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Čech cocycles for differential characteristic classes: an ∞-Lie theoretic construction

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Abstract

What are called *secondary characteristic classes* in Chern–Weil theory are a refinement of ordinary characteristic classes of principal bundles from cohomology to differential cohomology. We consider the problem of refining the construction of secondary characteristic classes from cohomology sets to cocycle spaces; and from Lie groups to higher connected covers of Lie groups by smooth ∞-groups, i.e., by smooth groupal A_∞-spaces. Namely, we realize differential characteristic classes as morphisms from ∞-groupoids of smooth principal ∞-bundles with connections to ∞-groupoids of higher $U(1)$-gerbes with connections. This allows us to

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study the homotopy fibres of the differential characteristic maps thus obtained and to show how these describe differential obstruction problems. This applies in particular to the higher twisted differential spin structures called *twisted differential string structures* and *twisted differential fivebrane structures.*

Summary

What are called *secondary characteristic classes* in Chern–Weil theory are a refinement of ordinary characteristic classes of principal bundles from cohomology to differential cohomology. We consider the problem of refining the construction of secondary characteristic classes from cohomology sets to cocycle spaces; and from Lie groups to higher connected covers of Lie groups by smooth ∞-groups, i.e., by smooth groupal A_∞-spaces. Namely, we realize differential characteristic classes as morphisms from ∞-groupoids of smooth principal ∞-bundles with connections to ∞-groupoids of higher $U(1)$-gerbes with connections. This allows us to study the homotopy fibres of the differential characteristic maps thus obtained and to show how these describe differential obstruction problems. This applies in particular to the higher twisted differential spin structures called *twisted differential string structures* and *twisted differential fivebrane structures.*

To that end we define for every L_∞-algebra $\mathfrak{g}$ a smooth ∞-group G integrating it, and define smooth G-principal ∞-bundles with connection. For every L_∞-algebra cocycle of suitable degree, we give a refined ∞-Chern–Weil homomorphism that sends these ∞-bundles to classes in differential cohomology that lift the corresponding curvature characteristic classes.

When applied to the canonical 3-cocycle of the Lie algebra of a simple and simply connected Lie group G this construction gives a refinement of the secondary first fractional Pontryagin class of G-principal bundles to cocycle space. Its homotopy fibre is the 2-groupoid of smooth String(G)-principal 2-bundles with 2-connection, where String(G) is a smooth 2-group refinement of the topological string group. Its homotopy fibres over non-trivial classes we identify with the 2-groupoid of *twisted differential string structures* that appears in the Green–Schwarz anomaly cancellation mechanism of heterotic string theory.

Finally, when our construction is applied to the canonical 7-cocycle on the Lie 2-algebra of the String-2-group, it produces a secondary characteristic map for String-principal 2-bundles which refines the second fractional Pontryagin class. Its homotopy fibre is the 6-groupoid of principal 6-bundles with 6-connection over the *Fivebrane 6-group.* Its homotopy fibres over non-trivial classes are accordingly *twisted differential fivebrane structures* that have been argued to control the anomaly cancellation mechanism in magnetic dual heterotic string theory.

Contents

1 Introduction

Classical Chern–Weil theory (see for instance [21, 28, 38]) provides a toolset for refining characteristic classes of smooth principal bundles from ordinary integral cohomology to differential cohomology.

This can be described as follows. For G a topological group and $P \to X$ a G-principal bundle, to any *characteristic class* $[c] \in H^{n+1}(BG, \mathbb{Z})$, there is associated a characteristic class of the bundle, $[c(P)] \in H^{n+1}(X, \mathbb{Z})$. This can be seen as the homotopy class of the composition

$$X \xrightarrow{P} BG \xrightarrow{c} K(\mathbb{Z}, n+1)$$

of the classifying map $X \xrightarrow{P} BG$ of the bundle with the *characteristic map* $BG \xrightarrow{c} K(\mathbb{Z}, n+1)$. If G is a compact connected Lie group, and with real coefficients, there is a graded commutative algebra isomorphism between $H^\bullet(BG, \mathbb{R})$ and the algebra $\mathrm{inv}(\mathfrak{g})$ of ad_G-invariant polynomials on the Lie algebra $\mathfrak{g}$ of G. In particular, any characteristic class c will correspond to such an invariant polynomial $\langle - \rangle$. The Chern–Weil homomorphism associates to a choice of *connection* ∇ on a G-principal bundle P the closed differential form $\langle F_\nabla \rangle$ on X, where F_∇ is the curvature of ∇. The de Rham cocycle $\langle F_\nabla \rangle$ is a representative for the characteristic class $[c(P)]$ in $H^\bullet(X, \mathbb{R})$. This construction can be carried out at a local level: instead of considering a globally defined connection ∇, one can consider an open cover $\mathcal{U}$ of X and local connections ∇_i on $P|_{U_i} \to U_i$; then the local differential forms $\langle F_{\nabla_i} \rangle$ define a cocycle in the Čech–de Rham complex, still representing the cohomology class of $c(P)$.

There is a refinement of this construction to what is sometimes called *secondary characteristic classes*: the differential form $\langle F_\nabla \rangle$ may itself be understood as the higher curvature form of a higher circle-bundle-like structure $\hat{\mathbf{c}}(\nabla)$ whose higher Chern-class is $c(P)$. In this refinement, both the original characteristic class $c(P)$ as well as its curvature differential form $\langle F_\nabla \rangle$ are unified in one single object. This single object has originally been formalized as a *Cheeger–Simons differential character*. It may also be conceived of as a cocycle in the Čech–Deligne complex, a refinement of the Čech–de Rham complex [28]. Equivalently, as we discuss here, these objects may naturally be described in terms of what we want to call *circle n-bundles with connection*: smooth bundles whose structure group is a smooth refinement — which we write $\mathbf{B}^n U(1)$ — of the topological group $B^n U(1) \simeq K(\mathbb{Z}, n+1)$, endowed with a smooth connection of higher order. For low n, such $\mathbf{B}^n U(1)$-principal bundles are known (more or less explicitly) as $(n-1)$-*bundle gerbes*.

The fact that we may think of $\hat{\mathbf{c}}(\nabla)$ as being a smooth principal higher bundle with connection suggests that it makes sense to ask if there is a general definition of smooth G-principal ∞-bundles, for smooth ∞-groups G, and whether the Chern–Weil homomorphism extends on those to an ∞-*Chern–Weil homomorphism.* Moreover, since G-principal bundles naturally form a parameterized groupoid — a stack — and circle n-bundles naturally form a parameterized n-groupoid — an $(n-1)$-stack, an n-truncated ∞-stack, it is natural to ask whether we can refine the construction of differential characteristic classes to these ∞-stacks. Motivations for considering this are threefold:

(1) The ordinary Chern–Weil homomorphism only knows about characteristic classes of classifying spaces BG for G a Lie group. Already before considering the refinement to differential cohomology, this misses useful cohomological information about *connected covering groups* of G.

For instance, for $G = \mathrm{Spin}$, the Spin group, there is the second Pontryagin class represented by a map $p_2 : B\mathrm{Spin} \to B^8\mathbb{Z}$. But on some Spin-principal bundles $P \to X$ classified by a map $g : X \to B\mathrm{Spin}$, this class may be further divisible: there is a topological group String, called the *String group*, such that we have a commuting diagram

$$\begin{array}{ccccc} & & B\mathrm{String} & \xrightarrow{\frac{1}{6}p_2} & B^8\mathbb{Z} \\ & \overset{\tilde{g}}{\nearrow} & \downarrow & & \downarrow{\cdot 6} \\ X & \xrightarrow{g} & B\mathrm{Spin} & \xrightarrow{p_2} & B^8\mathbb{Z} \end{array}$$

of topological spaces, where the morphism on the right is given on $\mathbb{Z}$ by multiplication with 6 [42]. This means that if P happens to admit a *String structure* exhibited by a lift $\tilde{g}$ of its classifying map g as indicated, then its second Pontryagin class $[p_2(P)] \in H^8(X, \mathbb{Z})$ is divisible by 6. But this refined information is invisible to the ordinary Chern–Weil homomorphism: while Spin canonically has the structure of a Lie group, String cannot have a finite-dimensional Lie group structure (because it is a $BU(1)$-extension, hence has cohomology in arbitrary high degree) and therefore the ordinary Chern–Weil homomorphism cannot model this fractional characteristic class.

But it turns out that String does have a natural smooth structure when regarded as a higher group — a 2-group in this case [8, 25]. We write $\mathbf{B}\mathrm{String}$ for the corresponding smooth refinement of the classifying space. As we shall show, there is an ∞-Chern–Weil homomorphism that does apply and produces for every smooth String-principal

2-bundle $\tilde{g} : X \to \mathbf{B}\mathrm{String}$ a smooth circle 7-bundle with connection, which we write $\frac{1}{6}\hat{\mathbf{p}}_2(\tilde{g})$. Its curvature 8-form is a representative in de Rham cohomology of the fractional second Pontryagin class.

Here and in the following:

- boldface denotes a refinement from continuous (bundles) to *smooth* (higher bundles);
- the hat denotes further *differential* refinement (equipping higher bundles with smooth connections).

In this manner, the ∞-Chern–Weil homomorphism gives cohomological information beyond that of the ordinary Chern–Weil homomorphism. And this is only the beginning of a pattern: the sequence of smooth objects that we considered continues further as

$$\cdots \to \mathbf{B}\mathrm{Fivebrane} \to \mathbf{B}\mathrm{String} \to \mathbf{B}\mathrm{Spin} \to \mathbf{B}SO \to \mathbf{B}O$$

to a smooth refinement of the *Whitehead tower* of BO. One way to think of ∞-Chern–Weil theory is as a lift of ordinary Chern–Weil theory along such smooth Whitehead towers.

(2) Traditionally the construction of secondary characteristic classes is exhibited on single cocycles and then shown to be independent of the representatives of the corresponding cohomology class. But this indicates that one is looking only at the connected components of a more refined construction that explicitly sends cocycles to cocycles, and sends coboundaries to coboundaries such that their composition is respected up to higher degree coboundaries, which in turn satisfy their own coherence condition, and so forth. In other words: a map between the full cocycle ∞-groupoids.

The additional information encoded in such a refined secondary differential characteristic map is equivalently found in the collection of the *homotopy fibres* of the map, over the cocycles in the codomain. These homotopy fibres answer the question: which bundles with connection have differential characteristic class equivalent to some fixed class, which of their gauge transformations respect the choices of equivalences, which of the higher gauge of gauge transformations respect the chosen gauge transformations, and so on. This yields refined cohomological information whose knowledge is required in several applications of differential cohomology, indicated in the next item.

(3) Much of the motivation for studies of differential cohomology originates in the applications this theory has to the description of higher gauge fields in physics. Notably the seminal article [28] that laid the basis of generalized differential cohomology grew out of the observation that this is the right machinery that describes subtle phenomena of quantum anomaly cancellation in string theory, discussed by Edward

Witten and others in the 1990s, further spelled out in [18]. In this context, the need for refined fractional characteristic classes and their homotopy fibres appears.

In higher analogy to how the quantum mechanics of a spinning particle requires its target space to be equipped with a Spin-structure that is differentially refined to a Spin-principal bundle with connection, the quantum dynamics of the (heterotic) superstring requires target space to be equipped with a differential refinement of a String structure. Or rather, since the heterotic string contains besides the gravitational Spin-bundle also a $U(n)$-gauge bundle, of a *twisted* String structure for a specified twist. We had argued in [44] that these differentially refined string backgrounds are to be thought of as twisted differential structures in the above sense. With the results of the present work this argument is lifted from a discussion of local ∞-connection data to the full differential cocycles. We shall show that by standard homotopy theoretic arguments this allows a simple derivation of the properties of untwisted differential string structures that have been found in [48], and generalize these to the twisted case and all the higher analogs.

Namely, moving up along the Whitehead tower of $O(n)$, one can next ask for the next higher characteristic class on String-2-bundles and its differential refinement to a secondary characteristic class. In [42], it was argued that this controls, in direct analogy to the previous case, the quantum super-fivebrane that is expected to appear in the magnetic dual description of the heterotic target space theory. With the tools constructed here the resulting *twisted differential fivebrane structures* can be analyzed in analogy to the case of string structures.

Our results allow an analogous description of twisted differential structures of ever higher covering degree, but beyond the fivebrane it is currently unclear whether this still has applications in physics. However, there are further variants in low degree that do:

for instance for every n there is a canonical 4-class on pairs of n-torus bundles and dual n-torus bundles. This has a differential refinement and thus we can apply our results to this situation to produce parameterized 2-groupoids of the corresponding higher twisted differential torus-bundle extensions. We find that the connected components of these 2-groupoids are precisely the *differential T-duality pairs* that arise in the description of differential T-duality of strings in [30].

This suggests that there are more applications of refined higher differential characteristic maps in string theory, but here we shall be content with looking into these three examples.

In this paper, we shall define connections on principal ∞-bundles and the action of the ∞-Chern–Weil homomorphism in a natural but maybe still somewhat *ad hoc* way, which here we justify mainly by the two main theorems about two examples that we prove, which we survey in a moment. The construction uses essentially standard tools of differential geometry. The construction can be derived from *first principles* as a model (in the precise sense of *model category theory*) for a general abstract construction that exists in ∞-*topos theory*. This abstract theory is discussed in detail elsewhere [41].

Note that our approach goes beyond that of [28] in two ways: the ∞-stacks we consider remember *smooth* gauge transformation and thus encode smooth structure of principal ∞-bundles already on cocycles and not just in cohomology; secondly, we describe *non-abelian* phenomena, such as connections on principal bundles for non-abelian structure groups, and more in general ∞-connections for non-abelian structure smooth ∞-groups, such as the String-2-group and the Fivebrane-6-group. This is the very essence of (higher) Chern–Weil theory: to characterize non-abelian cohomology by abelian characteristic classes. Since [28] work with spectra, nothing non-abelian is directly available there. On the other hand, the construction we describe does not as easily allow differential refinements of cohomology theories represented by non-connective spectra.

We now briefly indicate the means by which we will approach these issues in the following.

