COLLINEATIONS IN A PLANE WITH INVARIANT QUADRIC OR CUBIC CURVES.

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A LINEAR substitution applied to a ternary form will change, in general, the ratios among its coefficients. Unless the substitution is a specialized one, there are three independent linear forms which it leaves unchanged save by a multiplicative constant, but no such forms of higher order except those that are reducible to products of these three linear factors. So much being premised, it is apparent that if any irreducible forms of higher order are unchanged by a particular linear substitution, it must be by reason of some relation among the coefficients of that substitution; and further, that such a relation must be unaltered when this first substitution is transformed by (*not* compounded with) a second. Such relations are expressible in fact by equations involving only the invariants of the first linear substitution. The expression of a conditional relation in invariant form when there is a quadric invariant of the substitution has been effected by integrating a differential equation to determine transcendental invariant forms, then discussing what relations among the parameters will reduce those forms to quadrics. This method is indeed exhaustive; but for the special problem an exhaustive method is not indispensable. It is possible to obtain the invariant conditional equation by quite elementary processes, not only when quadric forms are to be left unchanged, but also when beside the quadrics there are proper cubic invariants. As preliminary to this main theme I will restate well-known formulæ concerning linear invariants and fundamental invariants of a linear substitution or collineation.

A ternary linear substitution or collineation may be represented by the equation (in Clebsch-Aronhold notation):

$$a_x u_\alpha = 0,$$

the equation in line coördinates (u) of the point into which any point (x) is transformed. More explicitly it is written:

$$y_1 = a_1\alpha_1 x_1 + a_2\alpha_1 x_2 + a_3\alpha_1 x_3$$
$$y_2 = a_1\alpha_2 x_1 + a_2\alpha_2 x_2 + a_3\alpha_2 x_3$$
$$y_3 = a_1\alpha_3 x_1 + a_2\alpha_3 x_2 + a_3\alpha_3 x_3.$$

Any invariant of the mixed form or connex $a_x u_\alpha$ is called an invariant of the collineation. There are three independent invariants, represented by i_1, i_2, i_3;*

$$i_1 = a_\alpha\,;\ i_2 = a_\beta b_\alpha\,;\ i_3 = a_\beta b_\gamma c_\alpha.$$

The three linear forms unaltered by the collineation are determined by the aid of the characteristic equation. If the equation of one invariant line were $A_x = 0$, and that of its transformed equivalent $A_y = 0$, since the two can differ only by a multiplicative constant λ, we must have identically

$$A_y = A_\alpha a_x \equiv \lambda A_x.$$

Separating this into three, and eliminating the (A), we find the characteristic equation:

$$\begin{vmatrix} a_1\alpha_1 - \lambda & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 - \lambda & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 - \lambda \end{vmatrix} = 0\,;$$

or reduced

$$\lambda^3 - i_1\lambda^2 + \tfrac{1}{2}(i_1^2 - i_2)\lambda - \tfrac{1}{6}(i_1^3 - 3i_1i_2 + 2i_3) = 0.$$

If the three roots are $\lambda_1, \lambda_2, \lambda_3$, the corresponding linear forms are found by solving the identical equation for A_x; or by factoring the bordered determinant:

$$\begin{vmatrix} a_1\alpha_1 - \lambda_i & a_1\alpha_2 & a_1\alpha_3 & u_1 \\ a_2\alpha_1 & a_2\alpha_2 - \lambda_i & a_2\alpha_3 & u_2 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 - \lambda_i & u_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = \varphi_i(x) \cdot \psi_i(u)$$

Excluding arbitrarily the case of degenerate collineation, where the absolute term in the characteristic equation would be zero, we have corresponding to the three roots three linearly independent forms $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$, and

* Clebsch and Gordan use the designations i, i_1, i_2, respectively. I adopt the subscript showing their degrees in the coefficients of the collineation.

may adopt them as independent variables x_1, x_2, x_3. The connex then has the simplified expression :

$$u_\alpha a_x = pu_1x_1 + qu_2x_2 + ru_3x_3.$$

We shall assume also that the quantities p, q, r are all different. These are obviously the roots of the characteristic equation, so that we have the relations:

$$p + q + r = i_1,$$
$$qr + rp + pq = \tfrac{1}{2}(i_1^2 - i_2),$$
$$pqr = \tfrac{1}{6}(i_1^3 - 3i_1i_2 + 2i_3).$$

The three invariants are seen to be the sums of first, second and third powers respectively of these three roots p, q, r.

