THE RELATIONS OF ANALYSIS AND MATHEMATICAL PHYSICS.*

AN ADDRESS BEFORE THE INTERNATIONAL CONGRESS OF MATHEMATICIANS, ZÜRICH, AUGUST 9, 1897.

BY PROFESSOR HENRI POINCARÉ.

I.

DOUBTLESS you are often asked what is the utility of mathematics and whether its nicely constructed theories, drawn entirely from the mind, are not artificial products of our caprice.

Among the persons who ask this question, I must make a distinction. The practical class demand of us nothing but means of getting money. These do not deserve to be answered. Rather ought it to be demanded of them what is the good of accumulating so much wealth and whether, in order to have time for its acquisition, it is necessary to neglect art and science, which alone render the soul capable of enjoying it,

Et, propter vitam, vivendi perdere causas.

Moreover, a science produced with a view single to its applications is impossible; truths are fruitful only if they are concatenated; if we cleave to those only of which we expect an immediate result, the connecting links will be lacking, and there will be no longer a chain.

The men who are most disdainful of theory find therein, without suspecting it, a daily aliment. Were they deprived of this aliment, progress would quickly be arrested, and we should very soon settle into Chinese immobility.

But we have sufficiently occupied ourselves with the uncompromising practicians; besides these there are those who are curious about Nature only and who ask us if we are in position to help them to a better comprehension of her. In response we have only to show them two monuments, already rough-hewn, celestial mechanics and mathematical physics. They would doubtless concede that these monuments are well worth the labor they have cost. But this is not enough.

Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore, it has

---

*Translated by permission from the *Revue Générale des Sciences*, vol. 8, No. 21, pp. 857–861, by Mr. C. J. KEYSER, Columbia University.

a philosophic end, and, I venture to say, an end esthetic. It ought to incite the philosopher to search into the notions of number, space, and time; and, above all, adepts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of numbers and of forms; they are amazed when a new discovery discloses for them an unlooked for perspective; and the joy they thus experience, has it not the esthetic character although the senses take no part in it? Only the privileged few are called to enjoy it fully, it is true; but is it not the same with all the noblest arts? Hence I do not hesitate to say that mathematics deserves to be cultivated for its own sake and that the theories not admitting of application to physics deserve to be studied as well as the others.

Even if the physical and esthetic ends were not conjoint, we ought to sacrifice neither the one nor the other. But these two ends are inseparable, and the best means of attaining the one is to aim at the other, or at least never to lose sight of it, a fact I shall now try to demonstrate by showing precisely the nature of the relations between pure science and its applications.

The mathematician ought not to be for the physicist a simple provider of formulæ; between the two there ought to be more intimate collaboration. Mathematical physics and pure analysis are not simply adjacent powers maintaining the relations of good neighborhood; they interpenetrate, and their spirit is the same. This we shall the better comprehend when I shall have shown what physics receives from mathematics and what mathematics, in return, borrows from physics.

II.

The physicist cannot demand of the analyst a revelation of new truth; the analyst can at best aid the physicist in the presentation of truth.

The time is past when people sought to anticipate experience, or to construct the world completely upon certain premature hypotheses. Of all the theories in which they delighted naïvely only a century ago, there remains to-day nothing but ruins.

Now all laws are derived from experience, but, to enunciate them, a special language is needed; ordinary language is too poor; it is besides too vague to express relations so delicate, so complex, and so precise. Here, then, is a prime reason why the physicist cannot dispense with mathematics; it alone furnishes him with an adequate language.

Neither is it a small matter that a language be fit: not to pass from the domain of physics, the unknown man who invented the word *chaleur* doomed many generations to error; heat has been treated as a substance simply because it was designated by a substantive, and it was believed to be indestructible. On the other hand, he who invented the word *électricité* has had the unmerited good fortune implicitly to endow physics with a new law, of the conservation of electricity, which, by a pure hazard, is found to be exact, at least up to the present time.