The construction that we discuss is the result of applying a refinement of the machine of ∞-*Lie integration* [20, 25] to the L_∞-algebraic structures discussed in [43, 44]:

For $\mathfrak{g}$ an L_∞-algebra, its *Lie integration* to a Lie ∞-group G with smooth classifying object $\mathbf{B}G$ turns out to be encoded in the simplicial presheaf given by the assignment to each smooth test manifold U of the simplicial set

$$\exp_\Delta(\mathfrak{g}) : (U, [k]) \mapsto \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet(U \times \Delta^k)_{\mathrm{vert}}),$$

where $\mathrm{CE}(\mathfrak{g})$ is the Chevalley–Eilenberg algebra of $\mathfrak{g}$ and 'vert' denotes forms which see only vector fields along Δ^k. This has a canonical projection $\exp_\Delta(\mathfrak{g}) \to \mathbf{B}G$, hence the name $\exp_\Delta(\mathfrak{g})$. One can think of this as saying that a U-parameterized smooth family of k-simplices in G is given by the *parallel transport* over the k-simplex of a flat $\mathfrak{g}$-valued vertical differential form on the trivial simplex bundle $U \times \Delta^k \to U$. This we discuss in detail in Section 4.2.

The central step of our construction is a *differential refinement* $\mathbf{B}G_{\mathrm{diff}}$ of $\mathbf{B}G$, where the above is enhanced to

$$\exp_\Delta(\mathfrak{g})_{\mathrm{diff}} : (U,[k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet(U\times\Delta^k)_{\mathrm{vert}} & \longleftarrow & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U\times\Delta^k) & \longleftarrow & \mathrm{W}(\mathfrak{g}) \end{array} \right\},$$

with $\mathrm{W}(\mathfrak{g})$ the Weil algebra of $\mathfrak{g}$. We also consider a simplicial sub-presheaf $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}} \hookrightarrow \exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ defined by a certain horizontality constraint. This may be thought of as assigning non-flat $\mathfrak{g}$-valued forms on the total space of the trivial simplex bundle $U \times \Delta^k$. The horizontality constraint generalizes one of the conditions of an *Ehresmann connection* [16] on an ordinary G-principal bundle. This we discuss in detail in Section 4.3.

We observe that an L_∞-algebra cocycle $\mu \in \mathrm{CE}(\mathfrak{g})$ in degree n, when we equivalently regard it as a morphism of L_∞-algebras $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ to the Eilenberg–MacLane object $b^{n-1}\mathbb{R}$, tautologically *integrates* to a morphism

$$\exp_\Delta(\mu) : \exp_\Delta(\mathfrak{g}) \to \exp_\Delta(b^{n-1}\mathbb{R})$$

of the above structures. What we identify as the ∞-Chern–Weil homomorphism is obtained by first extending this to the differential refinement

$$\exp_\Delta(\mu)_{\mathrm{diff}} : \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \to \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{diff}}$$

in a canonical way — this we shall see introduces *Chern–Simons elements* — and then descending the construction along the projection $\exp(\mathfrak{g})_{\mathrm{diff}} \to \mathbf{B}G_{\mathrm{diff}}$. This quotients out a lattice $\Gamma \subset \mathbb{R}$ and makes the resulting higher bundles with connection be circle n-bundles with connection, which represent classes in differential cohomology. This we discuss in Section 5.

Finally, in the last part of Section 5 we discuss two classes of applications and obtain the following statements.

Theorem 1.0.1. *Let X be a paracompact smooth manifold and choose a good open cover $\mathcal{U}$.*

Let $\mathfrak{g}$ be a semisimple Lie algebra with normalized binary Killing form $\langle -,-\rangle$ in transgression with the 3-cocycle $\mu_3 = \frac{1}{2}\langle -,[-,-]\rangle$. Let G be the corresponding simply connected Lie group.

- **1.** *Applied to this μ_3, the ∞-Chern–Weil homomorphism*

$$\exp(\mu)_{\mathrm{conn}} : \check{C}(\mathcal{U}, \mathbf{B}G_{\mathrm{conn}}) \to \check{C}\left(\mathcal{U}, \mathbf{B}^3 U(1)_{\mathrm{conn}}\right)$$

from Čech cocycles with coefficients in the complex that classifies G-principal bundles with connection to Čech–Deligne cohomology in degree 4 is a fractional multiple of the Brylinski–McLaughlin construction [5] of Čech–Deligne cocycles representing the differential refinement of the characteristic class corresponding to $\langle -, - \rangle$.

In particular, in cohomology it represents the refined Chern–Weil homomorphims

$$\frac{1}{2}\hat{\mathbf{p}}_1 : H^1(X, G)_{\mathrm{conn}} \to \hat{H}^4(X, \mathbb{Z})$$

induced by the Killing form and with coefficients in degree 4 differential cohomology. For $\mathfrak{g} = \mathfrak{so}(n)$, this is the differential refinement of the first fractional Pontryagin class.

Next let $\mu_7 \in \mathrm{CE}(\mathfrak{g})$ be a 7-cocycle on the semisimple Lie algebra $\mathfrak{g}$ (this is unique up to a scalar factor). Let $\mathfrak{g}_{\mu_3} \to \mathfrak{g}$ be the L_∞-algebra-extension of $\mathfrak{g}$ classified by μ_3 (the string Lie 2-algebra. Then μ_7 can be seen as a 7-cocycle also on $\mathfrak{g}_{\mu_3}$.

- **2.** *Applied to μ_7 regarded as a cocycle on $\mathfrak{g}_\mu$, the ∞-Chern–Weil homomorphism produces a map*

$$\check{C}(\mathcal{U}, \mathbf{B}\mathrm{String}(G)_{\mathrm{conn}}) \to \check{C}\left(\mathcal{U}, \mathbf{B}^7 U(1)_{\mathrm{conn}}\right)$$

from Čech cocycles with coefficients in the complex that classifies String(G)*-2-bundles with connection to degree 8 Čech–Deligne cohomology. For $\mathfrak{g} = \mathfrak{so}(n)$ this gives a fractional refinement of the ordinary refined Chern–Weil homomorphism*

$$\frac{1}{6}\hat{\mathbf{p}}_2 : H^1(X, \mathrm{String})_{\mathrm{conn}} \to \hat{H}^8(X, \mathbb{Z})$$

that represents the differential refinement of the second fractional Pontryagin class on Spin *bundles with* String *structure.*

These are only the first two instances of a more general statement. But this will be discussed elsewhere.

2 A review of ordinary Chern–Weil theory

We briefly review standard aspects of ordinary Chern–Weil theory whose generalization we consider later on. In this section we assume the reader is familiar with basic properties of Chevalley–Eilenberg and of Weil algebras; the unfamiliar reader can find a concise account at the beginning of Section 4.

2.1 The Chern–Weil homomorphism

For G a Lie group and X a smooth manifold, the idea of a *connection* on a smooth G-principal bundle $P \to X$ can be expressed in a variety of equivalent ways: as a distribution of horizontal spaces on the tangent bundle total space TP, as the corresponding family of projection operators in terms of local connection 1-forms on X or, more generally, as defined by Ehresmann [16], and, ultimately, purely algebraically, by Cartan [10, 11].

Here, following this last approach, we review how the Weil algebra can be used to give an algebraic description of connections on principal bundles, and of the Chern–Weil homomorphism.

We begin by recalling the classical definition of connection on a G-principal bundle $P \to X$ as a $\mathfrak{g}$-valued 1-form A on P which is G-equivariant and induces the Maurer–Cartan form of G on the fibres (these are known as the Cartan–Ehresmann conditions). The key insight is then the identification of $A \in \Omega^1(P, \mathfrak{g})$ with a differential graded algebra morphism; this is where the Weil algebra $\mathrm{W}(\mathfrak{g})$ comes in. We will introduce Weil algebras in a precise and intrinsic way in the wider context of Lie ∞-algebroids in Section 4.1; so we will here content ourselves with thinking of the Weil algebra of $\mathfrak{g}$ as a perturbation of the Chevalley–Eilenberg cochain complex $\mathrm{CE}(\mathfrak{g})$ for $\mathfrak{g}$ with coefficients in the polynomial algebra generated by the dual $\mathfrak{g}^*$. More precisely, the Weil algebra $\mathrm{W}(\mathfrak{g})$ is a commutative dg-algebra freely generated by two copies of $\mathfrak{g}^*$, one in degree 1 and one in degree 2; the differential d_{W} is the sum of the Chevalley–Eilenberg differential plus σ, the shift isomorphism from $\mathfrak{g}^*$ in degree 1 to $\mathfrak{g}^*$ in degree 2, extended as a derivation.

A crucial property of the Weil algebra is its freeness: dgca morphisms out of the Weil algebra are uniquely and freely determined by graded vector space morphism out of the copy of $\mathfrak{g}^*$ in degree 1. This means that a $\mathfrak{g}$-valued 1-form A on P can be equivalently seen as a dgca morphism

$$A : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(P)$$

to the de Rham dg-algebra of differential forms on P. Now we can read the Cartan–Ehresmann conditions on a $\mathfrak{g}$-connection as properties of this dgca morphism. First, the Maurer–Cartan form on G, i.e., the left-invariant $\mathfrak{g}$-valued form θ_G on G induced by the identity on $\mathfrak{g}$ seen as a linear morphism $T_eG \to \mathfrak{g}$, is an element of $\Omega^1(G, \mathfrak{g})$, and so it defines a dgca morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(G)$. This morphism actually factors through the Chevalley–Eilenberg algebra of $\mathfrak{g}$; this is the algebraic counterpart of the fact that the curvature 2-form of θ_G vanishes. Therefore, the first Cartan–Ehresmann condition on the behaviour of the connection form A on the fibres of $P \to X$ is encoded in the commutativity of the following diagram of differential graded commutative algebras:

$$\begin{array}{ccc} \Omega^\bullet(P)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(P) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \end{array} .$$

In the upper left corner, $\Omega^\bullet(P)_{\mathrm{vert}}$ is the dgca of *vertical* differential forms on P, i.e., the quotient of $\Omega^\bullet(P)$ by the differential ideal consisting of differential forms on P which vanish when evaluated on a vertical multivector field.

Now we turn to the second Cartan–Ehresmann condition. The symmetric algebra $\mathrm{Sym}^\bullet(\mathfrak{g}^*[-2])$ on $\mathfrak{g}^*$ placed in degree 2 is a graded commutative subalgebra of the Weil algebra $\mathrm{W}(\mathfrak{g})$, but it is not a dg-subalgebra. However, the subalgebra $\mathrm{inv}(\mathfrak{g})$ of $\mathrm{Sym}^\bullet(\mathfrak{g}^*[-2])$ consisting of $\mathrm{ad}_{\mathfrak{g}}$-invariant polynomials is a dg-subalgebra of $\mathrm{W}(\mathfrak{g})$. The composite morphism of dg-algebras

$$\mathrm{inv}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g}) \xrightarrow{A} \Omega^\bullet(P)$$

is the evaluation of invariant polynomials on the *curvature* 2-form of A, i.e., on the $\mathfrak{g}$-valued 2-form $F_A = dA + \frac{1}{2}[A, A]$. Invariant polynomials are d_{W}-closed as elements in the Weil algebra, therefore, their images in $\Omega^\bullet(P)$ are closed differential forms. Assume now G is connected. Then, if $\langle - \rangle$ is an $\mathrm{ad}_{\mathfrak{g}}$-invariant polynomial, by the G-equivariance of A it follows that the closed differential form $\langle F_A \rangle$ descends to a closed differential form on the base X of the principal bundle. Thus, the second Cartan–Ehresmann condition on A implies the commutativity of the diagram

$$\begin{array}{ccc} \Omega^\bullet(P) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(X) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) \end{array} .$$

Since the image of $\mathrm{inv}(\mathfrak{g})$ in $\Omega^\bullet(X)$ consists of closed forms, we have an induced graded commutative algebras morphism

$$\mathrm{inv}(\mathfrak{g}) \to H^\bullet(X, \mathbb{R}),$$

the *Chern–Weil homomorphism.* This morphism is independent of the particular connection form chosen and natural in X. Therefore, we can think of elements of $\mathrm{inv}(\mathfrak{g})$ as representing universal cohomology classes, hence as *characteristic classes*, of G-principal bundles. And indeed, if G is a compact connected finite-dimensional Lie group, then we have an isomorphism of graded commutative algebras $\mathrm{inv}(\mathfrak{g}) \cong H^\bullet(BG, \mathbb{R})$, corresponding to $H^\bullet(G, \mathbb{R})$ being isomorphic to an exterior algebra on odd-dimensional generators [12], the indecomposable Lie algebra cohomology classes of $\mathfrak{g}$. The isomorphism $\mathrm{inv}(\mathfrak{g}) \cong H^\bullet(BG, \mathbb{R})$ is to be thought as the universal Chern–Weil homomorphism. Traditionally, this is conceived of in terms of a smooth manifold version of the universal G-principal bundle on BG. We will here instead refine BG to a smooth ∞-groupoid $\mathbf{B}G$. This classifies not just equivalence classes of G-principal bundles but also their automorphisms. We shall argue that the context of smooth ∞-groupoids is the natural place (and *place* translates to *topos*) in which to conceive of the Chern–Weil homomorphism.

2.2 Local curvature 1-forms

Next we focus on the description of $\mathfrak{g}$-connections in terms of local $\mathfrak{g}$-valued 1-forms and gauge transformations. We discuss this in terms of the local transition function data from which the total space of the bundle may be reconstructed. It is this local point of view that we will explicitly generalize in Section 4. More precisely, in Section 4.3 we will present algebraic data which encode an ∞-connection on a trivial higher bundle on a Cartesian space $\mathbb{R}^n$, and will then globalize this local picture by *descent/ stackification.*

To prepare this general construction, let us show how it works in the case of ordinary $\mathfrak{g}$-connections on G-principal bundles. For that purpose, consider a Cartesian space $U = \mathbb{R}^n$. Every G-principal bundle on U is equivalent to the trivial G-bundle $U \times G$ equipped with the evident action of G on the second factor, and under stackification this completely characterizes G-principal bundles on general spaces. A connection on this trivial G-bundle is given by a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(U, \mathfrak{g})$. An isomorphism $A \xrightarrow{g} A'$ from the trivial bundle with connection A to that with connection A' is given by a function $g \in C^\infty(U, G)$ such that the equation

$$A' = g^{-1} A g + g^{-1} dg \tag{1}$$

holds. Here, the first term on the right denotes the adjoint action of the Lie group on its Lie algebra, whereas the second term denotes the pullback of the Maurer–Cartan form on G along g to U.

