IS CONTINUITY OF SPACE NECESSARY TO EUCLID'S GEOMETRY?

BY WENDELL M. STRONG, M.A.

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DEDEKIND has said that continuity of space is not essential to the constructions of Euclid and has defined a discontinuous space in which, "so far as he sees," the constructions can be made.* To prove this it is only necessary to show that the constructions will never lead *out* of the given discontinuous space if we start in it—in other words, if we start with a figure of the discontinuous space, the construction will lead always to other figures of that space. If the constructions can be made in a given space the theorems will be true in it.

It is not safe to assume in the demonstration that discontinuity may not render invalid principles which are postulated for continuous space; the question of the truth of the postulates in a given discontinuous space can be decided only by a knowledge of the properties of the space. For instance, in a space consisting of the points whose rectangular coördinates with a given system of axes are rational in a given unit of distance, not all figures can be moved about; parts of a figure may disappear entirely as the result of an allowable change in the position of other parts; thus a square of unit side may be constructed in this space, but if its diagonal is brought into coincidence with one of the axes, one

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* "Was sind und was sollen die Zahlen," Vorwort, p. XII.

"für einen grossen Theil der Wissenschaft vom Raume die Stetigkeit seiner Gebilde gar nicht einmal eine nothwendige Voraussetzung ist, ganz abgesehen davon, dass sie in den Werken über Geometrie zwar wohl dem Namen nach beilaüfig erwähnt, aber niemals deutlich erklärt, also auch nicht für Beweise zugänglich gemacht wird. Um dies noch näher zu erläutern, bemerke ich beispielsweise Folgendes. Wählt man drei nicht in einer Gerade liegende Puncte A, B, C nach Belieben, nur mit der Beschränkung, dass die Verhältnisse ihrer Entfernungen AB, AC, BC algebraische Zahlen sind, und sieht man im Raume nur diejenigen Puncte M als vorhanden an, fur welche die Verhältnisse von AM, BM, CM zu AB ebenfalls algebraische Zahlen sind, so ist der aus diesen Puncten M bestehende Raum, wie leicht zu sehen, überall unstetig; aber trotz der Unstetigkeit, Lückenhaftigkeit dieses Raumes sind in ihm, so viel ich sehe, alle Constructionen welche in Euklid's Elementen auftreten, genau ebenso ausführbar, wie in dem vollkommen stetigen Raume: die Unstetigkeit dieses Raumes würde daher in Euklid's Wissenschaft gar nicht bemerkt, gar nicht empfunden werden."

extremity at the origin, two of its sides will no longer exist as lines of the space. In a space consisting of the points whose rectangular coördinates are transcendental, two straight lines which intersect if considered as continuous straight lines may pass through each other without intersecting.

To make precise the conception of a line or surface of discontinuous space we shall define them thus: A line or surface of ordinary space is a line or surface of a given discontinuous space if the points of that space are infinitely thick—in Cantor's language, form a dense multiplicity—throughout its extent.

There are many anomalies besides those already mentioned which, so far as intuition is concerned, might present themselves as a result of discontinuity. Planes as well as lines might pass through each other without intersecting, or might intersect in a single point; circles might intersect in one point only; an arc of a circumference might exist when the whole circumference could not; a circumference might not have a center; a perpendicular to a plane at a point in that plane might not exist. On the other hand the question arises: if the points of a discontinuous space form a dense multiplicity throughout continuous space, will not every line or surface of continuous space be a line or surface of the discontinuous?

The space we shall adopt may be defined thus: Let a real number which can be obtained from the integers by a finite number of rational operations and extractions of square roots be called a *quadratic number*. A, B, C are any three points not in a straight line such that AC and BC are quadratic in terms of AB. The points whose distances from each of the three points A, B, C are quadratic in AB constitute the space.

Henceforth the word "quadratic" will denote quadratic in AB and the space will be called a *quadratic space*. Figures referred to will be figures of the quadratic space unless it is expressly stated that they are of continuous space.

