamics during the half century ending with 1825 (the period of Laplace's activity) completely overshadowed all other sciences and retarded to some extent the progress of astronomy itself. The stupendous work of Laplace was chiefly theoretical, however, and he was content in most cases with observational data which accorded with theory to terms of the first order of approximation only. He was probably not so profoundly impressed as men of science at this end of the century are with the necessity of testing a theory by the most searching observational means available. In fact, in elaborating his methods and in applying his theories to the members of the solar system, it has been essential for his disciples to display a degree of ingenuity and a persistence of industry worthy of the master himself. But the prerequisite to progress in celestial mechanics consisted not so much in following up immediately the lines of

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$$ru = ru_0 - \frac{2r_0u_0}{\sqrt{\pi}} \sum_{n=0}^{n=\infty} \int_{(2n+1)m_0-m}^{(2n+1)m_0+m} e^{-z^2}dz.$$

In these u is the temperature at a distance r from the center of the sphere at any time t; u_0 is the initial uniform temperature of the sphere; r_0 is the radius of the surface of the sphere; a^2 is the diffusivity, supposed constant; and $m = r/(2a\sqrt{t})$, $m_0 = r_0/(2a\sqrt{t})$. It will be observed that when the first series (which is Fourier's) converges very slowly the second converges very rapidly, and vice versa. It will be seen, also, that the series refuse, as they should, to give values of the temperature corresponding to negative values of the time.

* "L'Astronomie, par la dignité de son objet et par la perfection de ses théories, est le plus beau monument de l'esprit humain, le titre le plus noble de son intelligence." Système du Monde, Ed., 1884, p. 486.

investigation laid down by Laplace, as in perfecting the methods and increasing the data of observational astronomy.

The development of this branch of science along with the development of the closely related science of geodesy, is a work essentially of the present century and must be attributed chiefly to the German school of astronomers led by Gauss and Bessel. It is to these eminent minds, as well known in pure as in applied mathematics, that we are indebted for the theories, and for the most advantageous methods of use, of instrumental appliances, and for the refined processes of numerical calculation which secure the best results from observational data. It is a fortunate circumstance, perhaps, considering the irreverence which some modern pure mathematicians show for numerical computations, that Gauss and Bessel began their careers long before the resistless advent of the theory of functions and the theory of groups.

The story of the opportune discovery of the planet Ceres, as related by Gauss himself in the preface to his Theoria Motus Corporum Cœlestium, is well known; but it is less well known that the merit of this magnificent work lies rather in the model groups of formulas presented for the precise numerical solution of intricate problems than in the facility afforded for locating the more obscure members of the solar system. Indeed, the works of Gauss and Bessel are everywhere characterized by a clear recognition of the important distinction between those solutions of problems which are, and those which are not, adapted to numerical calculation. They showed astronomers how to systematize, to expedite, and to verify arithmetical operations in ways which were alone adequate to the accomplishment of the vast undertakings which have since been completed in mathematical geodesy and in sidereal astronomy.

Among the most important contributions of these authors to geodesy and astronomy in particular, and to the precise observational sciences in general, is that branch of the theory of probability called the 'method of least squares.'* No single adjunct has done so much as this to perfect plans of observation, to systematize schemes of reduction, and to give definiteness to computed results. The effect of the general adoption of this method has been somewhat like the

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* Gauss's fundamental paper in this subject is "Theoria combinationis observationum erroribus minimis obnoxiae," and dates from 1821. Werke, Band IV.

Bessel's numerous contributions to this subject are found in his Abhandlungen cited above, p. 142.

effect of the general adoption by scientific men of the metric system; it has furnished common modes of procedure, common measures of precision, and common terminology, thus increasing to an untold extent the availability of the priceless treasures which have been recorded in the century's annals of astronomy and geodesy.

When we pass from the field of observational astronomy to the more restricted but more intricate field of dynamical astronomy, it is apparent that Laplace and his contemporaries quite underestimated the magnitudes of the mathematical tasks they bequeathed to their successors. Laplace, almost unaided, had performed the unparalleled feat of laying down a complete outline of the 'system of the world'; but the labor of filling in the details of that outline, of bringing every member of the solar system into harmony at once with the simple law of gravitation and with the inexorable facts of observation, is a still greater feat which has taxed the combined efforts of the most acute analysts and the most skillful computers of the preceding and present generation.