We wish to amplify a specific way to understand this formula as the Lie integration of a path of infinitesimal gauge transformations: write $\Delta^1 = [0, 1]$ for the standard interval regarded as a smooth manifold (with boundary) and consider a smooth 1-form $A \in \Omega^1(U \times \Delta^1, \mathfrak{g})$ on the product of U with Δ^1. If we think of this as the trivial interval bundle $U \times \Delta^1 \to U$ and are inspired by the discussion in Section 2.1, we can equivalently conceive of A as a morphism of dg-algebras

$$A : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(U \times \Delta^1)$$

from the Weil algebra of $\mathfrak{g}$ into the de Rham algebra of differential forms on the total space of the interval bundle. It makes sense to decompose A as the sum of a horizontal 1-form A_U and a vertical 1-form $\lambda\, dt$, where $t \colon \Delta^1 \to \mathbb{R}$ is the canonical coordinate on Δ^1:

$$A = A_U + \lambda\, dt.$$

The vertical part $A_{\mathrm{vert}} = \lambda\, dt$ of A is an element of the completed tensor product $C^\infty(U) \hat{\otimes} \Omega^1(\Delta^1, \mathfrak{g})$ and can be seen as a family of $\mathfrak{g}$-connections on a trivial G-principal bundle on Δ^1, parameterized by U. At any fixed $u_0 \in U$, the 1-form $\lambda(u_0, t)\, dt \in \Omega^1(\Delta^1, \mathfrak{g})$ satisfies the Maurer–Cartan equation by trivial dimensional reasons, and so we have a commutative diagram

$$\begin{array}{ccc} \Omega^\bullet(U \times \Delta^1)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U \times \Delta^1) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \end{array}$$

By the discussion in Section 2.1, this can be seen as a *first Cartan–Ehresmann condition in the* Δ^1*-direction*; it precisely encodes the fact that the 1-form A on the total space of $U \times \Delta^1 \to U$ is flat in the vertical direction.

The curvature 2-form of A decomposes as

$$F_A = F_{A_U} + F_{\Delta^1},$$

where the first term is at each point $t \in \Delta^1$ the ordinary curvature $F_{A_U} = d_U A_U + \frac{1}{2}[A_U, A_U]$ of A_U at fixed $t \in \Delta^1$ and where the second term is

$$F_{\Delta^1} = \left(d_U \lambda + [A_U, \lambda] - \frac{\partial}{\partial t} A_U \right) \wedge dt.$$

We shall require that $F_{\Delta^1} = 0$; this is the *second Ehresmann condition in the* Δ^1*-direction* . It implies that we have a commutative diagram

$$\begin{array}{ccc} \Omega^\bullet(\Delta^1 \times U) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}). \end{array}$$

The condition $F_{\Delta^1} = 0$ is equivalent to the differential equation

$$\frac{\partial}{\partial t} A_U = d_U \lambda + [A_U, \lambda],$$

whose unique solution for given boundary condition $A_U|_{t=0}$ specifies $A_U|_{t=1}$ by the formula

$$A_U(1) = g^{-1} A_U(0) g + g^{-1} dg,$$

where

$$g := \mathcal{P} \exp \left(\int_{\Delta^1} \lambda dt \right) : U \to G$$

is, pointwise in U, the parallel transport of λdt along the interval. We may think of this as exhibiting formula (1) for gauge transformations as arising from *Lie integration* of infinitesimal data.

Globalizing this local picture of connections on trivial bundles and gauge transformations between them now amounts to the following. For any (smooth, paracompact) manifold X, we may find a *good* open cover $\{U_i \to X\}$, i.e., an open cover such that every non-empty n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ for all $n \in \mathbb{N}$ is diffeomorphic to a Cartesian space. The cocycle data for a G-bundle with connection relative to this cover is in degree 0 and 1 given by diagrams

$$\begin{array}{ccc} 0 & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U_i) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U_i) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega^\bullet(\Delta^1 \times U_{ij})_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(\Delta^1 \times U_{ij}) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U_{ij}) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) \end{array}, \qquad (2)$$

where the latter restricts to the former after pullback along the two inclusions $U_{ij} \to U_i, U_j$ and along the face maps $\Delta^0 = \{*\} \rightrightarrows \Delta^1$. This gives a collection of 1-forms $\{A_i \in \Omega^1(U_i, \mathfrak{g})\}_i$ and of smooth function $\{g_{ij} \in C^\infty(U_i \cap U_j, G)\}$, such that the formula

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij}$$

for gauge transformation holds on each double intersection $U_i \cap U_j$. This is almost the data defining a $\mathfrak{g}$-connecton on a G-principal bundle $P \to X$, but not quite yet, since it does not yet constrain the transition functions g_{ij} on the triple intersections $U_i \cap U_j \cap U_k$ to obey the cocycle relation $g_{ij} g_{jk} = g_{ik}$. But since each g_{ij} is the parallel transport of our connection along a vertical 1-simplex, the cocycle condition precisely says that parallel transport along the three edges of a vertical 2-simplex is trivial, i.e., that the vertical parts of our connection forms on $U_{ijk} \times \Delta^1$ are the boundary data of a connection form on $U_{ijk} \times \Delta^2$ which is *flat* in the vertical direction. In other words, the collection of commutative diagrams (2) is to be seen as the 0 and 1-simplices of a simplicial set whose 2-simplices are the commutative diagrams

$$\begin{array}{ccc} \Omega^\bullet(\Delta^2 \times U_{ijk})_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(\Delta^2 \times U_{ijk}) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U_{ijk}) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) \end{array}.$$

Having added 2-simplices to our picture, we have finally recovered the standard description of connections in terms of local differential form data. By suitably replacing Lie algebras with L_∞-algebras in this derivation, we will obtain a definition of connections on higher bundles in Section 4.2. As one can expect, in the simplicial description of connections on higher bundles, simplices of arbitrarily high dimension will appear.

3 Smooth ∞-groupoids

In this section, we introduce a central concept that we will be dealing with in this paper, smooth ∞-groupoids, as a natural generalization of the classical notion of Lie groups.

A Lie groupoid is, by definition, a groupoid internal to the category of smooth spaces and smooth maps. It is a widely appreciated fact in Lie groupoid theory that many features of Lie groupoids can be usefully thought of in terms of their associated groupoid-valued presheaves on the category of manifolds, called the *differentiable stack* represented by the Lie groupoid. This is the perspective that immediately generalizes to higher groupoids.

Since many naturally appearing smooth spaces are not manifolds — particularly the spaces $[\Sigma, X]$ of smooth maps $\Sigma \to X$ between two manifolds — for the development of the general theory it is convenient to adopt a not too strict notion of 'smooth space' . This generalized notion will have to be more flexible than the notion of manifold but at the same time not too far from that. The basic example to have in mind is the following: every smooth manifold X of course represents a sheaf

$$X : \mathrm{SmoothManifolds}^{\mathrm{op}} \to \mathrm{Sets}$$
$$U \mapsto C^\infty(U, X).$$

on the category of smooth manifolds. But since manifolds themselves are by definition glued from Cartesian spaces $\mathbb{R}^n$, all the information about X is in fact already encoded in the restriction of this sheaf to the category of Cartesian spaces and smooth maps between them:

$$X : \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Sets}.$$

Now notice that also the spaces $[\Sigma, X]$ of smooth maps $\Sigma \to X$ between two manifolds naturally exist as sheaves on CartSp, given by the assignment

$$[\Sigma, X] : U \mapsto C^\infty(\Sigma \times U, X).$$

Sheaves of this form are examples of generalized smooth spaces that are known as *diffeological spaces* or *Chen smooth spaces*. While not manifolds, these smooth spaces do have an underlying topological space and behave like smooth manifolds in many essential ways.

Even more generally, we will need to consider also ‘smooth spaces’ that do not have even an underlying topological space. The central example of such is the sheaf of (real valued) closed differential n-forms

$$U \mapsto \Omega^n_{cl}(U),$$

which we will need to consider later in the paper. We may think of these as modelling a kind of smooth Eilenberg–MacLane space that support a single (up to scalar multiple) smooth closed n-form. A precise version of this statement will play a central role later in the theory of Lie ∞-integration that we will describe in Section 4.

Thus we see that the common feature of generalized smooth spaces is not that they are *representable* in one way or other. Rather, the common feature is that they all define sheaves on the category of the archetypical smooth spaces: the Cartesian spaces. This is a special case of an old insight going back to Grothendieck, Lawvere and others: with a category $\mathcal{C}$ of test spaces fixed, the correct context in which to consider generalized spaces modelled on $\mathcal{C}$ is the category $\mathrm{Sh}(\mathcal{C})$ of *all* sheaves on $\mathcal{C}$: the sheaf topos [29]. In there we may find a hierarchy of types of generalized spaces ranging from ones that are very close to being like these test spaces, to ones that are quite a bit more general. In applications, it is good to find models as close as possible to the test spaces, but for the development of the theory it is better to admit them all.

Now if the manifold X happens, in addition, to be equipped with the structure of a Lie group G, then it represents more than just an ordinary sheaf of sets: from each group we obtain a simplicial set, its *nerve*, whose set of k-cells is the set of k-tuples of elements in the group, and whose face and degeneracy maps are built from the product operation and the neutral element in the group. Since, for every $U \in \mathrm{CartSp}$, also the set of functions $C^\infty(U, G)$ forms a group, this means that from a Lie group we obtain a *simplicial presheaf*

$$\mathbf{B}G : U \mapsto \left\{ \cdots \rightrightarrows C^\infty(U, G \times G) \rightrightarrows C^\infty(U, G) \rightrightarrows * \, , \right\},$$

where the degeneracy maps have not been displayed in order to make the diagram more readable.

The simplicial presheaves arising this way are, in fact, special examples of presheaves taking values in *Kan complexes*, i.e., in simplicial sets in which every *horn* — a simplex minus its interior and minus one face — has a completion to a simplex; see for instance [23] for a review. It turns out (see Section 1.2.5 of [34]) that Kan complexes may be thought of as modelling ∞-groupoids: the generalization of groupoids where one has not only morphisms between objects, but also 2-morphisms between morphisms and generally $(k+1)$-morphisms between k-morphisms for all $k \in \mathbb{N}$. The traditional theory of Lie groupoids may be thought of as dealing with those simplicial presheaves on CartSp that arise from nerves of Lie groupoids in the above manner

This motivates the definitions that we now turn to.

3.1 Presentation by simplicial presheaves

Definition 3.1.1. A *smooth ∞-groupoid A* is a simplicial presheaf on the category CartSp of Cartesian spaces and smooth maps between them such that, over each $U \in \mathrm{CartSp}$, A is a Kan complex.

Much of ordinary Lie theory lifts from Lie groups to this context. The reader is asked to keep in mind that smooth ∞-groupoids are objects whose smooth structure may be considerably more general than that of a Kan complex internal to smooth manifolds, i.e., of a simplicial smooth manifold satisfying a horn filling condition. Kan complexes internal to smooth manifolds, such as for instance nerves of ordinary Lie groupoids, can be thought of as representable smooth ∞-groupoids.

Example 3.1.2. The basic example of a representable smooth ∞-groupoids are ordinary Lie groupoids; in particular smooth manifolds and Lie groups are smooth ∞-groupoids. A particularly important example of representable smooth ∞-groupoid is the *Čech ∞-groupoid*: for X a smooth manifold and $\mathcal{U} = \{U_i \to X\}$ an open cover, there is the simplicial manifold

$$\check{C}(\mathcal{U}) := \left\{ \cdots \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \{U_{ijk}\} \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \{U_{ij}\} \; \substack{\longrightarrow\\[-1ex]\longrightarrow} \{U_i\} \right\}$$

which in degree k is the disjoint union of the k-fold intersections $U_i \cap U_j \cap \cdots$ of open subsets (the degeneracy maps are not depicted). This is a Kan complex internal to smooth manifolds in the evident way.

While the notion of simplicial presheaf itself is straightforward, the correct concept of morphism between them is more subtle: we need a notion of morphisms such that the resulting category — or ∞-category as it were — of our smooth ∞-groupoids reflects the prescribed notion of gluing of test objects. In fancier words, we want simplicial presheaves to be equivalent to a higher analog of a sheaf topos: an ∞-topos [34]. This may be achieved by equipping the naive category of simplicial presheaves with a *model category structure* [27]. This provides the information as to which objects in the category are to be regarded as equivalent, and how to resolve objects by equivalent objects for purposes of mapping between them. There are some technical aspects to this this that we have relegated to the appendix. For all details and proofs of the definitions and propositions, respectively, in the remainder of this section see there.

Definition 3.1.3. Write $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ for the global projective model category structure on simplicial presheaves: weak equivalences and fibrations are objectwise those of simplicial sets.

This model structure presents the ∞-category of ∞-*presheaves* on CartSp. We impose now an ∞-sheaf condition.

Definition 3.1.4. Write $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ for the left Bousfield localization (see for instance Section A.3 of [34]) of $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ at the set of all Čech nerve projections $\check{C}(\mathcal{U}) \to U$ for $\mathcal{U}$ a differentiably good open cover of U, i.e., an open cover $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of U such that for all $n \in \mathbb{N}$ every n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ is either empty or *diffeomorphic* to $\mathbb{R}^{\dim U}$.

This is the model structure that presents the ∞-category of ∞-*sheaves* or ∞-*stacks* on CartSp. By standard results, it is a simplicial model category with respect to the canonical simplicial enrichment of simplicial presheaves, see [15]. For X, A two simplicial presheaves, we write

- $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](X, A) \in \mathrm{sSet}$ for the simplicial hom-complex of morphisms;
- $\mathbf{H}(X, A) := [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](Q(X), P(A))$ for the right derived hom-complex (well defined up to equivalence) where $Q(X)$ is any local cofibrant resolution of X and $P(A)$ any local fibrant resolution of A.

Notice some standard facts about left Bousfield localization:

- every weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is also a weak equivalence in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$;
- the classes of cofibrations in both model structures coincide.
- the fibrant objects of the local structure are precisely the objects that are fibrant in the global structure and in addition satisfy *descent* over all differentiably good open covers of Cartesian spaces. What this means precisely is stated in corollary A.2 in the appendix.

- the localization right Quillen functor

$$\mathrm{Id} : [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}} \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$$

  presents ∞-sheafification, which is a left adjoint left exact ∞-functor [34], therefore all homotopy colimits and all finite homotopy limits in the local model structure can be computed in the global model structure.

In particular, notice that an acyclic fibration in the global model structure will not, in general, be an acyclic fibration in the local model structure; nevertheless, it will be a weak equivalence in the local model structure.

Definition 3.1.5. We write

- $\xrightarrow{\simeq}$ for isomorphisms of simplicial presheaves;
- $\xrightarrow{\sim}$ for weak equivalences in the global model structure;
- $\xrightarrow{\sim_{\mathrm{loc}}}$ for weak equivalences in the local model structure;

(Notice that each of these generalizes the previous.)

- $\twoheadrightarrow$ for fibrations in the global model structure.

We do not use notation for fibrations in the local model structure.

