GEOMETRY OF SOME DIFFERENTIAL EXPRESSIONS IN HEXASPHERICAL COÖRDINATES.

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THIS paper is to be regarded as an appendix to my dissertation: Ueber die linearen Complexe der Lie'schen Kugelgeometrie (Göttingen, Kaestner, 1895).

It will simply give an outline of differential geometry, and show its application to the quadratic complex.

Let $x_1, x_2 \cdots x_6$ be any six variables satisfying a homogeneous quadratic identity; these variables may be regarded as the six homogeneous coördinates of the sphere. The form here assumed will be

$$\Pi(x) \equiv \sum_{i=1}^{6} x_i^2 = 0, \tag{1}$$

where two of the coördinates must be imaginary to represent a real sphere.

The geometric meaning of the variables x_i is essentially the same as the $\xi, \eta, \zeta \cdots$ in my classification of Dupin's Cyclides,* from which they can be derived by a linear transformation.

A homogeneous equation of degree n among the variables $x_1 \cdots x_6$,

$$f(x) = 0. \tag{2}$$

in connection with (1) will define a spherical complex of degree n.

In the neighborhood of any given sphere x', belonging to the complex, it can be replaced by a linear complex

$$\sum_{i=1}^{6} \frac{\partial f}{\partial x_i'} y_i = 0, \tag{3}$$

where y_i are running coördinates.

This complex is called a linear tangent complex to $f(x)$ at x'; every sphere has a corresponding tangent complex, and in fact a whole pencil of them. $f(x) = 0$ is not changed by adding any multiple of $\Pi(x)$ to it, hence

* Criteria for nodes in Dupin's cyclides, with a corresponding classification, *Annals of Mathematics,* vol. 11, No. 5, p. 137, June, 1897.

$$\sum_{i=1}^{6}\left(\lambda\frac{\partial f}{\partial x_i'}+\mu x_i'\right)y_i=0 \tag{4}$$

will represent such a complex for every value of $\lambda:\mu$. This pencil has a linear congruence of spheres in common; its directrices are defined by

$$\lambda^2\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i'}\right)^2+2\lambda\mu\sum_{i=1}^{6}x_i'\frac{\partial f}{\partial x_i'}+\mu^2\Pi(x_i')=0 \tag{5}$$

but as $\Pi(x)$ is identically zero, and

$$\sum_{i=1}^{6}x_i'\frac{\partial f}{\partial x_i'}=0 \text{ when } f(x')=0,$$

the congruence is special when

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i'}\right)^2\neq 0,$$

and degraded when

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i'}\right)^2=0.$$

In the former case the congruence is composed of those spheres which touch x' in its circle of intersection with the fundamental sphere of (3), to which it belongs.

This circle is called the *trajectory circle* of x'; every sphere of the complex has a trajectory circle, and spheres of the complex that are infinitesimally close to a given sphere and touch it, must touch it in points of its trajectory circle.*

The expression

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2 \tag{6}$$

is a differential invariant† and has an important geometric meaning. When it vanishes, (3) is a special complex and the congruence defined by (4) becomes degraded. The trajectory circle of such a sphere reduces to a point, and the sphere itself is called *singular*. There are ∞^2 spheres fulfilling the conditions

$$f(x)=0,\ \Sigma\left(\frac{\partial f}{\partial x}\right)^2=0,$$

which envelop a surface called the *surface of singularities* of $f(x)=0$.‡

* Cf. Lie, in *Mathematische Annalen*, vol. 5, p. 207.

† See *e. g.* Koenigs : La géométrie reglée, p. 76.

‡ Klein: Höhere Geometrie, vol. 1, p. 485.

When (6) vanishes identically, when $f(x)=0$, $\Pi(x)=0$, then every directrix of the degraded congruence (4) belongs to $f(x)$ * and every sphere of $f(x)$ is singular; the complex consists of ∞^2 tangent pencils of spheres, and their points of tangency envelope a surface. Such a complex is called *special*, and may be said to have an envelope.

Consider the two complexes of orders n, m,

$$f(x)=0, \quad \varphi(x)=0.$$

Let x' be a sphere of the congruence defined by these complexes; the congruence can be replaced in the neighborhood of x' by the system of ∞^2 linear complexes or the ∞^2 linear congruences,

$$\Sigma\left(\lambda\frac{\partial f}{\partial x_i'}+\mu x_i'\right)y_i=0$$

$$\Sigma\left(\rho\frac{\partial \varphi}{\partial x_i'}+\sigma x_i'\right)y_i=0.$$

These congruences have a linear series in common, defined by the equations

$$(7) \qquad \Sigma\frac{\partial f}{\partial x_i'}y_i=0; \qquad \sum_{i=1}^{6}\frac{\partial \varphi}{\partial x_i'}\cdot y_i=0; \qquad \sum_{i=1}^{6}x_i'y_i=0,$$

but since the combinant of these forms vanishes identically, the envelope of the series is a gauche quadrilateral of minimum lines.

