The precise position of these p intervals can be determined when k is an integer either by Van Vleck's method or by the method explained at the beginning of this paper. If, for instance, $k = 3$ we may proceed as follows. We easily find that R_n satisfies the relation:

$$\xi^2 R_n''' + [\xi - (n+1)(n+2)] R_n' - (n+2) R_n = 0.$$

At two successive points where $R_n = 0$ R_n''' will therefore have opposite signs unless between the points is question $\xi = (n+1)(n+2)$; and we have the theorem:

$J_{n+3}(x)$ vanishes once and only once between two successive positive roots of $J_n(x)$ except between the two roots which include between them the point $x = 2\sqrt{(n+1)(n+2)}$ in which interval $J_{n+3}(x)$ does not vanish at all.

Bessel's equation is clearly only a first example to which the methods of Sturm, which we have discussed, can be profitably applied. Further considerations of this sort, however, with reference especially to Bessel's functions with negative subscripts and to the theory of hypergeometric functions I will reserve for a future occasion. I shall be satisfied if the foregoing discussion helps to emphasize the importance of Sturm's paper.

HARVARD UNIVERSITY,
January 4, 1897.

ON THE TRANSITIVE SUBSTITUTION GROUPS WHOSE ORDERS ARE THE PRODUCTS OF THREE PRIME NUMBERS.

BY DR. G. A. MILLER.

[Read at the January meeting of the Society, 1897.]

ALL the regular groups of these orders have been determined by Cole and Glover and by Hölder.* It is the object of this paper to determine all the transitive groups that are simply isomorphic to these regular ones. As every substitution group of a given order is simply isomorphic to one and only one regular group, we shall thus find all the possible non-regular transitive groups whose orders are the products of any three prime numbers. At the same time we shall be

* A regular substitution group may be said to be determined by the simply isomorphic abstract or operation group and *vice versa*.

able to observe a number of important properties of the given regular groups. We shall first develop the theorem upon which our method is based.

Let $a_1, a_2, a_3, \cdots, a_n$ and $1, s_2, s_3, \cdots, s_g$ represent the elements and the substitutions, respectively, of a transitive group G. The g substitutions may be arranged as follows:

$$\begin{array}{llllll} 1 & s_2 & s_3 & \cdots\cdots & s_m & a_1 \\ t_2 & s_2 t_2 & s_3 t_2 & \cdots\cdots & s_m t_2 & a_2 \\ t_3 & s_2 t_3 & s_3 t_3 & \cdots\cdots & s_m t_3 & a_3 \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots \\ t_n & s_2 t_n & s_3 t_n & \cdots\cdots & s_m t_n & a_n \end{array}$$

Where the line represented by a_α contains all the substitutions of G that replace a_1 by a_α. In particular, the substitutions of the first line do not contain a_1. It should be observed that the same letters are used to represent the elements of G, and also the lines in this arrangement of its substitutions. The reason for this will appear presently.

By multiplying all the substitutions of the line a_α into any one of the substitutions which replace a_α by a_β, we obtain each of the substitutions of the line a_β. Hence we obtain merely a new arrangement of the lines by multiplying all the substitutions of G into any one of its substitutions.* This new arrangement may be obtained from the given arrangement by transforming the a's representing the lines by the substitution which has been multiplied. By multiplying each of the substitutions of G by all its substitutions, we do not only obtain the identical substitution group in the a's representing the lines, but *we obtain identical substitution in every case*, that is, the rearrangement of the a's is always exactly represented by the substitution which has been multiplied.

It may be observed that we have not assumed that the line a_1 contains all the other elements of G. Our remarks, therefore, apply to all the cases when the first line is represented by any letter that does not occur in it, and the following lines by the letters which replace this in any one of their substitutions. By choosing one of the possible letters to represent one line, we determine all the others. When the group represented by a_1 is of degree $n - 1$, it is more convenient to represent its n conjugates by the given letters and notice that each of the substitutions of G transforms these n conjugates in exactly the same way as it transforms its elements. As this notation is not as general as the pre-

* DYCK, *Mathematische Annalen*, vol. 22, p. 90.

ceding, we shall not employ it, and shall therefore not consider it at this place.