Since the category CartSp has fewer objects than the category of all manifolds, we have that the conditions for simplicial presheaves to be fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ are comparatively weak. For instance

$$\mathbf{B}G : U \mapsto N((C^\infty(U, G) \rightrightarrows *)$$

is locally fibrant over CartSp but not over the site of all manifolds. This is discussed below in Section 3.2. Conversely, the condition to be cofibrant is stronger over CartSp than it is over all manifolds. But by a central result by [15], we have fairly good control over cofibrant resolutions: these include notably Čech nerves $\check{C}(\mathcal{U})$ of *differentiably good* open covers, i.e., the Čech nerve $\check{C}(\mathcal{U}) \to X$ of a differentiably good open cover over a paracompact smooth manifold X is a cofibrant resolution of X in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$, and so we write

$$\check{C}(\mathcal{U}) \xrightarrow{\sim_{\mathrm{loc}}} X.$$

Note that in the present paper these will be the only local weak equivalences that are not global weak equivalences that we need to consider.

In the practice of our applications, all this means that much of the technology hidden in Definition 3.1.4 boils down to a simple algorithm: after solving the comparatively easy tasks of finding a version A of a given smooth ∞-groupoid that is fibrant over CartSp, for describing morphisms of smooth ∞-groupoids from a manifold X to A, we are to choose a differentiably good open cover $\mathcal{U} = \{U_i \to X\}$, form the Čech nerve simplicial presheaf $\check{C}(\mathcal{U})$ and then consider spans of ordinary morphisms of simplicial presheaves of the form

$$\begin{array}{ccc} \check{C}(\mathcal{U}) & \xrightarrow{g} & A \, . \\ \big\downarrow {\scriptstyle \simeq_{\mathrm{loc}}} & & \\ X & & \end{array}$$

Such a diagram of simplicial presheaves presents an object in $\mathbf{H}(X, A)$, in the hom-space of the ∞-topos of smooth ∞-groupoids. As discussed below in Section 3.2, here the morphisms g are naturally identified with cocycles in non-abelian Čech cohomology on X with coefficients in A. In Section 4.2, we discus that we may also think of these cocycles as transition data for A-principal ∞-bundles on X [44].

For discussing the ∞-Chern–Weil homomorphism, we are crucially interested in composites of such spans: a *characteristic map* on a coefficient object A is nothing but a morphism $\mathbf{c} : A \to B$ in the ∞-topos, presented itself by a span

$$\begin{array}{ccc} \hat{A} & \longrightarrow & B \, . \\ \big\downarrow {\scriptstyle \wr} & & \\ A & & \end{array}$$

The evaluation of this characteristic map on the A-principal bundle on X encoded by a cocycle $g : \check{C}(\mathcal{U}) \to A$ is the composite morphism $X \to A \to B$ in the ∞-topos, which is presented by the composite span of simplicial presheaves

$$\begin{array}{ccccc} QX & \longrightarrow & \hat{A} & \longrightarrow & B \, . \\ \big\downarrow {\scriptstyle \wr} & & \big\downarrow {\scriptstyle \wr} & & \\ \check{C}(\mathcal{U}) & \longrightarrow & A & & \\ \big\downarrow {\scriptstyle \simeq_{\mathrm{loc}}} & & & & \\ X & & & & \end{array}$$

Here $QX \to \check{C}(\mathcal{U})$ is the pullback of the acyclic fibration $\hat{A} \to A$, hence itself an acyclic fibration; moreover, since $\check{C}(\mathcal{U})$ is cofibrant, we are guaranteed that a section $\check{C}(\mathcal{U}) \to QX$ exists and is unique up to homotopy. Therefore the composite morphism $X \to A \to B$ is encoded in a cocylce $\check{C}(\mathcal{U}) \to B$ as in the diagram below:

$$
\begin{array}{ccccc}
QX & \longrightarrow & \hat{A} & \longrightarrow & B\,. \\
\uparrow\!\!\!\big(\;\; \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} & & \\
\check{C}(\mathcal{U}) & \longrightarrow & A & & \\
\downarrow{\scriptstyle \wr_{\mathrm{loc}}} & & & & \\
X & & & &
\end{array}
$$

Our main theorems will involve the construction of such span composites.

3.2 Examples

In this paper we will consider three main sources of smooth ∞-groupoids

- Lie groups and Lie groupoids, leading to Kan complexes via their nerves; examples of this kind will be the smooth ∞-groupoid $\mathbf{B}G$ associated with a Lie group G, and its refinements $\mathbf{B}G_{\mathrm{diff}}$ and $\mathbf{B}G_{\mathrm{conn}}$;
- complexes of abelian groups concentrated in non-negative degrees, leading to Kan complexes via the Dold–Kan (DK) correspondence; examples of this kind will be the smooth ∞-groupoid $\mathbf{B}^n U(1)$ associated with the chain complex of abelian groups consisting in $U(1)$ concentrated in degree n, and its refinements $\mathbf{B}^n U(1)_{\mathrm{diff}}$ and $\mathbf{B}^n U(1)_{\mathrm{conn}}$;
- Lie algebras and L_∞-algebras, via flat connections over simplices; this construction will produce, for any Lie or L_∞-algebra $\mathfrak{g}$, a smooth ∞-groupoid $\exp_\Delta(\mathfrak{g})$ integrating $\mathfrak{g}$; other examples of this kind are the refinements $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ and $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ of $\exp_\Delta(\mathfrak{g})$.

In the following sections, we will investigate these examples and show how they naturally combine in ∞-Chern–Weil theory.

Smooth ∞-groups. With a useful notion of *smooth ∞-groupoids* and their morphisms thus established, we automatically obtain a good notion of *smooth ∞-groups.* This is accomplished simply by following the general principle by which essentially all basic constructions and results familiar from classical homotopy theory lift from the archetypical ∞-topos Top of (compactly generated) topological spaces (or, equivalently, of *discrete*

∞-groupoids) to any other ∞-topos, such as our ∞-topos $\mathbf{H}$ of smooth ∞-groupoids.

Namely, in classical homotopy theory a *monoid* up to higher coherent homotopy is a topological space $X \in \mathrm{Top} \simeq \infty\mathrm{Grpd}$ equipped with A_∞-structure [45] or, equivalently, an E_1-structure, i.e., a homotopical action of the little 1-cubes operad [36]. A *groupal* A_∞-space — an ∞-*group* — is one where this homotopy-associative product is invertible, up to homotopy. Famously, *May's recognition theorem* identifies such ∞-groups as being precisely, up to weak homotopy equivalence, loop spaces. This establishes an equivalence of *pointed* connected spaces with ∞-groups, given by looping Ω and delooping B:

$$\infty\mathrm{Grp} \underset{B}{\overset{\Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \infty\mathrm{Grpd}_* \, .$$

Lurie shows in Section 6.1.2 of [34] (for ∞-groups) and in theorem 5.1.3.6 of [35] that these classical statements have direct analogues in any ∞-topos. We are thus entitled to think of any (pointed) connected smooth ∞-groupoid X as the delooping $\mathbf{B}G$ of a smooth ∞-group $G \simeq \Omega X$

$$\mathrm{Smooth}\infty\mathrm{Grp} \underset{\mathbf{B}}{\overset{\Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathbf{H}_* = \mathrm{Smooth}\infty\mathrm{Grpd}_* \, ,$$

where we use boldface $\mathbf{B}$ to indicate that the delooping takes place in the ∞-topos $\mathbf{H}$ of smooth ∞-groupoids.

The most basic example for this, we have already seen above: for G any Lie group the Lie groupoid $\mathbf{B}G$ described above is precisely the delooping of G — not in Top but in our $\mathbf{H}$.

In this paper most smooth ∞-groups G appear in the form of their smooth delooping ∞-groupoids $\mathbf{B}G$. Apart from Lie groups, the main examples that we consider will be the higher line and circle Lie groups $\mathbf{B}^n U(1)$ and $\mathbf{B}^n\mathbb{R}$ that have arbitrary many delooping, as well as the non-abelian smooth 2-group String and the non-abelian smooth 6-group Fivebrane, which are smooth refinements of the higher connected covers of the Spin-group.

3.2.1 $\mathbf{B}G$, $\mathbf{B}G_{\mathrm{conn}}$ and principal G-bundles with connection

The standard example of a stack on manifolds is the classifying stack $\mathbf{B}G$ for G-principal bundles with G a Lie group. As an illustration of our setup, we describe what this looks like in terms of simplicial presheaves over the site CartSp. Then, we discuss its differential refinements $\mathbf{B}G_{\mathrm{diff}}$ and $\mathbf{B}G_{\mathrm{conn}}$.

Definition 3.2.1. Let G be a Lie group. The smooth ∞-groupoid $\mathbf{B}G$ is defined to associate to a Cartesian space U the nerve of the action groupoid $*//C^\infty(U,G)$, i.e., of the one-object groupoid with $C^\infty(U,G)$ as its set of morphisms and composition given by the product of G-valued functions.

Remark 3.2.2. Often this object is regarded over the site of all manifolds, where it is just a pre-stack, hence not fibrant. Its fibrant replacement over that site is the stack $G\mathrm{Bund} : \mathrm{Manfd}^{\mathrm{op}} \to \mathrm{Grpd}$ that sends a manifold to the groupoid of G-principal bundles over it. We may think instead of $\mathbf{B}G$ as sending a space to just the *trivial* G-principal bundle and its automorphisms. But since the site of Cartesian spaces is smaller, we have:

Proposition 3.2.3. *The object* $\mathbf{B}G \in [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *is fibrant.*

On the other hand, over the site of manifolds, every manifold itself is cofibrant. This means that to compute the groupoid of G-bundles on a manifold X in terms of morphisms of stacks over all manifolds, one usually passes to the fibrant replacement $G\mathrm{Bund}$ of $\mathbf{B}G$, then considers $\mathrm{Hom}(X, G\mathrm{Bund})$ and uses the 2-Yoneda lemma to identify this with the groupoid $G\mathrm{Bund}(X)$ of principal G-bundles on X. When working over CartSp instead, the situation is the opposite: here $\mathbf{B}G$ is already fibrant, but the manifold X is in general no longer cofibrant! To compute the groupoid of G-bundles on X, we pass to a cofibrant replacement of X given according to Proposition 2 by the Čech nerve $\check{C}(\mathcal{U})$ of a differentiably good open cover and then compute $\mathrm{Hom}_{[\mathrm{CartSp}^{\mathrm{op}},\mathrm{sSet}]}(\check{C}(\mathcal{U}), \mathbf{B}G)$. To see that the resulting groupoid is again equivalent to $G\mathrm{Bund}(X)$ (and hence to prove the above proposition by taking $X = \mathbb{R}^n$) one proceeds as follows:

The object $\check{C}(\mathcal{U})$ is equivalent to the homotopy colimit in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ over the simplicial diagram of its components

$$\begin{aligned}\check{C}(\mathcal{U}) &\simeq \mathrm{hocolim}\left(\cdots \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \coprod_{i,j,k} U_{ijk} \substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow} \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \right) \\ &\simeq \int^{[k]\in\Delta} \Delta[k] \cdot \coprod_{i_0,\ldots,i_k} U_{i_0,\ldots,i_k}.\end{aligned}$$

(Here in the middle we are notationally suppressing the degeneracy maps for readability and on the second line we display for the inclined reader the formal coend expression that computes this homotopy colimit as a weighted colimit [27]. The dot denotes the tensoring of simplicial presheaves over simplicial sets). Accordingly $\mathrm{Hom}(\check{C}(\mathcal{U}), \mathbf{B}G)$ is the homotopy limit

$$\begin{aligned}
&\mathrm{Hom}(\check{C}(\mathcal{U}), \mathbf{B}G) \\
&\quad \simeq \mathrm{holim}\left(\cdots \Lleftarrow \{\mathbf{B}G(U_{ijk})\} \Lleftarrow \{\mathbf{B}G(U_{ij})\} \Leftarrow \{\mathbf{B}G(U_i)\} \right) \\
&\quad \simeq \int_{[k]\in\Delta} \mathrm{Hom}\left(\Delta[k], \prod_{i_0,\ldots,i_k} \mathbf{B}G\left(U_{i_0,\ldots,i_k}\right) \right)
\end{aligned}.$$

The last line tells us that an element $g : \check{C}(\mathcal{U}) \to \mathbf{B}G$ in this Kan complex is a diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow\uparrow\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow\uparrow\uparrow \\
\Delta[2] & \xrightarrow{g^{(2)}} & \prod_{i,j,k} \mathbf{B}G(U_{ijk}) \\
\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow \\
\Delta[1] & \xrightarrow{g^{(1)}} & \prod_{i,j} \mathbf{B}G(U_{ij}) \\
\uparrow\uparrow & & \uparrow\uparrow \\
\Delta[0] & \xrightarrow{g^{(0)}} & \prod_{i} \mathbf{B}G(U_{i})
\end{array}$$

of simplicial sets. This is a collection $(\{g_i\}, \{g_{ij}\}, \{g_{ijk}\}, \ldots)$, where

- g_i is a vertex in $\mathbf{B}G(U_i)$;
- g_{ij} is an edge in $\mathbf{B}G(U_{ij})$;
- g_{ijk} is a 2-simplex in $\mathbf{B}G(U_{ijk})$
- etc.

such that the kth face of the n-simplex in $\mathbf{B}G(U_{i_0,\ldots,i_n})$ is the image of the $(n-1)$-simplex under the kth face inclusion $\mathbf{B}G(U_{i_0,\ldots,\hat{i}_k,\ldots,i_n}) \to \mathbf{B}G$ $(U_{i_0,\ldots,i_n})$. (And similarly for the coface maps, which we continue to disregard for brevity.) This means that an element $g : \check{C}(\mathcal{U}) \to \mathbf{B}G$ is precisely an element of the set $\check{C}(\mathcal{U}, \mathbf{B}G)$ of *non-abelian Čech cocycles* with coefficients in $\mathbf{B}G$. Specifically, by definition of $\mathbf{B}G$, this reduces to

- a collection of smooth maps $g_{ij} : U_{ij} \to G$, for every pair of indices i, j;
- the constraint $g_{ij}g_{jk}g_{ki} = 1_G$ on U_{ijk}, for every i, j, k (the *cocycle* constraint).

These are manifestly the data of transition functions defining a principal G-bundle over X.

Similarly working out the morphisms (i.e., the 1-simplices) in $\mathrm{Hom}(\check{C}(\mathcal{U}), \mathbf{B}G)$, we find that their components are collections $h_i : U_i \to G$ of smooth functions, such that $g'_{ij} = h_i^{-1} g_{ij} h_j$. These are precisely the gauge transformations between the G-principal bundles given by the transition functions $(\{g_{ij}\})$ and $(\{g'_{ij}\})$. Since the cover $\{U_i \to X\}$ is *good*, it follows that we have indeed reproduced the groupoid of G-principal bundles

$$\mathrm{Hom}(\check{C}(\mathcal{U}), \mathbf{B}G) = \check{C}(\mathcal{U}, \mathbf{B}G) \simeq G\mathrm{Bund}(X).$$

Two cocycles define isomorphic principal G-bundles precisely when they define the same element in Čech cohomology with coefficients in the sheaf of smooth functions with values in G. Thus we recover the standard fact that isomorphism classes of principal G-bundles are in natural bijection with $H^1(X, G)$.

