THE MECHANICAL AXIOMS OR LAWS OF MOTION.

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THE three laws of motion were called by Newton *Axiomata, sive Leges Motus.* Professors Thomson and Tait in their *Natural Philosophy,* section 243, say: "An axiom is a proposition, the truth of which must be admitted as soon as the terms in which it is expressed are clearly understood. But, as we shall show in our chapter on 'Experience,' physical axioms are axiomatic to those only who have sufficient knowledge of the action of physical causes to enable them to see their truth." They then proceed to give Newton's three laws, remarking that "these laws must be considered as resting on convictions drawn from observation and experiment, *not* on intuitive perception."

Whether this be accepted as a proper definition of a physical axiom or not, it is at least desirable to include among the axioms of mechanics the smallest basis of postulated principles upon which it is possible to construct the science by rigid mathematical reasoning.

The laws of motion, in the classic form given them in the *Principia,* admirably express such a basis of postulated principles, although the charge of redundancy has been brought against the first and second laws; but, in the case of the third law, the tendency has been, on the other hand, to make it include too much, and to assume, either under its authority or directly, as axiomatic, principles like the "impossibility of perpetual motion" which ought rather to be shown to follow directly from the three laws of motion.

It is here proposed to discuss the question of the mechanical axioms, beginning with an examination of the laws as presented by Newton.

The first law is that "Every body keeps in its state of rest or of moving uniformly in a straight line, except to the extent in which it is compelled by forces acting on it to change its state."*

Newton had already defined *vis impressa* as an action from without changing a body's state of rest or of moving uniformly in a straight line, † and in the scholium to the *Definitiones* had pointed out that our measure of time depends upon the assumption of the first law of motion; so

* Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

† Definitio IV. Vis impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum.

that with respect to a single body there is a logical "vicious circle." We define those intervals of time as equal in which equal spaces are described when there is no action: we say there is no action when equal spaces are described in equal intervals of time. But the physical truth expressed is that *all* bodies undisturbed by action from without describe equal spaces in the successive intervals in which any one such body describes equal spaces. We then define these intervals as equal, and the action, which in any other body causes a departure from this normal state, as a force.

The second law is that "Change of motion is proportional to the moving force acting, and takes place in the direction of the straight line in which the force acts."*

The simplest form of the physical truth involved is this: — *A given force acting upon a given body produces the same acceleration in its own direction, in equal intervals of time, no matter how the body may be moving, nor what other forces may be acting at the same time.* The law as expressed by Newton involves the obvious deductions that the force is proportional to the acceleration produced in a given interval, and that to produce a given acceleration the force must be proportional to the mass; for motion had been defined as proportional to mass and velocity conjointly. The *vis motrix* is the whole force acting on the body, and is defined as "proportional to the motion which it produces in a given time," † in distinction from *vis acceleratrix* which is defined as proportional to the velocity produced in a given time. These definitions imply the second law, just as those of *vis insita* and *vis impressa* imply the first law. But the essential point is the constancy and independence of the effects of force, which effects are therefore suitable to be taken as measures of the force.

It has been objected that the first law is unnecessary because it is included in the second which implies that *no* force produces *no* change of motion. It appears inevitable that the expression of the second law should thus include the first; but it is nevertheless fitting that the normal state of the body suffering no action from without, a departure from which constitutes the "change of motion" when action takes place, should be stated in a separate axiom. The first law is, in fact, far more axiomatic than the second, or, in the language of the definition quoted above, requires a much smaller knowledge of the action of physical causes to enable one

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* Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

† Definitio VIII. Vis centripetæ quantitas motrix est ipsius mensura proportionalis motui, quem dato tempore generat.

to see its truth. It is only necessary to get a clear notion of the absence of force to be in the mental state to admit the truth of the first law: but it requires a considerable familiarity with geometrical notions even to apprehend the manner in which the effect of a force upon a moving body is to be compared with its effect when the body is at rest, and the manner in which the effects of forces acting at an angle with one another are to be separated. Yet clear conceptions in these matters must be obtained before an intelligent assent can be given to the second law of motion.

The most axiomatic proposition involved in the second law is that two opposite equal forces acting upon a body at rest do not produce motion, which in some old treatises is taken as the first proposition in statics, and in others as the definition of the equality of forces.

