LOBACHÈVSKY AS ALGEBRAIST AND ANALYST.

BY PROF. A. VASILIEV.

THE mathematical genius of Lobachèvsky manifested itself not in geometry alone. His early study of Gauss's *Disquisitiones arithmeticæ*, under the direction of Professor Bartels, led him in 1813 to write a memoir (which has never been published and seems to be lost) on the resolution of the binomial equation $x^n - 1 = 0$ in the case $n = 4p + 1$. At a later period, in 1834, he returned to these studies and carefully examined the case $n = 8p + 1$ (see his paper "Reduction of the degree of the binomial equation when the exponent, diminished by 1, is divisible by 8," written in Russian and published in the first volume (1834) of the Scientific Memoirs of the University of Kazàn).

Next to geometry, Lobachèvsky's favorite subject was the systematic exposition of, algebra. He considered algebra as intended to lay the rigorous foundations for mathematical science, and this idea he carried out in a work published in 1833 under the title "Algebra, or calculus of finite quantities" (Russian). This extensive work is remarkable alike for the rigor of its definitions and proofs, and for the width of its scope. Thus we find here treated not only the solution of numerical equations and Gauss's theory of the resolution of the equation $x^n - 1 = 0$, but also the calculus of finite increments. Through his geometrical researches Lobachèvsky had been led to the necessity of defining the trigonometric functions independently of all geometric considerations; this idea he introduced into his algebra, in which the trigonometric functions $\cos z$ and $\sin z$ are defined by means of the series $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$ and $\frac{z}{1} - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$, and all their properties are derived from this definition. I believe that Lobachèvsky was the first to expound systematically the theory of trigonometric functions on this basis.*

* Professor A. Macfarlane in his interesting paper "On the definitions of the trigonometric functions" mentions De Morgan as expressing the same idea in his "Trigonometry and Double Algebra." Professor Mansion attributes the idea to Mr. Seidel (*Crelle's Journal für Math.*, vol. 73, 1871).

[This must be a misunderstanding, as Seidel's paper deals with products and not with series. Cauchy, in his "Analyse algébrique," 1821, p. 309, distinctly defines $\sin z$ and $\cos z$ by means of the infinite series. —A. Z.]

This point of view naturally led Lobachèvsky to the question of the convergence of infinite series ; he invented a special method for testing the convergence of a series and explained it in several memoirs :

(1) " On the convergence of trigonometric series " (Russian), in the *Scientific Memoirs of the University of Kazàn*, vol. I, 1834 ;

(2) " A method for ascertaining the convergence of infinite series and for approximating the values of functions of very large numbers," ib., vol. II, 1835 ;

(3) "Ueber die Convergenz der unendlichen Reihen," Kasan, 1841, 4to.

Lobachèvsky's method is based on the following considerations. Let there be given an infinite series

$$\sum_{i=1}^{i=\infty} f(i) = f(1) + f(2) + \ldots + f(i) + \ldots ; \quad (1)$$

every term of this series can evidently be expanded as a fraction in the form

$$\frac{\lambda_1}{2} + \frac{\lambda_2}{2^2} + \frac{\lambda_3}{2^3} + \ldots,$$

where each λ is equal to 1 or to 0.* Suppose now we have found a number μ such that

$$f(\mu) \geqq 2^{-\lambda} \text{ and } f(\mu + 1) < 2^{-\lambda}; \quad (2)$$

then not more than μ terms can have the fraction $1/2^{\lambda}$ in their expansion, and in the sum of the series this fraction $1/2^{\lambda}$ cannot have a coefficient exceeding μ.

It follows that the sum of the series cannot exceed $\sum_{\lambda=0}^{\lambda=\infty} \mu\, 2^{-\lambda}$. Thus an upper limit is found for the sum of the series, and this may lead to the determination of its convergence. The difficulty lies in determining μ so as to satisfy the inequalities (2).

It is worthy of notice that Lobachèvsky applied this method, based on the determination of an upper limit of the

* This is merely an adaptation of the idea of decimal fractions to the binary system of numeration.

sum, even to the simple exponential series $\Sigma \frac{x^i}{i!} = f(x)$, for the purpose of proving the property

$$f(x) \cdot f(y) = f(x + y).$$

He seems to have been guided here by the very considerations which, later on, led mathematicians to the distinction between uniform and non-uniform (" gleichmässige " and " ungleichmässige ") convergence.

