which the author would have regarded as available by which this problem can be attacked: *first*, the method suggested by Briot and Bouquet (théorie des fonctions elliptiques §130), or the modification of this method suggested by C. Neumann (Abelsche Integrale, Chap. VI., §2); and *second* the theory of implicit functions of two real variables involving the use of Jacobians.* It is to be regretted that the author did not include a brief and elementary account of this last mentioned method, which has so many other important applications, rather than some of the more difficult parts of Chapter III.

After all has been said however the volume before us remains an excellent treatment of the subject; good as an introduction, in so far as it does not prove too difficult; excellent for the mature student who already knows something of the subject; and invaluable to the teacher.

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DARBOUX'S ORTHOGONAL SYSTEMS.

Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes. Par GASTON DARBOUX, Membre de l'Institut, Doyen de la Faculté des Sciences et Professeur de Géométrie Supérieure à l'Université de Paris. Tome I, Paris, Gauthier-Villars et fils, 1898. 8vo, i+338 pp.

THE present volume is the fifth which Professor Darboux has prepared during the last decade for the Course in Geometry of the Faculty of Sciences of the University of Paris. This new work is to be devoted to the exposition of a theory which has its origin in the writings of Lamé and which has been the subject of a large number of researches in recent years. It is a direct sequel to the author's admirable treatise on the theory of surfaces in which he presented incidentally various properties of orthogonal systems and curvilinear coördinates, but reserved the organic and systematic development of these theories for a separate treatise of which the above is the first volume. The work glistens with originality both in material and in modes of presentation; the exposition exhibits the elegance and clearness characteristic of the author's writings, and the volume, as

* Cf. Jordan, Cours d'Analyse, vol. 1, pp. 80-89.

a whole, reveals abundant additional proof of how worthily the mantle of Chasles is worn by Darboux.

BOOK I. THE EQUATION OF THE THIRD ORDER.

1. The families of Lamé and Dupin's theorem with its reciprocal occupy the first chapter. The parameters α, β, γ of three orthogonal families, considered as functions of the rectilinear coördinates x, y, z of a point in space, satisfy a certain system of partial differential equations; a general theorem of Cauchy shows that the general solution of this system depends on three arbitrary functions of two variables. As an application of this theorem Darboux shows that we can determine a triply orthogonal system by the condition that the three families of surfaces which compose it intercept on a given surface given curves subject to the condition that the curves shall not cut at right angles. The elimination of one of the parameters leads immediately to the theorem of Dupin; surfaces composing a triply orthogonal system mutually intersect along their lines of curvature. The reciprocal of this theorem, namely the theorem: If we have two families of surfaces cutting orthogonally along lines which are lines of curvature for the surfaces of one of the two families, there exists a third family completing the orthogonal system, permits of proving that the parameter of every family forming part of a triply orthogonal system ought to satisfy a partial differential equation of the third order; this differential equation is at the same time necessary and sufficient, *i. e.*, every solution of it leads to a triple system. Every family of surfaces which can constitute a part of a triple system Darboux calls a *family of Lamé*. The existence of the defining differential equation of the third order was first recognized by Darboux* in 1866; its developed form was given a few years later by Cayley.† In order to facilitate the formation of this differential equation Darboux introduces various properties of the differential operator

$$\delta_u v = \delta_v u = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}. \qquad (1)$$

* Darboux, "Sur les surfaces orthogonales," *Annales de l'École Normale*, 1st series, vol. 3 (1866), pp. 97–141.

† Cayley, "Sur la condition pour qu'une famille de surfaces donnée puisse faire partie d'un système triple orthogonal," *Comptes rendus*, vol. 75 (1872), pp. 116, 177, 246, 324, 381, 1800; Collected Mathematical Papers, vol. 8.

He then establishes by an entirely new process that the whole difficulty of the problem of orthogonal systems is referred to the integration of the equation of the third order, which Darboux writes in the form of a very simple determinant of the sixth order. After verifying the results obtained previously by Bouquet* relative to families represented by an equation of the form $u = \varphi(x) + \psi(y) + \chi(z)$ and by V. Puiseux† relative to a particular system of axes, Darboux brings the differential equation under a new and irrational form by introducing the derivatives of the function H defined by the relation

$$\frac{1}{H} = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}. \qquad (2)$$

The chapter concludes with an application of the theory of characteristics of partial differential equations and of systems of such equations, for the case of three independent variables, to the problem in hand. The results furnish the theorem that we can always determine a triple system by the condition that the surfaces (A), which compose one of its families, cut a surface (Σ) along given curves (C) and have, along these curves, a contact of the second order with certain surfaces (S) which contain these curves, unless the curves (C) are lines of curvature of the surfaces (S), or the surfaces (S) are orthogonal to the surface (Σ); in these exceptional cases the problem becomes either impossible or indeterminate.

