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OCTONARY NUMERATION.

BY PROF. W. WOOLSEY JOHNSON.

THE comparatively small progress toward universal acceptance made by the metric system seems to be due not altogether to aversion to a change of units, but also to a sort of irrepressible conflict between the decimal and binary systems of subdivision.

Before the introduction of decimal fractions, about 1585, no connection would be felt to exist between the established scale of numeration and the method of subdividing physical units, and it would probably never occur to any one to subdivide a unit into tenths. The natural method is to bisect again and again. The mechanic prefers to divide the inch into halves, quarters, eighths, and sixteenths. The retailer of dry goods, whose unit is the yard, divides it into halves, quarters, and eighths, totally ignoring the inch. The mariner not only divides the horizontal angular space in which his course is laid down into quarters, thus recognizing the right angle as the natural unit,* but divides the space between the cardinal points of the compass into halves, quarters, and eighths. Where decimal money has been introduced quarters are insisted on in spite of their inconsistency with the decimal system. We are compelled to coin quarter dollars, and prices are very commonly quoted in eighths and even sixteenths of a dollar. Great Britain is compelled to coin eighths of a pound sterling, though half a crown contains a fraction of a shilling. The French divide the litre into quarters. The broker expresses prices in halves, quarters, and eighths of one per cent.

This irrepressible conflict would, of course, never have

* The uncompromising advocate of the metric system will not content himself with the centesimal division of the degree, but insists upon the centesimal division of the quadrant, although it is difficult to see how the latter could possess any advantage in the way of facilitating numerical computations. But why do they not go further and advocate the centesimal division of the whole circumference?

existed, but all would have been harmony, if the radix of our system of numeration had been a power of two. Mr. Alfred B. Taylor published in 1887 a very interesting pamphlet on "Octonary Numeration," being a paper read before the American Philosophical Society. After an extended review of the question, with many interesting historical notes, he argues in favor of the octonary system, and then proceeds to "develop the scale of notation thus selected, and to derive from it an ideal system of weights and measures."

This is not the place to consider the merits of a system of weights and measures; we propose therefore to consider only the theoretical merits of the octonary system. We regret that in his ingenious paper Mr. Taylor has caused his system to wear an outlandish look by employing new names, not only for his units of weights and measures, but also for the numbers from one to eight, and even new characters for the seven digits. We see no necessity for changing the characters or the names of the digits, although it would be necessary, in order to avoid the use of an old name in a new meaning, to replace the suffix *-ty* by a new one to denote the second place (which Mr. Taylor, having changed the names of the digits, did not find necessary). We might use the suffix *-ate*—thus the octonary 40 would be read *fourate*, that is, four eights; 56 would be read *fivate-six*, that is five eights and six.

The only advantage of a large radix, *quâ* large, over a smaller one is in diminishing the number of figures required on the average to express a given number. The number of figures is inversely as the logarithm of the radix; and, in passing from the radix ten to the radix eight, it increases only in the ratio of 10 : 9. The ratio increases rapidly for smaller radices, until for the binary system it becomes 10 : 3, as compared with the denary, and 3 : 1 as compared with the octonary system. To set against this we have, in favor of the smaller radix, the simplicity due to dispensing with superfluous characters; but of far more importance is the simplifying of the multiplication table. For example, the octonary multiplication table stands thus:

1	2	3	4	5	6	7
2	4	6	10	12	14	16
3	6	11	14	17	22	25
4	10	14	20	24	30	34
5	12	17	24	31	36	43
6	14	22	30	36	44	52
7	16	25	34	43	52	61

In comparing the labor with which this table could be committed to memory with that required by the denary table, it would seem fair to disregard in both cases not only the line and column corresponding to 1 (although our German friends insist upon *ein mal eins*), but also those corresponding to 2 and to half the radix, on account of their simplicity. Thus the difficulty would be about as 6^2 to 4^2; indeed, it seems safe to say that the difficulty experienced by children in acquiring the multiplication table, and that of older people in retaining it in a condition fresh enough to be used without an effort of thought, would be reduced more than one-half even by this slight decrease in the magnitude of the radix.