It is impossible within the limits of a semi-popular address to do more than mention in the most summary way the extraordinary contributions to dynamical astronomy made especially during the present half century, contributions formidable alike by reason of their bulk and by reason of the complexity of their mathematical details. An account of the theory of the perturbative function, or of the theory of the moon, for example, would alone require space little short of a volume.* To mention only the most conspicuous names, there is the pioneer and essentially prerequisite work of the illustrious Gauss and the incomparable Bessel. There is the remarkable work of the brilliant Leverrier (1811–1877) and the not less brilliant Adams (1819–1892),† well known to popular fame by reason of what may be called their mathematical discovery of the planet Neptune. Then came the monumental Tables de la Lune ‡ from the arithmetical laboratory of the indefatigable Hansen; and this marvellous production was quickly followed (1860) by the equally ponderous, and mathematically more important,

* A good account of the progress in dynamical astronomy from 1842 to 1867 is given by Delaunay in his "Rapport sur les progrès de l'Astronomie," Paris, 1867.

† The papers of Adams have been edited by Professor W. G. Adams and supplied with a biographical memoir by Professor J. W. L. Glaisher, under the title: Scientific Papers of John Couch Adams, Cambridge, At the University Press, vol. 1, 1896.

‡ Published by the British Government in 1857.

Théorie du Mouvement de la Lune * from the pen of the admirably fertile and industrious Delaunay. And finally, there is the still more elaborate work, bringing this great problem of the solar system well-nigh to completeness of solution, which, by common consent, is credited to the two preceding presidents of the AMERICAN MATHEMATICAL SOCIETY. †

Probably no mathematico-physical undertakings of the century have yielded so many definite, quantitative, results to the permanent stock of knowledge as the researches in dynamical astronomy. But notwithstanding the astonishing degree of perfection to which this science has been brought, there are still some outstanding discrepancies which indicate that the end of investigation is yet a long way off. The moon, which has given astronomers, as well as other people, more trouble than any other member of the solar system, is still devious to the extent of a few seconds in a century. The earth, also, it is suspected, is irregular as a time keeper by a minute but sensible amount ‡; while it has been proved recently, by the exquisite precision of modern observations, that the earth's axis of rotation wanders in a complex way through small but troublesome angles from its mean position, thus causing variations in the astronomical latitude of a place. §

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* *Mémoires de l'Académie des Sciences de l'Institut Impérial de France*, Tomes XXVIII., XXIX.

† For an account of the more recent work of Gyldén and Poincaré, reference is made to the presidential address of Dr. G. W. Hill, "Remarks on the progress of celestial mechanics since the middle of the century"; BULLETIN, 2d series, vol. 2, no. 5, p. 125.

‡ The effect of tidal friction on the speed of rotation of the earth appears to have been first explained by Ferrel in a "Note on the influence of the tides in causing an apparent acceleration of the moon's mean position." This paper was read before the American Academy of Arts and Sciences, in December, 1864, only a few weeks before Delaunay read a similar paper before the French Academy. See Ferrel's autobiography cited on p. 144. See also Delaunay's account of his own work in "Rapport sur les progrès de l'Astronomie," Paris, 1867.

§ The cause of such variations is found in the relative mobility of the parts of the earth, especially in the mobility of the oceans and atmosphere. Three types of variation may occur, namely: 1st, that due to sudden changes in the relative positions of parts of the earth's mass; 2d, that due to secular changes in position of those parts; and 3d, that due to periodic shiftings of those parts. Of these the most important appears to be the periodic type. A surprising, and as yet not fully explained, discrepancy brought to light by the discovery of latitude variations is the fact that the instantaneous axis of rotation of the earth makes a complete circuit around the axis of figure in about 428 days instead of in about 305 days as has been supposed from the time of Euler down to the present decade. The discovery of this discrepancy is due to