2. The second chapter studies triple systems containing a family of planes or a family of spheres. Every family of planes or of spheres is capable of forming a part of a triply orthogonal system. Those triple systems which contain a family of planes are obtained by drawing two families of rectangular curves in a plane and rolling this plane on any developable surface. Darboux shows also that all consideration of rolling may be avoided and these systems constructed without integration. After giving Lamé's definition of curvilinear coördinates and deriving the form of the lineal element of space for the preceding system, the author establishes equations capable of defining the system and containing no signs of quadrature. If the family of planes is given *a priori*, the determination of the orthogonal

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* Bouquet, "Note sur les surfaces orthogonales," *Liouville's Journal*, 1st series, vol. 11 (1846), p. 446.

† V. Puiseux, "Note sur les systèmes de surfaces orthogonales," *Liouville's Journal*, 2d series, vol. 8 (1863), p. 335.

trajectories, and consequently that of the corresponding triple systems, depends on three quadratures. Those families of spheres, for which the problem of orthogonal trajectories leads to the same differential equations, Darboux calls similar systems of spheres. The families similar to a given family of spheres can be constructed without integration. Among the families thus determined there exists an infinite number composed of spheres passing through a fixed point; a simple quadrature constructs these particular families if an orthogonal trajectory of the original family is known. By virtue of the preceding theorems we can pass, without integration, from triple systems containing a family of planes to those containing a family of spheres. The determination of the orthogonal trajectories of a given family of spheres is referred to the integration of two equations of Riccati, of which each trajectory furnishes a particular solution. A beautiful geometrical construction of triple systems containing a family of spheres is followed by Darboux's extension of the preceding results to space of n dimensions. The extension is an immediate consequence of Darboux's theorem that by giving a suitable form to the functions $a_1, a_2, \cdots, a_n$ and r, we can assign the general integrals of the system

$$\frac{dx_1}{x_1 - a_1} = \frac{dx_2}{x_2 - a_2} = \cdots = \frac{dx_n}{x_n - a_n}, \tag{3}$$

in a real form and without the sign of quadrature, $a_1, \cdots, a_n, r$ being functions of a parameter t connected with $x_1, \cdots, x_n$ by the relation

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = r^2. \tag{4}$$

The above system (3) of differential equations enters into various investigations in geometry. Notably for $n = 4$ in the work of Serret* on surfaces having spherical lines of curvature and in a note of Bonnet's,† in which he shows that the functions $a_1, \cdots, a_n, r$ can be brought to such a form that the integration can be effected by simple quadratures. Darboux proves that the same result can be reached without the sign of integration and without introducing the imaginaries made use of in Bonnet's investigations.

3. The partial differential equation of the third order

* Serret, J. A., "Sur les surfaces dont les lignes de l'une des courbures sont sphériques," *Comptes rendus*, vol. 42 (1856), p. 109.

† Bonnet, O., "Note sur l'intégration d'une certaine classe d'équations différentielles simultanées," *Comptes rendus*, vol. 53 (1861), p. 971.

which determines the families of Lamé admits of particular solutions defined by the equation of the first order

$$H = \varphi_0(u)(x^2 + y^2 + z^2) + \varphi_1(u)x + \varphi_2(u)y + \varphi_3(u)z + \varphi_4(u). \tag{5}$$

The third chapter is devoted to the study of these solutions. The equation (5) includes as particular cases the equation which characterizes parallel surfaces and also that which characterizes the surfaces derived by inversion from a family of parallel surfaces. The orthogonal trajectories in the first case are straight lines, in the second, circles passing through a fixed point. In order to lighten the discussion of the general case Darboux begins by studying the particular problem in which the mutual ratios of the five functions $\varphi_i(u)$ reduce to constants. The corresponding families of Lamé are then defined by the following construction : We construct the circles normal to any surface (Σ) and to a fixed sphere (S) ; all these circles are normal to the surfaces (Σ') which compose the family sought ; we construct each surface (Σ') point by point by determining the point upon each circle where the circle is normal to (Σ), the two points where it is normal to (S), and by constructing the fourth point which forms a constant anharmonic ratio with the preceding points always taken in the same order. The two other families of surfaces which complete the triple system are evidently formed of surfaces one of whose systems of lines of curvature is made up of circles. If this construction be applied to a cyclide of Dupin, a triple system is obtained composed exclusively of cyclides. After studying the special case, remarked by W. Roberts,* where the cyclides are of the third degree, Darboux shows that the preceding general construction gives a contact transformation characterized by the conservation of lines of curvature. Lie has determined the forms of all contact transformations possessing this property ; analytically they are equivalent to an orthogonal linear substitution performed on the six coördinates of a sphere ; Darboux presents an elegant geometrical résumé of the results of Lie's researches. Returning to the most general case of the equation (5) Darboux finds that the general integral can be given without the sign of quadrature. In order to interpret the solution geometrically the author gives certain fundamental properties of the function

*Roberts, W., "Application des coordonnées elliptiques à la recherche des surfaces orthogonales," *Crelle's Journal*, vol. 62 (1863), p. 57.