The third law is that "There is always a reaction opposite and equal to an action: or the actions of two bodies upon one another are always equal and oppositely directed."*

The physical truth expressed is that *every force acting upon a body is the action upon it of another body which in turn is acted upon equally by the first body, the action taking place in the straight line which joins the two bodies.* In other words, the law asserts that no forces exist which consist simply of tendencies in certain directions, as the ancients supposed in the case of the "gravity" of certain bodies and the "levity" of others. This granted, the equality of the two phases of the action follows readily, by the aid of the notion of the transmissibility of force which is intimately connected with this law. Compare Newton's remarks on attraction, quoted below. But the language of some writers concerning the "numerous applications of this law" indicates the view that something more than the equality of pressures is implied in it.

Of the illustrations which in the *Principia* immediately follow the third law, the first is that of a stone pressed by the finger, when in turn the finger is pressed by the stone.† Here there is no intervening body between the two bodies in question. The resistance of the stone to the finger is, by the second law, equal to the force communicated to the finger tip by the muscles because it prevents its motion. By the third law, this is the same as the force communicated by the finger to the stone. If we go a step further and consider the equilibrium of the stone, the resistance of the support upon which it rests is equal to the last named force, and is the reaction of the

* Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

† Si quis lapidem digito premit, premitur & hujus digitus a lapide.

same force regarded as the action of the stone upon the support. Thus the force is transmitted through the substance of the stone.

In the next illustration, that of a horse drawing a stone attached to a rope, there is a body through which the force is transmitted. "The rope stretched both ways will by the same endeavour to relax itself urge the horse toward the stone and the stone toward the horse; and will impede the progress of the one as much as it promotes the progress of the other." *

The next illustration is from impact. "If any body impinging upon another body, by its own force in any manner changes the motion of that body, it will also in turn suffer in its own motion (on account of the equality of mutual pressure) the same change in the contrary direction." † Here the third law is cited in the clause in parenthesis, and the equality of the actions is indicated by the effects which are produced in accordance with the second law. Newton proceeds in fact to say, "By these actions equal changes are made not of velocities but of motions." These comments on the third law close with the words: "This law holds also in attractions, as will be proved in the next scholium."

The passage in the scholium here alluded to is as follows:—"In attractions I thus briefly show the matter. Any two bodies *A* and *B* mutually attracting each other, conceive some obstacle to be interposed, by which their approach is prevented. If either one of the bodies *A* is drawn more toward the other body *B* than the other *B* toward the first *A*, the obstacle will be urged more by the pressure of the body *A* than by the pressure of the body *B*, and hence will not remain in equilibrium. The stronger pressure will prevail, and will cause the system of the two bodies and the obstacle to move in a straight line in the direction toward *B*, and by an ever accelerated motion in free space to pass to infinity. Which is absurd and contrary to the first law." ‡

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* Funis utrinque distentus eodem relaxandi se conatu urgebit equum versus lapidem, ac lapidem versus equum; tantumque impediet progressum unius quantum promovet progressum alterius.

† Si corpus aliquod in corpus aliud impingens, motum ejus vi sua quomodocunque mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem pressionis mutuæ) subibit.

‡ In attractionibus rem sic breviter ostendo. Corporibus duobus quibusvis *A*, *B* se mutuo trahentibus, concipe obstaculum quodvis interponi, quo congressus eorum impediatur. Si corpus alterutrum *A* magis trahitur versus corpus alterum *B*, quam illud alterum *B* in prius *A*, obstaculum magis urgebitur pressione corporis *A* quam pressione corporis *B*; proindeque non manebit in æquibrio. Prævalebit pressio fortior, facietque ut systema corporum duorum & obstaculi moveatur in directum in partes versus *B*, motuque in spatiis liberis semper accelerato abeat in infinitum. Quod est absurdum & legi primæ contrarium.

In this passage Newton, so far from making the third law imply anything more than equality of pressures, shows that this equality of the two phases of an action follows from the simple assumption that in a system of bodies preserving their relative positions the mutual actions of any two cannot result in a tendency to motion. The axiomatic portion of the law consists in this assumption. A tendency to motion is in Newton's system always the action of an external body.

We find, however, in Thomson and Tait the following passage: "Of late there has been a tendency to split the second law into two, called respectively the second and third, and to ignore the third entirely, though using it *directly* in every dynamical problem; but all who have done so have been forced *indirectly* to acknowledge the completeness of Newton's system, by introducing as an axiom what is called D'Alembert's principle, which is really Newton's rejected third law in another form. Newton's own interpretation of his third law directly points out not only D'Alembert's principle, but also the modern principles of work and energy."*

In support of this the authors remark further on,† after commenting upon the third law, "In the scholium appended, he makes the following remarkable statement, introducing another description of actions and reactions subject to his third law, the full meaning of which seems to have escaped the notice of commentators:"—[Here follows the passage from the scholium, of which the authors give the following translation, in which "activity" and "counter-activity" are put for *actio* and *reactio*.]