We now consider a differential refinement of $\mathbf{B}G$.

Definition 3.2.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The smooth ∞-groupoid $\mathbf{B}G_{\mathrm{conn}}$ is defined to associate with a Cartesian space U the nerve of the action groupoid $\Omega^1(U, \mathfrak{g})//C^\infty(U, G)$.

This is over U the groupoid $\mathbf{B}G_{\mathrm{conn}}(U)$

- whose set of objects is the set of smooth $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(U, \mathfrak{g})$;
- whose morphisms $g : A \to A'$ are labelled by smooth functions $g \in C^\infty(U, G)$ such that they relate the source and target by a *gauge transformation*

$$A' = g^{-1} A g + g^{-1} dg,$$

  where $g^{-1}Ag$ denotes pointwise the adjoint action of G on $\mathfrak{g}$ and $g^{-1}dg$ is the pullback $g^*(\theta)$ of the Maurer–Cartan form $\theta \in \Omega^1(G, \mathfrak{g})$.

With X and $\check{C}(\mathcal{U})$ as before we now have:

Proposition 3.2.5. *The smooth ∞-groupoid $\mathbf{B}G_{\mathrm{conn}}$ is fibrant and there is a natural equivalence of groupoids*

$$\mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}}) \simeq G\mathrm{Bund}_{\mathrm{conn}}(X),$$

where on the right we have the groupoid of G-principal bundles on X equipped with connection.

This follows along the above lines, by unwinding the nature of the simplicial hom-set $\check{C}(\mathcal{U}, \mathbf{B}G_{\mathrm{conn}}) := \mathrm{Hom}(\check{C}(\mathcal{U}), \mathbf{B}G_{\mathrm{conn}})$ of non-abelian Čech cocycles with coefficients in $\mathbf{B}G_{\mathrm{conn}}$. Such a cocycle is a collection $(\{A_i\}, \{g_{ij}\})$ consisting of

- a 1-form $A_i \in \Omega^1(U_i, \mathfrak{g})$ for each index i;
- a smooth function $g_{ij} : U_{ij} \to G$, for all indices i, j;
- the gauge action constraint $A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij}$ on U_{ij}, for all indices i, j;
- the cocycle constraint $g_{ij} g_{jk} g_{ki} = 1_G$ on U_{ijk}, for all indices i, j, k.

These are readily seen to be the data defining a $\mathfrak{g}$-connection on a principal G-bundle over X.

Notice that there is an evident "forget the connection" -morphism $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$, given over $U \in \mathrm{CartSp}$ by

$$(A \xrightarrow{g} A') \mapsto (\bullet \xrightarrow{g} \bullet).$$

We denote the set of isomorphism classes of principal G-bundles with connection by the symbol $H^1(X, G)_{\mathrm{conn}}$. Thus, we obtain a morphism

$$H^1(X, G)_{\mathrm{conn}} \to H^1(X, G).$$

Finally, we introduce a smooth ∞-groupoid $\mathbf{B}G_{\mathrm{diff}}$ in between $\mathbf{B}G$ and $\mathbf{B}G_{\mathrm{conn}}$. This may seem a bit curious, but we will see in Section 4.3 how it is the degree one case of a completely natural and noteworthy general construction. Informally, $\mathbf{B}G_{\mathrm{diff}}$ is obtained from $\mathbf{B}G$ by freely decorating the vertices of the simplices in $\mathbf{B}G$ by elements in $\Omega^1(U, \mathfrak{g})$. More formally, we have the following definition.

Definition 3.2.6. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The smooth ∞-groupoid $\mathbf{B}G_{\mathrm{diff}}$ is defined to associate with a Cartesian space U the nerve of the groupoid

(1) Whose set of objects is $\Omega^1(U, \mathfrak{g})$;
(2) A morphism $A \xrightarrow{(g,a)} A'$ is labelled by $g \in C^\infty(U, G)$ and $a \in \Omega^1(U, \mathfrak{g})$ such that

$$A = g^{-1} A' g + g^{-1} dg + a;$$

(3) Composition of morphisms is given by

$$(g, a) \circ (h, b) = \left(gh, h^{-1} a h + h^{-1} dh + b\right).$$

Remark 3.2.7. This definition intentionally carries an evident redundancy: given any A, A' and g the element a that makes the above equation hold does exist uniquely; the 1-form a measures the failure of g to constitute a morphism from A to A' in $\mathbf{B}G_{\mathrm{conn}}$. We can equivalently express the redundancy of a by saying that there is a natural isomorphism between $\mathbf{B}G_{\mathrm{diff}}$ and the direct product of $\mathbf{B}G$ with the codiscrete groupoid on the sheaf of sets $\Omega^1(-;\mathfrak{g})$.

Proposition 3.2.8. *The evident forgetful morphism $\mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$ factors through $\mathbf{B}G_{\mathrm{diff}}$ by a monomorphism followed by an acyclic fibration (in the global model structure)*

$$\mathbf{B}G_{\mathrm{conn}} \hookrightarrow \mathbf{B}G_{\mathrm{diff}} \overset{\sim}{\twoheadrightarrow} \mathbf{B}G.$$

3.2.2 $\mathbf{B}G_2$, and non-abelian gerbes and principal 2-bundles

We now briefly discuss the first case of G-principal ∞-bundles after ordinary principal bundles, the case where G is a *Lie 2-group*: G-principal 2-bundles.

When $G = \mathrm{AUT}(H)$ the *automorphism 2-group* of a Lie group H (see below) these structures have the same classification (though are conceptually somewhat different from) the smooth version of the H-banded *gerbes* of [22] (see around Definition 7.2.2.20 in [34] for a conceptually clean account in the modern context of higher toposes): both are classified by the *non-abelian cohomology* $H^1_{\mathrm{Smooth}}(-, \mathrm{AUT}(H))$ with coefficients in that 2-group. But the main examples of 2-groups that we shall be interested in, namely *string 2-groups*, are not equivalent to $\mathrm{AUT}(H)$ for any H, hence the 2-bundles considered here are strictly more general than Giraud's gerbes. The literature knows what has been called *non-abelian bundle gerbes*, but despite their name these are not Giraud's gerbes, but are instead models for the total spaces of what we call here *principal 2-bundles.* A good discussion of the various equivalent incarnations of principal 2-bundles is in [39].

To start with, note the general abstract notion of smooth 2-groups:

Definition 3.2.9. A *smooth 2-group* is a 1-truncated group object in $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{CartSp})$. These are equivalently given by their (canonically pointed) delooping 2-groupoids $\mathbf{B}G \in \mathbf{H}$, which are precisely, up to equivalence, the connected 2-truncated objects of $\mathbf{H}$.

For $X \in \mathbf{H}$ any object, $G2\mathrm{Bund}_{\mathrm{smooth}}(X) := \mathbf{H}(X, \mathbf{B}G)$ is the 2-groupoid of smooth G-principal 2-bundles on G.

While nice and abstract, in applications one often has — or can get — hold of a *strict model* of a given smooth 2-group. The following definitions can be found recalled in any reference on these matters, for instance in [39].

Definition 3.2.10. (1) A smooth *crossed module* of Lie groups is a pair of homomorphisms $\partial : G_1 \to G_0$ and $\rho : G_0 \to \mathrm{Aut}(G_1)$ of Lie groups, such that for all $g \in G_0$ and $h, h_1, h_2 \in G_1$, we have $\rho(\partial h_1)(h_2) = h_1 h_2 h_1^{-1}$ and $\partial\rho(g)(h) = g\partial(h)g^{-1}$.

(2) For $(G_1 \to G_0)$ a smooth crossed module, the corresponding *strict Lie 2-group* is the smooth groupoid $G_0 \times G_1 \rightrightarrows G_0$, whose source map is given by projection on G_0, whose target map is given by applying ∂ to the second factor and then multiplying with the first in G_0, and whose composition is given by multiplying in G_1.

This groupoid has a strict monoidal structure with strict inverses given by equipping $G_0 \times G_1$ with the semidirect product group structure $G_0 \ltimes G_1$ induced by the action ρ of G_0 on G_1.

(3) The corresponding one-object strict smooth 2-groupoid we write $\mathbf{B}$ $(G_1 \to G_0)$. As a simplicial object (under Duskin nerve of 2-categories) this is of the form

$$\mathbf{B}(G_1 \to G_0) = \mathrm{cosk}_3 \left(G_0^{\times 3} \times G_1^{\times 3} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} G_0^{\times 2} \times G_1 \rightrightarrows G_0 \longrightarrow * \right).$$

Examples.

(1) For A any abelian Lie group, $A \to 1$ is a crossed module. Conversely, for A any Lie group $A \to 1$ is a crossed module precisely if A is abelian. We write $\mathbf{B}^2 A = \mathbf{B}(A \to 1)$. This case and its generalizations is discussed below in Section 3.2.3.

(2) For H any Lie group with automorphism Lie group $\mathrm{Aut}(H)$, the morphism $H \overset{\mathrm{Ad}}{\to} \mathrm{Aut}(H)$ that sends group elements to inner automorphisms, together with $\rho = \mathrm{id}$, is a crossed module. We write $\mathrm{AUT}(H) := (H \to \mathrm{Aut}(H))$ and speak of the *automorphism 2-group* of H, because this is $\simeq \mathrm{Aut}_{\mathbf{H}}(\mathbf{B}H)$.

(3) For G an ordinary Lie group and $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3 U(1)$ a morphism in $\mathbf{H}$ (see Section 3.2.3 for a discussion of $\mathbf{B}^n U(1)$), its homotopy fibre $\mathbf{B}\hat{G} \to \mathbf{B}G$ is the delooping of a smooth 2-group $\hat{G}$. If G is compact, simple and simply connected, then this is equivalent ([41], Section 5.1) to a strict 2-group $(\hat{\Omega} G \to PG)$ given by a $U(1)$-central extension of the loop group of G, as described in [8]. This is called the *string 2-group* extension of G by $\mathbf{c}$. We come back to this in Section 5.1.

Observation 3.2.11. *For every smooth crossed module, its delooping object* $\mathbf{B}(G_1 \to G_0)$ *is fibrant in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$.

Proof. Since $(G_1 \to G_0)$ induces a strict 2-group, there are horn fillers defined by the smooth operations in the 2-group: we can always solve for the missing face in a horn in terms of an expression involving the smooth composite-operations and inverse-operations in the 2-group. □

Proposition 3.2.12. *Suppose that the smooth crossed module $(G_1 \to G_0)$ is such that the quotient $\pi_0 G = G_0/G_1$ is a smooth manifold and the projection $G_0 \to G_0/G_1$ is a submersion.*

Then $\mathbf{B}(G_1 \to G_0)$ is fibrant also in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Proof. We need to show that for $\{U_i \to \mathbb{R}^n\}$ a good open cover, the canonical descent morphism

$$B\left(C^\infty(\mathbb{R}^n, G_1) \to C^\infty(\mathbb{R}^n, G_0)\right) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U}), \mathbf{B}(G_1 \to G_0)\right)$$

is a weak homotopy equivalence. The main point to show is that, since the Kan complex on the left is connected by construction, also the Kan complex on the right is.

To that end, notice that the category CartSp equipped with the open cover topology is a *Verdier site* in the sense of Section 8 of [13]. By the discussion there it follows that every hypercover over $\mathbb{R}^n$ can be refined by a split hypercover, and these are cofibrant resolutions of $\mathbb{R}^n$ in both the global and the local model structure $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$. Since also $\check{C}(\mathcal{U}) \to \mathbb{R}^n$ is a cofibrant resolution and since $\mathbf{B}G$ is fibrant in the *global* structure by observation 3.2.11, it follows from the existence of the global model structure that morphisms out of $\check{C}(\mathcal{U})$ into $\mathbf{B}(G_1 \to G_0)$ capture all cocycles over any hypercover over $\mathbb{R}^n$, hence that

$$\pi_0[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U}), \mathbf{B}(G_1 \to G_0)\right) \simeq H^1_{\mathrm{smooth}}\left(\mathbb{R}^n, (G_1 \to G_0)\right)$$

is the standard Čech cohomology of $\mathbb{R}^n$, defined as a colimit over refinements of covers of equivalence classes of Čech cocycles.

Now by Proposition 4.1 of [39] (which is the smooth refinement of the statement of [7] in the continuous context) we have that under our assumptions on $(G_1 \to G_0)$ there is a topological classifying space for this smooth Čech cohomology set. Since $\mathbb{R}^n$ is topologically contractible, it follows that this is the singleton set and hence the above descent morphism is indeed an isomorphism on π_0.

Next we can argue that it is also an isomorphism on π_1, by reducing to the analogous local trivialization statement for ordinary principal bundles:

a loop in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\big(\check{C}(\mathcal{U}), \mathbf{B}(G_1 \to G_0)\big)$ on the trivial cocycle is readily seen to be a $G_0//(G_0 \ltimes G_1)$-principal groupoid bundle, over the action groupoid as indicated. The underlying $G_0 \ltimes G_1$-principal bundle has a trivialization on the contractible $\mathbb{R}^n$ (by classical results or, in fact, as a special case of the previous argument), and so equivalence classes of such loops are given by G_0-valued smooth functions on $\mathbb{R}^n$. The descent morphism exhibits an isomorphism on these classes.

Finally the equivalence classes of spheres on both sides are directly seen to be smooth $\ker(G_1 \to G_0)$-valued functions on both sides, identified by the descent morphism. □

Corollary 3.2.13. *For $X \in \mathrm{SmoothMfd} \subset \mathbf{H}$ a paracompact smooth manifold, and $(G_1 \to G_0)$ as above, we have for any good open cover $\{U_i \to X\}$ that the 2-groupoid of smooth $(G_1 \to G_0)$-principal 2-bundles is*

$$\begin{aligned}&(G_1 \to G_0)\mathrm{Bund}(X)\\ &\quad := \mathbf{H}(X, \mathbf{B}(G_1)) \simeq [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\big(\check{C}(\mathcal{U}), \mathbf{B}(G_1 \to G_0)\big)\end{aligned}$$

and its set of connected components is naturally isomorphic to the non-abelian Čech cohomology

$$\pi_0 \mathbf{H}\left(X, \mathbf{B}(G_1 \to G_0)\right) \simeq H^1_{\mathrm{smooth}}(X, (G_1 \to G_0)).$$

3.2.3 $\mathbf{B}^n U(1)$, $\mathbf{B}^n U(1)_{\mathrm{conn}}$, circle n-bundles and Deligne cohomology

A large class of examples of smooth ∞-groupoids is induced from chain complexes of sheaves of abelian groups by the DK correspondence [23].