The given permutations of the lines in the given arrangement of the substitutions of G are independent of the elements by which G may be represented. We obtain the same permutations if we use for the s's the corresponding substitutions in any simple isomorphic group G', the simple isomorphism having been established in any one of the possible ways. The subgroup of G' which corresponds to the line a_1 can evidently not contain any operations that generate any self-conjugate subgroup of G' besides identity and the substitution which corresponds to t_α and is used to generate a new line in the arrangement of the substitutions G' must differ from any of the preceding substitutions in this arrangement. Conversely, if the substitutions of any group are so arranged that the first line contains a subgroup which includes no substitutions that generate any self-conjugate subgroup besides identity and if the following lines are formed in the given manner, these lines will be permuted according to a simply isomorphic substitution group when each of the substitutions is multiplied by every substitution of the group.* Hence we have the

THEOREM I. *The number of the transitive substitution groups that are simply isomorphic to a given group is equal to the number of its systems of subgroups that satisfy the two conditions:* (1) *Each system includes all the subgroups that are brought in correspondence when the group is made simply isomorphic to itself in every possible way, and* (2) *A subgroup that belongs to such a system does not include any operations that generate any self-conjugate subgroup with the exception of identity. All these transitive groups may be derived from the given group.*

In order that the applications which we shall make of this theorem may be more easily followed we shall employ it to determine the transitive groups that are simply isomorphic to (abc) all (de). There are only two systems of subgroups that satisfy the given conditions, viz.: 1 and the system which includes (ab). To obtain the group that corresponds to the former, *i. e.*, the simply isomorphic regular group we may write the substitutions as follows:

1	a_1	ab	a_4	de	a_7	$ab.de$	a_{10}
abc	a_2	ac	a_5	$abc.de$	a_8	$ac.de$	a_{11}
acb	a_3	bc	a_6	$abc.de$	a_9	$bc.de$	a_{12}

* DYCK, loc. cit.

By multiplying all of these substitutions into each of the two generators *ab*, *ac*, *de* we obtain the following two arrangements, a substitution (line) being represented by the corresponding letter:

$$\begin{array}{cccc} a_4 & a_1 & a_{10} & a_7 \\ a_6 & a_3 & a_{12} & a_9 \\ a_5 & a_2 & a_{11} & a_8 \end{array} \qquad \begin{array}{cccc} a_{11} & a_8 & a_5 & a_2 \\ a_{10} & a_7 & a_4 & a_1 \\ a_{12} & a_9 & a_6 & a_3 \end{array}$$

The substitutions that correspond to these arrangements are respectively:

$$a_1a_4 \cdot a_2a_6 \cdot a_3a_5 \cdot a_7a_{10} \cdot a_8a_{12} \cdot a_9a_{11}$$
$$a_1a_{11} \cdot a_2a_{10} \cdot a_3a_{12} \cdot a_4a_8 \cdot a_5a_7 \cdot a_6a_9$$

Hence these are generators of the regular group which is simply isomorphic to (*abc*) all (*de*). To obtain the transitive group which corresponds to the latter of the two systems we may write the substitutions of (*abc*) all (*de*) as follows:

$$\begin{array}{lll} 1 & ab & a_1 \\ abc & ac & a_2 \\ acb & bc & a_3 \end{array} \qquad \begin{array}{lll} de & ab.de & a_4 \\ abc.de & ac.de & a_5 \\ acb.de & bc.de & a_6 \end{array}$$

By multiplying all these substitutions into the same two generators as before, we obtain the following two arrangements:

$$\begin{array}{cc} a_1 & a_4 \\ a_3 & a_6 \\ a_2 & a_5 \end{array} \qquad \begin{array}{cc} a_5 & a_2 \\ a_4 & a_1 \\ a_6 & a_3 \end{array}$$

The substitutions of the lines that correspond to these arrangements are:

$$a_2a_3 \cdot a_5a_6 \qquad a_1a_5 \cdot a_2a_4 \cdot a_3a_6$$

Hence these are generators of the transitive group of degree six which is simply isomorphic to (*abc*) all (*de*).