A question of intense interest to astronomers in the early part of the century is that of the stability of the solar system. Lagrange, Laplace, and Poisson thought they had demonstrated that, whatever may have been the origin of this system, the existing order of events will go on indefinitely. This conclusion seems to have been alike satisfactory to scientific and unscientific men. But with the growth of the doctrine of energy and with the developments of thermodynamics, it has come to appear highly probable that the solar system has not only gone through a long series of changes in the past, but is destined to undergo a similarly long series of vicissitudes in the future. In other words, while our predecessors of a century ago thought the 'system of the world' stable, our contemporaries are forced to consider it unstable.*

But interesting as this question of stability still is, there is no pressing necessity, fortunately, for a determination of the ulterior fate of our planet. A more important question lies close at hand, and merits, it seems to me, immediate and serious investigation. This question is the fundamental one whether the beautifully simple law of Newtonian attraction is exact or only approximate. No one familiar with celestial mechanics, or with the evidence for the law of gravitation as marshalled by Laplace in his Système du Monde can fail to appreciate the reasons for the profound conviction, long held by astronomers, that this law is exact. But on the other hand, no one acquainted with the obstinate properties of matter can now be satisfied with the Newtonian law until it is proved to hold true to a much higher degree of approximation than has been attained hitherto.* For, in spite of the superb experimental investigations made particularly during the past quarter of a century by Cornu

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Dr. S. C. Chandler, and was announced in the *Astronomical Journal*, no. 248, November, 1891. For the mathematical theory of this subject and for titles of the principal publications bearing on this theory, reference may be made to the author's paper on "Mechanical interpretation of the variations of latitudes," *Astronomical Journal*, no. 345, May, 1895; and to a paper by S. S. Hough on "The rotation of an elastic spheroid," *Philosophical Transactions*, no. 187, 1896.

* See a review of this subject by M. H. Poincaré, "Sur la stabilité du système solaire," in *Annuaire du Bureau des Longitudes* for 1898.

† As to the degree of precision with which the Newtonian law is established by astronomical data see Professor Newcomb's "Elements of the four inner planets and the fundamental constants of astronomy," Supplement to American Ephemeris and Nautical Almanac for 1897, Washington, 1895.

and Baille,* Poynting,† Boys,‡ Richarz and Krigar-Menzel,§ and Braun,‖ it must be said that the gravitation constant is uncertain by some units in the fourth significant figure, and possibly even by one or two units in the third figure;¶ thus falling along with the sun's parallax, the annual stellar aberration, and the moon's mass amongst the least well determined constants of the solar system. Here then is a fruitful field for research. The direct measurement of the gravitation constant to a much higher degree of precision seems to present insuperable obstacles; but may not the result be reached by indirect means; or, may it not be possible to make the solar system break its Sphinx-like reticence of the centuries and disclose the gravitational mechanism itself?

Just as the theories of astronomy and geodesy originated in the needs of the surveyor and navigator, so has the theory of elasticity grown out of the needs of the architect and engineer. From such prosaic questions, in fact, as those relating to the stiffness and the strength of beams, has been developed one of the most comprehensive and most delightfully intricate of the mathematico-physical sciences. Although founded by Galileo, Hooke, and Mariotte in the seventeenth century, and cultivated by the Bernoullis and Euler in the last century, it is, in its generality, a peculiar product of the present century.** It may be said to be the

* *Comptes rendus*, LXXVI. (1873).

† The Mean Density of the Earth, by J. H. Poynting, London, Charles Griffin & Co. (1894).

‡ *Philosophical Transactions*, no. 186 (1895).

§ *Sitzungsberichte*, Berlin Academy, Band 2 (1896).

‖ *Denkschriften*, Math. Natur. Classe, Vienna Academy, Bd. LXIV. (1897).

¶ The results of the investigators mentioned for the gravitation constant are, in C. G. S. units, as follows, the first result having been computed from data given by MM. Cornu and Baille in the publication referred to:

Cornu and Baille (1873)	6668×10^{-11}
Poynting (1894)	6698×10^{-11}
Boys (1894)	6657×10^{-11}
Richarz and Krigar-Menzel (1896)	6685×10^{-11}
Braun (1897)	6658×10^{-11}

Regarding these as of equal weight, their mean is 6673×10^{-11} with a probable error of ± 5 units in the fourth place, or 1/1330th part. This is of about the same order of precision as that deduced by Professor Newcomb from astronomical data.

