Real curve	$u' = .22314$
X	$u = .5596$
$\cosh 2u$	1.6945
$-\cosh 2u'$	-1.1012
$-\cosh(1.2528 + u)$	-3.1441
$+\cosh(1.5316)$	2.4209
	$-.1299$
Y	
$\sinh 2u$	1.3679
$-\sinh 2u$	$-.4612$
$-\sinh(1.2528 + u)$	-2.9808
$\sinh(1.53157)$	2.2047
	.1306

Imaginary curve	$u' = .5$	
X'	$u - .4$	
	or	
$\sinh(3.6931 + u + u')$	$+\sin h\, 1.5931$	2.3591
$-\sinh(1.2523 + u')$	$-\sin h\, 1.7528$	2.7983
		$-.4392$
Y'	or	
$\cosh(.6931 + u + u')$	$\cos h\, 1.5931$	2.5622
$-\cosh(1.2528 + u)$	$-\cos h\, 1.7528$	-2.9711
		$-.4089$

LIE'S GEOMETRY OF CONTACT TRANSFORMATIONS.*

SOPHUS LIE.—*Geometrie der Berührungstransformationen.* Dargestellt von SOPHUS LIE und GEORG SCHEFFERS. Erster Band. Mit Figuren im Text. Leipzig, Druck und Verlag von B. G. Teubner. 1896. 8vo. Pp. xi+693.

Lie was asked in conversation once what constitute the necessary and sufficient characteristics of a mathematician. He replied forthwith: "Phantasie, Energie, Selbstvertrauen,

* Lie uses in the German *Berührungstransformation;* his French pupils translate it *transformation de contact*, Forsyth translates it *tangential transformation*, and Klein *contact transformation.* The latter English translation is retained here as a more precise designation, since by such a transformation not only is the property of tangency, but also in general the order of contact preserved.

Selbstkritik." The illustrious Norwegian thereby unconsciously drew an impressionistic portrait of his own genius, as the striking originality and wonderful scope of his mathematical discoveries testify. Lie's phenomenal creative power and indomitable energy are evidenced not only by the numerous memoirs which have appeared from his pen in the journals and proceedings of learned societies of Norway, France and Germany during the last twenty-five years, but even more emphatically by the fact that the above-named volume is the last of six* tomes aggregating over four thousand royal octavo pages that have come from his hand in the interval between 1888 and 1896, the first five of which appeared in the years 1888–'93.

The last one may with all propriety be called Lie's Geometry or the Geometry of Lie's contact transformations. Transformations applied to certain given differential equations had appeared in the science of mathematics as early as Euler, Lagrange and Legendre, but none of these mathematicians studied the transformations in themselves or established general propositions concerning the particular transformations used, much less concerning general categories of such transformations. Ampère studied more general transformations than those of Euler and Lagrange which he applied to partial differential equations, but the idea of a contact transformation appears neither in his writings nor in those of his successors up to the time of Lie's activity. None of Lie's predecessors gave a definition of a contact transformation, much less a systematic theory of contact transformations. To him is due both the credit of inventing the idea of a contact transformation and that of establishing the theory of such transformations in an independent existence. He has also given a precise development of the theory of infinitesimal contact transformations

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* SOPHUS LIE: Theorie der Transformationsgruppen. Bearbeitet von Sophus Lie unter Mitwirkung von Prof. Dr. Engel. i. Abschnitt, x+632, 1888; ii. Abschnitt, vi+554, 1890; iii. Abschnitt, xxvii+830, 1893. Leipzig, Teubner.

SOPHUS LIE: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. Bearbeitet und herausgegeben von Dr. Scheffers, xiv+568, 1891. Leipzig, Teubner.

SOPHUS LIE: Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen. Bearbeitet und herausgegeben von Dr. Scheffers, xii+810, 1893. Leipzig, Teubner.

Vol. 1 of the Theory of Transformation-Groups was reviewed by Dr. Chapman in the BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY, vol. 2, No. 4, January, 1893; the Lectures on Continuous Groups, by Mr. Brooks in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, vol. 1, No. 10, July, 1895.

and groups of contact transformations, these constituting but a part of his more general and beautiful theory of continuous groups. With great injustice to Lie these transformations have been referred to as the contact transformations of Jacobi and Lie. As well, in fact with better propriety could we speak of the method of fluxions of Wallis and Newton or the infinitesimal calculus of Fermat and Leibnitz. All honor to Jacobi for his additions to our knowledge of the means of transforming the so-called canonical systems of partial differential equations in dynamics. It is true that Jacobi used a particular contact transformation. So did Apollonius of Perga two hundred years before the present era. But we have no evidence that either Jacobi or Apollonius knew it, let alone the conception of the most general transformation of this kind; the idea of a contact transformation is Lie's.

To be completed in two volumes the work is intended to give an extended account of Lie's first geometrical investigations * in the years 1869–72. The text of the first volume before us is preceded by a preface of five pages and a table of contents, and followed by an analytical alphabetical index of subjects and also one of the references to more than a hundred mathematicians of the past and present; the marginal references indicating concisely the contents of the paragraphs constitute another marked feature in the way of indices.

The preface is more than an indication of the contents and scope of the sequel. It presents in precise form Lie's ideas relative to the notions that have been of most power in the development of mathematics; his oft-repeated regrets that the science has of late years been broken into two distinct branches—analysis and geometry, to such an extent that one phase almost asserts its independence of the other, and a plea for a kind of reciprocity between analysis and geometry by which they will advance side by side and enrich each other step by step as in the earlier days. Mathematics is broader than geometry, and deeper than analysis, and while Lie is preëminently a geometer, and professionally a professor of geometry,† yet no living mathematician pleads more earnestly for the union of the two great forces in mathematics, and he has taken particular pains to develop

* The best known of these are: "Om en Classe geometriske Transformationer," *Ges. der Wiss. zu Christiania*, October, 1870. Ueber Complexe, insbesondere Linien- und Kugelcomplexe mit Anwendung auf die Theorie partieller Differentialgleichungen," *Math. Ann.*, vol. 5, 1872.

† Professor der Geometrie an der Universität Leipzig.

a rigorous analytical theory * of contact transformations alongside the more obvious geometrical theory.

It is characteristic of Lie's lectures that every advance synthetically is duplicated analytically and *vice versa.* Not the least important result of the dual development of his theory of continuous groups is its wide application to many apparently heterogeneous subjects and hence its great usefulness as a unifying principle in the science. This demand that analysis become the handmaid of geometry and geometry the servant of analysis, was the theme of Lie's inaugural lecture at Leipzig in 1886. How far this theme has been the burden and inspiration of his own mathematical investigations is best shown by the following sentence from the preface, "Charakteristich für meine Richtung dürfte es besonders sein, das ich nach dem Vorbilde von Monge die geometrischen Begriffe, namentlich die von Poncelet und Plücker eingeführten, für die Analysis verwertet und andererseits Lagrange's, Abel's and Galois' Ideen über die Behandlung der algebraischen Gleichungen auf die Geometrie und besonders auf die Theorie der Differentialgleichungen ausgedehnt habe."

The first volume falls into the three grand divisions: I. Contact Transformations in the Plane, chapters 1–5, pp. 1–176; II. Geometry of Lineal Elements in Space, chapters 6–10, pp. 177–480; III. Introduction to the Geometry of Surface Elements. Partial Differential Equations of the First Order, chapters 11–14, pp. 480–687.