H, which, multiplied by du, represents the shortest distance between two infinitesimally adjacent surfaces in any family. It amounts to the same thing to give H for every point of a surface or to give in these points the osculating circles of the curves which are the orthogonal trajectories of the family. The families discussed in this chapter are characterized by the property that the osculating circles of the orthogonal trajectories at the points where they cut one of the surfaces, are orthogonal to the same sphere, which varies with the surface. Darboux closes the chapter with a paragraph indicating how these families may be generated by means of infinitesimal transformations. The reader will observe that a typographical error makes two consecutive sections on pages 66 and 67 bear the same number, "39."

4. The author proves that the partial differential equation of the third order can be obtained by expressing that the shortest distance between two infinitesimally adjacent surfaces of the family is a particular solution of a point equation relative to the conjugate system formed by the lines of curvature. This theorem, together with the geometrical interpretation given at the close of the preceding chapter, leads to the theorem of Ribaucour :—The osculating circles of the orthogonal trajectories at the points where they cut a determinate surface of the family form a cyclic system. After a beautiful geometrical demonstration of Ribaucour's theorem and its reciprocal, Darboux proposes to complete the study of cyclic systems made in his Theory of Surfaces by attacking the two new problems : 1° the determination of those families of Lamé for which the osculating planes of the orthogonal trajectories at the point where they cut one of the surfaces of the family intersect in a point ; 2° the determination of the cyclic systems formed by circles whose planes envelop a developable surface ; the solution of the latter problem is due to Ribaucour.* Maurice Lévy† has given a remarkable form to the equation of the third order by taking as independent variables the parameter u and two rectangular coördinates ; Darboux derives this form from the general equation and applies it to finding those surfaces of invariable form which are capable of generating families of Lamé by movement. When the move-

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* Ribaucour, A., "Mémoire sur la théorie générale des surfaces courbes," *Liouville's Journal*, 4th series, vol. 7 (1891), p. 264.

† Lévy, M., "Sur une réduction de l'équation à différences partielles du troisième ordre qui régit les familles de surfaces susceptibles de faire partie d'un système triple orthogonal," *Comptes rendus*, vol. 77 (1873), p. 1435.

ment of the surface is unique and determinate it is necessarily helicoidal. The case where the surface may produce a family of Lamé by several different movements has not been completely solved. From a result obtained by Lucien Lévy* Darboux deduces that the spheres and planes are the only surfaces which can generate a Lamé family by all possible displacements, a theorem established also by Goursat.† Bertrand‡ has shown by geometry that if the movements include all translations the surface is a sphere or a cylinder. Adam§ has shown analytically that this result obtains if the movements reduce to two different translations, and has thus given the most extended result yet known in the solution of the problem. The partial differential equation of the surface sought expresses that the straight line of the plane joining the centers of geodesic curvature of two lines of curvature belongs to a linear complex ; we owe this elegant geometrical interpretation to Petot.|| How faithfully Darboux has brought his work up to date is indicated by the fact that E. Cosserat's theorem,¶ appearing after the body of the work was in type, is included in an extensive footnote. The chapter concludes with the formation of the general differential equation for the case in which the family is determined by an implicit equation $\varphi(x, y, z, u) = 0$, and the development of the equation for subsequent application.

5. The fifth chapter deals with the families of Lamé formed by quadrics. By seeking the condition that quadrics having a unique center and the same principal planes form a Lamé family, Darboux finds a differential relation between the axes found by M. Lévy** in 1867. The author

* Lévy, L., "Note sur le déplacement d'une figure de forme invariable," *Bulletin des Sciences mathématiques*, 2d series, vol. 15 (1891), p. 76.

† Goursat, E., "Sur les systèmes orthogonaux," *Comptes rendus*, vol. 121 (1895), p. 883.

‡ Bertrand, J , "Note sur un théorème de Géométrie," *Comptes rendus*, vol. 121 (1895), p. 921.