Nay, to pursue the comparison, writers who embellish language, who treat it as an object of art, at the same time make it a suppler instrument, fitter to render the finer shades of thought. We see, therefore, how the analyst, who pursues a purely esthetic end, contributes thus to create a language better suited to the needs of the physicist.

But this is not all; law issues from experience but it does not do so immediately. Experience is individual, the law derived therefrom is general; experience is only approximate, law is precise or at least pretends to be. Experience is produced under conditions always complex; the enunciation of the law eliminates these complications. This is what is termed "elimination of systematic errors."

In a word, in order to derive law from experience, it is necessary to generalize, a necessity that imposes itself upon the most circumspect observer. But how generalize? Every particular truth can evidently be generalized in an infinity of ways. Among the thousand paths that open before us, it is necessary to make a choice, at least a provisional one; in this choice what shall guide us?

Analogy alone. But how vague this word! Primitive man takes cognizance of only rude analogies, those that strike the senses, analogies of color and sound. He would never have attempted to connect, for example, light and radiant heat.

What has taught us to discern those genuine, profound analogies that the eye does not see and only the reason divines, if not the mathematical spirit disregarding matter in order to attach itself to pure form? It is this spirit that has directed us to name with the same name things differing in respect to matter only; to name with the same name, for example, the multiplication of quaternions and that of whole numbers.

If quaternions, of which I just spoke, had not been so promptly utilized by English physicists, many persons would doubtless have regarded them as only an idle dream ; nevertheless, by teaching us to connect things separated by appearances, they would have already rendered us the apter to penetrate the secrets of nature.

Such are the services the physicist may expect from analysis ; but in order that this science may render them, it must be cultivated on the largest scale, free from immediate preoccupation with utility ; the mathematician must have labored as an artist. What we demand of him is to aid us in seeing, in discerning our way in the labyrinth that presents itself to us. He that sees best is he that has risen to the highest level. Examples abound, I will confine myself to the most striking.

The first will show us how it suffices to change language in order to perceive generalizations that were not at first suspected. When the Newtonian law was substituted for that of Kepler, only the elliptic motion was known. Now, so far as this motion is concerned, the two laws differ in form only ; we pass from the one to the other by a simple differentiation. And yet from the law of Newton may be deduced, by an immediate generalization, all the effects of perturbations and the whole of celestial mechanics. Never, on the other hand, had we kept to Kepler's statement, would we have regarded the orbits of the disturbed planets—those complicated curves whose equations have never been written—as the natural generalization of the ellipse. The progress of observation would have served only to produce belief in chaos.

The second example deserves equal consideration. When Maxwell began his labors, the then recognized laws of electrodynamics, accounted for all the known facts. It was not a new experience that invalidated them, but, viewing them under a new aspect, Maxwell saw that the equations became more symmetric on the addition of a term, and, on the other hand, that this term was too small to produce appreciable effects by the old methods. We know that the *a priori* views of Maxwell awaited an experimental confirmation twenty years ; or, if you prefer, Maxwell was twenty years in advance of experience. How was this triumph achieved ?

Maxwell was profoundly impregnated with the sense of mathematical symmetry ; would the case have been the same, had not others, before him, pursued this symmetry for its own beauty ?

Maxwell was accustomed "to think in vectors," but vectors are introduced into analysis by the theory of imaginaries. And those who invented imaginaries little suspected that these would be turned to account in the study of the real world, a fact sufficiently proved by the name they gave them.

Maxwell, in a word, was perhaps not a skilled analyst but this skill would have been for him a useless and cumbrous baggage. On the other hand, he had in the highest degree a fine sense of mathematical analogies. On that account he became a thorough mathematical physicist.

The example of Maxwell teaches us yet another thing. How should the equations of mathematical physics be treated? Ought we simply to deduce from them all the consequences, and regard them as intangible realities? Far from it; that which they ought especially to teach us is that we can and must transform them, for thus shall we derive something useful from them.