"*If the activity of an agent be measured by its amount and its velocity conjointly; and if, similarly, the counter-activity of the resistance be measured by the velocities of its several parts and their several amounts conjointly, whether these arise from friction, cohesion, weight, or acceleration;—activity and counter-activity, in all combinations of machines, will be equal and opposite.*" ‡

Again Professor Tait in the article *Mechanics* in the Encyclopædia Britannica, 9th edition, quotes the passage and remarks: "This may be looked upon as a fourth law. But, in strict logic, the first law is superfluous. . . . Hence there are virtually only three laws, so far as Newton's system is concerned."

The "scholium to law III." is afterward referred to as

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* *Natural Philosophy*, Section 242. † Section 263.

‡ Nam si æstimetur agentis actio ex ejus vi & velocitate conjunctim; & similiter resistentis reactio æstimetur conjunctim ex ejus partium singularum velocitatibus & viribus resistendi ab earum attritione, cohæsione, pondere, & acceleratione oriundis; erunt actio & reactio, in omni instrumentorum usu, sibi invicem semper æquales.

giving us the principle of the "transference of energy from one body or system to another."

We shall be better prepared to estimate the import of the passage last quoted if we briefly consider the connection in which it stands in the *Principia*. The comments on the third law, quoted nearly in full above, are followed by six corollaries and a scholium, which complete the chapter entitled *Axiomata sive Leges Motus*, and immediately precede the treatise *De Motu Corporum*. Cor. I. gives the parallelogram of forces as deduced from laws II. and I. Cor. II. states the composition and resolution of forces. "Which composition and resolution is abundantly confirmed by the theory of machines." * The resolution of forces is then applied to prove that the efficiency of a force to turn a wheel is the product of the force and its arm—"the well known property of the balance, the lever, and the wheel"—and so on for the other simple machines, which are thus cited to confirm the truth of the laws of motion. Cor. III. proves by laws III. and II. that the quantity of motion "directed toward the same parts" is not altered by internal actions between the bodies of a system. Cor. IV. derives the conservation of the motion of the center of gravity. Cor. V. shows that the relative motions of a system of bodies enclosed in a given space are the same, whether the space be at rest or moving uniformly in a straight line, and cor. VI. extends this to the case in which the bodies are also acted upon by equal accelerating forces in the direction of parallel lines.

The scholium which follows is not a scholium to the third law exclusively, but is occupied with the experimental verifications of the laws; beginning with the discovery by Galileo, by means of the first two laws, that "the descent of heavy bodies is in the duplicate ratio of the time, and that the motion of projectiles takes place in a parabola, experiment confirming, except so far as these motions are somewhat retarded by the resistance of the air." † Then follows an account of experiments on the impact of bodies, showing that experience agrees with deductions drawn from the three laws, which ends with the words, "And this being established, the third law so far as impacts and reflexions are concerned is confirmed by a theory which plainly agrees with experience." ‡

* Quæ quidem compositio & resolutio abunde confirmatur ex mechanica.

† Descensum gravium esse in duplicata ratione temporis, & motum projectilium fieri in parabola; conspirante experientia, nisi quatenus motus illi per aëris resistentiam aliquantulum retardantur.

‡ Atque hoc pacto lex tertia quoad ictus & reflexiones per theoriam comprobata est, quæ cum experientia plane congruit.

The paragraph concerning attractions, quoted above, comes next, and is followed by a statement of Newton's own experiments with magnetic attraction, directly confirming the third law.

The paragraphs concerning attraction are followed in the scholium by one opening with these words : "As, in impacts and reflexions, those bodies have the same efficiency of which the velocities are reciprocally as the innate forces : so in mechanical instruments for producing motion, those agents have the same efficiency, and by opposite endeavours sustain one another, of which the velocities estimated in the direction of the forces are reciprocally as the forces." * The *vires insitæ* or *vires inertiæ* are proportional to the masses, as explained in *Definitio* III., so that the meaning is this :—Just as bodies having equal momenta are of equal efficiency in the case of impact, so, in machines, agents are of equal power (and if opposed produce equilibrium) when the products of the force and the velocity of the point of application in the direction of the force are the same for each.

This is nothing more nor less than the "principle of virtual velocities,"—a succinct statement of "the whole theory of machines diversely demonstrated by various authors," already cited in the second corollary as a confirmation of the laws of motion, because on the one hand deducible from them, and, on the other hand, in agreement with experience.

