sult from the notion of homography. But this notion is of no avail when it is a question of a space possessed only of a metric correspondence by reciprocal polars; this is the case for real space supposed non-euclidean, and it is for this reason, Fontené avers, that the metric properties of a general correlation in real space have not been studied. The author studies them in hyperspace; his theory is readily applicable to real space by introducing the notions parameter of a ray and parameter of an axis. This second interpretation of the theory of the memoir is indicated in the last chapter of the work.

EDGAR ODELL LOVETT.

PRINCETON, NEW JERSEY.

STAHL'S ABELIAN FUNCTIONS.

Theorie der Abel'schen Functionen. Von DR. HERMANN STAHL, Professor of Mathematics in the University of Tübingen. Leipzig, Teubner, 1896. 8vo, 354 pp.

DURING the last few years the literature on abelian functions has been enriched by three important treatises. The extent and scope is different in each case, so that no one of them will supplant another.

In the work under review, the author presupposes a ready knowledge of elliptic, and some familiarity with hyperelliptic functions; and an extensive knowledge of higher plane curves. His aim is to construct a serviceable bridge from the older to the newer parts of the theory, thus filling up a decided gap between the older books and the later memoirs.

The book is divided into two parts of about equal extent; the first deals with the algebraic function and its integral, while the second is concerned with inversion. Each part is then divided into four chapters. The author introduces his subject by giving a brief summary of the problems in elliptic functions, and shows how each has a natural generalization.

The first chapter is devoted to the treatment of the n branches of the algebraic function $F(x, y) = 0$, and a description of the associated Riemann's surface. The functional element is derived in the vicinity of various kinds of points, under the same restriction as was made by Clebsch and Gordan as to the nature of the branch points. Throughout the chapter the matter is presented in a neat, forcible and connected way, but so brief and concise that to one unfamiliar with the treatment it would be too difficult reading. The only problem that is treated at length is the transformation of the surface into the normal form (Clebsch-Lüroth theorem). The procedure is essentially the same as that given by Riemann, whose treatment the author follows (nearly) throughout. This is a valuable epitome after one has read the corresponding chapter in more elementary works.

The second chapter is devoted to rational functions of x and y. The proofs are purely algebraic; but in language the functions represent curves, the zero points being defined as points of intersection with the fundamental curve $F(x, y) = 0$. The fundamental curve is supposed to have no higher singularities than double points with distinct tangents. Then follow the criteria which a rational function must satisfy to be an integral function, some of them being given without proof, but in such cases a reference is usually given to a memoir containing the proof. The rest of this chapter widely departs from Riemann's treatment, following that given by Brill and Noether in volume 7 of the *Mathematische Annalen*.

The adjoint functions $\varphi(x, y)$ are defined as curves which pass once through all of the double points of $F(x, y)$, and the two cases are considered separately, according as the order of φ is greater or equal to $n - 3$. The treatment of the set of points of intersection and the series of coresidual curves is rather obscure; especially as the reader is not given any hint of their use until later.

After proving the Riemann-Roch theorem algebraically, the author proceeds with the construction of a rational function when its poles and residues are given. The treatment here is admirable; the problem is clearly stated at the outset, and comparisons drawn from the similar problem with one variable. Finally, the determination of the p relations among the coördinates of the poles and the residues is also obtained algebraically, incidentally giving the p independent integrands of the first species. This seems decidedly clearer than Riemann's method of obtaining this result from integrals of the second kind.

The third chapter is devoted to the abelian integrals. The moduli of periodicity are introduced so briefly and in such an abstract manner as to convey little meaning to a beginner. An illustrative example would greatly assist in fixing the idea in the reader's mind.

The classification of the integrals, their normal forms, the residue theorem, the relation between the moduli of periodicity and the coördinates of the singular point and the interchange of argument and parameter are made very easy by means of the matter in the previous chapter. The integrals of the second kind are derived by differentiating the normal integral of the third kind, at its singular point. The reference to Riemann is here misleading, because in his treatment the process is reversed.

The chapter closes with a proof of Abel's theorem; here the particular cases are stated more explicitly and in greater detail than those of any other part.

The last chapter (IV) in the first part treats of birational transformation; it commences by stating what constants and functions remain invariant by such transformation. The matter is arranged more didactically, and it is quite easy to follow the argument. Riemann's proof for the invariance of p is outlined, then a fuller proof is given, more closely following the method of Clebsch and Gordan. Three associated (adjoint) φ functions are used as transforming functions; the number of moduli is derived from the Jacobian of the three φ curves. The quotient of two associated functions is shown to be invariant, and by means of this, a normal invariant form of the fundamental equation $F(x, y) = 0$ is found. Finally, a brief outline of an extension is given when the defining curve is a curve in space of p dimensions; the differentials are expressed in homogeneous coördinates.