That a quadric A_x^2 may be transformed into itself by the collineation $u_\alpha a_x = 0$, the condition must be identically fulfilled

$$A_\alpha A_\beta a_x b_x \equiv \lambda A_x^2.$$

Resolving this identity into six equations and eliminating the unknown coefficients A_{ik}, we have the characteristic equation for the factor λ:

$$(p^2 - \lambda)\ (q^2 - \lambda)\ (r^2 - \lambda)\ (qr - \lambda)\ (rp - \lambda)\ (pq - \lambda) = 0.$$

If the roots of this equation are all different, there are six determinate quadric forms A_x^2, and no more. These six, however, are obviously nothing more than products of the invariant linear forms. For example,

to $\lambda = p^2$ corresponds $A_x^2 = x_1^2$,

to $\lambda = pr$ corresponds $A_x^2 = x_1x_3$, etc.

How then can any irreducible quadric occur with the invariant property? Certainly only when two roots of this characteristic equation are equal. In that case there will be not only one, but a simple infinity of quadric forms satisfying the condition. Of possible equalities among the roots, the following comprise all types :

(1) $p^2 = q^2$, (3) $pq = qr$,

(2) $p^2 = pr$, (4) $pq = r^2$.

These we will consider in their order.

1. Corresponding to $p^2 = q^2$ we should find the system of quadrics with arbitrary parameter m:

$$x_1^2 - mx_2^2,$$

all of which are reducible.

2 and 3. If $p^2 = pr$, or $pq = qr$, it follows that $p = r$, and the corresponding quadrics are the two systems, all reducible:

$$Ax_1^2 + Bx_1x_3 + Cx_3^2,$$
$$Dx_1x_2 + Ex_2x_3$$

4. Accordingly if any non-singular collineation can leave invariant any irreducible quadric, it must satisfy a condition of the type $pq = r^2$. The quadrics corresponding contain one arbitrary parameter m:

$$x_1x_2 - mx_3^2.$$

These are certainly irreducible except for $m = 0$ or $m = \infty$. Rationalizing now this typical condition $pq = r^2$ in terms of the three invariants, we find

$$pqr = \left(\frac{pq + qr + rp}{p + q + r}\right)^3, \text{ or}$$

$$3(i_1^2 - i_2)^3 - 4i_1^3(i_1^3 - 3i_1i_2 + 2i_3) = 0.*$$

Since also the argument is reversible, the result may be formulated thus:

If any proper conic is left unaltered by a non-singular collineation of the plane, then every conic of a simply infinite sheaf must share the invariant property; and the necessary and sufficient condition for the occurrence of such invariable conics is the following relation among the three rational invariants of the collineation:

$$3(i_1^2 - i_2)^3 - 4i_1^3(i_1^3 - 3i_1i_2 + 2i_3) = 0.$$

Before taking up cubic curves, it is useful to review a part of the above argument in geometric language. Consider as before a collineation that leaves unaltered the three lines of a non-vanishing triangle. If also some proper conic is to be transformed into itself, then the poles of these three

* The coefficient 4 of the second term of this condition equation is erroneously omitted in Clebsch-Lindeman, Vorlesungen über Geometrie, I., p. 994.

lines with respect to the conic must be invariant points. Such points cannot lie on a line, they must be therefore the three vertices of the invariant triangle. Further, this self-polar triangle must have two sides tangent to the conic; for if none were tangent, the six points in which they cut the conic would need to lie in involution, whence their three join-lines must meet in one point, contrary to our hypothesis. One side accordingly must be tangent to the conic, and since the triangle is self-polar a second side also is tangent, while the third is the chord of contact. The conic referred to this triangle has the equation:

$$Ax_1x_2 + Bx_3^2 = 0$$

while the collineation is of the form:

$$y_1 = px_1, \quad y_2 = qx_2, \quad y_3 = rx_3.$$

Substituting in the equation of the conic, we find as before the condition for its invariance,

$$pq = r^2,$$

and this gives the same relation among the invariants.