** An admirable history of this science, dealing with its technical aspects, was projected by Professor Isaac Todhunter and completed by Professor Karl Pearson, under the title, A History of the Theory of Elasengineers' contribution of the century to the domain of mathematical physics, since many of its most conspicuous devotees, like Navier, Lamé, Rankine, and Saint-Venant, were distinguished members of the profession of engineering; and it is a singular circumstance that the first of the great originators in this field, Navier, should have been the teacher of the greatest of them all, Barré de Saint-Venant.* Other great names are also prominently identified with the growth of this theory and with the recondite problems to which it has given rise. The acute analysts, Poisson, Cauchy, and Boussinesq, of the French school of elasticians; the profound physicists, Green, Kelvin, Stokes, and Maxwell, of the British school; and the distinguished Neumann (Franz Ernst, 1798–1895), Kirchhoff (1824–1887), and Clebsch (1833–1872), of the German school, have all contributed heavily to the aggregate of concepts, terminology, and mathematical machinery which make this at once the most difficult and the most important of the sciences dealing with matter and motion.

The theory of elasticity has for its object the discovery of the laws which govern the elastic and plastic deformation of bodies or media. In the attainment of this object it is essential to pass from the finite and grossly sensible parts of media to the infinitesimal and faintly sensible parts. Thus the theory is sometimes called molecular mechanics, since its range extends to infinitely small particles of matter if not to the ultimate molecules themselves. It is easy, therefore, considering the complexity of matter as we know it in the more elementary sciences, to understand why the theory of elasticity should present difficulties of a formidable character and require a treatment and a nomenclature peculiarly its own.

While it would be quite inappropriate on such an occasion to go into the mathematical details of this subject, I would recall your attention for a moment to the surprisingly simple and beautiful concepts from which the mathematical in-

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ticity and the Strength of Materials from the time of Galilei to the present time. Cambridge: at the University Press—

Vol. I., Galilei to Saint-Venant, 1886;

Vol. II., Parts I. and II., Saint-Venant to Lord Kelvin, 1893.

A capital though abridged history of the science is given by Saint-Venant in his annotated edition of Navier's Résistance des Corps Solides, Troisième édition, Paris, 1864.

The history of Todhunter and Pearson is dedicated to Saint-Venant, who has been fitly called 'the dean of elasticians.'

* And this illustrious master has left a worthy pupil in M. J. Boussinesq, Professor in the Faculty of Sciences, Paris.

vestigation proceeds rapidly and unerringly to the equations of equilibrium or motion of a particle of any medium. The most important of these are the concept which relates to the stresses on the particle arising from its connection with adjacent parts of the medium, and the concept with regard to the distortions of the particle which result from the stresses. If the particle be a rectangular parallelopiped, for example, these stresses may be represented by three pressures or tensions acting perpendicularly to its faces together with three tensions acting along, or tangentially to, those faces. Thus the adjacent parts of the medium alone contribute six independent force components to the equations of equilibrium or motion; and the stresses, or the amounts of force per unit area, which produce these components are technically known as tractions or shears according as they act perpendicularly to or tangentially along the sides of the particle.* Corresponding to these six components there are six different ways in which the particle may undergo distortion. That is, it may be stretched or squeezed in the three directions parallel to its edges; or, its parallel faces may slide in three ways relatively to one another. These six different distortions, which lead in general to a change in the relative positions of the ends of a diagonal of the parallelopiped, are measured by their rates of change, technically called strains, and distinguished as stretches or slides according as they refer to linear or angular distortion.†

It is from such elementary dynamical and kinematical considerations as these that this theory has grown to be not only an indispensable aid to the engineer and physicist, but one of the most attractive fields for the pure mathematician. As Pearson has remarked, "There is scarcely a branch of physical investigation, from the planning of a gigantic bridge to the most delicate fringes of color exhibited by a crystal, wherein it does not play its part."‡ It is, indeed, fundamental in its relations to the theory of structures, to the theory of hydromechanics, to the elastic solid theory of light, and to the theory of crystalline media.