Having called attention to the fundamental ideas employed in the following investigation we proceed to state in what sense we shall use several terms which are not always used in exactly the same sense. The term subgroup shall not include either identity or the entire group. Two subgroups of the same group are said to be transform subgroups when one can be transformed into the other by any operation that transforms the group into itself. With respect to regular groups or operation groups this is equivalent to saying that two subgroups are transform when they correspond in any one of the possible simple isomorphisms of the groups to themselves.* Two or more self-conjugate subgroups may, therefore, be transform. The concepts, transform subgroups and characteristic subgroups, are clearly corresponding extensions of the concepts, conjugate subgroups and self-conjugate subgroups respectively.

It should be observed that the two definitions which Frobenius gives † for a characteristic subgroup (which are equivalent with respect to regular and operation groups), are not equivalent with respect to substitution groups in general. As may be inferred from what has been stated above, we adopt the former of these two definitions for the characteristic subgroup in substitution groups. It may happen that the characteristic and transform subgroups of a given group are identical with its self-conjugate and conjugate subgroups. This is, for instance, the case in the only non-regular transitive group, whose order is the product of two prime factors, as is evident from the fact that this group contains only one self-conjugate subgroup and that all its other subgroups constitute a single system of conjugate subgroups.

Our problem is clearly not as general as that embraced in the given theorem, since we have only to find the transitive groups which are simply isomorphic to given regular groups. It may, therefore, be convenient to replace the theorem by the corollary which we shall directly employ, using the terms as they have been defined above.

COROLLARY. *The number of transitive groups that are simply isomorphic to a regular group (G) is equal to the number of such systems of transform subgroups of G as do not include any self-conjugate subgroup of G. All these groups may be directly obtained from G.*

The non-regular transitive groups whose order is p^3.

Since the degree of these groups is p^2 they are included in Sylow's determination‡ of all the transitive groups of degree p^2 and order p^a. For the sake of completeness and simplicity we shall re-determine them by means of the given corollary. In the special case when $p = 2$, it is well known that $(abcd)_8$ is the only group of this kind. In other words, there is only one regular group of order 8 that contains subgroups which do not include substitutions that generate a

* FROBENIUS, *Sitzungsberichte der Berliner Akademie*, 1895, p. 184.

† Ibid, pp. 183 and 184.

‡ *Acta Mathematica*, vol. 11, p. 202. Cf. *Annals of Mathematics*, 1896, p. 156.

self-conjugate subgroup, and all these subgroups belong to the same system of transforms; in fact, they are even conjugate.

When $p > 3$ the two non commutative regular groups contain substitutions of order p which are not commutative to all the substitutions of the entire group. Since a self-conjugate subgroup of order p in a group of order p^a contains no substitution that is not commutative to all the substitutions of the entire group, each of the two non-commutative regular groups must be simply isomorphic to at least one transitive group of degree p^2. It remains to prove that each of them is simply isomorphic to only one such group, or, in other words, that each of them contains only one system of transform subgroups that does not include any self-conjugate subgroup.

Each of these groups may be supposed to contain the non-cyclical group of order p^2 as head. The tail of one of them is composed of substitutions of order p^2, and that of the other is composed of substitutions of order p. Since each of them must be isomorphic to a commutative group with respect to its only self-conjugate subgroup of order p,* each of the substitutions of the other subgroups of order p must be transformed, by all the substitutions of the group, into itself multiplied by all the substitutions of the self-conjugate subgroup. Hence we see that any substitution in either of these groups transforms one of the substitutions of every non-self-conjugate subgroup of order p, to which it is not commutative, in exactly the same way as it transforms any given substitution of such a subgroup.

We can now see directly by means of the following elementary theorem that each of the given non-commutative regular groups of order p^3 contains only one system of transform subgroups, satisfying the given condition, and is, therefore, simply isomorphic to only one transitive group of degree p^2.