Proposition 3.2.14. *The DK correspondence is an equivalence of categories*

$$\mathrm{Ch}_\bullet^+ \underset{N_\bullet}{\overset{\mathrm{DK}}{\rightleftarrows}} \mathrm{sAb}\ ,$$

between non-negatively graded chain complexes and simplicial abelian groups, where $N_\bullet$ forms the normalized chains complex of a simplicial abelian group A_Δ. Composed with the forgetful functor $\mathrm{sAb} \to \mathrm{sSet}$ and prolonged to a functor on sheaves of chain complexes, the functor

$$\mathrm{DK} : [\mathrm{CartSp}^{op}, \mathrm{Ch}_\bullet^+] \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$$

takes degreewise surjections to fibrations and degreewise quasi-isomorphisms to weak equivalences in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

We will write an element $(A_\bullet, \partial)$ of $\mathrm{Ch}_\bullet^+$ as

$$\cdots \to A_k \to A_{k-1} \to \cdots \to A_2 \to A_1 \to A_0$$

and will denote by [1] the "shift on the left" functor on chain complexes defined by $(A_\bullet[1])_k = A_{k-1}$, i.e., $A_\bullet[1]$ is the chain complex

$$\cdots \to A_{k-1} \to \cdots \to A_2 \to A_1 \to A_0 \to 0.$$

Remark 3.2.15. The reader used to cochain complexes, and so to the shift functor $(A^\bullet[1])^k = A^{k+1}$ could at first be surprised by the minus sign in the shift functor on chain complexes; but the shift rule is actually the same in both contexts, as it is evident by writing it as $(A_\bullet[1])_k = A_{k+\deg(\partial)}$.

For A any abelian group, we can consider A as a chain complex concentrated in degree zero, and so $A[n]$ will be the chain complex consisting of A concentrated in degree n.

Definition 3.2.16. Let A be an abelian Lie group. Define the simplicial presheaf $\mathbf{B}^n A$ to be the image under DK of the sheaf of complexes $C^\infty(-, A)[n]$:

$$\mathbf{B}^n A : U \mapsto \mathrm{DK}\left(C^\infty(U, A) \to 0 \to \cdots \to 0\right),$$

with $C^\infty(U, A)$ in degree n. Similarly, for $K \to A$ a morphism of abelian groups, write $\mathbf{B}^n(K \to A)$ for the image under DK of the complex of sheaves of abelian groups

$$\left(C^\infty(-, K) \to C^\infty(-, A) \to 0 \to \cdots \to 0\right)$$

with $C^\infty(-, A)$ in degree n; for $n \geq 1$ we write $\mathbf{E}\mathbf{B}^{n-1}A$ for $\mathbf{B}^{n-1}(A \xrightarrow{\mathrm{Id}} A)$.

Proposition 3.2.17. *For $n \geq 1$ the object $\mathbf{B}^n A$ is indeed the delooping of the object $\mathbf{B}^{n-1}A$.*

Proof. This means that there is an ∞-*pullback* diagram [34]

$$\begin{array}{ccc} \mathbf{B}^{n-1}A & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}^n A \end{array} .$$

This is presented by the corresponding homotopy pullback in [CartSp$^{\mathrm{op}}$, sSet]. Consider the diagram

$$\begin{array}{ccccc} \mathbf{B}^{n-1}A & \longrightarrow & \mathbf{E}\mathbf{B}^{n-1}A & \xrightarrow{\sim} & * \,, \\ \downarrow & & \downarrow & & \\ * & \longrightarrow & \mathbf{B}^n A & & \end{array}$$

The right vertical morphism is a replacement of the point inclusion by a fibration and the square is a pullback in [CartSp$^{\mathrm{op}}$, sSet] (the pullback of presheaves is computed objectwise and under the DK-correspondence may be computed in $\mathrm{Ch}_\bullet^+$, where it is evident). Therefore this exhibits $\mathbf{B}^{n-1}A$ as the homotopy pullback, as claimed. □

Proposition 3.2.18. *For $A = \mathbb{Z}, \mathbb{R}, U(1)$ and all $n \geq 1$ we have that $\mathbf{B}^n A$ satisfies descent over* CartSp *in that it is fibrant in* [CartSp$^{\mathrm{op}}$, sSet]$_{\mathrm{proj,loc}}$.

Proof. One sees directly in terms of Čech cocycles that the homotopy groups based at the trivial cocycle in the simplicial hom-sets $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\big(\check{C}(\mathcal{U}), \mathbf{B}^n A\big)$ and $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](U, \mathbf{B}^n A)$ are naturally identified. Therefore it is sufficient to show that

$$* \simeq \pi_0[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](U, \mathbf{B}^n A) \to \pi_0[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\big(\check{C}(\mathcal{U}), \mathbf{B}^n A\big)$$

is an isomorphism. This amounts to proving that the nth Čech cohomology group of U with coefficients in $\mathbb{Z}$, $\mathbb{R}$ or $U(1)$ is trivial, which is immediate since U is contractible (for $U(1)$ one uses the isomorphism $H^n(U, U(1)) \simeq H^{n+1}(U, \mathbb{Z})$ in Čech cohomology). □

Definition 3.2.19. For X a smooth ∞-groupoid and $QX \to X$ a cofibrant replacement, we say that

- for $X \xleftarrow{\sim_{\mathrm{loc}}} QX \xrightarrow{g} \mathbf{B}^n A$ a span in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$, the corresponding $(\mathbf{B}^{n-1})A$*-principal* n*-bundle* is the ∞-pullback

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}^n A. \end{array}$$

Hence, the ordinary pullback in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$

$$\begin{array}{ccc} P & \longrightarrow & \mathbf{E}\mathbf{B}^{n-1}A\ . \\ \downarrow & & \downarrow \\ QX & \xrightarrow{g} & \mathbf{B}^n A \\ \downarrow{\scriptstyle \simeq_{\mathrm{loc}}} & & \\ X & & \end{array}$$

- the Kan complex

$$(\mathbf{B}^{n-1}A)\mathrm{Bund}(X) := \mathbf{H}(X, \mathbf{B}^n A)$$

is the *n-groupoid* of smooth $\mathbf{B}^{n-1}A$-principal n-bundles on X.

Proposition 3.2.20. *For X a smooth paracompact manifold, the n-groupoid $(\mathbf{B}^{n-1}A)\mathrm{Bund}(X)$ is equivalent to the n-groupoid $\check{C}(\mathcal{U}, \mathbf{B}^n A)$ of degree n Čech cocycles on X with coefficients in the sheaf of smooth functions with values in A. In particular*

$$\pi_0(\mathbf{B}^{n-1}A)\mathrm{Bund}(X) = \pi_0\mathbf{H}(X, \mathbf{B}^n A) \simeq H^n(X, A)$$

is the Čech cohomology of X in degree n with coefficients in A.

Proof. This follows from the same arguments as in the previous section given for the more general non-abelian Čech cohomology. □

We will be interested mainly in the abelian Lie group $A = U(1)$. The exponential exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$ induces an acyclic fibration (in the global model structure) $\mathbf{B}^n(\mathbb{Z} \hookrightarrow \mathbb{R}) \xrightarrow{\sim} \mathbf{B}^n U(1)$, and one has the long fibration sequence

$$\begin{array}{ccccc} & & & & \mathbf{B}^n(\mathbb{Z} \hookrightarrow \mathbb{R}) \longrightarrow \mathbf{B}^{n+1}\mathbb{Z} \cdots \\ & & & & \downarrow{\scriptstyle \wr} \\ \cdots \to \mathbf{B}^n\mathbb{Z} & \longrightarrow & \mathbf{B}^n\mathbb{R} & \longrightarrow & \mathbf{B}^n U(1) \end{array}$$

from which one recovers the classical isomorphism $H^n(X, U(1)) \simeq H^{n+1}(X, \mathbb{Z})$. Next, we consider differential refinements of these cohomology groups.

Definition 3.2.21. The smooth ∞-groupoid $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is the image via the DK correspondence of the *Deligne complex* $U(1)[n]^\infty_D$, i.e., of the chain complex of sheaves of abelian groups

$$U(1)[n]^\infty_D := \left(C^\infty(-, U(1)) \xrightarrow{\mathrm{dlog}} \Omega^1(-, \mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-, \mathbb{R}) \right)$$

concentrated in degrees $[0, n]$. Similarly, the smooth ∞-groupoid $\mathbf{B}^n(\mathbb{Z} \hookrightarrow \mathbb{R})_{\mathrm{conn}}$ is the image via the DK correspondence of the complex of sheaves of abelian groups

$$\mathbb{Z}[n+1]^\infty_D := \left(\mathbb{Z} \hookrightarrow C^\infty(-, \mathbb{R}) \xrightarrow{d} \Omega^1(-, \mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-, \mathbb{R}) \right),$$

concentrated in degrees $[0, n+1]$.

The natural morphism of sheaves of complexes $\mathbb{Z}[n+1]^\infty_D \to U(1)[n]^\infty_D$ is an acyclic fibration and so we have an induced acyclic fibration (in the global model structure) $\mathbf{B}^n(\mathbb{Z} \hookrightarrow \mathbb{R})_{\mathrm{conn}} \overset{\sim}{\twoheadrightarrow} \mathbf{B}^n U(1)_{\mathrm{conn}}$. Therefore, we find a natural isomorphism

$$H^0(X, U(1)[n]^\infty_D) \simeq H^0(X, \mathbb{Z}[n+1]^\infty_D)$$

and a commutative diagram

$$\begin{array}{ccc} H^0(X, U(1)[n]^\infty_D) & \longrightarrow & H^n(X, U(1)) \\ \downarrow \wr & & \downarrow \wr \\ H^0(X, \mathbb{Z}[n+1]^\infty_D) & \longrightarrow & H^{n+1}(X, \mathbb{Z}). \end{array}$$

Definition 3.2.22. We denote the cohomology group $H^0(X, \mathbb{Z}[n]^\infty_D)$ by the symbol $\hat{H}^n(X, \mathbb{Z})$, and call it the nth *differential cohomology* group of X (with integer coefficients). The natural morphism $\hat{H}^n(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ will be called the *differential refinement* of ordinary cohomology.

Remark 3.2.23. The reader experienced with gerbes and higher gerbes will have recognized that $H^n(X, U(1)) \simeq H^{n+1}(X, \mathbb{Z})$ is the set of isomorphism classes of $U(1)$-$(n-1)$-gerbes on a manifold X, whereas $H^0(X, U(1)[n]^\infty_D) \simeq \hat{H}^{n+1}(X, \mathbb{Z})$ is the set of isomorphism classes of $U(1)$-$(n-1)$-gerbes with connection on X, and that the natural morphism $\hat{H}^{n+1}(X, \mathbb{Z}) \to H^{n+1}(X, \mathbb{Z})$ is 'forgetting the connection', see, e.g., [19].

The natural projection $U(1)[n]_D^\infty \to C^\infty(-, U(1)[n]$ is a fibration, so we have a natural fibration $\mathbf{B}^n U(1)_{\mathrm{conn}} \twoheadrightarrow \mathbf{B}^n U(1)$, and, as for the case of Lie groups, we have a natural factorization

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \hookrightarrow \mathbf{B}^n U(1)_{\mathrm{diff}} \overset{\sim}{\twoheadrightarrow} \mathbf{B}^n U(1)$$

into a monomorphism followed by an acyclic fibration (in the global model structure).

The smooth ∞-groupoid $\mathbf{B}^n U(1)_{\mathrm{diff}}$ is best defined at the level of chain complexes, where we have the well known “cone trick” from homological algebra to get the desired factorization. In the case at hand, it works as follows: let $\mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]_D^\infty(U))$ be the mapping cone of the inclusion of the kernel of $\pi : U(1)[n]_D^\infty \to C^\infty(-, U(1)[n]$ into $U(1)[n]_D^\infty$, i.e., the chain complex

$$\begin{array}{ccccccccc}
C^\infty(-, U(1)) & \xrightarrow{d\log} & \Omega^1(-) & \xrightarrow{d} & \Omega^2(-) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(-) \\
\oplus & \nearrow^{\mathrm{Id}} & \oplus & \nearrow & \oplus & & \oplus & \nearrow^{\mathrm{Id}} & \\
\Omega^1(-) & \xrightarrow[d]{} & \Omega^2(-) & \longrightarrow & \cdots & \longrightarrow & \Omega^n(-) & \longrightarrow & 0
\end{array}$$

Then $U(1)[n]_D^\infty$ naturally injects into $\mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]_D^\infty)$, and π induces a morphism of complexes $\pi : \mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]_D^\infty) \to C^\infty(-, U(1)[n]$ which is an acyclic fibration; the composition

$$U(1)[n]_D^\infty \hookrightarrow \mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]_D^\infty) \xrightarrow{\pi} C^\infty(-, U(1)[n]$$

is the sought for factorization.

Definition 3.2.24. Define the simplicial presheaf

$$\mathbf{B}^n U(1)_{\mathrm{diff}} = \mathrm{DK}\left(\mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]_D^\infty)\right)$$

to be the image under the DK equivalence of the chain complex of sheaves of abelian groups $\mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]_D^\infty)$.

The last smooth ∞-groupoid we introduce in this section is the natural ambient for curvature forms to live in. As above, we work at the level of sheaves of chain complexes first. So, let $\flat\mathbb{R}[n]_{\mathrm{dR}}^\infty$ be the truncated de Rham complex

$$\flat\mathbb{R}[n+1]_{\mathrm{dR}}^\infty := \left(\Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n+1}_{\mathrm{cl}}(-)\right)$$

seen as a chain complex concentrated in degrees $[0, n]$.

There is a natural morphism of complexes of sheaves, which we call the *curvature map*,

$$\mathrm{curv} : \mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]^\infty_D) \to \flat\mathbb{R}[n+1]^\infty_{\mathrm{dR}}$$

given by the projection $\mathrm{cone}(\ker \pi \hookrightarrow U(1)[n]^\infty_D) \to \ker \pi[1]$ in degrees $[1, n]$ and given by the de Rham differential $d : \Omega^n(-) \to \Omega^{n+1}_{\mathrm{cl}}(-)$ in degree zero. Note that the preimage of $(0 \to 0 \to \cdots \to \Omega^{n+1}_{\mathrm{cl}}(-))$ via curv is precisely the complex $U(1)[n]^\infty_D$, and that for $n = 1$ the induced morphism

$$\mathrm{curv} : U(1)[1]^\infty_D \to \Omega^2_{\mathrm{cl}}(-)$$

is the map sending a connection on a principal $U(1)$-bundle to its curvature 2-form.