I. In the first division of the work is introduced the notion *contact transformation in the plane*, and methods are developed for finding all transformations of this kind. The most important properties of contact transformations are exhibited and their usefulness in geometry and ordinary differential equations illustrated by simple but instructive examples. The infinitesimal contact transformations are defined and all forms of them determined. Applications of these infinitesimal contact transformations to optics and mechanics are given, and finally, their importance in geometry is further shown by a beautiful application in the solution of the difficult problem—to find the form of the elements of arc of all surfaces for which the differential equation of the third order of the geodesic circles† written in

* See vol. 2 of the Theory of Transformation-Groups.

†These so-called geodesic circles are the curves having constant geodesic curvature and were first studied by Minding. They must not be confused with those geodesic circles studied by Gauss under the definition that their points have a constant geodesic distance from fixed points of the

Gaussian coördinates (x, y) admits* of one or more infinitesimal contact transformations of the two-dimensional manifoldness (x, y).

1. As already remarked, long before Lie introduced the general notion of a contact transformation, the notion appeared in the science in special forms. The first chapter is devoted to some of these special forms. It is composed of four sections. The first section is occupied with several historic point transformations† which have played important rôles as correspondences, namely, *orthogonal projection*, the *projective transformation* and *inversion*. Another no less important application of point transformations is their use as auxiliaries in investigations in the theory of invariants; their utility in this direction is shown here by reference to the problem: How can it be determined whether a given ordinary differential equation of the second order in x, y, can be reduced by means of a point transformation to the form $y''=0$?‡ It may be remarked *en passant* that it is always possible to bring an ordinary differential equation of the first order to the form $y'=0$ by a point transformation; one of the second order to the form $y''=0$ by a contact transformation; and a partial differential equation of the first order to the form $z=0$ by means of a contact transformation properly chosen.

Lie defines a *lineal element* to be the aggregate of a point and a straight line through the point; hence a lineal element is determined by three quantities—the coördinates x, y of the point and the direction y' of the line. A lineal element then has three coördinates x, y, y', and there are accordingly ∞^3 lineal elements in the plane; the plane, a two-

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surface. Either property completely defines a circle in the plane, but they define different classes of circles on a curved surface.

* A differential equation is said to admit of a transformation when the family of its integral curves admits of the transformation. A family of curves is said to admit of a transformation when every curve of the family is changed by the transformation into a curve belonging to the same family.

† By a point transformation is meant, geometrically, an operation by which a point is carried into the position of some point. A point transformation of the xy plane into itself is expressed analytically by two equations of the form $x_1=X(x, y)$, $y_1=Y(x, y)$, where X, Y are independent analytic functions in the Weierstrassian sense. The so-called *first extension* of this point transformation is given by the equations

$$x_1=X, \quad y_1=Y, \quad y_1'=\frac{Y_x+Y_y\,y'}{X_x+X_y\,y'}\,.$$

‡ Lie solved this problem in 1883. It appears that when the solution is possible, for which criteria were established, the reduction depends on the integration of a certain *linear* differential equation of the third order. *Archiv for Math. og Naturv.*, Christiania, 1883, vol. 9, pp. 371 et seq.

dimensional space, thus becomes a three-dimensional manifoldness. A curve determines not only ∞^1 points and ∞^1 tangents, but also ∞^1 lineal elements which are defined to be the lineal elements of the curve. Point transformations are shown to be transformations of the lineal elements of the plane, and an analytical definition of *extended* point transformations is established which is used in the sequel to prove them the simplest forms of contact transformations.

The historic transformations considered up to this point, change point into point and curve into curve; on the other hand there are operations long known which change curve into curve without transforming point into point, such as the *dilatations* which change a curve into its parallel curves, the *pedal transformations* and the *transformations by reciprocal polars*. Sections 2 and 3 are taken up with a discussion of these as transformations of lineal elements and their properties as principles of corrspondencee.

In the fourth and last section the change from point coördinates to line coördinates is shown to be a transformation of lineal elements and the resulting transformation is applied to Clairaut's equation. The section concludes with an application of the transformation by reciprocal polars to the integration of the *generalized* Clairaut equation $y - x\varphi(y') - f(y') = 0$.

2. The second chapter bears the heading "Definition and Determination of the Contact transformations of the Plane." The reader is introduced in the first of the five sections of this chapter to the notion—*element association*.* By an element association Lie means a family of lineal elements (x, y, y') which fulfils the Pfaff equation $dy - y'dx = 0$. He proves that such a family consists of exactly ∞^1 lineal elements; and that points and curves considered as loci of ∞^1 lineal elements are cases of this general notion *element association*. Two neighboring elements (x, y, y'), $(x + dx, y + dy, y' + dy')$ of the family $x = X(t)$, $y = Y(t)$, $y' = P(t)$ are said to be *associated* when $Y' - PX'$ vanishes; or geometrically when to terms of the second order the point of the second lies on the line of the first.

In section 2 there is presented a new conception of the problem of integrating an ordinary differential equation of the first order. To integrate a given ordinary differential equation of the first order in x, y, $F(x, y, y') = 0$ is to find all element associations whose elements satisfy the equation

* Lie uses in the German *Element-Mannigfaltigkeit* and *Elementverein*; the terms *multiplicité* and *association d' éléments* are employed in the French.

$F = 0$.* In this conception an ordinary algebraic equation $F(x, y) = 0$ has integral configurations, namely the points of the curve $F(x, y) = 0$ are general integral configurations while the curve itself is a singular integral configuration of the given algebraic equation. An advantage of this new conception of the problem is that *all* integral configurations of a differential equation are thus made to reappear as integral configurations of the transformed equation when the original is subjected to a transformation ; such is not necessarily the case in the classic conception of the integration problem.

The general *contact transformation* in the plane is defined in the third section. Geometrically, a contact transformation is a transformation of the lineal elements which changes an element association into an element association †; analytically, it is a transformation of the variables x, y, y' into the variables x_1, y_1, y_1' by virtue of which there exists an identity of the form

$$dy_1 - y_1' \, dx_1 = \rho(x, y, y') \; (dy - y'dx).$$

If S and T are contact transformations then the product $S\,T$ is also a contact transformation. By the *product* of two transformations is meant the transformation equivalent to their successive application. These transformation products do not follow exactly the same laws as ordinary algebraic products ; they always obey the associative law but do not of necessity obey the commutative law.

Contact transformations in the plane fall into two classes : 1° *extended* point transformations which change point into point ; 2° *proper* contact transformations, which change point into curve or, more strictly speaking, which transform the lineal elements of a point into the lineal elements of a curve. The fourth section is devoted to the determination of all proper contact transformations in the plane by processes of elimination and differentiation with the result that every contact transformation which is not a mere extended point transformation is completely determined by a system of equations

$$\Omega\,(x, y, x_1, y_1) = 0, \quad \Omega_x + y'\Omega_y = 0, \quad \Omega_{x_1} + y'_1\,\Omega_{y_1} = 0,$$

where the equation $\Omega = 0$ is subject to but one condition that the determinant

* This conception of the problem, and in fact in n dimensions, was given by Lie in the *Göttinger Nachrichten*, October, 1872, p. 481.