THEOREM II. *If two groups are placed in simple isomorphism we may add two new corresponding operations (s_1 and s_2), one to each, and thus obtain a simple isomorphism between two larger groups, provided: (1) s_1 and s_2 transform corresponding generating operations of the two simply isomorphic groups into corresponding operations, and (2) the first power of either s_1 or s_2 that occurs in one of the isomorphic groups corresponds to the same power of the other.*

Since each of the non-commutative groups of order p^3 is

* *Quarterly Journal of Mathematics*, vol. 28, p. 267.

generated by any two of its substitutions that are not commutative, we may obtain generating substitutions of each of the two possible non-regular transitive groups of order p^3 by multiplying all the substitutions of each regular non-commutative group of order p^2 into two of its substitutions that are non-commutative and observing how the lines, with respect to some non-self-conjugate subgroup of order p, are permuted by this operation. If we arrange the work so that the first line contains one of the multiplied substitutions we evidently obtain a substitution whose degree is less than p^2 as one of the two generators.

The non-regular transitive groups whose order is pq^2.

Every group of order pq^2 contains either a self-conjugate subgroup of order q^2 or it contains such a subgroup of order p. When the same group contains a self-conjugate subgroup of each of these orders it is commutative and can therefore not be represented as a non-regular transitive group. We shall first determine the non-regular transitive groups which contain a self-conjugate subgroup of order q^2.

According to Sylow's theorem each of these groups contains only one system of conjugate subgroups of order p. From this and the given corollary we have that *every regular group of order pq^2 that contains no self-conjugate subgroup of order p is simply isomorphic to one and only one transitive group of degree q^2.* As none of the required groups can be of a prime degree it remains only to find those which are of degree pq. In other words, it remains to determine the number of systems of transform groups of order q in the regular groups of order pq^2 that contain no self-conjugate subgroup of order p.

As the subgroups of order q must be contained in the self-conjugate subgroup of order q^2* there can be only one such subgroup when the given self-conjugate subgroup is cyclical. That is, there is no non-regular group of degree pq and order pq^2 that contains a self-conjugate subgroup of order q^2. This is also directly evident. When the given self-conjugate subgroup of order q^2 is non-cyclical it contains $q + 1$ subgroups of order q. We need to consider only the two cases when the remaining substitutions of the groups permute either $q - 1$ or all of these subgroups. By means of Theorem II we readily find that there is in each of these cases only one system of transform subgroups of order q that does not include any self-conjugate subgroup.

* FROBENIUS, *Crelle's Journal*, vol. 101, p. 284.

Hence each of these regular groups is simply isomorphic to one and only one non-regular transitive group of degree pq.

We have now determined the numbers of the non-regular transitive groups that are simply isomorphic to the different regular groups of order pq^2 that contain a self-conjugate subgroup of order q^2. By employing the lists of the regular groups we may state results as follows:

Of degree q^2, *p being odd*, there are $\frac{1}{2}(p+5)$ groups when $q-1$ is divisible by p and there is only one such group when $q+1$ is divisible by p. When $p=2$ there are 3 such groups.

Of degree pq, p being odd, there are $\frac{1}{2}(p+1)$ groups when $q-1$ is divisible by p and there is only one group when $q+1$ is divisible by p. When $p=2$ there is only one such group. When neither $q=1$ nor $q+1$ is divisible by p there are no such groups.

It remains to find the non-regular transitive groups that are simply isomorphic to the non-commutative regular groups of order pq^2 which contain a self-conjugate subgroup of order p. Since p has primitive roots, there must be one and only one such group of degree p, when $p-1$ is divisible by q^2. When this condition is not satisfied there is no group of this degree. It now remains only to find the possible groups of degree pq.

One of the three regular groups which satisfy the given. conditions contains no substitution of order q^2. This contains only two systems of transform subgroups of order q. One of these systems consists of a characteristic subgroup. Hence this regular group is simply isomorphic to only one transitive group of degree pq. The other two of the given regular groups contain substitutions of order q^2.