In closing these very inadequate allusions to this intensely

* The terminology here used is that of Todhunter and Pearson, History of the Theory of Elasticity and Strength of Materials, vol. I., note B.

† The terminology and symbology of the theory of elasticity appear to be more highly developed than those of any other mathematical science. A comparison of the terms and symbols of elasticity with those of the older subject of hydromechanics as shown in part on next page is instructive.

‡ History of Elasticity, etc., vol. 1, p. 872.

IN ELASTICITY.

Stresses.

Tractions $\begin{cases} p_{xx} \\ p_{yy} \\ p_{zz} \end{cases}$

Shears $\begin{cases} p_{yz} \\ p_{zx} \\ p_{xy} \end{cases}$

Strains.

Stretches $\begin{cases} s_x \\ s_y \\ s_z \end{cases}$

Slides $\begin{cases} \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{cases}$

Shifts, or components of displacement $\begin{cases} u \\ v \\ w \end{cases}$

Shift-fluxions, or space rates of change of shifts $\begin{cases} \frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial u}{\partial z}, \\ \frac{\partial v}{\partial x}, \text{ etc.} \\ \frac{\partial w}{\partial x}, \text{ etc.} \end{cases}$

Dilatation, $\theta = s_x + s_y + s_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

Twists $\begin{cases} \tau_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \tau_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \tau_{xz} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{cases}$

Displacement potential in irrotational, or pure, strain.

IN HYDROMECHANICS.

Fluid pressure
p

Component velocities $\begin{cases} u \\ v \\ w \end{cases}$

Space rates of change of component velocities $\begin{cases} \frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial u}{\partial x}, \\ \frac{\partial v}{\partial x}, \text{ etc.} \\ \frac{\partial w}{\partial x}, \text{ etc.} \end{cases}$

Expansion, $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

Component spins, or components of molecular rotation $\begin{cases} \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{cases}$

Velocity potential in irrotational motion.

practical and abstrusely mathematical science, it is fitting that attention should be called to the magnificent labors of the 'dean of elasticians,' M. Barré de Saint-Venant. It was the object of his life-work to make the theory of elasticity serve the utilitarian purposes of the engineer and at the same time to divest it so far as possible of all empiricism. His epoch-making memoir * of 1853 on the torsion of prisms is not only a classical treatise from the practical point of view but one of equal interest and importance in its theoretical aspects. His investigations are everywhere delightfully interesting and instructive to the physicist; and many parts of them are replete, as observed by Kelvin and Tait,† with "astonishing theorems of pure mathematics, such as rarely fall to the lot of those mathematicians who confine themselves to pure analysis or geometry, instead of allowing themselves to be led into the rich and beautiful fields of mathematical truth which lie in the way of physical research." More important still in a didactic sense, are his annotated edition of Navier's Résistance des Corps Solides of 1864, and his annotated edition of the French translation of the Theorie der Elasticität fester Körper of Clebsch, which appeared in 1883. These monumental works, whose annotations and explanatory notes quite overshadow the text of the original authors, must remain for a long time standard sources of information as to the history, theory, methods, and results of this complex subject. The luminous exposition, the penetrating insight into physical relations, and the scientific candor in his criticism of other authors, render the work of Saint-Venant worthy of the highest admiration.

Closely allied to the theory of elasticity, though historically much older, is the science of hydromechanics. The latter is, indeed, included essentially in the former; and probably the great treatises of the next century will merge them under the title of molecular mechanics. It may seem somewhat singular that the mathematical theory of solids should have originated so many centuries later than the theory of fluids; for at first thought, tangible though flexible solids would appear much more susceptible of investigation than mobile liquids and invisible gases. But a little reflection leads one to the conclusion that it was, in fact, much easier to observe the data essential to found a theory of hydromechanics than it was to discover the principles

* "Mémoire sur la torsion des prismes, etc.," published in *Mémoires des Savants étrangers*, Tome XIV., 1855.

