A wise moderation has been exercised in the selection of topics, and the arrangement of the rich material is pedagogical as well as logical. The concept of the analytic function takes the central place, everything else is subservient to it, either by preparing for it or by illustrating it. By this means, a harmony and equilibrium between the different parts is attained, which impart to the whole in a remarkable degree the character of an organic unity.

Thus the authors have succeeded in producing not only a work of high scientific and pedagogical value but at the same time of a singular beauty and elegance. But there are numerous beauties of detail as well, for which, however, the reader must be referred to the book itself. A certain freshness and originality pervade the whole, even in places where the authors follow along beaten tracks, and give at every turn evidence of the complete mastery of the subject with which the book is written.

Oskar Bolza.

University of Chicago,
June 30, 1899.

McAULAY'S OCTONIONS.

Octonions. By A. McAulay, M.A. Cambridge, The University Press, 1898. 8vo, 253 pp.

This treatise is a development of Clifford's biquaternions with applications to ordinary space. The starting point of the analysis is quaternions, combined later with methods from Grassman's Ausdehnungslehre.

The development is open to criticism, as a work for beginners, because of its extremely refined formal character. This is perhaps unavoidable because the book was first compiled as a scientific paper, and, as the author says, he did not feel justified in recasting the whole appropriately. Discussions of formal laws and expert reasoning upon terms that are imperfectly defined, relying upon subsequent developments to bring out their full meaning, are not conducive to clearness of apprehension on the part of a learner. Perhaps as good a review, therefore, as can be made of the book, which is indeed an extensive and thorough development of what must prove to be a valuable analysis, is to give a brief and clear exposition of the octonion system.

The octonion is a quantity defined by four numbers called its *tensor*, *scalar*, *convert*, and *pitch*, and an axis having a fixed position in space. This makes in all eight numbers, upon which the octonion depends, and hence its name. Let q, r be two quaternions, and O a given point. Then q, r, O determine the octonion as follows: the four numbers, as named above, are Tq, Sq, Sr, and $S{\cdot}r/q$; the axis is a line parallel to Vq, whose vector distance from O is $V{\cdot}Vr/Vq$. McAulay expresses this octonion by $Q = q + r\Omega$, where Ω is a unit of separation; but it is better, because different points may be employed in the definition of the same octonion, to exhibit the point of reference, say $Q = qO + rO\Omega$; and we shall presently see that we may write $O\Omega = \Omega$.

If instead of O, we employ another point P, at a vector distance ρ from O, certain changes must be made in r to give the same octonion; q evidently will remain unchanged since the numbers Tq, Sq, and the direction of the axis (parallel to Vq), completely determine q. Since Sr remains unchanged, r can change only by a vector σ, and since $S{\cdot}r/q$ remains unchanged, therefore $S{\cdot}\sigma/q = 0$, or σ is a vector perpendicular to Vq. Finally, $\sigma/Vq =$ increase of perpendicular distance of the axis $=$ component of PO perpendicular to $Vq = - V\rho Vq{\cdot}/Vq$, and hence $\sigma = - V\rho Vq$. Thus,

$$Q = qO + rO\Omega = qP + (r - V\rho Vq)P\Omega.$$

When $Vq = 0$, the axis is either at infinity ($Vr \neq 0$) or indeterminate ($Vr = 0$); in either case the quaternion components of Q are independent of the reference point, so that it may be omitted. In particular, $rO\Omega = rP\Omega = r\Omega$ say; so that in effect $O\Omega = P\Omega = \Omega$.

Addition is defined by adding coefficients of corresponding units, *viz.*, if

$$Q = qO + r\Omega = qP + (r - V\rho Vq)\Omega, \quad Q' = q'O + r'\Omega = \text{etc.},$$

then, $$Q + Q' = (q + q')O + (r + r')\Omega = (q + q')P + [r + r' - V\rho V(q + q')]\Omega.$$

Here we add Q and Q' as to the reference point O, and also, as to the reference point P, and find the two sums to be equal. Addition is thus seen to be independent of the reference point O or P, and this means, of course, that the sum is determined wholly by the numbers and axis that define each term of the sum. Addition is also, from quaternion laws, associative and commutative.

Multiplication is defined by making it distributive and the units O, Ω commutative with everything, with the table $O^2 = O$, $\Omega^2 = 0$, and $O\Omega = \Omega O = \Omega$ as found above. It follows that multiplication is also independent of the reference point, viz.:

$$QQ' = qq'O + (qr' + rq')\,\Omega = qq'P + (qr' + rq' - V\rho\,Vqq')\Omega,$$
$$= qq'P + (qr' + rq' - q\,V\rho\,Vq' - V\rho\,Vq\cdot q')\,\Omega,$$
since $$V\cdot\rho\,Vqq' = q\,V\rho\,Vq' + V\rho\,Vq\cdot q'.$$

Multiplication is also associative, but not in general commutative. An octonion scalar $x + y\Omega$ is commutative with everything; nonaxised octonions are commutative with each other; a nonaxised and an axised octonion are commutative only when the (secondary) direction (Vr) of the first is parallel to the axis of the second; axised octonions are commutative only when their axes coincide.