† This is the original definition given by Lie in the *Göttinger Nachrichten*, October, 1872, p. 480.

$$\begin{vmatrix} 0 & \Omega_x & \Omega_y \\ \Omega_{x_1} & \Omega_{xx_1} & \Omega_{xy_1} \\ \Omega_{y_1} & \Omega_{xy_1} & \Omega_{yy_1} \end{vmatrix}$$

shall not vanish by virtue of $\Omega = 0$. The theorem is established both synthetically and analytically, and it is observed that the results are wholly independent of the particular system of point coördinates used.

This theorem is applied to numerous examples in the fifth section. Among these are the reciprocal polars, the principle of duality and the dilatations. The forms of contact transformations commutative with certain point transformations are found. Notable among these results are that the dilatations are the only contact transformations commutative with all rotations around the origin and with all translations at the same time; and that those contact transformations commutative with all rotations around the origin and with all uniform extensions from the origin form an infinite group of which the pedal transformations are a particular case.

3. Contact transformations are defined by means of differential equations in Chapter iii. The necessary and sufficient conditions that a transformation of lineal elements

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p)$$

be a contact transformation are found to be

$$[XY] \equiv 0, \quad [PX] \equiv \rho, \quad [PY] \equiv \rho P,\text{*} \quad \rho(x, y, p) \neq 0,$$

where the symbol used since Lagrange and Poisson is introduced, namely,

$$[XY] \equiv X_p(Y_x + pY_y) - Y_p(X_x + pX_y).$$

Two functions X and Y satisfying the condition $[XY] \equiv 0$ are said to be in involution. In the second section of the chapter three interpretations are given to this relation of involution and it is applied to establish categories of differential equations of the first order integrable by elimination.

That ordinary differential equations of the second order have no properties invariant by contact transformations is proved in the third section, *i. e.*, every ordinary differential equation of the second order may be transformed into every other ordinary differential equation of the second order by contact transformations properly chosen. In particular,

* LIE: *Verh. d. Ges. d. Wiss. zu Christiania*, 1872–73; *Math. Ann.*, vol. 8, 1874.

the equation may be transformed into itself, and the condition for this is applied to the equation $y''=0$ to determine the most general contact transformation which changes straight line into straight line. Similar theorems do not obtain for ordinary differential equations of a higher order; in fact, it is remarked that there are differential equations of the third order which do not admit of contact transformations.

The fourth section terminates the chapter with a short historical résumé of points in the writings of Lagrange and Plücker, which were touched incidentally in the developments of the chapter. These historical summaries, of which more occur later, constitute one of the most valuable features of the book to the general mathematical reader.

4. The infinitesimal contact transformations * are powerful implements in the theory of contact transformations and their applications. By an infinitesimal contact tranformation is meant one by virtue of which the lineal element suffers an infinitesimally small change of position. These infinitesimal contact transformations are studied in the fourth chapter. The notion of a one-parameter group of contact transformations stands in close relationship to the notion of an infinitesimal contact transformation. In Lie's terminology a one-parameter group is a family of ∞^1 transformations which possesses the property that the product of any two transformations of the family is a member of the family. Some of the properties of these interesting groups are presented in the first section. Among others that every infinitesimal transformation generates a one-parameter continuous group and conversely, every one-parameter continuous group contains one and but one infinitesimal transformation. These theorems hold in particular when the transformations are contact transformations.

All infinitesimal contact transformations are determined in the second section with the remarkable result that it is possible to give explicitly the general form of an infinitesimal contact transformation in the plane and that this form contains an arbitrary function of the coördinates x, y, p of the lineal elements together with the first derivatives of this function with regard to x, y, p. The result is the more surprising since it is by no means possible to give explicitly the general form of a finite contact transformation, but only to develop methods for their determination. This function $W(x, y, p)$ is called the *characteristic function* of the trans-

* The fundamental notion of an infinitesimal contact transformation Lie introduced in *Verh. der Ges. der Wiss. zu Christiania*, 1872, p. 25.

formation. By virtue of the transformation, x, y, p receive the respective increments:

$$\delta x = W_p \,\delta t, \quad \delta y = (pW_p - W)\,\delta t, \quad \delta p = -\,(W_x + pW_y)\,\delta t;$$

and an arbitrary function $f(x, y, p)$ the increment:

$$\delta f = \{[Wf] - Wf_y\}\,\delta t.$$

The last expression divided by δt is adopted as the *symbol* of the infinitesimal contact transformation.

Lie interprets these transformations mechanically as the propagation of a wave motion in an elastic medium and shows that Huygens' principle in optics is a physical expression of the fact that all dilatations form a one-parameter continuous group. Particular categories of infinitesimal contact transformations are examined in this section. Among these are the infinitesimal contact transformations of optics which are defined by the characteristic function $W = \pi(p)$; and those of mechanics for which the characteristic function has the form $W = \Omega(x, y)\sqrt{1 + p^2}$. The section is terminated by an account of ordinary differential equations invariant by known infinite simal contact transformations.

Sections 3 and 4 have todo with the differential invariants of infinitesimal contact transformations and their determination. Any function of x, y, p invariant by an infinitesimal contact transformation is defined to be a *differential invariant of the first order.* One of the important theorems developed is to the effect that if one differential invariant of the first order of an infinitesimal contact transformation is known the most general differential invariant of the first order is found by a quadrature. Differential invariants of a higher order and the question of integrating a differential equation of the nth order admitting of a known infinitesimal contact transformation conclude the section.

The fifth and last section deals with commutative infinitesimal contact transformations, and their applications. One beautiful theorem is that two contact transformations C_1f, C_2f can be brought by new variables introduced by a contact transformation, to the form of translations $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, only in the case when they are independent and commutative. If commutative they satisfy the equation

$$C_1(C_2f) - C_2(C_1f) = 0.$$

5. It is the object of the fifth chapter to show the importance of the preceding notions by solving the following difficult problem: To find the square of the element of arc on all surfaces, on which the family of ∞^3 geodesic circles (in Minding's sense) admits of an infinite number of contact transformations. The problem reduces itself at once to that of finding what form the square of the element of arc must possess on those surfaces for which a certain differential equation of the third order admits of one or more infinitesimal contact transformations. We find here an elegant exposition of those beautiful investigations* of Lie relative to geodesic circles which bear a striking analogy to some theorems of Beltrami, Dini and Lie in the theory of geodesic lines. The notation and formulæ of the theory of surfaces are assumed known in this chapter and the first section formulates the problem analytically. The differential equation of the third order of all ∞^3 circles of the surface is reduced to the form

$$w'' - \frac{R}{Z} w + \frac{T}{Z} w^3 = 0,$$

$$\text{where } w = y'^{-\frac{1}{2}}, Z = \frac{1}{\sqrt{z(x, y)}}, \; Z_{xx} = R, Z_{yy} = T.$$

The second section reduces the characteristic function to the form

$$W = 2\,\Omega\,(x, y)\sqrt{y'} + \xi\,(x)\,y' - \eta\,(y).$$

Sections 3 and 4 are occupied with the solution of the two cases $\Omega \neq 0$, $\Omega \equiv 0$, respectively. The following are some of the interesting theorems deduced: the family of all ∞^3 geodesic circles of a surface of constant curvature admits of ten independent infinitesimal contact transformations and the family may always be written in the form

$$(x-a)^2 + (y-b)^2 = c^2.$$

If the family admits of only infinitesimal *point* transformations, these are conformal and there are two independent or only one. In the first case

$$ds^2 = \frac{dx\,dy}{[A(x+y)^{\frac{1}{2}+n} + B\,(x+y)^{\frac{1}{2}-n}]^2},$$

* See Lie, *Archiv for Math. og Naturv.*, vol. 9, 1884, pp. 40 et seq.