Definition 3.2.25. The smooth ∞-groupoid $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}$ is

$$\flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R} = \mathrm{DK}\left(\Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n+1}_{\mathrm{cl}}(-)\right),$$

the image under DK of the truncated de Rham complex.

The above discussion can be summarized as

Proposition 3.2.26. *In* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *we have a natural commutative diagram*

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\mathrm{conn}} & \longrightarrow & \Omega^{n+1}_{\mathrm{cl}}(-) \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1)_{\mathrm{diff}} & \xrightarrow{\mathrm{curv}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R} \\ \downarrow\!\wr & & \\ \mathbf{B}^n U(1) & & \end{array} .$$

whose upper square is a pullback and whose lower part presents a morphism of smooth ∞-groupoids from $\mathbf{B}^n U(1)$ to $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}$. We call this morphism the curvature characteristic map.

Remark 3.2.27. One also has a natural $(\mathbb{Z} \hookrightarrow \mathbb{R})$ version of $\mathbf{B}^n U(1)_{\mathrm{diff}}$, i.e., we have a smooth ∞-groupoid $\mathbf{B}^n(\mathbb{Z} \hookrightarrow \mathbb{R})_{\mathrm{diff}}$ with a natural morphism

$$\mathbf{B}^n(\mathbb{Z} \hookrightarrow \mathbb{R})_{\mathrm{diff}} \overset{\sim}{\twoheadrightarrow} \mathbf{B}^n U(1)_{\mathrm{diff}}$$

which is an acyclic fibration in the global model structure.

4 Differential ∞-Lie integration

As the notion of L_∞-algebra generalizes that of Lie algebra so that of *Lie ∞-group* generalizes that of Lie group. We describe a way to *integrate* an L_∞-algebra $\mathfrak{g}$ to the smooth delooping $\mathbf{B}G$ of the corresponding Lie ∞-group by a slight variant of the construction of [25]. (Recall from the introduction that we use "Lie" to indicate generalized smooth structure which may or may not be represented by smooth manifolds). Then we generalize this to a *differential* integration: an integration of an L_∞-algebroid $\mathfrak{g}$ to smooth ∞-groupoids $\mathbf{B}G_{\mathrm{diff}}$ and $\mathbf{B}G_{\mathrm{conn}}$. Cocycles with coefficients in $\mathbf{B}G$ give G-principal ∞-bundles; those with coefficients in $\mathbf{B}G_{\mathrm{diff}}$ support the ∞-Chern–Weil homomorphism, those with coefficients in $\mathbf{B}G_{\mathrm{conn}}$ give G-principal ∞-bundles with connection.

4.1 Lie ∞-algebroids: cocycles, invariant polynomials and CS-elements

We summarize the main definitions and properties of L_∞-algebroids from [32, 33], and their cocycles, invariant polynomials and Chern–Simons elements from [43, 44].

Definition 4.1.1. Let R be a commutative $\mathbb{R}$-algebra, and let $\mathfrak{g}$ be a chain complex of finitely generated (in each degree) R-modules, concentrated in non-negative degree. Then a (reduced) L_∞-algebroid (or Lie ∞-algebroid) structure on $\mathfrak{g}$ is the datum of a degree 1 $\mathbb{R}$-derivation $d_{\mathrm{CE}(\mathfrak{g})}$ on the exterior algebra

$$\wedge^\bullet_R \mathfrak{g}^* := \mathrm{Sym}^\bullet_R(\mathfrak{g}^*[-1])$$

(the free graded commutative algebra on the shifted dual of $\mathfrak{g}$), which is a differential (i.e., squares to zero) compatible with the differential of $\mathfrak{g}$.

A chain complex $\mathfrak{g}$ endowed with an L_∞-algebroid structure will be called a *L_∞-algebroid*. The differential graded commutative algebra

$$\mathrm{CE}(\mathfrak{g}) := \left(\wedge^\bullet_R \mathfrak{g}^*, d_{\mathrm{CE}(\mathfrak{g})}\right)$$

will be called the *Chevalley–Eilenberg algebra* of the L_∞-algebroid $\mathfrak{g}$. A morphism of L_∞-algebroids $\mathfrak{g}_1 \to \mathfrak{g}_2$ is defined to be a dgca morphism $\mathrm{CE}(\mathfrak{g}_2) \to \mathrm{CE}(\mathfrak{g}_1)$

Since all L_∞-algebroids which will be met in this paper will be reduced, we will just say Lie ∞-algebroid to mean reduced Lie ∞-algebroid in what follows.

Remark 4.1.2.

- Lie ∞-algebroids could be more intrinsically defined as follows: the category $L_\infty\mathrm{Algd} \subset dg\mathrm{Alg}^{op}$ of *L_∞-algebroids* is the full subcategory of the opposite of that of differential graded commutative $\mathbb{R}$-algebras on those dg-algebras whose underlying graded-commutative algebra is free on a finitely generated graded module concentrated in positive degree over the commutative algebra in degree 0.
- The dual $\mathfrak{g}^* = \mathrm{Hom}_R^{-\bullet}(\mathfrak{g}, R)$ is a *cochain* complex concentrated in non-negative degrees. In particular the shift to the right functor $[-1]$ changes it into a cochain complex concentrated in strictly positive degrees.
- The restriction to finite generation is an artifact of dualizing $\mathfrak{g}$ rather than working with graded alternating multilinear functions on $\mathfrak{g}$ as the masters (Chevalley–Eilenberg–Koszul) did in the original ungraded case. In particular, the direct generalization of their approach consists in working with the cofree connected cocommutative coalgebra cogenerated by $\mathfrak{g}[1]$, see, e.g., [46]. At least for L_∞-algebras, there are alternate definitions and conventions as to bounds on the grading, signs, etc. cf. [32, 33] among others.
- Given an L_∞-algebroid $\mathfrak{g}$, the degree 0 part $\mathrm{CE}(\mathfrak{g})_0$ of $\mathrm{CE}(\mathfrak{g})$ is is a commutative $\mathbb{R}$-algebra which we think of as the formal dual to the space of objects over which the Lie ∞-algebroid is defined. If $\mathrm{CE}(\mathfrak{g})_0 = \mathbb{R}$ equals the ground field, we say we have an ∞ algebroid over the point, or equivalently that we have an *L_∞-algebra*.
- The underlying algebra in degree 0 can be generalized to an algebra over some Lawvere theory. In particular in a proper setup of higher differential geometry, we would demand $\mathrm{CE}(\mathfrak{g})_0$ to be equipped with the structure of a C^∞-ring.

Example 4.1.3.
- For $\mathfrak{g}$ an ordinary (finite-dimensional) Lie algebra, $\mathrm{CE}(\mathfrak{g})$ is the ordinary Chevalley–Eilenberg algebra with coefficients in $\mathbb{R}$. The differential is given by the dual of the Lie bracket,

$$d_{\mathrm{CE}(\mathfrak{g})} = [-,-]^*$$

  extended uniquely as a graded derivation.
- For a dg-Lie algebra $\mathfrak{g} = (\mathfrak{g}_\bullet, \partial)$, the differential is

$$d_{\mathrm{CE}(\mathfrak{g})} = [-,-]^* + \partial^*.$$

- In the general case, the total differential is further determined by (and is equivalent to) a sequence of higher multilinear brackets [33].
- For $n \in \mathbb{N}$, the L_∞-algebra $b^{n-1}\mathbb{R}$ is defined in terms of $\mathrm{CE}(b^{n-1}\mathbb{R})$ which is the dgc-algebra on a single generator in degree n with vanishing differential.
- For X a smooth manifold, its *tangent Lie algebroid* is defined to have $\mathrm{CE}(TX) = (\Omega^\bullet(X), d_{dR})$ the de Rham algebra of X. Notice that $\Omega^\bullet(X) = \wedge^\bullet_{C^\infty(X)}\Gamma(T^*X)$.

  We shall extensively use the tangent Lie algebroid $T(U \times \Delta^k)$ where $U \in \mathrm{CartSp}$ and Δ^k is the standard k-simplex.

Definition 4.1.4. For $\mathfrak{g}$ a Lie ∞-algebroid and $n \in \mathbb{N}$, a *cocycle* in degree n on $\mathfrak{g}$ is, equivalently

- an element $\mu \in \mathrm{CE}(\mathfrak{g})$ in degree n, such that $d_{\mathrm{CE}(\mathfrak{g})}\, \mu = 0$;
- a morphism of dg-algebras $\mu : \mathrm{CE}(b^{n-1}\mathbb{R}) \to \mathrm{CE}(\mathfrak{g})$;
- a morphism of Lie ∞-algebroids $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$.

Example 4.1.5.
- For $\mathfrak{g}$ an ordinary Lie algebra, a cocycle in the above sense is the same as a Lie algebra cocycle in the ordinary sense (with values in the trivial module).
- For X a smooth manifold, a cocycle in degree n on the tangent Lie algebroid TX is precisely a closed n-form on X.

For our purposes, a particularly important Chevalley–Eilenberg algebra is the Weil algebra.

Definition 4.1.6. The *Weil algebra* of an L_∞-algebra $\mathfrak{g}$ is the dg-algebra

$$\mathrm{W}(\mathfrak{g}) := \left(\mathrm{Sym}^\bullet\left(\mathfrak{g}^*[-1] \oplus \mathfrak{g}^*[-2]\right), d_{W(\mathfrak{g})}\right),$$

where the differential on the copy $\mathfrak{g}^*[-1]$ is the sum

$$d_{W(\mathfrak{g})}|_{\mathfrak{g}^*} = d_{CE(\mathfrak{g})} + \sigma,$$

with $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*[-1]$ is the grade-shifting isomorphism, i.e., it is the identity of $\mathfrak{g}^*$ seen as a degree 1 map $\mathfrak{g}^*[-1] \to \mathfrak{g}^*[-2]$, extended as a graded derivation, and where

$$d_{W(\mathfrak{g})} \circ \sigma = -\sigma \circ d_{W(\mathfrak{g})}.$$

Proposition 4.1.7. *The Weil algebra is a representative of the free differential graded commutative algebra on the graded vector space* $\mathfrak{g}^*[-1]$ *in that* *there exist a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{dgca}}\left(W(\mathfrak{g}),\Omega^\bullet\right)\simeq\mathrm{Hom}_{\mathrm{gr-vect}}\left(\mathfrak{g}^*[-1],\Omega^\bullet\right),$$

for $\Omega^\bullet$ *an arbitrary dgca. Moreover, the Weil algebra is precisely that algebra with this property for which the projection morphism* $i^*:\mathfrak{g}^*[-1]\oplus\mathfrak{g}^*[-2]\to\mathfrak{g}^*[-1]$ *of graded vector spaces extends to a dg-algebra homomorphism*

$$i^*:\mathrm{W}(\mathfrak{g})\to\mathrm{CE}(\mathfrak{g}).$$

Notice that the free dgca on a graded vector space is defined only up to isomorphism. The condition on i^* is what picks the Weil algebra among all free dg-algebras. A proof of the above proposition can be found, e.g., in [43].

Equivalently, one can state the freeness of the Weil algebra by saying that the dgca-morphisms $A:\mathrm{W}(\mathfrak{g})\to\Omega^\bullet$ are in natural bijection with the degree 1 elements in the graded vector space $\Omega^\bullet\otimes\mathfrak{g}$.

Example 4.1.8.

- For $\mathfrak{g}$ an ordinary Lie algebra, $\mathrm{W}(\mathfrak{g})$ is the ordinary Weil algebra [10]. In that paper, H. Cartan defines a $\mathfrak{g}$-algebra as an analog of the dg-algebra $\Omega^\bullet(P)$ of differential forms on a principal bundle, i.e., as a dg-algebra equipped with operations i_ξ and L_ξ for all $\xi\in\mathfrak{g}$ satisfying the usual relations, including

  $$L_\xi=di_\xi+i_\xi d.$$

  Next, Cartan introduces the Weil algebra $\mathrm{W}(\mathfrak{g})$ as the universal $\mathfrak{g}$-algebra and identifies a $\mathfrak{g}$-connection A on a principal bundle P as a morphism of $\mathfrak{g}$-algebras

  $$A:\mathrm{W}(\mathfrak{g})\to\Omega^\bullet(P).$$

  This can in turn be seen as a dgca morphism satisfying the Cartan–Ehresmann conditions, and it is this latter point of view that we generalize to an arbitrary L_∞-algebra.
- The dg-algebra $\mathrm{W}(b^{n-1}\mathbb{R})$ of $b^{n-1}\mathbb{R}$ is the free dg-algebra on a single generator in degree n. As a graded algebra, it has a generator b in degree n and a generator c in degree $(n+1)$ and the differential acts as $d_{\mathrm{W}}:b\mapsto c$. Note that, since $d_{\mathrm{CE}}b=0$, this is equivalent to $c=\sigma b$.

Remark 4.1.9. Since the Weil algebra is itself a dg-algebra whose underlying graded algebra is a graded symmetric algebra, it is itself the CE-algebra of an L_∞-algebra. The L_∞-algebra thus defined we denote $\mathrm{inn}(\mathfrak{g})$:

$$\mathrm{CE}\,(\mathrm{inn}(\mathfrak{g})) = \mathrm{W}(\mathfrak{g}).$$

Note that the underlying graded vector space of $\mathrm{inn}(\mathfrak{g})$ is $\mathfrak{g} \oplus \mathfrak{g}[1]$. Looking at $\mathrm{W}(\mathfrak{g})$ as the Chevalley–Eilenberg algebra of $\mathrm{inn}(\mathfrak{g})$ we therefore obtain the following description of morphisms out of $\mathrm{W}(\mathfrak{g})$: for any dgca $\Omega^\bullet$, a dgca morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^\bullet$ is the datum of a pair (A, F_A), where A and F_A are a degree 1 and a degree 2 element in $\Omega^\bullet \otimes \mathfrak{g}$, respectively, such that (A, F_A) satisfies the Maurer–Cartan equation in the L_∞-algebra $\Omega^\bullet \otimes \mathrm{inn}(\mathfrak{g})$. The Maurer–Cartan equation actually completely determines F_A in terms of A; this is an instance of the freeness property of the Weil algebra stated in Proposition 4.1.7.

