FOCAL PROPERTIES OF SURFACES OF THE SECOND ORDER.

Die Focaleigenschaften der Flächen zweiter Ordnung. Ein neues Kapitel zu den Lehrbüchern der analytischen Geometrie des Raumes. Von Professor Dr. OTTO STAUDE. Leipzig, Teubner, 1896. 8vo, viii + 186 pp.

THIS is a text-book which has grown from four articles by the author, published in : *Berichte der K. Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, 1882, p. 5, Mar. 3 ; 1895, p. 483, July 1 ; *Mathematische Annalen*, vols. 20 and 27.

The focal properties of the conic in the plane are so familiar that it is natural to enquire if there be not their counterpart in space. Investigations in this line for certain particular theorems were published by : Dupin (1813), Développement de géométrie ; Steiner (1826), *Crelle* 1 ; Jacobi (1834), *Crelle* 12 ; Chasles (1835), *Mémoires de l'Institut*, and (1837), Notes ; MacCullagh (1836), *Proceedings of the Irish Academy*; Lamé (1837), *Liouville* 2 ; Salmon (1842) ; Amiot (1843), *Liouville* 7, and (1845), 10 ; Plücker (1846), System der analytischen Geometrie des Raumes ; Liouville (1847), *Liouville* 12 ; Townsend (1848), *Cambridge and Dublin Mathematical Journal* 3. This list of names makes it all the more strange that these masters of geometric analysis did not carry through the reasoning and find the focal properties. It was reserved for the author to simplify much that had gone before and to develop in a simple way the general theory by the use of the now common idea of a movable coördinate triheder. This development is also particularly valuable in that it uses throughout either the elliptic or parabolic coördinates, giving a most excellent presentation of these systems. And not the least important feature of the book is the care with which these exceptional cases are treated as part of the organic whole.

The first step in the development consists in using the equation to the cone from any point in space to each of the focal conics of a confocal system. This is found to be as follows : λ, μ, ν being the parameters of the three surfaces of the system through the point P; ξ, η, ζ, coordinates of any point referred to the three normals at P to these surfaces as axes ; β, γ, the parameters corresponding to the two focal conics ; then the equations of the cones from P (λ, μ, ν) to the two focal conics are derived in the form (either for a central or a parabolic system) :

$$(\gamma-\lambda)(\gamma-\mu)(\gamma-\nu)\left(\frac{\xi^2}{\gamma-\lambda}+\frac{\eta^2}{\gamma-\mu}+\frac{\zeta^2}{\gamma-\nu}\right)=0,$$

$$(\beta-\lambda)(\beta-\mu)(\beta-\nu)\left(\frac{\xi^2}{\beta-\lambda}+\frac{\eta^2}{\beta-\mu}+\frac{\zeta^2}{\beta-\nu}\right)=0.$$

The lines common to these cones are called the *focal lines* of P. There are four real lines, and, in general, they are distinct. If the point P be on the other focal conic the cone is right circular. Thence Professor Staude deduces many theorems, of which some had been known previously but not included in the general theory.

Possibly the most interesting, although not the most general of the focal properties which Professor Staude deduces, are the following. (The "focal distance" from any point to the focus of the one of a pair of focal conics is defined as the shortest distance on the broken line to a point on that conic, and thence to the adjacent focus).

In an ellipsoid, for any point on the surface, the sum of the focal distances to a focus of the focal ellipse and to the opposite focus of the focal hyperbola is a constant. And the normal to the surface at the point bisects the angle of the two lines. This theorem is the basis of the well-known wire model, No. 110 of the Brill collection, which represents the two focal curves of the system by wires rigidly connected, and which has a string fastened so as to represent the sum of the focal distances of the theorem.

In an hyperboloid of one sheet, for any point on the surface, the difference between the focal distances to a focus of the focal hyperbola and to the adjacent focus of the focal ellipse is a constant. And the normal to the surface at the point bisects the exterior angle of the two lines.

In an hyperboloid of two sheets, for any point on the surface, the difference between the focal distances to the two foci of a focal conic is a constant. And the normal to the surface at the point bisects the exterior angle of the two lines. It is easy to see how a string can be adjusted on the Brill model No. 110 to give the generation of the two hyperboloids.

For the confocal system of hyperboloids many new properties are developed. The property of the focal paraboloids corresponding to Dupin's focal properties is derived. (The plane having the properties of the directrix is called the principal directrix plane.)

In an elliptic paraboloid, for any point on the surface, the focal distance to the focus of the inner focal parabola is equal to the normal upon the corresponding principal directrix plane. And the normal to the surface at the point bisects the exterior angle of the two lines.

In an hyperbolic paraboloid, for any point on the surface, the difference between the focal distances to the foci of the two focal parabolas is a constant. And the normal to the surface at the point bisects the exterior angle of the two lines.

The reader of this book will hope with Professor Staude "that the focal properties of the conicoid will attain the same recognition that the corresponding properties in the plane have long since enjoyed."

H. D. THOMPSON.

PRINCETON,
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NOTE ON THE THEORY OF CONTINUOUS GROUPS.

IN a "Note on the fundamental theorems of Lie's theory of continuous groups," contributed to the BULLETIN in November, 1897, by Dr. Lovett, attention is drawn to an error or misapprehension in a paper which I had the honor of contributing to the *Proceedings of the London Mathematical Society* (vol. 23, p. 381–390).

The theorem to which objection is taken is: "If $x_1', \cdots, x_n'$ is a point obtained from the point $x_1, \cdots, x_n$ by the operation

$$1 + X + \frac{X^2}{2\,!} + \cdots;$$

and $x_1'', \cdots, x_n''$ is a point obtained from the point $x_1', \cdots, x_n'$ by the operation

$$1 + Y' + \frac{Y'^2}{2\,!} + \cdots,$$

where $$X \equiv \lambda_1 X_1 + \cdots + \lambda_r X_r,$$

and $$Y \equiv \mu_1 X_1 + \cdots + \mu_r X_r,$$

X_k denoting the linear operator

$$\sum_{i=1}^{i=n} \xi_{ki}(x_1, \cdots, x_n) \frac{\partial}{\partial x_i};$$

then $x_1'', \cdots, x_n''$ can be directly derived from the point $x_1, \cdots, x_n$ by the operation