ELLIOTT'S ALGEBRA OF QUANTICS.

An Introduction to the Algebra of Quantics. By EDWIN BAILEY ELLIOTT, M.A., F.R.S. Oxford, Clarendon Press, 1895. 8vo, xiii + 423 pp.

No single book is likely soon to supersede Salmon's Higher Algebra, but it will be replaced gradually by a combination of others, each of narrower scope. Professor Elliott's book is likely to be one of these, and to become equally well known. So far as binary forms are concerned, it worthily signalizes the advance made by English investigators during the past quarter century. To the memoirs of French and German writers before Hilbert there has been ready access through the standard works of Clebsch, Gordan, and Deruyts. There was needed a manual which should introduce the student to the researches of Cayley and Sylvester since 1870, and of Franklin, Hammond, Forsyth, MacMahon, and others of the English school.

A cursory inspection of the book shows that its style is that of the lecture, and therefore easy for the reader. Topical headings are supplied to nearly all sections. Frequent openings are made for geometrical application or illustration; and every student will thank the author for the occasional explicit warnings against the most probable errors and misconceptions. As to the matter, of 415 pages all but 45 are given to binary quantics, although incidental references to forms in more than two variables occur throughout the introductory chapters. Eleven chapters (259 pages) treat questions pertaining to binary forms of all orders; four chapters (110 pages) are given to particular forms and systems, and a final chapter (45 pages) to ternary forms and particularly the cubic. Well-chosen examples, more than 350 in number, distributed through the book double its value.

The first four chapters are entitled: Principles and Direct Methods, Essential Qualities of Invariants, Essential Qualities of Covariants, Cogredient and Contragredient Quantities. The first defines invariants, and gives abundant elementary examples, eliminants, Jacobians, Hessians, discriminants, catalecticants, canonizants. In the second and third, homogeneity and isobarism are shown to be necessary properties. Concomitants of one quantic are made to yield joint concomitants of two or more by substituting for the quantic f a linear aggregate of the same order : $f + \lambda\varphi + \psi\mu$

+ etc., and developing the corresponding concomitant in powers of the parameters λ, μ, etc. The third closes with the theorem, so nearly self-evident yet always perplexing to the beginner, that a covariant of covariants is a covariant of the fundamental quantics. The fourth chapter exhibits the relation of certain concomitants to others as emanants (polars), whereby invariants of a quantic of any order give rise to covariants of quantics of all higher orders. Next the invariance of the differential operator

$$\left(\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial y_2}-\frac{\partial}{\partial x_2}\cdot\frac{\partial}{\partial y_1}\right)^r$$

is proven, introducing Cayley's hyper-determinant notation as a means of discovering invariants and covariants in unlimited number; and mention is made of Aronhold's symbolic notation. Here we notice a curious slip of the pen (p. 79): "It is symbolic *ab initio*, denoting a binary quantic $(a_0, a_1, a_2, \cdots, a_p)(x, y)^p$ by $(ax + a'y)^p$, where $a^r a'^{p-r}$ means a_r, and $a^r a'^s$ has no meaning unless $r + s$ is a multiple of p." It should read, of course: "Unless $r + s = p$." Another (p. 78) is more humorous, where the reader is referred to Cayley's memoirs for the "reduction of irreducible systems."

It is not too much to say that these first four chapters are admirable both in plan and in execution, and especially so if regarded from a pedagogic point of view. By reason of their very excellence we may be pardoned if we venture two critical strictures. (a) The first is this: The invariant property is in this theory (not so in the Galois theory of equations) a result purely of the form as distinguished from the content. This view can hardly be emphasized too strongly or too often. It is only partially and imperfectly brought out in this book. When by the formation of emanants it is seen that several sets of cogredient variables can figure in a covariant as well as a single set, the student can be led to perceive that the cogredience, hence the linear character of the transformation, is the fundamental premise. When, again, by a process in every way similar (though the similarity is not here exploited), "intermediate" invariants and covariants are derived, the same thesis should be expounded a second time. When finally the hyper-determinant calculus is presented, it need be but a short step to replace each fundamental quantic of pth order, f_p, by its equivalent according to Euler's theorem for homogeneous functions :

$$f_p = \frac{1}{p!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^p f.$$

The concomitants of the quantic appear then as concomitants solely of the *linear operators in the symbolic quantic :*

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^p;$$

or, by application of the emanation process, as concomitants solely of the *linear symbolic quantics* in the operator product :

$$\left(x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y}\right)\left(x_2\frac{\partial}{\partial x} + y_2\frac{\partial}{\partial y}\right)\cdots\left(x_p\frac{\partial}{\partial x} + y_p\frac{\partial}{\partial y}\right).$$

At this point the student would of necessity comprehend the purely formal nature of invariants. At the same time he would see the connection, or better, the identity, between Cayley's hyper-determinant notation and the Aronhold symbolic ; and this is certainly the point at which the latter could be explained with least effort.