For a further decrease of radix, the difficulty of the multiplication table decreases rapidly: for the binary system no multiplication table exists, but even for the radix four the difficulty has practically disappeared.

But this advantage of a very small radix is, as mentioned above, attended by a rapid increase in the number of figures required to express a given number; and the inconvenience arising from this source has, we think, been frequently underestimated. Binary arithmetic, in which the characters 1 and 0 alone are used, has even been proposed by some enthusiasts as a substitute for logarithmic computation. Mr. Taylor, in commenting upon this system, after mentioning the absence of tables to be committed to and retained in the memory, says: "Every form of calculation would be resolved into simple numeration and notation. In fact, calculation as an effort of mathematical thought might be said to be entirely dispensed with, and the labor of the brain to be all transferred to the eye and hand. A perfect familiarity with the notation of the scale, and with the simple rules of position, *would enable the operator to determine in every case by mere inspection, whether the next figure should be a* 1 *or a* 0. It follows that the only errors possible in such a work would be the merely clerical ones of the eye or hand; * * * and it may well be doubted whether, in all important and lengthy calculations, the binary system would not be found to afford a real economy of labor, instead of an increase as has been generally supposed."

Now it is to be remarked that the number of figures used in calculation would increase at a rate much greater than that of the number of figures used in expressing results. For example, in performing a multiplication in the binary notation, the number of figures to be written down (after making due allowance for the greater proportion saved by the occurrence of ciphers in the multiplier) would be about five times, instead of three times, the number occurring in the same operation performed in the octonary notation.

Again, whenever the columns to be added are of consider-able length their summation, though executed by mere counting and the determination of the numbers "to carry," would require fixed attention, and involve liability to error; so much so, that the words we have italicized in the quotation appear hardly justified. The numbers to carry would be inconveniently large, especially if mentally expressed in the binary system. Indeed, counting in this system would obviously be very much more liable to error than in the denary (or in the octonary) system, which gives highly distinctive names to all such numbers as have to be carried in the mind in the course of calculation.

The same objection exists, though to a less degree, to the quaternary system, so that the labor of accurate calculation in this system, although perhaps less than in the denary system, would probably exceed that which would be required in the octonary system.

The conclusion appears to be inevitable that, considering only the two features which depend upon the mere size of the radix, ten is decidedly too large and four too small a radix, so that the ideal radix in this respect is about six or eight.

Passing now to the intrinsic character of the radix, it is desirable that the radix should be divisible by simple factors. Thus it is universally admitted that an uneven radix would be quite out of the question. It is indispensable for a multitude of purposes that even the least instructed persons should be familiar with the distinction between even and uneven numbers, and able to recognize at a glance to which class a given number belongs. It was formerly the custom to extol twelve as an ideal radix, because of its divisibility by two, three, four, and six. Divisibility by three, although incomparably less important than divisibility by two, would no doubt be a great convenience, much more so than divisibility by five; but it is doubtful whether much weight should be given to divisibility of the first power of the radix by four, so long as we do not adopt a purely binary system (that is, two or a power of two for radix). We ought rather to consider only the prime factors of the radix, so that six would possess all the advantages of twelve, and since on the other score twelve is far too large a radix, six would be far preferable to it. (The number of figures used to express a given number would for six exceed that for twelve only in the ratio 7: 5, and would exceed that for ten only in the ratio 9: 7.)

Against this advantage of divisibility by different prime factors we have to set the advantages of a purely binary system. Theoretical considerations here point in the same direction as the practical ones rehearsed in the first part of this paper. Owing to the unique character of the number two, it must be admitted that the expression of a given number in powers of two gives a better notion of its intrinsic character than expression in powers of any other number. Accordingly the binary system has always been regarded as theoretically the ideal system, although for practical purposes the great number of figures used in expressing numbers is an insuperable objection. Now it is to be noticed that if the radix is a power of two, we have virtually all the advantages of the binary system. For example, if we have a number expressed in the octonary system, we have only to substitute for the characters 0, 1, 2, ...7 their binary equivalents 000, 001, 010, ...111 to obtain the number in the binary system.