§ Adam, P., "Sur les systèmes orthogonaux," *Comptes rendus*, vol. 121 (1895), p. 812.

|| Petot, A., "Sur certains systèmes de coordonnées sphériques et sur les systèmes triples orthogonaux correspondants," *Comptes rendus*, vol. 112, p. 1426 ; "Sur les surfaces susceptibles d'engendrer, par un déplacement hélicoïdal, une famille de Lamé," *Comptes rendus*, vol. 118, p. 1409.

¶ E. Cosserat's theorem : The cyclide of Dupin can generate a family of Lamé by an infinite number of different movements ; these movements result from the composition of two rotations about the straight lines D, Δ, through which the planes of the two series of lines of curvature of the surface pass. *Comptes rendus*, vol. 124 (June, 1897), p. 1426.

** Lévy, M., "Mémoire sur les coordonnées curvilignes orthogonales et en particulier sur celles qui comprennent une famille quelconque de surfaces du second degré," *Journal de l'École Polytechnique*, vol. 43 (1867), p. 157.

presents also Lévy's direct derivation and geometrical interpretation, namely, that the differential relation expresses that any one of the umbilical lines, *i. e.*, any one of the lines described by the umbilics, is normal to the surfaces composing the family. It follows from this that the families of quadrics sought are determined as soon as any plane line is given *a priori* which will serve as an umbilical line. The particular case where one of the umbilical lines reduces to a straight line has been found by G. Humbert;* then the eleven other umbilical lines are straight lines and the surfaces which compose the family are tangent to eight isotropic planes and form part of a point sheaf; we can determine the two other families which complete the triple system. After an interesting discussion of those cases in which the umbilical lines are circles, conics, etc., Darboux returns to the general problem and shows that it can be solved without using the umbilical line and by a direct method which can be attached to a general principle. The proposition relative to umbilical lines does not apply to families composed of surfaces of the second degree alone, but it can be shown that it is true for any surfaces forming a family of Lamé; the demonstration given by Darboux rests on the consideration of the form of the lines of curvature in the neighborhood of an umbilic. This general proposition leads to the result that, if quadrics having unequal axes form a family of Lamé, the principal planes of these surfaces necessarily coincide; hence the preceding investigations are not as particular in character as we should have supposed, but make known all families of Lamé formed of quadrics whose axes are unequal. Darboux demonstrates and applies to surfaces of the second degree a much more general theorem, namely the theorem that if a family of Lamé is composed of surfaces each of which possesses planes of symmetry, the planes of symmetry ought to coincide. He quotes a similar theorem† relative to anallagmatic surfaces, and then turns his attention to the concluding problem of the chapter. We know that the surfaces which are the loci of points such that the sum or the difference of their distances to two fixed surfaces (A), (B) is constant, form a doubly orthogonal system, *i. e.*, they are distributed into two dis-

* Humbert, G., "Sur les normales aux quadriques," *Comptes rendus*, vol. 3 (1890), p. 963.

† Darboux, "Mémoire sur la théorie des coordonnées curvilignes et des systèmes orthogonaux," *Annales de l'École Normale*, 2d series, vol. 7 (1878), pp. 136–138.

tinct families of orthogonal surfaces; the problem then is to find in what cases the system can be completed by adjoining to two different families a third family composed of surfaces cutting the preceding at right angles. Darboux demonstrates geometrically that if we neglect the solutions already studied, the third family ought to be composed of quadrics whose right line generators are normal to the surfaces (A), (B). The axes of these quadrics satisfy certain differential equations which appear in the theory of elliptic functions and which were first integrated by Halphen;* Darboux gives a new method of integrating these equations. In the case of surfaces without centers, the elliptic functions are replaced by logarithms, except in one case remarked by Serret in which the solution is algebraic: in this exceptional case the system is represented by the equations

$$yz = ux, \quad \sqrt{y^2 + x^2} + \sqrt{z^2 + x^2} = v, \quad \sqrt{y^2 + x^2} - \sqrt{z^2 + x^2} = w; \tag{6}$$

Darboux characterizes this result as the most beautiful one of the theory and shows how it may also be obtained by geometry. The chapter closes with the proposal of the problem, still unsolved, of finding the lines of curvature of a surface which is the locus of points for which is constant the algebraic sum of their distances from two straight lines which are in a relation less particular than that of being rectangular and cutting; the case which this problem seeks to generalize is solved in the text.