and the surfaces are applicable to surfaces of revolution; if in addition the family of ∞^2 geodesic lines of the surface admits of the two independent infinitesimal transformations

$$ds^2 = \frac{dx\,dy}{(x+y)^m}.$$

In the second case that the family admits of but one infinitesimal point transformation the surfaces are applicable to spiral surfaces and ds^2 can be brought to the form

$$ds^2 = \omega(x+y)e^{ax}\,dxdy.$$

In the fifth and last section of the chapter Lie proposes a generalization of ordinary geodesic representation by seeking the cases in which two surfaces may be made to correspond in such a manner that the geodesic circles of the one shall correspond to the geodesic circles of the other. He finds that the representation is conformal and that surfaces between which the correspondence exists are applicable to surfaces of revolution; further that the ds^2 of one determines the ds^2 of its correspondent. We have thus between certain surfaces of revolution a generalization of stereographic projection such that meridian corresponds to meridian and parallel to parallel, while the zones between corresponding parallels are transformed projectively.

II. As in the plane a lineal element in space is defined to be the ensemble of a point and a straight line through the point. The position of the point is determined by its three coördinates x, y, z; and the direction of the line is fixed by the ratios of the increments dx, dy, dz. Hence there are ∞^5 lineal elements in space. A family of ∞^4 will be defined by an equation of the form

$$\Omega\ (x, y, z;\ dx, dy, dz) = 0, \tag{1}$$

which is homogeneous in dx, dy, dz. Lie calls such an equation a Monge equation. If (1) is linear in dx, dy, dz it is called a Pfaff equation and has the form

$$Xdx + Ydy + Zdz = 0. \tag{2}$$

A curve in space is considered to be the locus of ∞^1 lineal elements. Those curves, the coördinates of whose lineal elements satisfy the equation (1) are defined to be the integral curves of the Monge equation. Two neighboring lineal elements are said to be *associated* when to terms of the second order the point of one lies on the line of the other. There is a corresponding extension of the notion element association of lineal elements to space which we shall have occasion to refer to later.

6. The mass of matter compressed between the covers of the volume is appalling. Even with the aid of the concise résumés with which each chapter is prefaced any account the reviewer may attempt must be inadequate and unsatisfactory. Chapter vi. is to constitute a connecting link between the first and second grand divisions of the book and bears the heading Pfaff Equations and Nullsystems.* Section 1 interprets the Pfaff equation

$$dy - pdx = 0 \tag{3}$$

in space. Regarding x, y, p, on the one hand, as coördinates of a point in ordinary space, and on the other hand, as coördinates of a lineal element in the xy plane, the simplest form of a Pfaff equation (3) establishes a correspondence between the points of space and the lineal elements of the plane of such a nature that to the integral curves of this Pfaff equation there correspond element associations in the plane. To every contact transformation of the plane there corresponds a point transformation in space which leaves the above Pfaff equation invariant. To integrate an ordinary equation of the first order $F\left(x, y, \frac{dy}{dx}\right) = 0$ is to find the integral curves of the Pfaff equation (3) which lie on the given surface $F(x, y, p) = 0$.

In section 2, Pfaff expressions in three variables are reduced to the canonical forms $d\varphi$, $\Phi d\varphi$, $d\psi + \Phi d\varphi$.

The third section is occupied with nullsystems. The theory of nullsystems or linear line complexes is identical with the study of the special Pfaff equation

$$A(ydz - zdy) + B(zdx - xdz) + C(xdy - ydx) + Ddx + Edy + Gdz = 0. \tag{4}$$

Lie shows that this equation may be transformed into one of the forms

$$xdy - ydx + dz = 0, \tag{5}$$

$$xdy - ydx = 0, \tag{6}$$

* A family of ∞^3 straight lines of which all passing any one point form a flat pencil. *Nullsystem* and *linear line complex* are identical in signification. The family of ∞^3 straight lines cutting a fixed straight line constitute a so-called *special nullsystem.*

by projective transformation. Every general nullsystem admits of a ten-parameter projective group and every special nullsystem an eleven-parameter group of projective transformations. The integral curves of the Pfaff equation (4) are defined as the curves of the nullsystem or linear line complex.*

These complex curves are studied in section 4. Among other remarkable properties all complex curves through a point have the same osculating plane and the same torsion at the point. The latter property is the more remarkable since a nullsystem is defined *projectively* and the notion of torsion is *metric*. The section concludes with a determination of all the asymptotic lines of a ruled surface generated by the lines of the nullsystem and a construction is given by which all the complex curves may be found.

Section 5 establishes a correspondence† between the ∞^3 lines of a nullsystem and the ∞^3 circles in the plane. The equation (5) is reduced to the form (3) and thereby such a transformation of the points of space into the lineal elements of the plane reached that to every straight line of the nullsystem there corresponds a parabola in the plane with axis parallel to a given direction. A certain contact transformation changes these parabolas into circles. Further, by virtue of the equations

$$x_1 = \frac{i(4yz + x)}{8y}, \quad y_1 = \frac{4yz - x}{8y}, \quad p_1 = i\frac{1 - 4y^2}{1 + 4y^2}, \qquad i = \sqrt{-1},$$

the points of space are so transformed into the lineal elements of the plane that to every curve of the nullsystem (5) there corresponds an element association in the plane. In particular to every line of the nullsystem there corresponds a circle in the plane; the ∞^{10} projective transformations which change null-line into null-line are transformed into the ∞^{10} contact transformations which change circle into circle.

7. Lie devotes the seventh chapter to Monge equations and Plücker's line-geometry. The theory of Monge equations is closely connected with that of partial differential equations of the first order. Lie's early researches in geometry showed that Plücker's line geometry falls under the theory of the same equations. These equations are

* LIE: Christiania *Verhandlungen*, 1871, 1883; *Math. Ann.*, vol. 5, 1872, pp. 154, 179.

† LIE: *Math. Ann.* 1872; *Göttinger Nachrichten*, 1874; Transformationsgruppen, vol. 2, chap. 24.

studied directly in the first section of the chapter. A Monge equation (1) as already observed, defines a family of ∞^4 lineal elements in space. Through every point (x, y, z) there pass ∞^1 lineal elements of the family; these form an infinitesimal cone which is defined to be the *elementary cone* of the point. The equation (1) makes correspond to every point an *elementary cone*. An integration problem of (1) then is to find all surfaces which are tangent at all their points to the *elementary cones* corresponding respectively to the points. In case the cone becomes a flat pencil the Monge equation reduces to a Pfaff equation. By introducing the notion *surface element* this problem may be otherwise formulated. A surface element is the ensemble of a point and a plane through the point. There are ∞^3 points in space, and ∞^2 planes through every point; hence there are ∞^5 *surface elements* in ordinary space. A Monge equation defines a family of ∞^4 *surface elements*. The above problem then becomes, to find all surfaces whose *surface elements* belong to this family of ∞^4 surface elements. The problem is solved by the integration of a certain partial differential equation of the first order. An integral curve of the equation (1) is a curve the coördinates of whose lineal elements satisfy the equation. In case ∞^3 straight lines are integral curves of this equation it has the form:

$$\Phi(ydz - zdy,\ zdx - xdz,\ xdy - ydx,\ dx,\ dy,\ dz) = 0. \quad (8)$$

Section 2 is a most valuable historical résumé in fine print of earlier investigations relative to families of straight lines in space. With characteristic clearness and conciseness the researches of Euler, Monge, Malus, Binet, Ampère, Giorgini, Möbius, Chasles, Hamilton, Kummer, Transon, Grassmann, Plücker and Cayley are put in proper perspective.