The second part, the inversion problem, is prefaced by a short review of the ϑ function of one variable, and a suggestion that each problem can be generalized.

The subject begins with the definition of the ϑ functions of p variables and their zero points; they are introduced independently of preceding matter by means of their defining functional equation. By a transformation of variables their pseudo-periods are brought to the form suggested in the previous chapters, then the p normal integrals of the first kind are introduced as arguments. This seems to me to be the clearest introduction extant of the form and definition of the general ϑ function.

Theta functions of the first order and having an integral characteristic are then discussed in greater detail, the procedure is outlined for the general case and that of the half-integer characteristic is quite fully treated. Several theorems for the general case are given without proof.

The number and the position of the zero points are next determined by means of Cauchy's theorem, applied to the dissected Riemann's surface.

Finally, several special cases are derived in which the ϑ function is identically zero, and the uniform solvability of the inversion problem is discussed. Many of the proofs of propositions hitherto generally known are given in a neat connected form that is quite new. The success of the author's undertaking is here particularly manifest.

The sixth chapter treats of the solution of the inversion problem, beginning with the zero points of the ϑ function whose arguments are integrals in the canonical form ; the radical functions $\sqrt{\phi(x, y)}$ are next introduced. The curve $x = 0$ is an adjoint curve of degree $n - 2$, which passes through the $n - 2$ points of intersection of $F(x, y) = 0$ and one of its tangents, and also touches F in p points. These curves are called contact curves.

The next few sections give relations between the square of the ratio of two ϑ functions and the ratio of two functions ϕ. The argument is confined to ϑ functions with half-integer characteristics ; two sets of theorems are usually given, to apply when the characteristic is even or odd. In the latter case, the function ϕ is replaced by $(ax)\,\varphi$, $(ax) = 0$ being the equation of the tangent to F at a. The language is usually geometric, but the proofs are uniformly algebraic.

A paragraph is devoted to the determination of ϕ, and its relation to the characteristic μ, which is very plain. An example for $p = 3$ follows. The chapter closes with a discussion of the relation between ϑ and the normal integrals of the first kind. A number of unsolved problems are here stated, with references to all the literature bearing upon them. This and the succeeding chapters show the influence of Stahl's own researches, even where frequent references are given to Riemann.

The seventh chapter deals with the representation of algebraic functions by means of ϑ functions. The general function having as many zeros as poles, which is continuous on the dissected surface, and assumes a constant factor when it crosses a cross cut, can be represented by ϑ functions and an exponential factor. When the constant factor is unity the function becomes rational. Conversely, every rational function can be represented by ϑ functions. When the arguments ν of the ϑ function are defined by

$$\nu_h \equiv \sum_{k=0}^{q} \int_{a_k}^{x_k} du_k \qquad (h = 1, 2 \cdots p)$$

(q independent of p), a rational function can not in general be represented by ϑ functions, but the mth power can, wherein m is the least common denominator of the elements of the characteristic. A rational function R_x of order σ is shown to be expressed by the relation

$$R_x - R_a = c\prod_{k=1}^{\sigma} \frac{E(x, a_k)}{E(x, b_k)}$$

where E is the prime function of Schottky. The abelian functions are shown to have an addition theorem. Finally, algebraic functions are expressed by means of ϑ functions and derivatives of the normal integrals of the second and third kind. The addition theorem for these integrals is derived in explicit form. This is the only use made of them in the second part of the book.

The last chapter (VIII) discusses the linear transformation of the ϑ functions. The first problem is to consider the effect of a linear transformation of the moduli of periodicity of the integrals on the form of the crosscuts on the associated surface. Every canonical transformation (*i. e.*, with determinant unity) is equivalent to passing from one system of cross-cuts to another, and conversely.

Several figures are shown to illustrate the effect of specific elementary transformations.

The effect of the transformation on the period characteristics is next obtained; then follows the linear tranformation of the thetafunctions and its effect on the theta characteristic. In the last discussion the two kinds of characteristics are compared and contrasted.

Many of the older theorems have more rigorous proofs than previous works contained; a large number of references makes comparison and future reading quite accessible.

Mechanically, the book contains one rather unusual feature, that of "hanging indentations" of a theorem or discussion, which greatly assists in showing the connection. The book is unfortunately not provided with an index.

VIRGIL SNYDER.

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