There can certainly be no doubt but that the keen genius of Lobachèvsky perceived the insufficiency of the assumption that every continuous function is necessarily capable of differentiation. It is well known that this assumption was made by all mathematicians until Weierstrass showed it to be untrue, by the example of the function

$$\sum_{n=0}^{n=\infty} b^n \cos \pi(a^n x)$$

(where a is an odd integer, b a positive proper fraction, and $ab > 1 + \frac{3}{2}\pi$), which though continuous is not "*differentiirbar.*" *

Lobachèvsky developed his ideas on this subject with particular clearness in his Russian memoir of 1835 (" A method for ascertaining, etc."), in which he says: " *The function is gradual* (postepènna) *if its increment diminishes to zero along with the increment of the variable x. The function is continuous* (neprerỳvna) *if the ratio of these increments, as the latter diminish, passes into a new function, which is the differential coefficient.* Integrals must always be divided into intervals in such a manner that the elements remain both gradual and continuous."

These words clearly show that Lobachèvsky was far in advance of his contemporaries in this question concerning the principles of the infinitesimal calculus, just as he was in advance of them on the foundations of geometry.

He had expressed these views more elaborately in his memoir of 1834 ("On the convergence of trigonometric series"). Here he gives his definition of the differential coefficient in the following terms: "Let $F(x)$ denote a function which, varying with x, increases from a certain

* This example was first published by P. du Bois-Reymond in his " Versuch einer Classification der willkürlichen Functionen " (*Crelle's Journal für Mathematik*, vol. 79)

value of x to $x = a$. We divide $a - x$ into i equal parts and put $(a - x)/i = h$. Let the quantities

$$F(x), \quad F(x+h), \quad F(x+2h), \quad \ldots \quad F(a)$$

be known for every small value of h, which decreases indefinitely as i increases. The ratio $\dfrac{F(x+h) - F(x)}{h}$ will vary with h. For $i' > i$, let $(a-x)/i'$ be equal to h'. If now the difference

$$\frac{F(x+h) - F(x)}{h} - \frac{F(x+h') - F(x)}{h'} = \epsilon,$$

or, which amounts to the same,

$$\frac{h'F(x+h) - hF(x+h') + (h-h')F(x)}{hh'} = \epsilon,$$

decreases simultaneously with h for every value of x and can be made as small as we please, the function $F(x)$ receives the name of a *continuous* function. The ratio $\dfrac{F(x+h) - F(x)}{h}$ has in this case a limit which is obtained by letting h decrease; and this limit is $\dfrac{dF(x)}{dx}$."

These are Lobachèvsky's ideas concerning the foundations of the differential calculus. As regards the integral calculus, it is well known that Lobachèvsky repeatedly insisted on the great value of his imaginary geometry for the evaluation of integrals. (See *N. I. Lobatcheffsky, Collection complète des œuvres géométriques,* vol. II, p. 613, 655 sq.) In 1836 he published a paper (in Russian) in the *Memoirs of the University of Kazàn* entitled "Application of the imaginary geometry to some integrals" (see ib., vol. I, pp. 121–218). This extensive memoir contains many results of great value in the integral calculus. Lobachèvsky insists in particular on the importance of the integral $L(x) = -\int_0^x dx \log \cos x$, to which a large number of very complicated integrals are reducible. Lobachèvsky's work in the integral calculus is still awaiting a full and careful appreciation and may perhaps be made the subject of another paper.

In conclusion I may perhaps be allowed to express the hope that the appreciation of the work of our great Russian mathematician, manifested by the mathematicians assembled at the

Chicago Congress through their generous contribution to the Lobachèvsky memorial fund, contains the promise of a still closer union between the mathematicians of America and Russia, and proves the solidarity of scientific interests among all nationalities.

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MACFARLANE'S ALGEBRA OF PHYSICS.

Principles of the Algebra of Physics. By A. MACFARLANE, Professor of Physics in the University of Texas, Austin, Texas. *Proceedings of the American Association for the Advancement of Science,* vol. 40, 1891, 53 pp.

On the Imaginary of Algebra. By A. MACFARLANE. *Ibid.,* vol. 41, 1892, 23 pp.

THE purpose of the first of the articles which are to form the subject of this review may most properly be stated in the author's own words: "The guiding idea in this paper is generalization. What is sought for is an algebra which will apply directly to physical quantities, will include and unify the several branches of analysis, and when specialized will become ordinary algebra."

A student who sets out to use Grassmann's algebra in geometrical work finds that it applies beautifully to projective problems in curves and surfaces of no higher order than the second, but beyond them he is confronted and stopped by difficulties which can be overcome only by the study of the ordinary theory of algebraic forms. In the same way quaternions work out many metrical properties of curves and surfaces with facility and grace, but I think every student who has tried to go far with them finds that he is at last brought back to the study of the equations and functions of ordinary analysis. There seems to be no way around the difficulties of the theories of forms and functions, and even when results have been attained by methods which appear to avoid them the mind is seldom convinced of their validity. As we shall see, Professor Macfarlane derives the formulas of trigonometry with great facility, but it seems almost certain that no analyst would dare to use them if they had no other foundation.

Passing by considerations of this kind which seem to make it doubtful whether or not any system of analysis other than the ordinary one can do much to advance mathematical science, we come to the author's first objection to quaternions