The one which transforms the substitutions of the self-conjugate subgroup of order p according to the cyclical group of order $q-1$ contains only commutative substitutions of order q. The other, which transforms the substitutions of the self-conjugate subgroup of order p according to the cyclical group of order q^2 contains one conjugate system of groups of order q. Hence this is simply isomorphic to a transitive group of degree pq while the preceding is not simply isomorphic to such a group. The last is also simply isomorphic to one group of degree p as has been observed above.

Hence we have that there are two transitive groups of degree pq and order pq^2 that contain a self-conjugate subgroup of order p and there is one such group of degree p, when $p-1$ is divisible by q^2. When $p-1$ is divisible by q

but not by q^2 only one of these three groups occurs. This is of degree pq. When $p-1$ is not divisible by q there is no such group. This completes the determination of the non-regular transitive groups of degree pq^2. The generating substitutions can be found by the method which has been explained above.

The non-regular transitive groups whose order is pqr.

We may assume $p > q > r$. Since every group of order pqr contains only one subgroup of order p * there can be no transitive group of degree qr and order pqr. All of the given groups that do not contain an operation of order qr include also a self-conjugate subgroup of order q.† From Sylow's theorem it follows that these groups contain only one subgroup of each of the orders p and q. Hence we have that *each of the non-commutative groups of order pqr that do not contain an operation of order qr is simply isomorphic to one and only one transitive group of a lower degree and that the degree of each of these groups is pq.* It remains to consider those groups of order pqr which contain operations of order qr.

Since p has primitive roots, there is one and only one group of degree p and order pqr, provided $p-1$ is divisible by qr. When this condition is not satisfied, there is no such group. We see directly that this group contains more than one subgroup of each of the orders q and r. It is therefore simply isomorphic to a transitive group of degree pr, and also to one of degree pq.

If we establish a simple isomorphism between r non-cyclical transitive groups of order pq, and also between q non-cyclical transitive groups of order pr, we obtain two heads, to each of which we may add a substitution which merely interchanges the systems of intransitivity, and thus obtain the two remaining non-commutative groups of order pqr that contain substitutions of order qr. From this construction it is evident that the former contains a self-conjugate subgroup of order r, and is simply isomorphic to only one transitive group of degree pr while the latter contains a self-conjugate subgroup of order q and is simply isomorphic to only one transitive group of degree pq. As the non-regular transitive groups occur under the same conditions as the simply isomorphic regular groups, the determination of the non-regular transitive groups of order pqr is complete.

* COLE and GLOVER, *American Journal of Mathematics*, vol. 15, p. 215.
† HÖLDER, *Mathematische Annalen*, vol. 43, p. 370.

Summary.

We have now found all the possible non-regular transitive groups whose orders are the products of three prime factors by employing, in most cases, the regular groups of these orders. The results are as follows:

Order.	Degree.	Number.	Conditions.
p^3	p^2	2	when $p > 2$.
		1	when $p = 2$.
pq^2	p	1	when $p - 1$ is divisible by q^2.
	q^2	$\frac{p+5}{2}$	when $p > 2$ and $q - 1$ is divisible by p.
		1	when $p > 2$ and $q + 1$ is divisible by p.
		3	when $p = 2$.
	pq	$\frac{p+1}{2}$	when $p > 2$ and $q - 1$ is divisible by p.
		2	when $p - 1$ is divisible by q^2.
		1	when $p - 1$ is divisible by q but not by q^2.
		1	when $p > 2$ and $q + 1$ is divisible by p.
		1	when $p = 2$.
pqr	p	1	when $p - 1$ is divisible by qr.
	pr	2	when $p - 1$ is divisible by qr.
		1	when $p - 1$ is divisible by q but not by r.
	pq	$r + 2$	when $p - 1$ is divisible by qr and $q - 1$ is divisible by r.
		$r + 1$	when $p - 1$ is divisible by r but not by q and $q - 1$ is divisible by r.
		2	when $p - 1$ is divisible by qr but $q - 1$ is not divisible by r.
		1	when $p - 1$ is divisible by r but not by q and $q - 1$ is not divisible by r.
		1	when $p - 1$ is not divisible by r but $q - 1$ is divisible by r.

PARIS, *December* 7, 1896.