INTRODUCTORY MODERN GEOMETRY.

Introductory Modern Geometry of Point, Ray, and Circle. By WILLIAM BENJAMIN SMITH, A.M., Ph.D. (Goett.), Professor of Mathematics and Astronomy, University of the State of Missouri. New York and London, Macmillan & Co., 1893. 16mo, pp. vii + 297.

IN reviewing a book, one of the canons of fair criticism is to regard its adaptation to the readers for whom the author himself designs it; but as a preliminary to this notice, we must object to the selection implied in the preface, where Professor Smith describes his book as intended "to present in simple and intelligible form a body of geometric doctrine acquaintance with which may fairly be demanded of candidates for the Freshman class," and then points out that *one year's study of geometry* is about as much as can be expected in schools. Our own conviction is that geometry may with great advantage be taught to children in their early school days. The simplest kind of geometry, of course; with few formal proofs, and depending more on the teacher than the text-book. But even when this early introduction has been omitted, the subject seems to be one that may be presented in formal guise to the average child of twelve or thirteen. It relates to that with which he is already practically familiar, illustrations may be drawn from his every-day experience, his unconscious perceptions of space relations may be appealed to and formulated; and thus it presents itself as a valuable discipline by which his reasoning faculties may be developed, and his vague disconnected perceptions organized, without burdening him with a mass of new and possibly uninteresting facts.

But just because geometry is so eminently fitted for the youthful mind it should be *at first* presented in such form as to be in accord with the general views of laymen, when these are not in direct opposition to the truth. It may well be that the ideas of modern geometry on such questions as that of the nature of space ought to be explained earlier and to more students than is now done; but if we accept W. K. Clifford's rule, "before teaching any doctrine, wait until the nature of the evidence for it can be understood," such discussions will not be put in the forefront of our geometrical teaching. They will be more easily and profitably treated when they can be founded on knowledge derived from a study of Euclidean space. "There is no time of reading a book better than when you need it, and when you are on the point of finding it out yourself if you were able," says J. Clerk Maxwell; why then should we thrust upon the student "the consoling hope that, after all, this other [view of space] may be the true state of things," until he is able to appreciate the relief thus offered, through having, with Clifford, suffered from "the dreary infinities of homaloidal space"?

But given this preliminary study of geometry, then comes the time for a concise systematic treatment of the subject with direct reference to modern ideas. Certainly all teachers of even the most elementary geometry ought to know wherein these modern views differ from the older ones; and to those that have neither opportunity nor inclination for abstruse mathematical studies, a book on the lines of the one before us will afford valuable assistance. To some such circle of readers, therefore, we prefer to regard this introductory modern geometry as addressed.

Professor Smith begins with a discussion of the nature of the properties of space, emphasizing Riemann's distinction between *infinite* and *boundless*. The order of development of any system of geometry depends on what we accept as fundamental ideas. Here, following Bolyai, the idea of the *distance* between two points is regarded as fundamental, and the conceptions of plane and straight line are deduced from this. We have also the idea of an *angle;* though the fact that this is just as much a fundamental assumption as the idea of distance is not sufficiently emphasized. The distances, as also the angles, are not regarded as capable of comparison, except as to being equal or unequal. Deduced from these fundamental conceptions we have then some of the ordinary propositions depending on a comparison of the simpler rectilinear figures, leading up to a careful and logical introduction of parallels, pp. 49–56. The exact nature of the assumptions made in the ordinary treatment of parallels is very clearly brought out. We have the theorem, "Either *no* triangle has the sum of its angles equal to two right angles, or *every* triangle has the sum of its angles equal to two right angles." The choice to be made is shown to depend on the answer to the question, Can two indefinite lines be drawn in a plane so as not to meet? And here experience yields no answer; we may adopt either hypothesis, deducing a set of axioms, and on these our system of geometry must be founded. Here the choice having been made, the result is formulated in two axioms expressing that through a given point to a given line one parallel and only one can be drawn.