Section 3 gives the fundamental principles of Plücker's geometry, and section 4 treats of the properties of pencils and nets of linear line-complexes and complexes in involution.

Section 5 investigates relations between line-geometry and differential equations. Here we find Lie's extension of the notion of a line complex to that of a *complex of curves; complex curve* and *complex cone* take the place of *complex line* and *elementary cone*. As in the case of nullsystems discussed in the preceding chapter, the theorems relative to the osculating plane and the torsion obtain here also. Lie proves in conclusion that the characteristics of the integral-surfaces are asymptotic lines, and that to find the integral-surfaces is to find the surfaces on which one family of asymptotic lines is made up of curves of the complex.

8. The eighth chapter bears the heading: "On the Transformation-theory of Tetrahedral Complexes." In this chapter it is proposed to illustrate the close connection between line-geometry and the notions lineal element, Pfaff equation and Monge equation, by a closer study of a remarkable line-complex of the second degree, the so-called *tetrahedral complex*, *i. e.*, the ensemble of all straight lines in space which cut the faces of a tetrahedron in the same anharmonic ratio. This aggregate is composed of ∞^3 straight lines, and since the anharmonic ratio may have ∞^1 values, every tetrahedron has ∞^1 tetrahedral complexes.

The first section contains the fundamental properties of these tetrahedral complexes and an account of Lie's early investigations* concerning them. The Monge equation of a tetrahedral complex has the form

$$(b - c)\, xdydz + (c - a)\, ydzdx + (a - b)\, zdxdy = 0.\dagger \quad (9)$$

The ∞^3 projective transformations which leave the vertices of the tetrahedron invariant are commutative in pairs, form a simply transitive continuous group, and leave also the ∞^1 tetrahedral complexes and their ∞^1 Monge equations invariant.

Section 2 is in fine print and comprises an exhaustive historical sketch of older researches on tetrahedral complexes, the additions to the theory made by Binet, Ampère, Dupin, Chasles, Kummer, von Staudt, de la Gournerie, Plücker, Reye and Lie are reviewed in succession.

Lie's own theory of curves of tetrahedral complexes is presented in the third section. A curve of the complex is one all of whose tangents cut the tetrahedron in the same anharmonic ratio. The tetrahedral complex and its Monge equation (9) admit of the ∞^3 projective transformations

$$x_1 = \lambda x, \quad y_1 = \mu y, \quad z_1 = \nu z. \qquad (10)$$

The ∞^3 lineal elements into which a lineal element is transformed by (10) are said to be of the same species.‡ Similarly a curve or surface is changed respectively into a family of ∞^3 curves or ∞^3 surfaces, invariant by the transformation (10); the ∞^3 curves or ∞^3 surfaces are said to be curves or surfaces of the same species. One class of curves

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* *Göttinger Nachrichten*, January, 1870.

† It is observed that this equation contains but one parameter.

‡ Gattung.

of the same species is made up of the tetrahedral symmetric* curves ; the latter are defined by the equations

$$\begin{aligned} Ax^m + By^m + Cz^m + D &= 0, \\ Ex^m + Fy^m + Gz^m + H &= 0. \end{aligned} \tag{11}$$

Section 4 considers certain transformations of the Monge equation of a tetrahedral complex into itself. All the developments not known before 1870 are Lie's. The Monge equation (9) admits of the transformations

$$x_1 = \lambda x^m, \quad y_1 = \mu y^m, \quad z_1 = \nu z^m ; \tag{12}$$

the fact of this invariance leads to a number of important results among which are the determination of the asymptotic lines of the tetrahedral symmetric surfaces, the anharmonic properties of secants of space curves of the third order and of the right line generatrices of surfaces of the second degree, and the fundamental property of Steiner's surface, namely, that every tangent plane cuts the surface in two conics.

In the fifth and last section† Lie introduces his *logarithmic representation :*

$$x_1 = \log x, \quad y_1 = \log y, \quad z_1 = \log z ; \tag{13}$$

by virtue of this transformation the ∞^1 Monge equations (9) of a tetrahedral complex are changed into the ∞^1 Monge equations

$$(b - c)\, dy_1 dz_1 + (c - a)\, dz_1 dx_1 + (a - b)\, dx_1 dy_1 = 0, \tag{14}$$

of the line complex of the second degree formed by all secants of a conic belonging to a pencil. Of the many correspondences established by means of this transformation only the following can be mentioned: 1° a plane is transformed into a *surface of translation*‡ which may be generated in ∞^1 ways; 2° a surface of the second degree is changed into a surface of translation which may be generated in four ways.

9. Certain Partial Differential Equations of the Second Order in Line-Geometry. The geometrical problems discussed in this chapter are expressed analytically by partial differ-

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*De la Gournerie—Recherches sur les surfaces réglées tétraédrales symétriques. Paris, 1867.

† The developments of this section were worked up by Lie in the winter of 1869–'70.

‡ By a *surface of translation* Lie means a surface containing ∞^1 curves congruent and similarly placed.

ential equations of the second order; they are solved by special methods so that the general theory of such equations is not assumed, and only so much of the theory of partial differential equations of the first order is used as has been developed in the preceding chapters.

The first section solves the problem of determining the surfaces, one of whose families of asymptotic lines belongs to a line complex, yielding the solution that the surfaces sought are integral surfaces of the partial differential equation of the second order

$$r + 2Ns + N^2t = 0, \tag{15}$$

where r, s, t have the usual signification and N is a function of x, y, z, p, q given by the system

$$\Phi = 0, \quad dz - pdx - qdy - 0, \quad dy - Ndx = 0,$$

by eliminating dx, dy, dz; where $\Phi = 0$ is the equation (8) of this review. The theory is illustrated by a number of beautiful examples.

The second section proposes the general problem*—Given a non-linear line-complex; to find all surfaces which contain two families of conjugate curves of the complex. Such surfaces are said to be conjugate with regard to the line-complex. The problem is solved for the particular case when the complex is composed of all the secants of a plane curve. The solution determines among other results all minimal surfaces in space. When the plane curve lies at infinity the conjugate surfaces are the integral surfaces of the partial differential equation of the second order

$$R(p, q)\, r + 2S(p, q)\, s + T(p, q)\, t = 0; \tag{16}$$

in this case the conjugate surfaces are either developable surfaces or surfaces of translation.