An essential feature of the quadratic space is that it is the least space in which the constructions of Euclid are possible; it contains the points which can be obtained by a finite number of these constructions and no others. It differs from Dedekind's space in that it consists of part of the points contained in his. The quadratic space is also a natural space to adopt, since the constructions of Euclid lead directly to it.

The consideration of the quadratic space may be much simplified by introducing a system of rectangular coördinates in which A shall be the origin; AB, the x axis; and the y axis shall be in the plane ABC; the z axis will then be perpendicular to that plane. The following theorems establish the relations which the coördinates of a point must fulfil if it be a point of the space.

(1) The coördinates of the three fundamental points A, B, C are quadratic.

This requires proof for the point C only. The coördinates of the three points are $(0, 0, 0)$, $(b_1, 0, 0)$, $(c_1, c_2, 0)$.

The equations,

$$c_1^2 + c_2^2 = \overline{AC}^2$$

$$(b_1 - c_1)^2 + c_2^2 = \overline{BC}^2$$

show that c_1 and c_2 are quadratic.*

(2) The coördinates of a point of continuous space which is at a quadratic distance from each of the three fundamental points are quadratic.

From the expression for distance we can readily derive

$$x^2 + y^2 + z^2 = d_1^2$$

$$2b_1x = d_1^2 - d_2^2 + b_1^2$$

$$2c_1x + 2c_2y = d_1^2 - d_3^2 + c_1^2 + c_2^2$$

Applying the same method to the xy plane we derive this theorem which will be useful later: A point of the continuous xy plane at quadratic distances from two of the fundamental points is a point of the quadratic space.

(3). If the coördinates of a point of continuous space are quadratic, its distances from the three fundamental points are quadratic and it is a point of the quadratic space.

We have thus established that *the quadratic space consists of the points whose coördinates, in the particular system chosen, are quadratic quantities.*

Important results follow immediately:

The axes are lines of the quadratic space, the coördinate planes are planes of that space.

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* The principles which enable us to deduce our results by inspection are:

The solution of n simultaneous equations of the first degree, or $n-1$ of the first and one of the second degree, in n variables, in which the parameters are quadratic, gives quadratic results.

If an equation containing linearly n parameters is satisfied by n independent sets of quadratic values of the variables, the parameters must be quadratic or the same multiple of quadratic quantities.

The quadratic space is everywhere discontinuous but its points form a dense multiplicity whose order is the same as the order of the multiplicity of the integral numbers. We can now answer the question whether all lines or surfaces of continuous space would also be lines or surfaces of the quadratic space. A one to one correspondence can be established between the straight lines of continuous space through a point and the points of a continuum; that is, the multiplicity of these lines is of the same order as the multiplicity of the points of a continuum. Hence, to speak loosely, the number of continuous lines through a point is infinitely greater than the number of points in the quadratic space; an infinite number of these lines must consequently fail to pass through any other point of the quadratic space. Cantor's theory shows further that the multiplicity of such lines is itself of the order of the continuum. This result applies as well to figures in general as to straight lines. Two cases are worthy of notice: only angles of certain magnitudes can exist in quadratic space; continuous circumferences which pass through no point of the space can be described about any center.

The two following theorems may be easily proved from the expression for the distance of two points:

(1) Any two points of the quadratic space are at a quadratic distance from each other.

(2) A point of continuous space at quadratic distances from three points of the quadratic space is a point of the quadratic space.

These theorems show that the three points A, B, C by means of which the quadratic space was defined, do not differ in their properties from its other points. Therefore, if a system of rectangular coördinates is defined with reference to three new non-collinear points in the same way as the system first chosen was defined with reference to A, B, and C, the quadratic space will consist of the points whose new coördinates are quadratic. This shows that such a change in the system of coördinates is allowable. Noting that the new x axis and the new xy plane will be a line and plane of the quadratic space we may state the transformation of coördinates thus: Any point may be taken as the origin, any line through that point as the x axis, any plane through that line as the xy plane. Hence the most general transformation of rectangular coördinates is possible. To change the position of the system of axes—provided this makes no change in the space or its relation to the coördinate system—leaving a given figure unchanged in position is equivalent to changing the position of the figure, that of the axes being fixed. We have, therefore, established this very important theorem :

Figures may be moved about in the quadratic space without change of size or shape.