After applying this principle of virtual velocities in detail to the several simple machines, Newton continues : "But to treat of mechanism does not belong to the present design. I wished only to show by these things how widely extends and how certain is the third law of motion." † Then follows the passage quoted by Thomson and Tait (see above) in which the only new idea involved is the inclusion of the resistance to acceleration among the "reactions."

The scholium shows indeed that Newton had a clear conception of what we now know as "D'Alembert's principle" ‡ as well as the "principle of virtual velocities," but does not, as it seems to me, indicate any intention to postulate a new axiom.

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* Ut corpora in concursu & reflexione idem pollent, quorum velocitates sunt reciproce ut vires insitæ · sic in movendis instrumentis mechanicis agentia idem pollent & conatibus contrariis se mutuo sustinent, quorum velocitates secundum determinationem virium æstimatæ, sunt reciproce ut vires.

† Cæterum mechanicam tractare non est hujus instituti. Hisce volui tantum ostendere, quam late pateat quamque certa sit lex tertia motus.

‡ It must be remembered, however, that we call such propositions "principles," not when they are presented simply as demonstrated theorems, but when they are made the basis of a systematic method of applying analysis to the solution of problems.

Professor Tait * regards the first words of the scholium—"Up to this, I have laid down principles received by mathematicians and confirmed by experiments in great number" †—as claiming for Newton the discovery of what, as stated above, he regards as a fourth law : as if Newton were about to proceed to some new axiom not yet known to the men of science of the day. Yet we have seen that the scholium treats of a variety of topics at great length, before coming to what is alleged to be the new axiom. The context rather shows that the matter new to mathematicians, to which Newton implicitly refers in the words quoted, is the body of the treatise *De Motu Corporum*, which immediately follows the introductory chapters—the *Definitiones*, and the *Axiomata* and Corollaries.

At the close of the article *Mechanics* Professor Tait summarizes the third law proper thus—"Every action between two bodies is a stress." He subsequently points out in the simple instance of a falling stone how force may be regarded either as "the *space-rate at which energy is transformed*," or "the *time-rate at which momentum is generated*," and says (§ 294) that these are "merely particular cases of Newton's two interpretations of action in the third law." He then proceeds to connect them analytically as follows:—"if s be the space described, v the speed of a particle,

$$\ddot{s} = \dot{v} = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\,\frac{dv}{ds}.$$

Hence the equation of motion (formed by the second law)

$$m\ddot{s} = m\dot{v} = f,$$

which gives f as the time-rate of increase of momentum, may be written in the new form

$$mv\,\frac{dv}{ds} = \frac{d}{ds}\,(\tfrac{1}{2}\,mv^2) = f,$$

giving f as the space-rate of increase of kinetic energy."

Is it not equally true in the general case that the so-called two interpretations of action, so far from being the subjects of separate axioms, are demonstrably equivalent by virtue of the equality of the two phases of a stress and the second law of motion ?

* Article *Mechanics*, Encyclopædia Britannica, § 12.

† Hactenus principia tradidi a mathematicis recepta & experientia multiplici confirmata.

Another instance of the unnecessary assumption of physical axioms occurs in the *Natural Philosophy.* The principle that "the perpetual motion is impossible" is introduced as an axiom to prove that "If the mutual forces between the parts of a material system are independent of their velocities, whether relative to one another, or relative to any external matter, the system must be dynamically conservative. For if more work is done by the mutual forces on the different parts of the system in passing from one particular configuration to another, by one set of paths than by another set of paths, let the system be directed, by frictionless constraint, to pass from the first configuration to the second by one set of paths and return by the other, over and over again forever. It will be a continual source of energy without any consumption of materials, which is impossible." *

Again in Williamson and Tarleton's *Dynamics* (p. 397), the same demonstration is given, closing with the words "This process may be repeated forever, and thus an inexhaustible supply of work can be obtained from permanent natural causes without any consumption of materials. The whole of experience teaches us that this is impossible."

Thus we find these authors appealing to the general principle of the conservation of energy in proof of what is really but its simplest form, namely the equivalent transference of energy from body to body of a material system, and from the kinetic to the potential form, a proposition which is easily shown to be a consequence of Newton's laws of motion. The appeal to experience is in fact only necessary to establish the hypothesis laid down in the above quotation from Thomson and Tait, namely, that the forces do not in any way depend upon velocities, or, let us say, that the stress between two bodies depends only upon the distance between them.