Spheres of the linear series break up, therefore into two tangent pencils; x' is the only sphere belonging to both. As x' varies, the points of intersection of two pairs of these minimum lines generate the two mantles of a surface, the *focal surface* of the congruence. *A congruence of spheres is usually composed of the spheres which are doubly tangent to a surface.*

If all the first minors of the combinant of (7) vanish, the two tangent pencils coincide; that is, for such values of x' that satisfy

$$(8) \qquad \Sigma\left(\frac{\partial f}{\partial x_i}\right)^2\Sigma\left(\frac{\partial \varphi}{\partial x_i}\right)^2-\Sigma\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial \varphi}{\partial x_i}\right)=0.$$

Such spheres have a contact of the second order with the focal surface; they are therefore principal spheres.

* Klein: Differentialgleichungen in der Liniengeometrie, *Mathematische Annalen*, vol. 5, p. 288.

If (8) is identically satisfied by all the spheres of the given congruence, the congruence is composed of the principal spheres of one generation, belonging to the focal surface.

If one of the complexes, φ, be not given, (8) will represent the differential equation of all the congruences of principal spheres contained in f.

The principal spheres of a surface cannot belong to a linear complex, for adjacent spheres must touch each other. If two adjacent spheres of a linear complex touch, the whole pencil must belong to the complex.

Consider the congruence formed by a special complex, and a linear one. If (8) is not satisfied, the congruence must also envelope another surface; if (8) is satisfied, the congruence consists of the ∞^1 tangent pencils which touch the given surface in the points of the curve of intersection with the fundamental sphere of the linear complex.

The spheres of the congruence cut the sphere at a constant angle; hence the curve of intersection, being a line of curvature on the sphere, must be a line of curvature on the surface.* Hence

Among the ∞^3 spheres which touch a given surface, there are ∞^2 which also cut a fixed sphere at a constant angle. These spheres will either envelope another surface or be arranged in ∞^1 pencils, touching the surface along the curve of intersection with the sphere, which is then a line of curvature of the given surface.†

The points of space form a linear complex; the point-spheres belonging to a general complex lie on a surface. The degree of this surface may be found as follows: take any two linear complexes which contain the plane at infinity; their fundamental spheres are planes, which are likewise the surfaces of point spheres of the complexes. Now find the number of spheres the given complex, the two linear complexes and the complex of points have in common—this gives $2n$; so an arbitrary line pierces the surface in $2n$ points. Further, by taking two complexes not containing the plane at infinity; the point spheres common to the congruence lies on a circle—by the same reasoning, an arbitrary circle will intersect the surface in $2n$ points, which necessitates that the surface contains the circle at infinity as an n fold line, hence:

The locus of the point-sphere in a spherical complex of degree n

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* Cf., *e. g.*, Knoblauch: Theorie der krummen Flächen, p. 263.

† Darboux, in his Théorie des Surfaces, vol. 1, p. 257, states the general theorem, having derived it by means of inversion of planes; he does not mention the exceptional case.

is a surface of degree $2n$, and containing the circle at infinity as an n fold line.

The planes of space form a linear complex, the planes contained in a general complex lie on a surface of class $2n$.

When

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2=0,$$

these two surfaces must be identical.

The spheres common to three complexes, $f(x) = 0$, $\varphi(x) = 0$, $\chi(x) = 0$ of order n, m, p form a series; they may envelope an annular surface, or a curve, or simply be contained in a hyper pencil, all the spheres containing a minimum line of one generation in common.

At any sphere x', the series can be replaced by ∞^1 Dupin's cyclides (linear series), all of which have two coincident spheres in common.

If the first minors of the vanishing combinant do not identically vanish, the series envelopes an annular surface, in which $3mnp(m + n + p - 3)$ spheres are touched by the consecutive one. Finally if the first minors vanish identically, the series envelopes a curve or is contained in a hyperpencil, as every sphere touches the consecutive one. The determinant expressing this condition may be regarded as the differential equation of a line of curvature on the focal surface of every congruence satisfying it.

A series contained in the complex of points defines the focal lines of the focal surface of the congruence. The series common to a general complex and the congruence of points and planes defines the singular line of curvature, which is also a minimum curve on the locus of points contained in the general complex.