The same problem is solved for the tetrahedral complex in the third section; the solution is embodied in the theorem that all surfaces conjugate to a given tetrahedral complex are integral surfaces of the partial differential equation of the second order

$$(b - c)\, xqr - [(b - c)\, xp + (c - a)\, yq + (a - b)\, z]\, s + (c - a)\, ypt = 0. \tag{17}$$

A construction for these surfaces is the *logarithmic* trans-

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* The investigations of this section appeared for the first time in a memoir of Lie's in vol. 2 (1877) of the *Archiv for Math. og Naturv.*, Christiania.

formation applied to them, their asymptotic lines are found by quadrature, and examples of algebraic surfaces of this class given, notably Steiner's surface and the so-called *special Plücker complex surface.**

The fourth and last section discusses the relations between the theory of surfaces of translation and Abel's theorem. Lie seeks the surfaces conjugate to two line complexes composed of the secants of a plane curve at infinity and finds that they must satisfy a system of two linear partial differential equations of the second order homogeneous in r, s, t, of the form (17); and further that the conditions of integrability of the system demand that the curve at infinity be an algebraic curve of the fourth order. The connection between Lie's theory of surfaces of translation and Abel's theorem on algebraic integrals of the first species is embodied in the following theorem: If $F(\xi, \eta) = 0$ is the equation of any curve of the fourth order and we put

$$\int \frac{\xi d\xi}{F_\eta} \equiv \Phi(\xi), \quad \int \frac{\eta d\xi}{F_\eta} \equiv \Psi(\xi), \quad \int \frac{d\xi}{F_\eta} \equiv X(\xi),$$

the equations

$$x = \Phi(\xi_1) + \Phi(\xi_2), \quad y = \Psi(\xi_1) + \Psi(\xi_2), \quad z = X(\xi_1) + X(\xi_2)$$

represent a surface generated in four ways by the translation of a curve. Some cases where the curve of the fourth order is reducible are studied in detail and the discussion leads to the surfaces of translation of the preceding chapter, to Scherck's minimal surface, to the minimal helicoid and to Cayley's ruled surface of the third order.†

10. In the introductory paragraph to the tenth chapter Lie characterizes it as the most important one of the volume and states that the investigations of this chapter will be given a wider extension in the second volume. The chapter is headed: Relation between Propositions concerning Straight Lines and Spheres. The first section deals with conformal point transformations in space and the representation of circles in the plane as points in space. This section furnishes the point of departure for the numerous re-

* The surface is $x = \alpha \frac{(a+u)^2}{a+v}, \quad y = \beta \frac{(b+u)^2}{b+v}, \quad z = \gamma \frac{(c+u)^2}{c+v}$.

† The three surfaces last named are respectively

$$e^z = \frac{\sin x}{\sin y}, \quad z = \text{arc tan} \frac{y}{x}, \quad y^3 + 2xy - 2z = 0.$$

searches of the chapter which culminate finally in Lie's wonderful contact transformation by which straight lines are changed into spheres. Liouville's theorem is established proving that there are ∞^{10} conformal point transformations in space and that all are one-to-one; further that every such transformation may be represented as the product of similitudinous transformations and transformations by reciprocal radii. The well known transformation studied by Chasles, Möbius, Laguerre, Cayley, and Darboux is then given, namely

$$x = X, \quad y = Y, \quad r = iZ, \tag{18}$$

by which to every point X, Y, Z of space there corresponds a circle

$$(x - X)^2 + (y - Y)^2 + Z^2 = 0$$

of the plane $Z = 0$. By means of this transformation Lie constructs a correspondence between the minimal lines of space and the lineal elements of the plane; the correspondence brings to light many properties of minimal curves and minimal developable surfaces.

The second section investigates the connection between the conformal point transformations of space and the contact transformations in the plane and deduces another interesting property of the transformation (18), namely, that by virtue of (18) to every conformal point transformation of space there corresponds a contact transformation of the plane by which circle is changed into circle.* The section concludes with a determination of all infinitesimal conformal point transformations in ordinary space. The most general infinitesimal conformal point transformation is formed by taking the sum of the following ten independent ones multiplied by constant coefficients:

$$p,\ q,\ r;\ yr - zq,\ zp - xr,\ xq - yp;\ xp + yq + zr;$$

$$(-x^2 + y^2 + z^2)p - 2xyq - 2xzr,\ -2yzp + (x^2 - y^2 + z^2)q$$

$$-2yzr,\ -2zxp - 2zyq + (x^2 + y^2 - z^2)r;$$

where p, q, r, are written for $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ respectively. The first six constitute the group of Euclidean motions and the

* Lie, *Göttinger Nachrichten*, May, 1871, p. 201; *Math. Ann.*, vol. 5, 1872, p. 186.

first seven the group of similitudinous transformations.

Lie examines the relations between the linear line complex and the complex of all minimal lines in section 3 and applies and extends some of the transformations considered in chapter VI. If we regard s, F, F' as the coördinates of a minimal line whose equations are

$$\begin{aligned} (1-s^2)\,X + i(1+s^2)\,Y + 2s\,Z + F &= 0, \\ 2sX - 2isY - 2Z - F' &= 0, \end{aligned} \tag{19}$$

we may verify that every conformal transformation of space transforms these coördinates in such a manner as to leave the Pfaff equation

$$dF - F'ds = 0 \tag{20}$$

invariant; and that the minimal lines passing through a point are given by a relation of the form

$$F = a + 2bs + cs^2.$$

If s, F, F' are now interpreted as the coördinates of a lineal element of another plane, the conformal transformations of space will correspond to the contact transformations of the plane which leave invariant the family of all parabolas having parallel axes; whence the transformation of the sixth chapter which changes these parabolas into circles. Lie combines this correspondence between the minimal lines of space and the lineal elements (s, F, F') of the plane with the transformation of chapter VI., which changes the lineal elements of the plane into the points of space. He thus obtains a one-to-one correspondence, defined by the equations

$$\begin{aligned} X + iY + xZ + z &= 0, \\ x(X - iY) - Z - y &= 0, \end{aligned} \tag{21}$$

between the lineal elements of the Pfaff equation

$$xdy - ydx + dz = 0 \tag{5}$$

and those of the Monge equation

$$dX^2 + dY^2 + dZ^2 = 0. \tag{22}$$

The points of the space (x, y, z) are changed into the minimal lines of the space (X, Y, Z) and the points of the space (X, Y, Z) are transformed into the lines of the linear complex (5).

This transformation is studied further in the fourth section which has for its subject a correspondence between the straight lines of one space and the spheres of another. Lie finds that, by virtue of the transformation (21), the straight lines of a ray system of the first order and first class, that belongs to the linear complex (5), *i. e.*, the straight lines of the space (x, y, z) cutting two reciprocal polars:

$$x = rz + \rho,\ y = sz + \sigma;\ x = -\frac{r}{\eta}z - \frac{\rho}{\eta},\ y = -\frac{s}{\eta}z - \frac{\sigma}{\eta},$$

$$(\eta \equiv s\rho - r\sigma),$$

correspond to the points (X, Y, Z) of a sphere in the space (X, Y, Z)

$$\left(X - \frac{\rho + s}{2r}\right)^2 + \left(Y + i\frac{\rho - s}{2r}\right)^2 + \left(Z - \frac{\eta - 1}{2r}\right)^2 = \left(\frac{\eta + 1}{2r}\right)^2.$$

The equations (21) determine a contact transformation of the surface elements of space by which the surface elements of a straight line and of its reciprocal polar correspond to the surface elements of a sphere. The most beautiful property of this transformation of Lie is that it changes the asymptotic lines of a surface into the lines of curvature of its transformed surface.