6. The sixth chapter discusses the systems completely orthogonal in a space of n dimensions, and extends the methods of the preceding chapters to such systems. The first example of orthogonal systems in n variables is Jacobi's system of general elliptic coördinates. Darboux generalizes in a similar manner the system of orthogonal and confocal cyclides and uses for the case of n dimensions variables analogous to pentaspheric coördinates. The fundamental problem of the chapter is: Given n functions forming a system completely orthogonal; to eliminate all the functions except one, and to form the necessary and sufficient partial differential equations which the latter function ought to satisfy. Darboux's method of attack is as follows: One of the functions u (say) being known, a first group of $(n-1)^2$ equations determines the mutual ratios of the first derivatives of the $n-1$ other functions $v, w, \cdots$; this first

* Halphen, "Sur un système d'équations différentielles," *Comptes rendus*, vol. 92 (1881), p. 1101.

group of equations is identical with that which we find when reducing two quadratic forms in n variables, the variables being connected by a linear relation, to sums composed of the same squares. The equations thus obtained for each of the functions $v, w, \cdots$, give rise to the conditions of integrability which should evidently contain the derivatives of u up to the third order inclusive. By attempting to construct these conditions in the simplest form we find two different groups of third order equations for u. The first equations, $\frac{1}{2}(n-1)(n-2)$ in number, are analogous to the equation formed in the case of three variables, and are possessed, as was also that equation, of the property of assuming a simple form by the aid of the second derivatives of the function

$$H = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \tag{7}$$

The equations of the second group, $\frac{1}{6}(n-1)(n-2)(n-3)$ in number, appear only when n is greater than 3. In order to interpret these results in geometric language Darboux introduces the generalized notion of a normal to a surface; a generalization which carries with it the extension of the definitions of principal directions and lines of curvature. In space of n dimensions, there are $n-1$ principal directions and consequently $n-1$ systems of lines of curvature for each surface. Two different principal directions are simultaneously orthogonal and conjugate. These notions lead at once to the following generalization of Dupin's theorem: If a surface is part of a completely orthogonal system, the surfaces which belong to other families admit of as normals at each point of the surface considered the principal directions of this surface; consequently, taken $n-2$ by $n-2$, they cut this surface along one of its lines of curvature. It is interesting to note a new property here, namely, that not every surface in a space of n dimensions, n being greater than 3, can be part of a completely orthogonal system; in order that this be possible it is necessary that its lines of curvature be *coördinated.* Darboux says that the lines of curvature are coördinated when we can find $n-1$ functions, of which *one only* varies upon each line of curvature. The second group of equations referred to above give the necessary and sufficient condition that the lines of curvature of the surface whose parameter is u be coördinated. The theorem, that if the lines of curvature of a surface are coördinated the surface can be a member of a completely orthogonal system, is established by generalizing the theory of parallel surfaces. The remainder of the chapter is occupied with the study of various orthogonal systems, among which are 1° those systems containing a family of the form

$$u = X_1 + X_2 + \cdots + X_n,$$

where u is an arbitrary constant and X_i a function of x_i alone, and 2° those possessing a family defined by the equation

$$u = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

in which the m's are constants; together with the determination of the lines of curvature of different surfaces among which we note the tetraedral symmetric surfaces, called attention to by Lamé* and studied extensively by De la Gournerie.† Any account of this chapter, however brief, would be incomplete without remarking a beautiful generalization of propositions due to Bouquet and Serret,‡ and the fact that the results of the chapter give the lines of curvature of a great number of surfaces of the third order in ordinary space.

This chapter concludes the first book of the work which is divided into two books of equal length and each of the same number of chapters. The first book is devoted to the equation of the third order as we have seen; the second book takes up the study of curvilinear coördinates. Before attempting a brief digest of the rich material of the second division of the volume it may be remarked that Darboux has spared no pains to make his work immediately available to the student. The carefully prepared analytical summaries preceding each chapter and the judicious use of italics throughout the text have greatly increased the usefulness of an otherwise valuable treatise. The reader will find the author's theory of surfaces almost indispensable as an auxiliary because of the numerous references to that classic work.

Book II. Curvilinear Coördinates.

1. It is the object of this chapter, the first one of the second book, to present the formulæ which Lamé has given in his

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* Lamé, "Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie," Paris, 1818.

† De la Gournerie, "Recherches sur les surfaces réglées tétraédrales symmétriques." Paris, 1867. This work consists of three memoirs which appeared in the *Recueil des Savants étrangers*, 1865–66.

‡ Serret, J. A., "Mémoire sur les surfaces orthogonales," *Liouville's Journal*, 1st series, vol. 12 (1847), p. 241.

