QUATERNIONS AS NUMBERS OF FOUR-DIMENSIONAL SPACE.

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ONE of my students* has shown that quaternions may be extended to four-dimensional space, the generalized versor operation of a quaternion being that of turning a directed line so that its projections on two given "right-angled" planes describe given angles in definite senses. One of these planes of a quaternion is its ordinary plane in three dimensions; the other is that plane which is the locus of all lines through the origin perpendicular to the ordinary plane. Mr. Philip shows further that Hamilton's method of assigning directed lines as the "indices" of numbers is an extended Argand method in which the index of 1 is the fourth dimensional unit. Hamilton himself showed that the fourth proportional to three mutually perpendicular unit lines was "a species of fourth unit in geometry" to which the number 1 might be assigned, but he did not further carry out this geometrical idea.

I have found that when a line is turned as Philip describes the line itself describes a *plane* angle of the same magnitude. This fact leads to an interesting theory of parallel great circles of a four-dimensional sphere, *viz.*, great circles that are everywhere equally distant with respect to great arc measurements. Two and only two great circles may be drawn through a given point of the sphere parallel to a given great circle; of these one may be excluded by a definite convention, leaving one and only one proper parallel to a given great circle through a given point. Denoting the versor operation of a given quaternion by the directed arc of its turn then the turning value of this arc is unaltered by translation in its own direction or parallel to itself. The associative law of products becomes thus a matter of instantaneous proof, since we may move the great arc of any factor parallel to itself so that it begins where the preceding great arc ends.

The algebraic theory of quaternions, which is based upon the multiplication table of the units 1, i, j, k, and is inde-

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pendent of geometry, lends itself most readily to the proof of the geometrical facts of four-dimensions.

Let OW, OX, OY, OZ, be four mutually perpendicular unit lengths, and let any directed line whose components parallel to these units are wOW, xOX, yOY, zOZ, be numbered $p(=w+xi+yj+zk)$. Then all lines of the same length and direction are numbered the same and we may conveniently limit ourselves to lines emanating from the origin O. Any line, *line* p, is Tp units in length. Also, *line* $(p+q)$ is the diagonal of the parallelogram whose sides are *line* p, *line* q, so that by taking the parallelogram law for adding directed lengths, we have, *line* p + *line* q = *line* $(p+q)$. Further, from the trigonometry with respect to lengths in a parallelogram, we have, if *line* p, *line* q are inclined at an angle θ: $T(p+q)^2 = Tp^2 + Tq^2 + 2\,Tp\cdot Tq\cos\theta$. Comparing this with the identity $T(p+q)^2 = Tp^2 + Tq^2 + 2SpKq$, we find, $\cos\theta = SpKq\,/\,TpTq$. Thus the condition that *line* p and *line* q are at right angles is $SpKq = 0$.

We next undertake to give meaning to the multiplication of a directed line by a number, by the definition,

$$p\cdot \textit{line}\ q = \textit{line}\ pq$$

We have at once, from the identity, $Tpq = Tp\cdot Tq$:

(*a*). *The multiplier* p *lengthens any line in the ratio* $Tp:1$. Also, since the angle between lines q, pq is $\cos^{-1}Sp/Tp = \angle p$, we have:

(*b*). *The multiplier* p *turns any line through a plane angle of magnitude* $\angle p$.

In particular, a positive or negative scalar leaves direction unaltered or reverses it. Thus if x', y', be any scalars, then $q' = x'q + y'pq$ is the number of any line in the plane of lines q, pq; also since $p\cdot$ *line* q' = *line* $[-y'Tp^2\cdot q + (x' + 2y'Sp)\cdot pq]$, this product is also a line of the same plane. Hence,

(*c*). *The multiplier* p *turns every line in the plane of one of its turns also in that plane, and always by the same amount, and in the same sense.*

Any multiplier p has therefore a system of planes of which one and only one contains a given line. Hence a line may be resolved in one and only one way into components in two given planes of p. Also, if *line* q', *line* q'' be such components of *line* q, then p *line* q', p *line* q'' are the like components of the product p *line* q. We thus arrive at a generalization of Philip's definition:

(*d*). *The multiplier p turns any line so that its components in two given planes of p describe the turns of p in those planes.*