The third example goes to show us how we may perceive mathematical analogies among phenomena which are neither apparently nor really so related physically that the laws of one phenomenon aid us in divining those of the other. A single equation, that of Laplace, is encountered in the theories of Newtonian attraction, of the motion of liquids, of the electrical potential, of magnetism, of the propagation of heat and in many others besides. What of it? These theories seem like images traced the one upon the other. They are mutually illuminated by each appropriating the language of the others; ask the electricians if they do not felicitate themselves on having invented the phrase, "flux de force," suggested by hydrodynamics and the theory of heat.

Thus mathematical analogies not only enable us to surmise physical analogies but are still useful when these latter are wanting.

To summarize, the end of mathematical physics is not merely to facilitate the numerical calculation of certain constants or the integration of certain differential equations. It is more, it is above all to disclose to the physicist the concealed harmonies of things by furnishing him with a new point of view.

Of all parts of analysis, those are the highest and purest, so to speak, which will be most productive in the hands of such as know how to use them.

III.

Let us now consider what analysis owes to physics.

We should quite ignore the history of science if we failed to recall how the desire to know nature has constantly exerted upon the development of mathematics their happiest influence.

In the first place, the physicist propounds problems whose solution he awaits at our hands. By proposing them to us he has paid largely in advance for the service we should render by their eventual solution. If I may be permitted to pursue my comparison with the fine arts, the pure mathematician who should forget the existence of the external world would be like a painter who knew how to combine colors and forms harmoniously but who had no models. His creative power would be quickly perverted.

Possible combinations of numbers and symbols form an infinite multitude. How shall we choose from this multitude such as are worthy to detain our attention? Shall we be guided solely by our caprice? This caprice, which, moreover, would ere long wear itself out, would doubtless lead us far asunder, and we should very soon cease to understand each other.

But this is only a minor phase of the question. Physics will doubtless prevent us from going astray, but it will preserve us from a much more formidable danger: that of revolving constantly in the same circle. History proves that physics has not only constrained us to choose among the hosts of problems that present themselves; it has forced upon us those which we had otherwise never attempted. How varied soever the imagination of man, nature is yet a thousand times richer. To pursue her, we should take paths that have been neglected, and these will often conduct to summits whence new landscapes will be revealed.

What more useful! It is with mathematical symbols as with physical realities; by comparing the different aspects of things we shall be able to comprehend their intimate harmony, which alone is beautiful and therefore worthy of our efforts.

The first example I will cite is so old that one is prone to forget it; it is, notwithstanding, the most important of all. The sole natural object of mathematical thought is the whole number. It is the external world that has imposed upon us the continuum, an invention doubtless of our own, but one that the external world forced us to make. Without it, there could be no infinitesimal analysis; all mathematical science would reduce to arithmetic or to the theory of substitutions.

On the other hand, we have devoted to the study of the continuum almost all our time and powers. Who would regret it? Who would pretend that this time and energy have been lost?

Analysis unfolds before us infinite perspectives of which arithmetic has no suspicion; it shows us at a glance a grand assemblage, of simple and symmetric order. In number theory, on the contrary, where the unexpected reigns, the view is, so to speak, arrested at every step.

Doubtless it will be said that, apart from the whole number, there is no rigor and consequently no mathematical truth; that everywhere the whole number is concealed, and that we should try to render the veils transparent, even at the expense of interminable repetition. Let us not be such purists but let us be grateful to the *continuum*, which, if everything proceeds from the whole number, was alone capable of causing so great issue therefrom.

Need I, moreover, recall the fact that Mr. Hermite has derived a striking advantage from the introduction of continuous variables into the theory of numbers? Thus, the domain itself of the whole number has been invaded, and as a result order has been established there where disorder prevailed. Such is our debt to the *continuum* and, by consequence, to physical nature.

The series of Fourier is a precious instrument continually employed by the analyst; but Fourier invented it to solve a physical problem; if his problem had not been naturally set, we should never have dared to render to *discontinuum* its rights; we should for yet a long time have regarded the continuous functions as the only genuine functions.

The notion of *function* has been thereby considerably extended and, at the hands of certain analyst logicians, has received an unforeseen development. These analysts have thus ventured into the regions of the purest abstraction and have departed as far as possible from the real world. It was a physical problem, however, that furnished the point of departure.