† Natural Philosophy, second ed., Part II., p. 249.

which lead to the theory of stress and strain; and the time-interval between Archimedes and Galileo may serve perhaps as a rough measure of the relative complexity of hydrostatics and the theory of flexure and torsion of beams. It must not be inferred, however, that the simplicity of the phenomena of fluids in a state of relative rest extends to the phenomena of fluids in a state of relative motion; for the gap which separates hydrostatics from hydrokinetics is one which has not yet been fully bridged even by the aid of the powerful resources of modern mathematics.

The elements of hydrokinetics, with which branch of hydromechanics this sketch is alone concerned, were laid down by Euler about the middle of the last century.* It is to him that we owe the equations of motion of a particle of a 'perfect fluid.' This is an ideal fluid, moving without friction, or subject, in technical terminology, to no tangential stress. But while no such fluids exist, it is easily seen that under certain circumstances this assumed condition approaches very closely to the actual condition; and, in accordance with the method of mathematico-physical science in disentangling the intricate processes of nature, progress has proceeded by successive steps from the theory of ideal fluids towards a theory of real fluids.

The history of the developments of hydromechanics during this century has been very carefully and completely detailed in the reports to the British Association for the Advancement of Science of Sir George Gabriel Stokes,† in 1846, and of Professor W. M. Hicks,‡ in 1881 and 1882. And the history of the subject has been brought down to the present time by the address of Professor Hicks before Section A of the British Association for the Advancement of Science in 1895, and by the report § of Professor E. W. Brown to Section A of the American Association for the Advancement of Science in 1898. It may suffice here, therefore, to glance rapidly at the salient points which mark

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* "Principes généraux du mouvement des fluides." *Histoire de l'Académie de Berlin*, 1755.

"De principiis motus fluidorum." *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ*, Tomus XLV., Pars I., pro anno 1759.

† "Report on recent researches in hydrodynamics," B. A. A. S. Report for 1846.

‡ "Report on recent progress in hydrodynamics," B. A. A. S. Reports for 1881, 1882.

§ "On recent progress towards the solution of problems in hydrodynamics," A. A. A. S. Report for 1898.

Reference should be made also to Professor A. E. H. Love's paper "On recent English researches in vortex-motion" in the *Mathematische Annalen*, Band 30, 1887.

the advances from the state of the science as it was left by Lagrange a hundred years ago.

The general problem of the kinetics of a particle of 'perfect fluid' is easily stated. It is this:* given for any time and for any position of the particle its internal pressure, its density, and its three component velocities, along with the forces to which it is subject from external causes; to find the pressure, density, and velocity components corresponding to any other time and any other position. There are thus, in general, five unknown quantities requiring as many equations for their determination. The usual six equations of motion, or the equations of d'Alembert, contribute only three to this required number, namely, the three equations of translation of the particle, since the three which specify rotation vanish by reason of the absence of tangential stress. A fourth equation comes from the principle of the conservation of mass, which is expressed by equating the time rate of change of the mass of the particle to zero. This gives what is technically called the equation of continuity. A fifth equation is usually found in the law of compressibility of the fluid considered.†

Now, the equations of rotation, as just stated, refuse to answer the question whether the particles proceed in their orbits without rotation or whether they undergo rotation along with their motion of translation. This was a critical question, for the failure to press and to answer it seems to have retarded progress for nearly half a century. Lagrange, and after him Cauchy and Poisson, knew that under certain

*The statement here given is that of the 'historical method,' which seeks to follow a particle of fluid from some initial position to any subsequent position and to specify its changes of pressure, density, and speed. What is known as the 'statistical method,' on the other hand, directs attention to some fixed volume in the fluid and specifies what takes place in that volume as time goes on.

† The five equations in question are

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{d\,(V\rho)}{dt} = 0,$$

$$\frac{dv}{dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{dw}{dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}, \qquad p = f\,(\rho);$$

in which p is the pressure and ρ is the density at the centroid (x, y, z) of the particles; V is its volume; u, v, w are its component velocities; and X, Y, Z are the force components per unit mass arising from external causes.

conditions* the differential equations of motion are integrable, but they do not appear to have understood the physical meaning of these conditions. It remained for Sir George Gabriel Stokes to show that the Lagrangian conditions of integrability correspond to the case of no molecular rotation, thus clearly distinguishing the two characteristic types of what we now call irrotational and rotational motion.† Such was the great step made by Stokes in 1845 ; and it furnishes a forcible illustration of the importance, in applied mathematics, of attending to the physical meaning of every symbol and every combination of symbols.