The last section of the chapter is devoted to associations of lineal elements in space. As in the plane, two lineal elements in space are said to be *associated* when to terms of the second order the line of the first element contains the point of the second. An *association* of lineal elements is a family of elements in which every element of the family is *associated* with all infinitesimally neighboring elements of the family. An *association* of lineal elements in space consists of the ∞^1 elements of a curve, the ∞^1 elements of an elementary cone, or the ∞^2 elements of a point; there are no other varieties. The theorem that the point transformations are the only ones by which every element association of lineal elements in space is transformed into an element association concludes the section.

III. It has already been observed that the third grand division of the volume contains an introduction to the geometry of surface elements and the theory of partial differential equations as a part of that geometry. The family of all ∞^5 surface elements of ordinary space is conceived to be a five dimensional manifoldness and the quantities x, y, z, p, q, are coördinates for the determination of an element of this manifoldness. One or more relations in x, y, z, p, q determine families of surface elements. If there is but one relation it takes the form of a partial differential equation of the first order:

$$F(x, y, z, p, q) = 0;$$

if this relation is free from p and q, it determines a family of ∞^4 surface elements whose points lie on the given surface

$$F(x, y, z) = 0$$

and whose planes are arbitrary. Two relations in x, y, z, p, q give a family of ∞^3 surface elements. This family may consist of the elements of the ∞^1 integral surfaces of a certain Pfaff equation, or of those whose points lie on a certain surface, or of those whose points generate a certain space curve. Families of ∞^2 surface elements will be given by three relations in x, y, z, p, q. Among the number of possibilities in this case the family of ∞^2 elements may consist of those of a surface, or of those of a curve or of those of a point; accordingly surfaces, curves and points appear as special forms of two dimensional manifoldnesses in the five dimensional manifoldness of all surface elements.

The geometry of surface elements is an analogue of Plücker's line geometry. Plücker regarded the family of all ∞^4 straight lines of ordinary space

$$x = rz + \rho, \quad y = sz + \sigma$$

as a four dimensional manifoldness, in which r, ρ, s, σ determine the elements, namely, the straight lines. According as there are given one, two or three relations in r, ρ, s, σ we have nullsystems, raysystems or ruled surfaces. Just as the *projective* transformations play a leading rôle in the *line geometry* so the *contact transformations* come to the front in the *geometry of surface elements*. This geometry of surface elements was founded by Lie in his first geometrical work in the year 1870. The last division of the first volume is intended to give an introduction to this geometry only so far as it relates to partial differential equations of the first order; *i. e.*, to the theory of families of ∞^4 surface elements. This geometry will be further developed in the second volume. In the first chapter of the division Lagrange's theory is given, in the second the theory from the standpoint of Lie's geometry, while the last two chapters are concerned with special categories of partial differential equations of the first order.

11. Lagrange's Theory of Partial Differential Equations of the First Order and its Geometrical Interpretation according to Monge. These classic theories are here reproduced in a concise and elegant form. The first section has do with linear partial differential equations and systems of *simultaneous* ordinary differential equations. Among other things, it contains a generalization for a system of $n-1$ equations in n variables and introduces the notion *integral manifoldness.*

The second section gives Lagrange's development of a general solution of a partial differential equation from a complete solution. Lie renders the classic treatment of this problem particularly simple and satisfactory by making continual use of the notion surface element. The general case is illustrated by a number of examples happily chosen.

The notion of the complete integral and the generation of integral surfaces by the method of envelopes leads to the definition of the *characteristics* and their fundamental properties given in the third section, which studies the generation of integral surfaces by the characteristics. The term characteristic is used here in the sense introduced by Monge. A non-linear partial differential equation of the first order has ∞^3 characteristics. Through every characteristic there passes a family of an infinite number of integral surfaces. For this reason the characteristics are called *curves of indetermination.**

This observation that, when the integral surfaces are to be determined, the characteristics are curves of indetermination, leads to the differential equations of the characteristics in the fourth section of the chapter. Sections 3 and 4 are illustrated by examples.

The fifth section ends the chapter with an extended historical recapitulation of the early investigations in the theory of partial differential equations of the first order. The contributions of Euler, d'Alembert, Lagrange, Jacobi, Lacroix, Charpit, Monge, Bonnet, Lie, du Bois-Reymond, Cauchy, Grassmann, Cayley and Riemann are successively reviewed with the conclusion that Lie was the first to apply the notions of the theory of manifoldnesses to the theory of partial differential equations.

12. The Theory of Partial Differential Equations of the First Order as a Part of the Geometry of Surface Elements. The notion *element association* of surface elements is intro-

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* Unbestimmtheitscurven.

duced and a new formulation of the integration problem presented in the first section. By an *element association* of surface elements Lie means a family of surface elements the coördinates x, y, z, p, q of whose elements satisfy the Pfaff equation

$$dz - pdx - qdy = 0.$$

Such a family consists of ∞^2 or ∞^1 surface elements; in the former case the *element association* is composed of the surface elements of a *point*, *curve* or *surface*; in the latter, either of those of an *elementary* cone, or of those of an *element band*,* where by an *element band* is meant such a family of ∞^1 surface elements along a curve that the points of the elements are points of the curve and the planes of the elements are tangent planes to the curve at the respective points. To integrate a partial differential equation of the first order

$$F(x, y, z, p, q) = 0$$

is to find all element associations whose elements belong to the family of ∞^4 elements defined by the equation. This generalized conception of the problem throws light on the meaning of the terms *solution*, *general solution*, *complete solution* and *singular solution*. By introducing his notion *characteristic bands* in place of Monge's *characteristic curves* Lie has also generalized the notion of a complete integral.

The second section is devoted to Lie's *characteristic bands* and their representation as lineal elements in the plane. These characteristic bands are defined in a manner analogous to the definition of Monge's characteristics by assuming the existence of a complete integral. Two correspondences are derived in this section by each of which the *characteristic bands* are represented as lineal elements in the plane and the integrals correspond to element associations of lineal elements in the plane. These transformations differ in that by one the integral configurations of a certain complete solution are represented as points, and by the other as ∞^2 curves in the plane. The correspondence between the two planes of these correspondences is effected by a contact transformation. The derivation of the ordinary differential equations of the characteristic bands terminates the section.

The third section presents the proof of the existence of a complete solution. Starting from the differential equations of the characteristic bands Lie demonstrates by a rigorous process which avoids the famous objection made to the

* Elementstreifen.

method of Cauchy, that the integration of this system of ordinary differential equations leads to a possible construction for a complete solution of the given partial differential equation of the first order. In what has gone before Lie has developed his theory of characteristic bands as a parellel to Monge's theory of characteristic curves; he concludes this section, however, with a *direct* theory of partial differential equations in terms of his notions *element associations* and *characteristic bands*, independent of the original development.

The relation of involution already introduced in chapter III is studied further in section 4 of this chapter. Two partial differential equations of the first order have ∞^1 integral configurations in common when the two are in involution. The problem of integrating the ∞^1 partial differential equations of the first order

$$F(x, y, z, p, q) = \text{constant},$$

is solved as soon as two independent functions Φ and Ψ of x, y, z, p, q are known which are in involution with each other and with F.