Let *line* r be perpendicular to the lines q, pq, *i.e.*, let $SrKq = 0$, $SrKpq = 0$; then *line* pr is also perpendicular to these lines, since $SprKq = 2SpSrKq - SrKpq = 0$ and $SprKpq = Tp^2SrKq = 0$. The plane (r, pr) is therefore right angled to the given plane (q, pq). Let these planes be directed each in the sense in which p turns; then when p, q, r change continuously in value these planes move continuously, and therefore move as a rigid figure since they are right-angled. We thus find (by taking $p = i$, $q = 1$, $r = j$):

(*e*). *The relative direction of the turns of p in any pair of its right-angled planes is the same as the relative direction of the turns from OW to OX and OY to OZ.*

There are two such relative directions, giving rise to two distinct systems of quaternions. We may regard the initial convention which selects one of these relative directions, as determining *the direction of positive rotation round a directed plane axis.*

It is now obvious that the geometric elements of a quaternion are four: one for its tensor, one for its angle, and two for its directed plane through any given line. The direction of the right-angled plane is fixed by initial convention, and the directed plane through any other line is fixed by (*d*). With such a system of quaternions, number any line *1*; then limit ourselves to the space perpendicular to this line (vector space) and we have Hamilton's system; limit ourselves to a plane through the unit line and we have Argand's system.

When a directed great arc is moved so as to remain the arc of a given quaternion either in the system founded upon the adopted or the contrary initial convention, we may call these different motions respectively *parallel* and *contra-parallel* translation. When a rigid great arc is moved so that all of its points are, at any instant, at equal great arc distances from their initial positions, we may call this motion *equi-distant* translation. Whatever is proved with respect to parallels holds also for contra-parallels, since we have only to change the initial convention in order to make parallels contra-parallels, and *vice versa.*

Lemma. If a plane of p intersect a plane of p′ in a line, then the plane angle of their diedral angle equals the angle between the lines Vp, Vp′.

For, let the line of intersection be *line* q; then, since Vp turns through a right angle in the same plane and sense as p, therefore *line* $Vp\ q$ is a line in *plane* p perpendicular to the edge, *line* q, and drawn in the direction in which p turns. Also, *line* $Vp'\cdot q$ is a similar line in plane p'; and the angle between these lines equals the angle between the lines Vp, Vp'.

(f). *Any great circle intersects two parallel great circles so that corresponding angles are equal.*

This follows immediately from the preceding lemma, the planes of the parallel great circles being planes of one quaternion and the intersecting plane the plane of the other.

(g). *Equi-distant translation is parallel or contra-parallel translation.*

Take radius of unit length so that the measures of radii are versors. Let radius $OQ =$ *line* q, *etc.* Let $Q'R'$ be any equi-distant translation of QR to Q'. Let $pq = r$, $p'q' = r'$. Then $< QR = < Q'R', = < p = < p' = \theta$, say. Thus $p = \cos\theta + \epsilon\sin\theta$, $p' = \cos\theta + \rho\sin\theta$, where ϵ, ρ are known and unknown unit vectors. The condition of equi-distant translation is $< RR' = < QQ'$ or $SrKr' = SqKq'$, for all values of θ. This gives,

$$S(\rho - \epsilon)\, VqKq' = 0, \quad S(\rho - \epsilon)\, V\epsilon qKq' = 0, \quad T\rho = 1,$$

the equations of two planes and a sphere, in vector space. Hence there are in general only two equi-distant translations of QR so that Q reaches Q'. Since $\rho = \epsilon$ or $p' = p$ is one solution, therefore a parallel translation is one equi-distant translation, and therefore a contra-parallel translation is the other. It might seem that ρ should be indeterminate when the two planes coincide; but this occurs only when $\epsilon \parallel VqKq'$, and then the two planes become a common tangent to the sphere, giving only one solution. This is also the only case in which the intersection of the two planes is tangent to the sphere and corresponds to Q' lying on the arc QR. In other words: *The parallel and contra-parallel of an arc coincide when and only when they are drawn from a point of that arc.*

It follows that in a quadrilateral formed by great arcs, *if two sides are equal and parallel, the other two sides are equal and contra-parallel; the adjacent angles of such a figure are supplementary; the great arc diagonals do not intersect each other.*

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