Thirteen years later came the remarkable memoir of Helmholtz (1821–1894) on the integrals of the equations of hydrokinetics for the case of rotational, or vortex, motion.‡ This memoir is alike wonderful for the directness with which the mathematical argument proceeds to its conclusions and for the clearness of insight it affords of the physical phenomena discussed. In short, it opened up a new realm and supplied the results, concepts, and methods which led the way to the important advances in the science made during the past three decades.

Another powerful impulse was given to hydrokinetics, and to all other branches of mathematical physics as well, by Kelvin and Tait's Natural Philosophy—the Principia of the nineteenth century—the first edition of which appeared in 1867. From this great work have sprung most of the ideas and methods appertaining to the theory of motion of solids in fluids, a theory which has yielded many interesting and surprising results through the researches of Kirchhoff, Clebsch, Bjerknes, Greenhill, Lamb, and others. Of prime importance also are the numerous contributions of Lord Kelvin to other branches of hydrokinetics, and particularly to the theory of rotational mo-

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* That is, when $udx + vdy + wdz$ is a perfect differential, u, v, w being velocity components ; or, when

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

which are the doubles of the components of molecular rotation, vanish, these latter being the conditions for the existence of a velocity potential.

† This discovery of Stokes was announced in his fundamental paper "On the theories of internal friction of fluids in motion, and of the equilibrium and motion of elastic solids," *Transactions of the Cambridge Philosophical Society*, vol. 8. Reprinted also in his Mathematical and Physical Papers, vol. I.

‡ "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle's Journal für die reine und angewandte Mathematik*, 1858.

tion.* In fact, every department of the entire science of hydromechanics, from the preliminary concepts up to his vortex atom theory of matter, has been illuminated and extended by his unrivalled versatility.

When we turn to the more intricate branch of the subject which deals with the motion of viscous fluids, or with the motion of solids in such fluids, it appears that the progress of the century is less marked, but still very noteworthy. This branch is closely related to the theory of elasticity, and goes back naturally to the early researches of Navier, Poisson, and Saint-Venant; but the revival of interest in this, as well as in the less intricate branch of the subject, seems to date from the fruitful memoir † of Stokes of 1845, and from his report to the British Association for the Advancement of Science of 1846. Since then many interesting and useful problems relative to the flow of viscous fluids and to the motion of solids in such media, have been successfully worked out to results which agree fairly well with experiment. But on the whole, nothwithstanding the searching investigations in this field of Stokes, Maxwell, Helmholtz, Boussinesq, Meyer, Oberbeck, and many others, it must be said that difficulties, both in theory and in experiment, of a formidable character remain to be surmounted.‡

Of all branches of hydromechanics there is none of so great practical utility and of such widely popular interest as the theory of tides and waves. These phenomena of the sea are appreciable to the most casual observer; and there has been no lack of impressive descriptions of their effects from the days of Curtius Rufus down to the present time. The mechanical theory of tides and waves is, however, a distinctly modern development whose perfection must be credited to the labors of the mathematicians of the present century. §

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* "On vortex motion," 1867. *Transactions of the Royal Society of Edinburgh*, vol. 25.

† Cited on p. 160.

‡ An extremely interesting method of experimental investigation has been recently applied with success by Professor Hele-Shaw. See a paper by him on "Stream-line motion of a viscous film," and an accompanying paper by Sir G. G. Stokes on "Mathematical proof of the identity of the stream lines obtained by means of a viscous film with those of a perfect fluid moving in two dimensions." Report of British Association for the Advancement of Science for 1898.

§ An excellent summary of the history and theory of tides, and of methods of observing and predicting them, is given by Dr. Rollin A. Harris in his Manual of Tides, published as Appendices 8 and 9 of the Report of the United States Coast and Geodetic Survey for 1897.