Many interesting properties of annular surfaces may be obtained by regarding the coördinates of a sphere, $x_1 \cdots x_6$ as functions of a new variable t; the corresponding study has been made of ruled surfaces by Koenigs, in his Géometrie Réglée, and all his results may be interpreted by spheres.

Application to the quadratic complex.

Let $n = 2$. A general quadratic equation in 6 variables can be transformed by a linear transformation to a sum of squares, as

$$f(x) = \sum_{i=1}^{6} a_i x_i^2 = 0,$$

and the quadratic identity simultaneously to the form

$$\Pi(x) = \sum_{i=1}^{6} x_i^2 = 0.*$$

The singular spheres will be determined by those values for x_i which satisfy $f = 0$, $\Pi = 0$, and

$$\sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i} \right)^2 \equiv \sum_{i=1}^{6} a_i^2 x_i^2 = 0.$$

Let $\frac{\partial f}{\partial x_i}$ be denoted by z_i; z_i are now the coördinates of a sphere which touches x (x a singular sphere) in its trajectory circle which is a point. z is also a tangent to the surface of singularities at the same point as x; any sphere of the tangent pencil is therefore expressible in the form

$$\lambda x + z$$

where λ is a parameter. When x successively takes the value of every singular sphere, this expression will represent all the ∞^3 tangent spheres to the surface of singularities.

Consider the equations

$$\sum x_i^2 = 0; \quad \sum a_i x_i^2 = 0; \quad \sum a_i^2 x_i^2 = 0.$$

The first one expresses that x is a sphere, the second, that it belongs to the given complex, and the third, that it is a singular sphere. Let a tangent of the surface of singularities be donoted by m,

$$m_i = \lambda x_i + z_i = \lambda x_i + a_i x_i.$$

Substitute these values for x in the equations

$$\sum m_i^2 = 0; \quad \sum x_i m_i = 0; \quad \sum x_i^2 = 0$$

that express that x_i, m are spheres of a tangent pencil. Hence

$$\sum_{i=1}^{6} \frac{m_i^2}{a_i + \lambda} = 0, \quad \sum \frac{m_i^2}{(a_i + \lambda)^2} = 0, \quad \sum m_i^2 = 0.$$

When λ is eliminated between these equations, that is, the λ discriminant of the first one is equated to zero, it will represent the surface of singularities in tangential coördinates. This equation is of degree 4 in λ, its discriminant is, therefore of degree 12 in m, $\psi(m) = 0$.

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* For the general discussion, see Weierstrass : Zur Theorie der bilinearen und quadratischen Formen, *Berliner Monatsberichte*, 1868, and for its application to 6 variables, cf. Klein, in *Mathematische Annalen*, vol. 23, p. 538, and Weiler, in *Mathematische Annalen*, vol. 7, p. 145. Only the general form will be considered here.

Let y, t be any two spheres not in contact ; $y + kt$ will be a sphere containing their circle of intersection.

$\psi(y + kt) = 0$ is an equation of degree 12 in k ; through any given circle 12 spheres can be passed, which touch the surface. Especially, when y, z are planes, $y + kt$ will represent a book of planes containing their line of intersection. From this it follows that *the surface of singularities of the quadratic spherical complex is of class* 12.*

The further study of the complex will be facilitated by comparing each theorem with the corresponding theorem in line geometry. All of these results are known, but it is desirable to show how each system can be derived from the other, independent of the methods of Darboux (l. c.), Loria† Bôcher,‡ and the smaller works of Reye, Moutard, Laguerre, Maxwell, Casey, etc.

The relation between the two spaces is that discussed by Lie§ and the notation employed will be that of Klein, in his manuscript lectures on Liniengeometrie.

Suppose $x_6 = 0$ is the equation of the point complex ; it is taken as one of the complexes of reference.

The six equations $x_i = 0$ represent six general linear complexes, mutually in involution ; one of them is the complex of points, hence being in involution necessitates their being orthogonal, and the fundamental sphere of each belongs to each of the others (except x_6 whose fundamental sphere is entirely illusory). This gives five mutually orthogonal spheres, discussed by Darboux in his chapter on Coordonnées Pentaspheriques. (Théorie des Surfaces, vol. 1.) These linear complexes, together with their congruences and series have been exhaustively studied by Domsch.||

Now consider the series of quadratic complexes

$$\sum_{i=1}^{6} \frac{x_i^2}{a_i + \lambda} = 0. \tag{9}$$

From the method in which it was derived it is at once evident that the whole set has a common surface of singularities; this surface is the same one as was discussed above,

* Darboux, in his "Sur une classe remarquable" obtains this theorem in practically this way, though he does not define the surface as *surface of singularities.* Cf. p. 276.