Any octonion may be resolved into four commutative factors in one and only one way, viz.:

A *tensor* Tq, an *additor* $1 + a\Omega$, a *versor* $Uq\cdot A$, and a *translator* $1 + b\,Vq\cdot\Omega$. The product of these four is, $Q = qA + r\Omega$, where A is a point of the axis, since $Vr \| Vq$. The above factors are denoted by T_1Q, T_2Q, U_1Q, U_2Q, respectively; also

$$TQ = T_1Q\,T_2Q = \textit{augmentor of } Q = \textit{octonion tensor},$$
$$UQ = U_1Q\cdot U_2Q = \textit{twister of } Q = \textit{unit octonion}.$$

In terms of q, r, A, we have

$$TQ = Tq(1 + S\cdot r/q\cdot\Omega),\ \ UQ = Uq\cdot A(1 + V\cdot r/q\cdot\Omega).$$

The above names are derived from the effects of these factors on a vector octonion or *motor*, whose axis intersects the axis of the multiplier at right angles. Any motor $\alpha A + \beta\Omega$ is characterized by a tensor ($T\alpha$), pitch ($S\cdot\beta/\alpha$), and axis ($V\beta/\alpha$ *from* A); of these Tq affects only $T\alpha$ (the tensor of a product = product of the tensors of the factors); $1 + a\Omega$ affects only the pitch, adding to it a, the pitch of the multiplier (pitch of product = sum of pitches of the factors); $1 + V\cdot r/q\cdot\Omega$ translates the axis of the motor the distance $V\cdot r/q$; $Uq\cdot A$ turns the axis of the motor round the axis of Q through the angle of q. Conversely, any octonion is the ratio of two motors, and in fact, octonion and motor stand in very much the relationship of quaternion and vector.

The symbols S, V, K have application as invariant operations, viz.:

$$SQ = Sq + Sr\Omega,\ VQ = Vq\cdot O + Vr\cdot\Omega,\ KQ = Kq\cdot O + Kr\cdot\Omega.$$

A complex scalar (SQ) is self-conjugate, and its square is a positive scalar (if the first term (Sq) determines the sign).

A complex vector, *i. e.*, a motor, is conjugate to its negative, and its square is a negative scalar. McAulay changes VQ to MQ because V does not connote with motor, but such change is unnecessary, and is not given weight in the equally important cases of T for augmentor, U for twister, s for convert, t for pitch, etc.

We have $Q = SQ + VQ$, $KQ = SQ - VQ$, $QKQ = KQ \cdot Q = (SQ)^2 - (VQ)^2 = (TQ)^2$ $K \cdot QQ' = KQ' \cdot KQ$, etc. If $QQ' = 0$, then either $Q = 0$ or $Q' = 0$ or $Q = r\Omega$, $Q' = r'\Omega$.

All results not depending upon divisions by a nil octonion $r\Omega$, are therefore identical in form with quaternion results, but with wider geometrical and physical meanings. For example, a motor $\alpha O + \beta\Omega = \alpha P + \beta'\Omega$, may be represented by the velocity system of a rigid body in which α is the vector angular velocity and β, β' the linear velocities at O, P. The axis is the instantaneous axis of rotation. This is what Ball calls a twist about a screw; the pitch of the motor is the pitch of the screw, which is right or left handed according as the pitch is positive or negative. If A be a point of the axis the motor takes the form $\alpha A + c\alpha\Omega$ where c is the pitch and $c\alpha$ the translation vector. The motor is also represented by a system of forces acting on a rigid body, which reduces to what Ball calls a wrench about a screw; α is the resultant force, and β, β' the moments of the system about O, P. It is also represented by a system of impulses, whose momentum is α, and moments of momentum about O, P are β, β'.

It will now be seen in what respect this analysis can prove valuable, and McAulay has not only developed this analysis very thoroughly in respect to linear functions of octonions, differentiations, nabla, and McAulayan differentiators, but he has applied it to develop and extend the results of Ball's theory of screws. It would seem that the "A process" which was presented by the reviewer before the Indiana academy of science in 1897, would prove a valuable addition to this theory. It is also possible that like quaternions (BULLETIN, November, 1897) the system may be extended to fourfold space with complete representation of the octonion.

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