Here, again, progress is measured from the advanced position occupied by Laplace, who was the first to attempt a solution of the tidal problem on hydrokinetic principles. After the fundamental contributions of Laplace, contained in the second and fifth volumes of the Mécanique Céleste, the next decisive advance was that made by Sir George Airy (1801–1892) in his article on Tides and Waves, which appeared in the Encyclopedia Metropolitana in 1842. A quarter of a century later came the renaissance, started undoubtedly by the great memoir of Helmholtz and by the Natural Philosophy of Kelvin and Tait, along with Lord Kelvin's inspiring communications on almost every phase of wave and tidal problems to scientific societies and journals. Then followed the decided theoretical improvements in tidal theory of Professor William Ferrel,* particularly in the development of the tide generating potential and in the determination of the effects of friction. And a little later there appeared the novel investigations of Professor G. H. Darwin, who, in addition to furnishing a complete practical treatment of terrestrial tides,† has extended tidal theory to the solar system and thrown an instructive light on the evolutionary processes whence the planets and their satellites have emerged and through which they are destined to pass in the future.‡

As we reflect on the progress which has been thus summarily and quite inadequately outlined, it will appear that the mathematicians of the nineteenth century have contributed a splendid aggregate of permanent accessions to knowledge in the domain of the more exact of the physical sciences. And as we turn from the certain past to the less certain future one is prone to conjecture whether this brilliant progress is to continue, and, if so, what part the American Mathematical Society may play in promoting further advances. With respect to these inquiries I should be loath to hazard a prediction or to offer advice. But there appears to be no reason for entertaining anything other than optimistic expectations. The routes along which ex-

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* Tidal Researches. Appendix to Report of United States Coast and Geodetic Survey for 1874. Washington, 1874.

† In article on Tides in Encyclopedia Britannica, 9th edition.

‡ Darwin's investigations are published in a series of papers in the *Philosophical Transactions of the Royal Society of London*, Parts I., II., 1879; Part II., 1880; Part II, 1881; Part I., 1882. They are republished in part in Appendix G, Thomson and Tait's Natural Philosophy, second edition. See also the capital semi-popular work, The Tides and Kindred Phenomena in the Solar System, by G. H. Darwin, Boston and New York, Houghton, Mifflin and Co., 1898.

ploration may proceed are numerous and attractive. We have only to follow the example set by Laplace, Poisson, Green, Gauss, Maxwell, Kirchhoff, Saint-Venant, Helmholtz, and their eminent contemporaries and successors. In commending the works of these great masters, to the younger members especially of the AMERICAN MATHEMATICAL SOCIETY, I would not be understood as urging the cultivation of pure mathematics less, but rather as suggesting the pursuit of applied mathematics more. The same sort of fidelity to research and the same sort of genius for infinite industry which enabled those masters to accomplish the grand results of the nineteenth century may be confidently expected to achieve equally grand results in the twentieth century.

THE STATUS OF IMAGINARIES IN PURE GEOMETRY.

(Read before the American Mathematical Society, October 28, 1899.)

IN teaching the elements of analytical geometry we are practically forced to allow, even to encourage, a slipshod identification of the field of geometry with the field of algebra. We must all have realized the disadvantages attendant on this course. If ever we have the chance of repairing the error—if error indeed it be at that stage—it is in teaching synthetic geometry; but we can repair it then only if we can establish the existence of imaginary elements without the slightest dependence on algebra. Many books refer to the analogy of algebra as affording sufficient basis, others openly rely on algebraic principles; Chasles, for instance, in the Géométrie Supérieure (pp. 54–57) relies essentially on quadratic equations, whose imaginary roots assure him of the existence of imaginary points.

The two chief books that deal with absolutely pure geometry are those of von Staudt and Reye. It is one of the axioms of modern mathematics that von Staudt placed the doctrine of imaginaries on a firm geometrical basis; but logical and convincing as his treatment is, when patiently studied in all its detail, it yet seems to me hardly practicable as a class-room method.

Von Staudt's primary domain is the visible universe; the elements of his geometry, together with the idea of direction, are an intellectual abstraction from the results of observation. He then extends his domain beyond the visible universe by formal definition; to replace the idea of direction he introduces a set of "ideal points," and