"Leçons sur les coördonnées curvilignes et leurs diverses applications" by applying and extending his method to completely orthogonal systems in n variables; the results have been given in part in Darboux's memoirs of 1866 and 1878 already referred to and cited in footnotes of this review. After defining the elements of an orthogonal linear substitution by the formulæ $X_i^k = H_i \frac{\partial \rho_i}{\partial x_k}$, and deducing the differential relations to which they give place, Darboux introduces the new quantities β_{ik} which verify two different systems of partial differential equations of the first order. In the process of developing the general method Darboux finds the following generalization of a theorem of Combescure:*—To every orthogonal system we can join an infinity of others which depend on n arbitrary functions of one variable. The general method of determining an orthogonal system consists of the following steps: 1° the integration of the equations for the functions β_{ik}; 2° that of the system for the functions H_i; 3° the integration of the system which determines the functions U_i; then 4° the integration of the expressions

$$du = H_1 U_1 d\rho_1 + \cdots + H_n U_n d\rho_n$$

will give the functions $x_1, \cdots, x_n$ and the solution of the problem will be achieved. The author applies this general method to find the solution of a fundamental problem, namely, the most general resolution of the equation

$$dx_1^2 + \cdots + dx_n^2 = \frac{1}{h^2}(d\rho_1^2 + \cdots + d\rho_n^2).$$

The solution is entirely similar to the one long known for the case of three variables and is furnished by a generalized inversion preceded or followed by a displacement. That to a system in elliptical coördinates we can make correspond an unlimited series of algebraic orthogonal systems is one of the remarkable results called to mind in the presentation of the different methods by which orthogonal systems in n or fewer variables can be obtained from an orthogonal system in n variables. Darboux derives a geometrical theorem which generalizes the Gaussian notion of spherical representation and uses it in the investigation of surfaces in an n-dimensional space whose lines of curvature are coördinated.

*Combescure, E., "Sur les déterminants fonctionnels et les coördonnées curvilignes," *Annales de l'École Normale*, 1st series, vol. 4 (1867), p. 93.

The latter problem demands the determination of all completely orthogonal systems in a space of $n - 1$ dimensions, a determination which is effected by methods analogous to those followed in seeking a surface having a given spherical representation. The chapter terminates with a generalization of the theory of cyclic systems and its extension to space of n dimensions.

2. This chapter studies the properties of triply orthogonal systems by the aid of the displacement of a trirectangular triedron (T) formed by the normals to the three coördinate surfaces which intersect in each point of space. Preparatory to making this application of the theory of the mobile triedron Darboux reproduces a body of results already obtained in his Theory of Surfaces, namely, the partial differential equations connecting the rotations and the translations, the determination of a triedron whose rotations and translations are given *a priori*, the linear system whose integration gives the direction cosines of the axes of this triedron, and the general formulæ which determine the displacement of a point defined by its coördinates relative to the moving triedron. As observed above, this general theory is applied to curvilinear coördinates by taking as moving triedron the one whose sides are normal respectively to the three coördinate surfaces; three of the rotations are zero, the others are expressible in terms of the coefficients H, H_1, H_2 of the lineal element, and of their derivatives; in developing the theory of this particular triedron the quantities β_{ik}, introduced in the preceding chapter, play an important part. The formulæ for the projections of a displacement on the axes of the triedon lead to a new demonstration of Dupin's theorem. After deriving the expressions of the radii of curvature of the coördinate surfaces and the law of variation of the six principal curvatures discovered by Lamé, Darboux defines Lamé's differential parameter of the first order $\varDelta U$, and derives its expression and that of the form $\varDelta\ (U, V)$ in curvilinear coördinates. He then uses the theorem of Stokes and Carl Neumann to find the differential parameter or linear invariant of the second order, and shows in a most interesting manner how the introduction of this invariant may be attached to the study of certain properties of general point transformations. For all transformations of this kind there are in general three lineal elements issuing from a point to which correspond parallel elements; if these three lineal elements are always to form a trirectangular triedron, Darboux finds it necessary and sufficient that the transformation shall be defined by the formulæ

$$X=\frac{\partial U}{\partial x},\ Y=\frac{\partial U}{\partial y},\ Z=\frac{\partial U}{\partial z},$$

where U is any function of x, y, z. There exists then an equation of the third degree which makes known the dilatations of elements whose directions are unaltered, and the roots of this equation are the invariants. Lamé's differential parameter of the second order is none other than the sum of the roots of this equation. The problem, to determine the transformations of the above nature for which the elements whose direction is not changed in space are normal to three families forming necessarily a triply orthogonal system of curvilinear coördinates, leads to the result that there are an infinite number of transformations of this kind corresponding to a triple system given *a priori*. The determination of the latter transformations depends on the integration of three linear partial differential equations which are satisfied by the same function. The final study of the chapter is a complement of the theory of the displacement of the triedron (T). If we consider the most general system of oblique curvilinear coördinates, the lineal element of space takes the form

$$ds^2 = \Sigma\Sigma A_{ik} d\rho_i d\rho_k;$$

it may be proposed to find all the differential relations which exist among the A_{ik}, differential relations analogous to those which we owe to Lamé for the case of orthogonal coördinates. This question can be solved by taking a triedron (T) occupying a particular position relative to the tangent planes of the coördinate surfaces, but Darboux employs a much more elegant method which rests upon the consideration of the different decompositions of the above quadratic form into squares and which leads to six equations simultaneously necessary and sufficient.