From this point the geometry is specifically and exclusively the geometry of Euclidean space. Much use is made of the principle of symmetry; and in connection with the two kinds of symmetry, axial and central, the principle of reciprocity is explained. This we should have liked to see introduced much earlier; how simply and naturally it can be done may be seen in Professor Henrici's admirable little book "Congruent Figures." Properties of symmetrical rectilinear figures having been established, the discussion of the symmetrical curve, the circle, logically follows; and here the vicious practice of so many American text-books now in use, of treating angles in a circle by means of their numerical measure, is departed from, and the original purely geometric method of proof is reverted to. In fact, all numerical properties are left to be treated together in a later section; and following the circle, we have propositions on the relation of areas, and on proportions in general, all with the satisfactory rigorous geometric treatment. A useful idea, rarely found in elementary works, is introduced—that of the generation of an area by the motion of a line; the use of this is possible, since the fact that an area, just as much as a line or an angle, is susceptible of sign is here explained. After this we have the taction problem of Apollonius, in whose date there is a misprint, 200 A.D. being given instead of 200 B.C. In the section on metric geometry the transference of numerical propositions to geometry is clearly and scientifically explained; and six or seven pages are devoted to the trigonometrical ratios, just sufficient to excite the student's interest in trigonometry.

This outline will show how logical is the sequence of the contents; the arrangement of the arguments and demonstrations will be found equally pleasing. And yet the book, with all its good points, is hopelessly marred by the author's persistent disregard of conventional nomenclature. If he had entirely invented the science, or at any rate this special development of it, we could not deny him the legal right of naming his own creations, even though we might deplore his unfortunate choice of such terms as *tract* (finite straight line), *perigon* (four right angles), *numeric,* used as a noun, and *finity,* in contradistinction to infinity. But all the ideas here set forth are the common property of the mathematical world; the science of geometry was not invented yesterday, and it is already provided with a fairly complete and satisfactory English vocabulary. It requires very cogent reasons to justify a scientific writer in deviating from established usage in nomenclature or notation. Why speak of the base circle in reciprocation as the *referee?* Why speak of the point of *tangence* rather than of the point of contact? Why, in short, disfigure the work with the multitude of strange and uncouth terms that we find here? Have not the most effectual reformers been conservative in trifles?

To some of Professor Smith's specialties we may object on even stronger grounds; as, for example, to his notation for sin α and cos α; to his employment of the word *dimension* for the actual length of the base (or altitude) of 'an areal figure'; and to his absolute use of the term *double point*, speaking of a double point in a correspondence as if it were a double point on the circle considered, and then defining consecutive points as points that "fall together in a double point."

If Professor Smith could be induced to translate his work into ordinary mathematical English, we feel sure that he would greatly increase its usefulness not only in aiding "the first upward steps of the climber," but also in preparing him for the "much higher ascent."

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WRONSKI'S EXPANSION.

BY PROF. W. H. ECHOLS.

IN 1810 Höené Wronski presented to the French Academy of Sciences the following formula, without demonstration,

$$fx = a_0 + a_1\omega_1 + a_2\omega_2 + \ldots \text{ ad. inf.}, \quad . \quad . \quad . \quad (1)$$

in which fx, ω_1, ω_2, ... are arbitrary functions of x, and a_0, a_1, ... are independent of x. This formula, or rather the law for the formation of the coefficients, he called *la loi suprême*.

Lagrange and Lacroix were appointed as a committee to examine Wronski's memoir and to report on it to the Academy. This report is an admirable production and in every way worthy of the distinguished names attached to it. It is especially noticeable for its conservative tone and yet its acknowledged recognition of the importance and possible future of the formula. The commissioners must have been very much impressed, to have repeated section III. in section IV., "Ce qui a frappé vos commissaires dans le mémoire de M. Wronski, c'est qu'il tire de sa formule toutes celles que l'on connaît pour le développement des fonctions, et qu'elles n'en sont que des cas très-particuliers." It would seem remarkable in view of this that nothing has been done toward developing his work and placing it on a sound scientific basis. The whole of Wronski's work and method of work appears to be purely qualitative; it is truly algorithmic, inasmuch as he