† Geometria della sfera, *Memorie di Torino*, II., 36, 1884.

‡ Reihenentwickelungen in der Potentialtheorie, Göttinger Preisschrift, 1891.

§ First discussed in *Mathematische Annalen*, vol. 5, and more in detail in his Geometrie der Berührungstransformationen, pp. 444–475.

|| Darstellung der Flächen 4ter Grades mit einem Doppeltkegelschnitt durch hyperelliptische Reihen, *Grunert's Archiv*, 2d Series, Part 2, 1885.

for the original complex $\sum a_i x_i^2 = 0$ belongs to this set, corresponding to $\lambda = \infty$. Clearing the equation of fractions,

$$\lambda^5 \sum x_i^2 + \lambda^4[(\sum a_i)(\sum x_i^2) - \sum a_i x_i^2] + \cdots = 0,$$

but $\sum x_i^2 \equiv 0$, and if the quartic has as an infinite root, $\sum a_i x_i^2 = 0$, which is the equation of the original complex.

If x_i be the coördinates of any sphere, in general 4 corresponding values of λ can be found, *i. e.*, any sphere of space belongs to four complexes of this set; when λ is a double root of the quartic equation, the corresponding sphere is a singular sphere of the corresponding complex, as was shown above. It then belongs to two other complexes, but not as singular sphere.

In every tangent pencil of spheres of the surface of singularities is a singular sphere corresponding to each value of λ, hence the ∞^3 tangent spheres are the singular spheres of the set of ∞^1 complexes

$$\sum_{i=1}^{6} \frac{x_i^2}{a_i + \lambda} = 0.$$

In a quadratic line complex, a singular line lies in a plane, whose complex-conic breaks up into two pencils. The surface of singularities is a Kummer surface; it is of the fourth order and fourth class. The vertices of the pencils are the points where the singular tangent line cuts the surface again. When the singular line is an inflexional tangent, three of its points of intersection coincide, hence in this case the whole pencil belongs to the quadratic complex. At every point there are two inflexional tangents, hence there are two complexes of the set which contain the entire pencil of tangents at that point. These lines are transformed, in spherical geometry, into the two principal spheres; the two complexes which contain one or the other of the two principal spheres contain the entire pencil of spheres at that point. This explains why any singular sphere belongs to two other complexes of the set, not as singular sphere. In case of a principal sphere, however, it can only belong to one other complex, the one containing the other principal sphere at that point as singular sphere.

This requires that λ is a triple root of (9); hence *the principal spheres of the surface of singularities of the quadratic complexes* (9) *are defined by the equations*

$$\sum_{i=1}^{6} \frac{x_i^2}{a_i + \lambda} = 0, \qquad \sum_{i=1}^{6} \frac{x_i^2}{(a_i + \lambda)^2} = 0, \qquad \sum_{i=1}^{6} \frac{x_i^2}{(a_i + \lambda)^3} = 0.$$

By giving λ a fixed value, these equations define the series of principal spheres contained in the corresponding complex.

In line geometry, the inflexional tangents envelop a curve of order 16 on the Kummer surface, which is an asymptotic line.* The principal spheres define a line of curvature on the surface. By varying λ all of the lines of curvature can be obtained.

Among the complexes of the set (9) are six, corresponding to $\lambda = -a_\nu$, which degrade to linear complexes counted twice. The singular spheres of such complexes are those that belong to the contiguous complex of the set, which is found to be

$$\sum_{i=1}^{6} \frac{x_i}{a_i - a_\nu} = 0. \quad [i \neq \nu].$$

This defines 6 congruences

$$x_\nu = 0 \quad \sum_{i=1}^{6} \frac{x_i^2}{a_i - a_\nu} = 0 \quad [i \neq \nu]$$

which have the surface of singularities as their common focal surface.

In line geometry, at every point of tangency on a Kummer surface, are six tangents which touch the surface again; these lines are lines of the tangent pencil which belong to the six complexes $x_\nu = 0$.

The focal surface of

$$\sum_{i=1}^{5} \frac{x_i^2}{a_i - a_6} = 0, \quad x_6 = 0$$

is the surface of points in the complex

$$\sum_{i=1}^{5} \frac{x_i^2}{a_i - a_6} = 0$$

and, as was shown above, is a surface of order 4, containing the circle at infinity as a double line. This surface was defined by Darboux† as a *cyclide*, hence

The surface of singularities of a quadratic spherical complex is a cyclide.