The collineation has three invariants, and if two conditions are imposed upon these, they determine its *absolute invariants*; no more conditions can be imposed without specializing the invariant triangle. We know that there are general collineations transforming a plane cubic into itself. Let it be required to determine their two invariant characteristics. If a proper non-singular cubic is transformed into itself, so are also necessarily both the conic polars and the linear polars of the three fixed points of the collineation. Two conic polars can never be doubly tangent while their poles are distinct, and so cannot belong to any such invariant sheaf as that discussed above. There must be three different sheaves of invariant conics, one in each sheaf being the conic polar of one vertex of the fixed triangle, and touching two sides where they are met by the third. Now two sorts of coördination are possible; either each vertex lies on its conic polar and on the cubic, or else each is the pole of the opposite conic and line; the other two hypotheses are excluded by the order of the cubic. Of the two possibilities the former covers two distinct coördinations of pole and polar, the latter of course but one.

For both cases alike, since an invariant conic occurs in each of the three possible systems, the three conditions must be fulfilled:

$$qr = p^2, \quad rp = q^2, \quad pq = r^2.$$

Of these only two are independent. From them result the three sets of equalities of third degree:

(1) $$p^2q = q^2r = r^2p,$$

(2) $$pq^2 = qr^2 = rp^2,$$

(3) $$p^3 = q^3 = r^3 = pqr.$$

The equivalent relation among the invariants is:

$$i_1 = 0, \quad i_2 = 0, \quad i_3 \neq 0 \quad (i_3 = 3p^3).$$

Corresponding to the three sets of equal roots of the characteristic equation for invariant cubics, there are three systems having the following equations:

(1) $$Ax_1^2x_2 + Bx_2^2x_3 + Cx_3^2x_1 = 0,$$

(2) $$Ax_1x_2^2 + Bx_2x_3^2 + Cx_3x_1^2 = 0,$$

(3) $$Ax_1^3 + Bx_2^3 + Cx_3^3 + Dx_1x_2x_3 = 0.$$

For each of the first two systems the triangle is simultaneously inscribed in and circumscribed about each cubic of the system, and these cubics are of the special kind called *æquianharmonic*.* For every cubic of the third system the triangle consists of three lines of inflexion. This result is recapitulated in the statement:

In order that a non-singular collineation may leave unaltered a non-singular plane cubic curve, it is necessary and sufficient that its invariants of first and second degree should vanish, while its invariant of third degree remains different from zero; and then three discrete systems of cubics enjoy the invariant property, the one containing a threefold infinity of general cubics, the other two containing each a twofold infinity of æquianharmonic cubics. All of the former system have their inflexions upon the fixed triangle, all of the latter systems are simultaneously inscribed in and circumscribed about the fixed triangle.

It will be remembered that every non-singular cubic has eight collineations into itself, besides the identity, as well

* Clebsch-Lindemann, Geometrie, I., p. 579.

as nine others of a sort excluded from the present consideration, namely perspective transformations. It is of interest to observe that the three other inflexional triangles of each invariant cubic of the third class are themselves members of the system, and are therefore transformed into themselves, not interchanged, by the eight non-perspective collineations of the cubic.

Of plane curves of higher order than the third, it is easily shown that only highly specialized classes are collinear with themselves, and that of these classes the groups are correspondingly small. It appears possible, however, to extend this method of inquiry to such interesting topics as these:

(1) What simultaneous invariant conditions must two non-singular collineations satisfy in order to belong to the group leaving a common conic unaltered?

(2) What invariant conditions are met by collineations which leave unaltered a quadric surface? a twisted cubic? a twisted quartic curve?

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A GENERATING FUNCTION FOR THE NUMBER OF PERMUTATIONS WITH AN ASSIGNED NUMBER OF SEQUENCES.

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§ 1. *André's Recurrence-formula.* In *Liouville's Journal*, 1895, and in earlier memoirs there referred to, M. André proves the formula

$$P_{n,s} = sP_{n-1,s} + 2P_{n-1,s-1} + (n-s)P_{n-1,s-2} \qquad (1)$$

where $P_{n,s}$ is the number of permutations of n things (say of the numbers $1, 2, \cdots n$) with s sequences; and shows that (taking the number of sequences as great as possible) the numbers $\frac{1}{2}P_{n+1,n}$ are the coefficients of $x^n/n!$ in $1/(1 - \sin x)$, when expressed as a Maclaurin series.

My object is to obtain a function of x and y which when developed in positive integer powers of x and y will have $P_{n,s}$ as the general coefficient.