The fifth section ends the chapter with a discussion of transformations of partial differential equations of the first order. By a point transformation the characteristic bands of an equation are changed into the characteristic bands of the transformed equation. The relation of involution is invariant by a point transformation. The representation of the characteristic bands as lineal elements in the plane leads to the beautiful result that there corresponds a contact transformation of the plane to every point transformation of space which leaves invariant the partial differential equation considered.

13. The thirteenth chapter treats of *partial differential equations of the first order which admit of infinitesimal point transformations.* The investigations of this chapter touch in many points Lie's theory of continuous groups and its applications to differential equations, but it is not his purpose to go into the general theory of groups in this work and accordingly he gives only some of his early (1870–71) reseaches in this direction as a preparation for the more general theories and as an indication of their importance to geometry.

The first section deals with categories of equations which admit of infinitesimal translations or rotations. If a partial differential equation of the first order admits of two independent infinitesimal translations $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, it is free from x and y, in involution with the equation $\frac{p}{q} = a$, and the complete solution is given by a quadrature. If it admits of an infinitesimal rotation about the z-axis it is in involution with the equation $xq - yp = 0$. Other beautiful theorems are deduced for infinitesimal screws and infinitesimal projective transformations.

In section 2 Lie finds classes of equations admitting of a known infinitesimal transformation. But a few of the novel theorems deduced may be mentioned: 1° partial differential equations of the first order admitting of an infinitesimal rotation around the z-axis have the general form:

$$\Omega(x^2 + y^2, z, p^2 + q^2, px + qy) = 0;$$

2° if a partial differential equation of the first order admits of

$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}, \tag{23}$$

it is in involution with the linear equation

$$\xi p + \eta q - \zeta = 0;$$

3° the most general line complex admitting of an infinitesimal rotation around the z-axis is

$$\Psi(r^2 + s^2, \rho^2 + \sigma^2, s\rho - r\sigma) = 0.$$

Section 3 is taken up with equations which admit of two commutative infinitesimal transformations. If a non-linear partial differential equation of the first order admits of two independent commutative infinitesimal transformations it is integrable by a quadrature.

The fourth and last section of the chapter has to do with equations admitting of two non-commutative infinitesimal transformations. Let a given non-linear partial differential equation of the first order, $F = 0$, admit of the two independent non-commutative infinitesimal point transformations U_1f and U_2f, where the Uf's have the form (23); the equation $F = 0$ is then in involution with

$$J \equiv \frac{\xi_2 p + \eta_2 q - \zeta_2}{\xi_1 p + \eta_1 q - \zeta_1} = \text{constant},$$

and $U_1' J$, $U_2' J$ are integrals of the system of simultaneous ordinary differential equations associated with $F = 0$, where $U'f$ is the *extension* of Uf which includes the increments of

p and q. There are four cases to consider: 1° $U_1' J = 0$; 2° $U_1' J =$ constant; 3° $U_1' J = \varphi(J)$; 4° $U' J$ independent of J. In the first case $U_1 f$ and $U_2 f$ are commutative; in the second $U_1(U_2 f) - U_2(U_1 f) \equiv U_1 f$; in the third, $U_1(U_2 f) - U_2(U_1 f) \equiv a U_1 f + b U_2 f$, where a and b are constants. In the first and fourth cases the original equation $F = 0$ is integrable by a quadrature, and in the second and third its integration is reduced to that of an ordinary differential equation of the first order and further reduction is impossible.

14. This, the last chapter of the book, discusses *certain classes of partial differential equations of the first order* which are of particular interest in the *geometry of curves on surfaces*. The first class of these are those whose characteristics are asymptotic lines on the integral surfaces; these occupy section 1. For this case $F = 0$ is subject to the condition that the equation

$$(F_x + F_z p) F_p + (F_y + F_z q) F_q = 0$$

shall exist as a consequence of $F = 0$. All partial differential equations of this class may be found by processes of differentiation and elimination.

The second section is devoted to the class whose characteristics are lines of curvature. Since Lie's transformation of straight lines into spheres changes the asymptotic lines of a surface into the lines of curvature of the transformed surface, the solution of this second problem may be obtained from the solution of the first above by an application of this transformation; for this reason the fundamental properties of contact transformations and of systems and complexes of spheres are presented in this section. Among other interesting details of the exposition are two tables of correspondences between configurations, problems and theorems, established by Lie's line sphere transformation. It appears that the characteristics are lines of curvature only in the case when $F = 0$ satisfies the condition

$$\begin{aligned}(F_y + F_z q) F_p - (F_x + F_z p) F_q \\ + (F_y p - F_x q + F_z pq)(F_p p + F_q q) = 0.\end{aligned}$$

In case the characteristics are asymptotic lines and lines of curvature at the same time the equation either has the form

$$F(p, q, z - xp - yq) = 0,$$

or it is linear and its characteristics are ∞^2 minimal right lines.

The third section seeks those equations whose characteristics are geodesic lines on the integral surfaces. A large class of these is very ingeniously deduced from the preceding by making use of the properties of evolutes* of surfaces; this class has the form

$$(p\,H_x+qH_y-H_z)^2-(H_x^2+H_y^2+H_z^2-1)(p^2+q^2+1)=0,$$

where H is an arbitrary function of x, y, z; Lie gives also a second determination of this same class by means of considerations based on his sphere geometry. For the solution of the general problem of this type $F=0$ is subject to a condition containing second derivatives, namely, that the determinant

$$\begin{vmatrix} p & q & -1 \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ \frac{d^2x}{dt^2} & \frac{d^2y}{dt^2} & \frac{d^2z}{dt^2} \end{vmatrix} = 0$$

in consequence of $F=0$.

The fourth and last section considers further categories of partial differential equations of the first order. The solution of the problem of determining all surfaces whose normals belong to a given line complex leads to the integration of an equation of the form

$$F(p, q, x+zp, y+zq) = 0.$$

These surfaces were first studied by Transon in 1861; Lie showed in 1871 that the solution of this normal problem finds its analytical expression in those partial differential equations of the first order which admit of an infinitesimal dilatation. The solution of the problem includes also the determination of the surfaces of which ∞^1 geodesic lines are curves of a given line complex. The section concludes the book with a résumé of the problems treated in this last chapter together with indications of other problems formed by combinations of the preceding ones, discussions of which are promised for the second volume.

Relative to the part taken by Dr. Scheffers in the preparation of this volume we can heartily emphasize the words of Lie in the preface: "Ich bin besonders glücklich darüber, dass Herr Scheffers dazu bereit war, mit mir zusammen meine Geometrie der Berührungstransformationen zu redigieren."

* Centraflächen.

As regards both the arrangement and the presentation of subject matter this is the best German text-book and treatise combined that has come to the writer's notice. The mechanical execution is excellent. The type is large and well spaced and leaded. The numerous figures are carefully drawn and the engravings are clear cut. It is a credit to the Teubner press.

The book is delightful reading both for the student of mathematics and for the general mathematical reader who is interested in mathematics but not making it a specialty. It is broadening, stimulating and suggestive; students will find suggestions for many an original note and teachers will pick up many a hint for class work with their more intelligent pupils. Previous acquaintance with Lie's theories is not necessary to an intelligent perusal of the book. Lie's marvelous power of concrete representation is here at its best and mathematicians will look forward with keen interest to the concluding volume of this treatise.

EDGAR ODELL LOVETT.

BALTIMORE,
March 31, 1897.