There are five systems of spheres which are doubly tangent to the surface; by reducing to point coördinates, their centres are found to lie on quadratic surfaces, and, as was

*Klein and Lie, *Berliner Monatsberichte*, 1870, reprinted in *Mathematische Annalen*, vol. 23, p. 579.

† Mémoire sur une classe de courbes et de surfaces, *Comptes rendus*, vol. 68, p. 1311.

shown above, those of each system cut a fixed sphere orthogonally. This is exactly the method of generation given by Darboux (*l. c*) (and expanded in his monogram of the same name, 1873).

Each sphere cuts the cyclides along a curve of the fourth order with two double points; hence the curve breaks up into two circles; hence, *on the cyclide are ten series of circles.* This surface is therefore included among those studied by Kummer,* that have one or more series of conics upon them.

The Kummer surface contains 16 double points, and 16 double planes; six of the double points lie in each of the double planes, and the same, dually. Hence upon the surface of the cyclide are 16 minimum lines, each of which will cut five others.

In each of the line congruences

$$x_\nu = 0, \quad \sum_{i=1}^{6} \frac{x_i^2}{a_i - a_\nu} = 0 \quad [i \neq \nu].$$

are ∞^1 lines whose two points of tangency on the Kummer surface become coincident so that the lines cut the surface in 4 coincident points; in this, as the general case, the whole tangent pencil belongs to the linear complex $x_\nu = 0$. The locus of these points is an asymptotic line of the order 8; there are six such curves on the surface.

The congruence

$$x_6 = 0, \quad \sum_{i=1}^{5} \frac{x_i^2}{a_i - a_6} = 0$$

becomes the line of curvature on the cyclide which is likewise a minimum curve. It is of order 8.

As the tangent planes at the points of each of the special asymptotic curves belong to the linear complex, the corresponding spherical congruence will contain a tangent pencil of spheres; this pencil will contain one point sphere—hence the curve is the curve of intersection of the cyclide and the five orthogonal spheres. These are the focal lines of the cyclide; they are of order 4, and are the curves of contact of the minimum developable circumscribed about the cyclide and the circle at infinity.

Finally, consider the ∞^1 congruences defined by

$$x_6 = 0, \quad \sum_{i=1}^{5} \frac{x_i^2}{a_i - \lambda} = 0,$$

* Ueber die Flächen vierten Grades auf welche Schaaren von Kegelschnitten liegen, *Berliner Monatsberichte*, 1863.

where λ is given, in succession, every value. The focal surfaces will be a system of ∞^1 Kummer surfaces, which touch each other along the six special asymptotic curves of order 8.*

An arbitrary line of $x_6 = 0$ belongs to three of the congruences; its three pairs of points of tangency define a double involution, *i. e.*, every pair is harmonic with the other two.†

Five of the congruences are linear, corresponding to $\lambda = a_1, a_2, \cdots a_5$; their focal surfaces are five pairs of straight lines, each pair being conjugate with regard to x_6.

This gives, in spherical geometry, ∞^1 quadratic congruences, contained in the complex of point spheres; *i. e.*, ∞^1 cyclides having common focal lines. Through each point pass three surfaces of the system. The six points on a tangent to three Kummer surfaces become six minimum lines through a point; each pair determines a tangent plane to a cyclide through a point; these planes make conjugate traces on the circle at infinity, hence the three cyclides passing through any point intersect each other orthogonally. Their curves of intersection are therefore lines of curvature and are curves of the 8 order.

Five of the cyclides reduce to spheres, the fundamental spheres of $x_1 = 0, \cdots x_5 = 0$. The surface of singularities of the complexes (9) is included in this system of confocal cyclides, corresponding to $\lambda = a_6$.

When

$$\sum_{i=1}^{6} \left(\frac{\partial f}{\partial x_i} \right)^2 = \sum_{i=1}^{6} a_i^2 x_i^2$$

is identically zero, the surface of singularities in line geometry becomes a quadric surface; in spherical geometry it coincides with the point locus and envelope of planes contained in the complex—hence a surface of the fourth order and fourth class; it is a Dupin's cyclide; hence

The Dupin cyclide is the only surface that can be the complete envelope of a non reducible special quadratic spherical complex.

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* This remark is made by Lie in *Mathematische Annalen*, vol. 5, p. 255, in a foot-note, inferring it from the results of spherical geometry. It is proved independently by Reye in his memoir Ueber die Singularitätenflächen quadratischer Strahlencomplexe und ihre Haupttangentencurven, *Crelle's Journal*, vol. 97, p. 247.

† Klein, in Linien- und Metrische Geometrie, *Mathematische Annalen*, vol. 5.