3. This chapter is devoted to the investigation of a particular triple system. Lamé has attached great importance in his Leçons to systems composed of three *isothermal* families. His definition and criterion, verified in the case of the system of confocal ellipsoids, show that there exists at least one triple system simultaneously orthogonal and isothermal. Lamé proposed to determine all systems of this kind. Bertrand* has shown that the surfaces which compose these particular systems are *isothermic*, *i. e.*, they are divided into infinitesimal squares by their lines of curvature; but this

*Bertrand, J., "Mémoire sur les surfaces isothermes orthogonales," *Liouville's Journal*, vol. 9 (1844), p. 117.

property is not characteristic, it belongs, for example, to the system of confocal cyclides, which is not isothermal. Hence by proposing to find all triple systems composed of isothermic systems, if the problem is successfully solved we are assured of obtaining not only the isothermal systems sought by Lamé, but also systems more general. Lamé has shown that if ρ, ρ_1, ρ_2 designate the elliptic coördinates of a point in space, the equation of heat admits of an infinity of solutions of the form $f(\rho)f_1(\rho_1)f_2(\rho_2)$. Darboux shows that all triply orthogonal systems, for which an analogous proposition can be formulated are composed of isothermic surfaces. All these remarks lead to an attack on the more general problem, that of the determination of the triple systems composed of isothermic surfaces. Darboux finds that the equations for determining H, H_1, and H_2 possess three types of solutions. The first type gives three triple systems: 1° a family of parallel planes and two families of isothermal cylinders; 2° a family of concentric spheres and two families of isothermal cones; 3° a family of planes passing through a straight line and two families of surfaces of revolution whose axis is the line and whose meridians form an orthogonal and isothermal system. To these systems should be added those obtained from them by inversion. The second type does not furnish a solution of the problem and should be rejected.

4. The third type of solution is the subject of the fourth chapter. It corresponds to the values

$$H = \frac{(\rho_1 - \rho)^{-h}(\rho - \rho_2)^{-h}}{M\sqrt{a}}, \quad H_1 = \frac{(\rho_2 - \rho_1)^{-h}(\rho_1 - \rho)^{-h}}{M\sqrt{a_1}},$$
$$H_2 = \frac{(\rho - \rho_2)^{-h}(\rho_2 - \rho_1)^{-h}}{M\sqrt{a_2}}.$$

In these formulæ M is any function, h is a constant and a_i is a function only of ρ_i. By expressing that certain equations of condition given in the preceding chapter are verified, Darboux obtains the following solutions: 1° $h = -\frac{1}{2}$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ are identical polynomials of the fifth degree; the corresponding triple system is that formed by the confocal cyclides and their varieties. 2° $h = \frac{1}{2}$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ are identical polynomials of the third degree; the corresponding system is transcendental.* 3° $h = 1$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ are polynomials of the second degree whose

* For the study of the hypotheses, $h = \frac{1}{2}$ and $h = 2$, the author refers the reader to the memoirs of his already cited.

sum is zero; the corresponding system is composed exclusively of those surfaces to which Darboux gave the name cyclides of Dupin. The author demonstrates that this system is identical with those studied in the third chapter of the first book. 4° $h = 2$; $a(\rho)$, $a_1(\rho_1)$, $a_2(\rho_2)$ reduce to constants whose sum is zero; the corresponding system is imaginary.* In the course of the discussion Darboux shows how the value of M corresponding to each solution may be determined.

5. Among the systems determined in the two preceding chapters there are two classes which are to be the objects of study in this chapter: 1° those composed of three isothermal families; 2° those for which the equation of heat admits of an infinity of solutions of the form

$$P(\rho, \rho_1, \rho_2)\, f(\rho) f_1(\rho_1) f_2(\rho_2),$$

P being a determinate function while the f's may be chosen in an infinite number of ways. It is proposed to determine these two particular classes. The author commences by seeking to determine the isothermal systems, considering successively the different solutions obtained in the two preceding chapters. The isothermal systems composed of a family of parallel planes and two families of isothermal cylinders, or of a family of concentric spheres and two families of isothermal cones, or a family of planes passing through an axis and two families of confocal quadrics of revolution, are first found, then the system of confocal quadrics and finally an imaginary system. This last isothermal system was pointed out by Combescure† but not determined by him; it plays no rôle in mathematical physics, but has a geometric interest; Darboux finds that it is formed by cyclides of Dupin which are imaginary, and of the third degree. The author follows an analogous method for the determination of the second class of triple systems noted above. By the use of a fundamental lemma relative to inversion, due to Lord Kelvin,‡ Darboux proves that the systems sought reduce to the single system of confocal cyclides and their varieties.§ He also establishes this prop-

* See footnote on the preceding page of this review.