(*b*) Two of the three processes just specified are well developed by Professor Elliott, but lose most of their theoretic value for lack of the third and for want of emphatic enunciation of the main thesis. Were this third process brought in, and the problem of the inner structure of invariants reduced to that of invariants of a system of *linear* quantics, then it would have been easy to strengthen the framework in a second point of prime importance. The chief merit of the hyper-determinant calculus is not its formal character, but the fact that its method is exhaustive. Not only does it produce a limitless number of covariants, but it produces all possible covariants. This converse theorem, that every rational covariant can be produced by an aggregate of hyper-determinant operators, ought evidently to be the first climactic theorem in an English treatise. The fact that Cayley did not in his early memoirs make this a cardinal point ought not to obscure its significance in the light of later knowledge.* Although we are told (p. 78) that this

* See Meyer : "Bericht über den gegenwärtigen Stand der Invariantentheorie," *Jahersberichte der deutschen Math. Vereinigung*, vol. 1, p. 98, first half-page.

method "did not, in its originator's form, succeed in establishing the finiteness of complete systems of irreducible covariants in general; that triumph was reserved for a later method, etc.," yet the discovery of it was none the less a noteworthy achievement, and its unrealized potency may most properly be acknowledged and elucidated.

This opportunity for a principal result being apparently neglected, an equivalent is given in the fifth chapter under the caption: "Invariant functions of the differences of roots" (p. 93). It is assumed as known from the elementary theory of equations, that any function of the roots which is not altered when all roots are increased or decreased equally is a function only of the differences of the roots. Every rational invariant is such a function multiplied by some power of the leading coefficient of the equation, and is symmetric in the roots. This lemma gives a system of Diophantine equations satisfied by the exponents of the differences in any term of the invariant function of roots. Hereupon Hermite's law of reciprocity could be employed to prove the sufficiency of hyper-determinants to represent all covariants, as the author points out somewhat later (p. 161).

From this point on, in the general theory, two principal results are to be reached. The first is Cayley's theorem, proved by Sylvester, concerning the exact number of independent covariants of given degree in the coefficients and of given order in the variables. Denoting weight, degree, and order of the fundamental quantic by w, i, p respectively, the exact number is denoted in partition-symbols by

$$(w\,;\, i,\, p) - (w-1\,;\, i,\, p).^*$$

The second is Gordan's theorem, that all covariants of any given set of binary quantics are rational integral functions of a finite number of such, constituting a "complete form-system."

The proof of the first of these theorems requires the discussion of the two differential equations that covariants must satisfy, and of the corresponding differential operators (annihilators of covariants) Ω and O, with their alternant: $\eta = \Omega O - O\Omega$. The name semivariants is given to all homogeneous isobaric functions of the coefficients of the quantic which are annihilated by Ω; and those are found to be the coefficients of leading terms of covariants, which in turn are uniquely determined from the semi-invariant lead-

* See Salmon's Higher Algebra, 4th ed., p. 132, where $p = n$ and $i = \theta$.

ers. Chapters VI and VII are devoted to this relation, and to the exact enumeration of semiinvariants of given degree and excess, whence follows the proof of the number of independent covariants.

In closing Chapter VII Hermite's law of reciprocity is neatly demonstrated. Chapter VIII is an introduction to the study of generating functions both for the number of covariants of given type, and for the number and types of those forming a complete system for a given quantic. The example most fully treated, that of the system of concomitants of a cubic, is so attractively presented as to make this difficult and neglected subject appear one of the most interesting. The same may be said of the Chapters X and XI, on Protomorphs and Perpetuants.

For the second principal theorem, the so-called Gordan theorem, the proof selected is that extremely simple one given by Hilbert in vol. 30 of *Math. Annalen.* As illustrations the invariants of the cubic and quartic are analyzed, where Hilbert's limit to the number of independent invariants is seen to be largely in excess of the true number. Although this mode of proof has no extension to ternary and higher quantics, its simplicity and elegance fully justify its adoption in preference either to Gordan's proof, or to Hilbert's own more general method.

Of the four chapters on particular quantics, one deals with canonical forms, canonizants, and catalecticants. Elliott's discussion is probably more elementary, but Salmon's chapter on the same topic should be read in connection with it. It is questionable whether the canonizant, for example, might not be introduced more appropriately by its invariant differential equation, reserving till later its relation to a canonical form. This would seem particularly desirable as a natural introduction of irrational covariants. The quintic and sextic have a chapter, simultaneous quantics another, and the invariants called by this author *Boolians* are accorded the closing place among binary quantics. A brief but valuable chapter on ternary differential operators and the invariants of a ternary cubic is added, raising the hope that the author may purpose a similar volume on ternary quantics, or at least the cubic and quartic, as a sequel to this. The appearance of any work of so high merits as this Algebra of quantics is certain to result in a renewed and more widespread activity in the domain traversed.

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