The determination of the new x axis by two points and of the new xy plane by three points shows that a continuous straight line passing through two points of the quadratic space is a line of that space, a continuous plane passing through three points is a plane of that space.

The above theorem makes it possible without loss of generality to consider figures in the most convenient position with reference to the axes, especially to consider plane figures in the xy plane. The following theorems are consequently evident : A continuous straight line is a line of the quadratic space, if it is parallel to a given straight line and passes through a given point ; if it is parallel to a given straight line and at a quadratic distance from it, the plane of the two being a plane of the space ; if it passes through a given point and makes a given angle with a given line ; if it passes through a given point and is perpendicular to a given plane. In the last two theorems the intersection of two lines in the same plane, and of a plane and a line has been assumed ; this will be justified later.

A continuous plane is a plane of the quadratic space, if it is parallel to a given plane and passes through a given point ; if it is parallel to a given plane and at a quadratic distance from it ; if it is perpendicular to a given line and passes through a given point.

The solution of the simultaneous equations involved shows that, if parallelism does not occur, two straight lines in the same plane intersect ; a straight line and a plane intersect ; two planes intersect in a straight line.

If a continuous circle passes through three points, it has a center, its radius is of quadratic length, and it is a circle of the quadratic space. The center and radius may be found by substituting the coördinates of the three points in the equation of a circle. Every point on the circumference for which x is quadratic has y quadratic also ; hence these points form a dense multiplicity.

By methods similar to those already employed we can prove the following theorems : A straight line and a circle or two circles intersect as in continuous space. If a continuous spherical surface pass through four points, it has a center, its radius is of quadratic length and it is a surface of the quadratic space. A spherical surface and a straight line or plane intersect as in continuous space.

The cylinder and cone may be treated by similar methods.

The preceding theorems suffice to show the validity of Euclid's constructions. It remains to show that the quadratic space is the least space which will suffice for these constructions, and that the restriction of AC and BC to quadratic values is not arbitrary. The rational operations are all required in the addition and subtraction of lines, and the division of lines into any number of equal parts; the operation of extracting any square root is equivalent to finding a mean proportional and is therefore essential. Moreover any point of the space can be reached by a finite number of operations. The reason that AC and BC should be restricted to quadratic values is: the function of C is to determine a plane through AB, while AB is the fundamental distance; if C is assumed to be a point of the space we ought to be able to reach C by Euclid's constructions, starting from A and B.

The antithesis of the quadratic space is a space consisting of the points whose rectangular coördinates, in a certain system, are transcendental quantities. It has an infinitely greater number of points than the quadratic space, their multiplicity being of the order of the continuum. Moreover every line or surface of continuous space, except certain cases of a plane, or lines in a plane, parallel to one of the coördinate planes is also a line or surface of this space. We might therefore expect all Euclid's constructions to be possible in it; such, however, is not the case, as we have already mentioned. Two striking peculiarities of this space are: two circumferences may intersect in a single point; a circumference may have no center.

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NIEWENGLOWSKI'S GEOMETRY.

Cours de Géométrie analytique, par B. NIEWENGLOWSKI; avec une *Note sur les transformations en géométrie*, par EMILE BOREL. Paris, Gauthier-Villars, 1894–96. Vol. I., 483 pp; Vol. II., 292 pp; Vol. III., 572 pp.

THE work before us, of which the first two volumes deal with plane analytic geometry, the third with analytic geometry of three dimensions, is intended for boys in the *Classes de mathematiques speciales* at the French *Lycées*, at one of the best known of which (the Lycée Louis-le-