The digital expression of a number in the octonary system would be much more suggestive of its intrinsic nature than expression in any non-binary system, for the highest power of two contained as a factor in a number is its most important characteristic. Again the distinction between numbers of the form $4n + 1$ and those of the form $4n + 3$ is of great importance in the theory of numbers, and in the octonary system it would be obvious at a glance to which of these classes a given uneven number belongs. So also with the distinction between "evenly even" and "unevenly even" numbers. It is interesting also to note that the square of every uneven number would end in 1, the preceding figures expressing a triangular number. Thus the uneven squares in octonary notation are 1, 11, 31, 61, 121....

We have seen above that, if divisibility by another prime factor besides two be regarded as the paramount desideratum, six would be preferable to ten as a radix. But the tests of divisibility by small divisors (such as the familiar one for nine or three) would always to a great extent serve the same purpose as the divisibility of the radix. These tests depend upon the lowest value of n for which $r^n - 1$ or $r^n + 1$ (r being the radix) is divisible by the divisor in question; and they consist in reducing the given number to one of n places which will give the same remainder when divided by the given divisor. This is done in the first case by the addition of periods of n figures each, beginning with the units; and in the second case, by the addition of periods of $2n$ figures, followed by subtraction of the second period of n figures from the first. For example, with the radix ten we can test for each of the divisors seven, eleven, and thirteen, which are factors of $10^3 + 1$, by reducing to six places by addition of periods of six, and then to three places by subtraction of the figures representing thousands from the first or unit period of three figures.

Let us see how the matter would stand in the octonary system. For seven we should add all the digits, and for nine (and therefore for three) we should add by periods of two. Again since $8^2 + 1 = 5 \times 13$, we should test for five and thirteen (or oneate-five) by reducing to four figures by addition, and then to two figures by subtraction. Among small primes, eleven is the least adapted to the octonary system, but for this divisor we might convert the given number to the binary system, then reduce to ten figures by addition, and to five by subtraction (since $2^5 + 1 = 3 \times 11$), and finally reconvert into an octonary number of two digits.

As there is no doubt that our ancestors originated the decimal system by counting on their fingers, we must, in view of the merits of the octonary system, feel profound regret that they should have perversely counted their thumbs, although nature had differentiated them from the fingers sufficiently, she might have thought, to save the race from this error.

THE TEACHING OF ELEMENTARY GEOMETRY IN GERMAN SCHOOLS.

Inhalt und Methode des planimetrischen Unterrichts. Eine vergleichende Planimetrie. Von Dr. Heinrich Schotten. Leipzig, B. G. Teubner, 1890. 8vo, pp. iv. + 370.

Whoever has followed the efforts of the Association for the Improvement of Geometrical Teaching in England in the course of the last ten years will have been struck by the slowness of the progress made and the paucity of the practical results attained. In Germany there exists no such society; but a powerful agitation for the reform of geometrical teaching has been in progress there for at least sixty years, and with particular force during the last two decades. And yet, even from Germany, with its well developed and highly centralized system of education, comes the complaint that progress is slow and much remains to be done.

Recent statistics have shown, in particular, that the most widely used text-books are far from being the best. Thus, while Hubert Müller's Geometry, which may be regarded as the best representative of the "modern school," reached its third edition in 1889, after a lapse of fifteen years from its first appearance, Kambly's very inferior text-book, whose faults and mistakes have frequently been exposed and complained of, appeared in 1884 in its 74th edition.

This book of Kambly's easily leads in the list of text-books used in various schools; it is adopted in 217 schools, the next in order being another rather inferior book, by Koppe,