After the series of Fourier, other analogous series were introduced into the domain of analysis; they enter by the same gate; they were conceived in view of applications. It is sufficient to cite those that have as elements the sphere functions or the functions of Lamé.

The theory of partial differential equations of the secsecond order has had an analogous history: it was specially developed by and for physics. If the analysts had been left to their natural tendencies, the following is probably how they would have viewed these equations, and how they would have chosen the limiting conditions:

Consider, for example, an equation between two variables x and y and a function F of these two variables. They would have assumed F and $\frac{dF}{dx}$ for $x = 0$. This was done, for example, by Mme. de Kowalewski in her celebrated memoir. But there is a host of other ways of putting the problem. F may be given along an entire closed contour, as in the problem of Dirichlet, or the ratio of F to $\frac{dF}{dn}$ may be given as in the theory of heat.

It is to physics that we owe all these ways of putting the problem. We may, then, say that without physics the theory of partial differential equations would not be known.

It is useless to multiply examples; I have said enough to warrant a conclusion. When the physicists require of us the solution of a problem, they do not thus impose drudgery upon us; on the contrary, we are under obligation to them.

IV.

But this is not all; physics does not merely furnish problems for solution; it aids us in finding means therefor; and that in two ways: it causes us to surmise the solution, it suggests the proof.

I have mentioned above the fact that the equation of Laplace is met with in a host of far separated physical theories. We find it again in geometry in the theory of conformal representation, and in pure analysis in that of imaginaries. Thus in the study of functions of complex variables the analyst, besides the geometric image which is his usual instrument, finds several physical images that can be used with equal success. Thanks to these images he can see at a glance what pure deduction could show him only in succession. He thus collects the separate elements of the solution, and by a sort of intuition divines before he can demonstrate.

Divine before demonstrate! Need I recall the fact that all important discoveries are thus made? What truths of which the physical analogies give us a presentiment, and which we are not yet in position to establish by rigorous argument! For example, mathematical physics introduces a great number of developments in series. That these series converge, no one doubts; but mathematical certainty is wanting. These are so many assured conquests for the investigators who shall come after us.

Physics, on the other hand, does not merely furnish us with solutions; it also in certain measure provides arguments It will be sufficient to recall how Mr. Klein, in a question respecting Riemann surfaces, had recourse to the properties of electric currents. Arguments of this class, it is true, are not rigorous in the sense that the analyst attaches to this word.

And, in this connection, arises the question : How a demonstration that is not rigorous enough for the analyst, can be sufficient for the physicist? It seems there can not be two rigors, that rigor is or it is not, and that there where it is not, argument can not exist.

We will best comprehend this apparent paradox by recalling the conditions of the applicability of number to natural phenomena. Whence arise, in general, the difficulties encountered when one seeks to give a rigorous demonstration? One strikes them almost always when attempting to establish that such a quantity tends towards such a limit, or that such a function is continuous, or that it has a derivative. Now, the numbers that the physicist measures by experience are known to him only approximately, and, on the other hand, any function whatever always differs by as little as we please from a discontinuous function, and, at the same time, it differs as little as we please from a continuous function.

The physicist may accordingly suppose, at pleasure, that the function studied is continuous or that it is discontinuous; that it has a derivative or that it has not; and this, without fear of contradiction by either present or future experience. One understands how, with such liberty, he makes play of the difficulties that detain the analyst. He may always reason as if all the functions occurring in his calculations were entire polynomials.

Thus the view that suffices for physics is not such reasoning as analysis requires. It does not follow that the one is not able to aid the other.

So many physical observations have been already transformed into rigorous demonstrations that this transformation is easy to-day. Examples abound, if I did not fear, in citing them, to weary your attention, and if this conference had not been already too long.

I hope I have said enough to show that pure analysis and mathematical physics may be reciprocally helpful without either entailing sacrifice upon the other, and that each should rejoice in whatever exalts its associate.