† *Vide loc. cit.*

‡ Thomson, W., "Extrait de deux lettres adressées à M. Liouville;" Liouville: "Note au sujet de l'article précédent," *Liouville's Journal*, 1st series, vol. 12 (1847), pp. 256 and 265.

§ This property was established by Wangerin for cyclides having principal planes; Darboux demonstrated the result directly and extended it to cyclides of the most general form. See Wangerin, A., "Ueber ein erty of confocal cyclides by a direct demonstration in which he employs pentaspherical coördinates and a remarkable form which the equation of heat assumes in this system of coördinates. The chapter ends with a remark pointing to an extension of some of the above results to systems of curvilinear coördinates the most general.

6. The sixth chapter of the second book and last chapter of the volume is devoted to those triple systems which Darboux has named the triple systems of Bianchi. These are the triple systems which contain a family of surfaces of constant total curvature, the curvature varying from surface to surface of the family according to any law. This application of the general method made by Bianchi has its origin in a theorem of Weingarten which foreshadows the existence of families of Lamé composed of surfaces of constant and equal total curvature; Weingarten's theorem is as follows: Given a surface (S) having the constant total curvature $1/k$, if we take upon the normal at each of its points an infinitesimal length MM' proportional to $\cos d/\sqrt{k}$, d designating the geodesic distance of M from a fixed point A of (S), the surface (S') described by the point M' is also of constant curvature $1/k$ and together with (S) is part of a family of Lamé. This theorem makes possible the construction one after another of surfaces having the same constant total curvature which depend upon four arbitrary functions and form a family of Lamé. After solving a problem in the theory of deformation of surfaces naturally suggested by Weingarten's theorem and remarking properties of certain cyclic systems attached to a surface applicable on a surface of revolution, Darboux presents an exposition of the researches of Bianchi. By discarding the special case in which the surfaces sought are surfaces of revolution, we obtain an elegant form of the lineal element; this form depends on a single function ω which must satisfy three partial differential equations of the second and third order. The study of this system of partial differential equations reveals the fact that its general integral depends upon five arbitrary functions of one variable. Among the general properties of the triple systems to which we are led we note that the transformation of Bäcklund can be applied to them as Bianchi has shown. After examining the systems of

dreifach orthogonales Flächensystem, gebildet aus gewissen Flächen vierter Ordnung," *Crelle's Journal*, vol. 82 (1876), p. 145; Darboux, "Sur l'application de méthodes de la physique mathématique à l'étude des corps terminés par des cyclides," *Comptes rendus*, vol. 83 (1876), pp. 1037, 1099.

Weingarten in particular, Darboux concludes the volume by determining all triple systems for which the nine quantities H_i, β_{ik} depend on a single variable α; the result is a system composed exclusively of helicoids having constant total curvature.

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THE NEW MATHEMATICAL ENCYCLOPAEDIA.

IT is an established belief that mathematics like the classics stopped growing long ago. The chemist, biologist, physicist, and other scientists look with complacency at the gigantic strides their branches of learning have taken in the last generation and rather pity their colleagues, the mathematicians. According to them, the golden age of our science dates back two thousand years ago when Euclid, Archimedes, Apollonius and Diophantes flourished. It is true that some time afterwards algebra and trigonometry, analytical geometry and the calculus were invented; they have, however, been long since perfected and mathematicians spend their time teaching this ancient body of facts, improving here and there a small detail and solving ingenious problems which they devise to test each other's skill. How different the real state of affairs is! No science presents a more intensely active and vigorous condition than ours. Indeed the growth of mathematics has been so rapid in the last century, the discoveries and inventions so numerous and their importance so far reaching that it is permitted only to a very few extraordinary minds still to overlook the whole field of mathematics as it stands to-day. One has only to recall a few of the great theories which have sprung up in these later days and the numberless special theories and ramifications they have given rise to. First of all that great leviathan, the function theory, embracing as special topics of limitless extent the theory of elliptic, hyperelliptic, abelian, modular, and automorphic functions not to mention others. Of still younger date is the theory of groups finite and infinite, with their far-reaching applications. We have then the modern theory of invariants, the new theory of algebraic numbers, the non-Euclidian geometry, the theory of algebraic transformations and correspondences, Cantor's theory of multiplicities, the partial