We conclude this examination of the physical axioms with a brief sketch of the steps by which the conservation of energy, in its mechanical forms of kinetic and potential energy of masses, may be established directly from the axioms of 'the independent accelerative action of force,' 'the duality of stress,' and 'the dependence of its intensity solely upon distance.' The steps 4 and 5, which establish the right to deal with the kinetic energy of relative motion, are developed more in detail, because the point does not seem to be sufficiently developed in the usual text-books.

1. Let the conservation of the motion of the centre of gravity be deduced, as in the *Principia,* from the second and third laws of motion.

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* *Natural Philosophy*, Art. 272.

2. The kinetic energy of a body may be decomposed into parts corresponding respectively to its component velocities in two *rectangular* directions.

3. A moving body acted upon by a force, directed to or from a fixed point and in magnitude a function of the distance of the body from that point, experiences a gain or loss of kinetic energy equal to the loss or gain of potential energy relative to the fixed point.

4. Although fixed centres of force do not exist, yet when a stress exists between two bodies (in magnitude a function of the distance), the centre of gravity being fixed, and dividing the distance between them in a fixed ratio, the actual change of potential energy is equal to the sum of the changes in the potential energy of the two bodies, each with reference to the centre of gravity as if it were a centre of force. Hence the sum of the potential energy and the kinetic energies of the two bodies is constant.

5. The total kinetic energy of a system of bodies may be decomposed as follows :—First, decompose the energy of each body into parts corresponding to the velocities perpendicular to and along the line in which the centre of gravity is moving. Put u for the first of these components, v for the second taken relatively to the centre of gravity, and V for the velocity of the centre of gravity. Then the total kinetic energy is

$$\tfrac{1}{2}\Sigma mu^2 + \tfrac{1}{2}\Sigma m(v + V)^2$$

From the property of the centre of gravity $\Sigma mv = 0$: therefore the total kinetic energy is

$$\tfrac{1}{2}\Sigma m(u^2 + v^2) + \tfrac{1}{2}\Sigma m \,.\, V^2$$

The first term is the sum of the kinetic energies corresponding to the motions relative to the centre of gravity, and the second is the kinetic energy corresponding to the total mass as if situated at and moving with the centre of gravity. Thus the total kinetic energy is equal to the *internal* kinetic energy of the system relative to the centre of gravity as a fixed point, and the *external* energy of the system due to the motion of the centre of gravity.

6. When a stress exists between two bodies whose centre of gravity is in motion, the stress causes at every instant the same gain of kinetic energy in one as loss in the other, if the distance is unchanged. But, when the distance is changing, we find by considering the external and internal energy of the system, that the former is unchanged by 1, and that the change in the latter is, by 4, compensated for by that in the potential energy connected with the stress, so that in either case the sum

of the two kinetic energies and the mutual potential energy is unchanged.

7. Hence in any system of bodies, between pairs of which stresses exist whose intensities depend solely upon the distances, the sum of the kinetic energies and the potential energy due to their relative positions is constant.

EIGHT-FIGURE LOGARITHM TABLES.

Tables des Logarithmes a huit décimales des nombres entiers de 1 à 120000, et des sinus et tangentes de dix secondes en dix secondes d'arc dans le système de la division centésimale du quadrant. Publiées par ordre du Ministre de la guerre. Paris, Imprimerie Nationale, 1891. 4to., pp. iv. + 628.

ADVOCATES of the decimal subdivision of the quadrant will be much pleased by the appearance of the above work, which contains the most extensive set of tables of the kind as yet issued. It is not intended in the present notice to enter upon the respective merits of the several systems of dividing the circle, but to consider the volume as a table of logarithms simply. As such it presents marked points of difference from the usual types. These differences are found almost exclusively in the trigonometric portion of the tables, that containing the logarithms of numbers being similar to the customary form. The logarithms of the four trigonometric functions appear on each double page in four separate tables, instead of the usual arrangement in parallel vertical columns. The interval of the argument is the same throughout the entire quadrant, no diminution being found near the beginning of the table. The auxiliary quantities for obtaining sines and tangents of small angles by means of the table of number logarithms are given; but they are placed upon the pages devoted to the trigonometric functions. It would probably be more convenient to find them as usual at the bottom of the pages containing the number logarithms.

The decimal progression of the argument allows the trigonometric tables to be arranged in the form usually adopted for the logarithms of numbers. But instead of ten columns headed with the digits 0 to 9, we find *eleven* columns, of which the first ten are headed 0 to 9. The eleventh, which has no heading, contains a repetition of the column headed 0. This makes it unnecessary to look back along the horizontal line, when we wish the difference between column 9 and the next one. Yet the size of the volume is somewhat increased by this system, and the tables containing the number logarithms