Definition 4.1.10. For X a smooth manifold, a $\mathfrak{g}$-*valued connection form* on X is a morphism of Lie ∞-algebroids $A : TX \to \mathrm{inn}(\mathfrak{g})$, hence a morphism of dg-algebras

$$A : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(X).$$

Remark 4.1.11. A $\mathfrak{g}$-valued connection form on X can be equivalently seen as an element A in the set $\Omega^1(X, \mathfrak{g})$ of degree 1 elements in $\Omega^\bullet(X) \otimes \mathfrak{g}$, or as a pair (A, F_A), where $A \in \Omega^1(U, \mathfrak{g})$, $F_A \in \Omega^2(U, \mathfrak{g})$, and A and F_A are related by the Maurer–Cartan equation in $\Omega^\bullet(X, \mathrm{inn}(\mathfrak{g}))$. The element F_A is called the *curvature form* of A.

Example 4.1.12. If $\mathfrak{g}$ is an ordinary Lie algebra, then a $\mathfrak{g}$-valued connection form A on X is a 1-form on X with coefficients in $\mathfrak{g}$, i.e., it is naturally a connection 1-form on a trivial principal G-bundle on X. The element F_A in $\Omega^2(X, \mathfrak{g})$ is then given by equation

$$F_A = dA + \frac{1}{2}[A, A],$$

so it is precisely the usual curvature form of A.

The last ingredient we need to generalize from Lie algebras to L_∞-algebroids is the algebra $\mathrm{inv}(\mathfrak{g})$ of invariant polynomials.

Definition 4.1.13. An *invariant polynomial* on $\mathfrak{g}$ is a $d_{\mathrm{W}(\mathfrak{g})}$-closed element $\langle - \rangle$ in $\mathrm{Sym}^\bullet(\mathfrak{g}^*[-2]) \subset \mathrm{W}(\mathfrak{g})$.

To see how this definition encodes the classical definition of invariant polynomials on a Lie algebra, notice that invariant polynomials are elements of $\mathrm{Sym}^\bullet(\mathfrak{g}^*[-2])$ that are both horizontal and ad-invariant ("*basic* forms"). Namely, for any $v \in \mathfrak{g}$ we have, for an invariant polynomial $\langle - \rangle$, the identities

$$\iota_v \langle - \rangle = 0 \quad \text{(horizontality)}$$

and

$$\mathcal{L}_v \langle - \rangle = 0 \quad \text{(ad-invariance)},$$

where $\iota_v : \mathrm{W}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$ the contraction derivation defined by v and $\mathcal{L}_v := [d_{\mathrm{W}(\mathfrak{g})}, \iota_v]$ is the corresponding Lie derivative.

We want to identify two indecomposable invariant polynomials which differ by a "horizontal shift". A systematic way of doing this is to introduce the following equivalence relation on the dgca of all invariant polynomials: we say that two invariant polynomials $\langle - \rangle_1, \langle - \rangle_2$ are *horizontally equivalent* if there exists ω in $\ker(\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g}))$ such that

$$\langle - \rangle_1 = \langle - \rangle_2 + d_{\mathrm{W}} \omega.$$

Write $\mathrm{inv}(\mathfrak{g})_V$ for the quotient graded vector space of horizontal equivalence classes of invariant polynomials.

Definition 4.1.14. The dgca $\mathrm{inv}(\mathfrak{g})$ is defined as the free polynomial algebra on the graded vector space $\mathrm{inv}_V(\mathfrak{g})$, endowed with the trivial differential.

Remark 4.1.15. A choice of a linear section to the projection

$$\{\text{invariant polynomials}\} \to \mathrm{inv}(\mathfrak{g})_V$$

gives a morphism of graded vector spaces $\mathrm{inv}(\mathfrak{g})_V \to \mathrm{W}(\mathfrak{g})$, canonical up to horizontal homotopy, that sends each equivalence class to a representative. This linear morphism uniquely extends to a dg-algebra homomorphism

$$\mathrm{inv}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g}).$$

Remark 4.1.16. The algebra $\mathrm{inv}(\mathfrak{g})$ is at first sight a quite abstract construction which is apparently unrelated to an equivalence relation on indecomposable invariant polynomials. A closer look shows that it is actually not so. Namely, only indecomposable invariant polynomials can be representatives for the non-zero equivalence classes. Indeed, if $\langle - \rangle_1$ and $\langle - \rangle_2$ are two non-trivial invariant polynomials, then since the cohomology of $\mathrm{W}(\mathfrak{g})$ is trivial in positive degree, there is cs_1 in $\mathrm{W}(\mathfrak{g})$ (not necessarily in $\ker(\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g}))$) such that $d_{\mathrm{W}}\mathrm{cs}_1 = \langle - \rangle_1$, but then $\mathrm{cs}_1 \wedge \langle - \rangle_2$ is a horizontal trivialization of $\langle - \rangle_1 \wedge \langle - \rangle_2$. One therefore obtains a very concrete description of the algebra $\mathrm{inv}(\mathfrak{g})$ as follows: one picks a representative indecomposable invariant polynomial for each horizontal equivalence class and considers the subalgebra of $\mathrm{W}(\mathfrak{g})$ generated by these representatives. The morphism $\mathrm{inv}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g})$ is then realized as the inclusion of this subalgebra into the Weil algebra. Different choices of representative generators lead to distinct but equivalent subalgebras: each one is isomorphic to the others via an horizontal shift in the generators.

Remark 4.1.17. For $\mathfrak{g}$ an ordinary reductive Lie algebra, Definition 4.1.14 reproduces the traditional definition of the algebra of $\mathrm{ad}_{\mathfrak{g}}$-invariant polynomials. Indeed, for a Lie algebra $\mathfrak{g}$, the condition $d_{\mathrm{W}(\mathfrak{g})}\langle - \rangle = 0$ is precisely the usual $\mathrm{ad}_{\mathfrak{g}}$-invariance of an element $\langle - \rangle$ in $\mathrm{Sym}^\bullet \mathfrak{g}^*[-2]$. Moreover, the horizonal equivalence on indecomposables is trivial in this case and it is a classical fact (for instance theorem I on page 242 in volume III of [21]) that the graded algebra of $\mathrm{ad}_{\mathfrak{g}}$-invariant polynomials is indeed free on the space of indecomposables.

Definition 4.1.18. For any dgca morphism $A : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet$, the composite morphism $\mathrm{inv}(\mathfrak{g}) \to \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet$ is the evaluation of invariant polynomials on the element F_A. In particular, if X is a smooth manifold and A is a $\mathfrak{g}$-valued connection form on X, then the image of $F_A : \mathrm{inv}(\mathfrak{g}) \to \Omega^\bullet(X)$ is a collection of differential forms on X, to be called the *curvature characteristic forms* of A.

Example 4.1.19. For the L_∞-algebra $b^{n-1}\mathbb{R}$, in the notations of Example 4.1.8, one has $\mathrm{inv}(b^{n-1}\mathbb{R}) = \mathbb{R}[c]$.

Definition 4.1.20. We say an invariant polynomial $\langle - \rangle$ on $\mathfrak{g}$ is *in transgression* with a cocycle μ if there exists an element $\mathrm{cs} \in \mathrm{W}(\mathfrak{g})$ such that

(1) $i^*\mathrm{cs} = \mu$;
(2) $d_{\mathrm{W}(\mathfrak{g})}\mathrm{cs} = \langle - \rangle$.

We call cs a *Chern–Simons element* for μ and $\langle - \rangle$.

For ordinary Lie algebras this reduces to the classical notion, for instance 6.13 in vol. III of [21].

Remark 4.1.21. If we think of $\mathrm{inv}(\mathfrak{g}) \subset \ker i^*$ as a subcomplex of the kernel of i^*, then this transgression exhibits the connecting homomorphism $H^{n-1}(\mathrm{CE}(\mathfrak{g})) \to H^n(\ker i^*)$ of the long sequence in cohomology induced from the short exact sequence $\ker i^* \to \mathrm{W}(\mathfrak{g}) \xrightarrow{i^*} \mathrm{CE}(\mathfrak{g})$.

If we think of

- $\mathrm{W}(\mathfrak{g})$ as differential forms on the total space of a universal principal bundle;
- $\mathrm{CE}(\mathfrak{g})$ as differential forms on the fibre;
- $\mathrm{inv}(\mathfrak{g})$ as forms on the base;

then the above notion of transgression is precisely the classical one of transgression of forms in the setting of fibre bundles (for instance Section 9 of [6]).

Example 4.1.22.

- For $\mathfrak{g}$ a semisimple Lie algebra with $\langle -,- \rangle$ the Killing form invariant polynomial, the corresponding cocycle in transgression is $\mu_3 = \frac{1}{2}\langle -, [-,-]\rangle$. The Chern–Simons element witnessing this transgression is $\mathrm{cs} = \langle \sigma(-), -\rangle + \frac{1}{2}\langle -, [-,-]\rangle$.
- For the Weil algebra $\mathrm{W}(b^{n-1}\mathbb{R})$ of Example 4.1.8, the element b (as element of the Weil algebra) is a Chern–Simons element transgressing the cocycle b (as element of the Chevalley–Eilenberg algebra $\mathrm{CE}(b^{n-1}\mathbb{R})$) to the invariant polynomial c.
- For $\mathfrak{g}$ a semisimple Lie algebra, $\mu_3 = \frac{1}{2}\langle -, [-,-]\rangle$ the canonical Lie algebra 3-cocycle in transgression with the Killing form, let $\mathfrak{g}_{\mu_3}$ be the corresponding *string Lie 2-algebra* given by the next Definition 4.1.23, and discussed below in 4.2.3. Its Weil algebra is given by

$$\begin{aligned} d_W t^a &= -\frac{1}{2} C^a{}_{bc} t^b \wedge t^c + r^a \\ d_W b &= h - \mu_3 \end{aligned}$$

and the corresponding Bianchi identities, with $\{t^a\}$ a dual basis for $\mathfrak{g}$ in degree 1, with b a generator in degree 2 and h its curvature generator in degree 3. We see that every invariant polynomial of $\mathfrak{g}$ is also an invariant polynomial of $\mathfrak{g}_{\mu_3}$. But the Killing form $\langle -,-\rangle$ is now horizontally trivial: let cs_3 be any Chern–Simons element for $\langle -,-\rangle$ in $\mathrm{W}(\mathfrak{g})$. This is not horizontal. But the element

$$\tilde{\mathrm{cs}}_3 := \mathrm{cs}_3 - \mu_3 + h$$

is in $\ker(W(\mathfrak{g}_{\mu_3}) \to \mathrm{CE}(\mathfrak{g}_{\mu_3}))$ and

$$d_W \tilde{\mathrm{cs}}_3 = \langle -,-\rangle.$$

Therefore $\mathrm{inv}(\mathfrak{g}_{\mu_3})$ has the same generators as $\mathrm{inv}(\mathfrak{g})$ except the Killing form, which is discarded.

Definition 4.1.23. For $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ a cocycle in degree $n \geq 1$, the *extension* that it classifies is the L_∞-algebra given by the pullback

$$\begin{array}{ccc} \mathfrak{g}_\mu & \longrightarrow & \mathrm{inn}(b^{n-2}\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\mu} & b^{n-1}\mathbb{R} \end{array} .$$

Remark 4.1.24. Dually, the L_∞-algebra $\mathfrak{g}_\mu$ is the pushout

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}_\mu) & \longleftarrow & \mathrm{W}(b^{n-2}\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathbb{R}) \end{array}$$

in the category dgcAlg. This means that $\mathrm{CE}(\mathfrak{g}_\mu)$ is obtained from $\mathrm{CE}(\mathfrak{g})$ by adding one more generator b in degree $(n-1)$ and setting

$$d_{\mathrm{CE}(\mathfrak{g}_\mu)} : b \mapsto -\mu.$$

These are standard constructions on dgc-algebras familiar from rational homotopy theory, realizing $\mathrm{CE}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g}_\mu)$ as a relative Sullivan algebra. Yet, it is still worthwhile to make the ∞-Lie theoretic meaning in terms of L_∞-algebra extensions manifest: we may think of $\mathfrak{g}_\mu$ as the homotopy fibre of μ or equivalently as the extension of $\mathfrak{g}$ classified by μ. In Section 5 we discuss how these L_∞-algebra extensions are integrated to extensions of smooth ∞-groups; the homotopy fibre point of view will be emphasized in Section 6.

Example 4.1.25. For $\mathfrak{g}$ a semisimple Lie algebra and $\mu = \frac{1}{2}\langle -, [-,-]\rangle$ the cocycle in transgression with the Killing form, the corresponding extension is the *string Lie 2-algebra* $\mathfrak{g}_\mu$ discussed in Section 4.2.3, [8, 25].

We may summarize the situation as follows: for μ a degree n cocycle which is in transgression with an invariant polynomial $\langle - \rangle$ via a Chern–Simons element cs, the corresponding morphisms of dg-algebras fit into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}) \\
\uparrow & & \uparrow \\
\mathrm{W}(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(b^{n-1}\mathbb{R}) \\
\uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{g}) & \xleftarrow{\langle - \rangle} & \mathrm{inv}(b^{n-1}\mathbb{R})
\end{array}
$$

In Section 4.3, we will see that under ∞-*Lie integration* this diagram corresponds to a universal circle n-bundle connection on $\mathbf{B}G$. The composition of the diagrams defining the cells in $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ (Section 4.3) with this diagram models the ∞-Chern–Weil homomorphism for the characteristic class given by $\langle - \rangle$.

4.2 Principal ∞-bundles

We describe the integration of Lie ∞-algebras $\mathfrak{g}$ to smooth ∞-groupoids $\mathbf{B}G$ in the sense of Section 3.

The basic idea is Sullivan's old construction [47] in rational homotopy theory of a simplicial set from a dg-algebra. It was essentially noticed by Getzler [20], following Hinich [26], that this construction may be interpreted in ∞-Lie theory as forming the smooth ∞-groupoid underlying the Lie integration of an L_∞ -algebra. Henriques [25] refined the construction to land in ∞-groupoids internal to Banach spaces. Here, we observe that the construction has an evident refinement to yield genuine smooth ∞-groupoids in the sense of Section 3 (this refinement has independently also been considered by Roytenberg in [40]): the integrated smooth ∞-groupoid sends each Cartesian space U to a Kan complex which in degree k is the set of smoothly U-parameterized families of smooth flat $\mathfrak{g}$-valued differential forms on the standard k-simplex $\Delta^k \subset \mathbb{R}^k$ regarded as a smooth manifold (with boundary and corners).

To make this precise, we need a suitable notion of smooth differential forms on the k-simplex. Recall that an ordinary smooth form on Δ^k is a smooth form on an open neighbourhood of Δ^n in $\mathbb{R}^n$. This says that the derivatives are well behaved at the boundary. The following technical definition imposes even more restrictive conditions on the behaviour at the boundary.

Definition 4.2.1. For any point p in Δ^k, let Δ_p be the lowest dimensional subsimplex of Δ^k the point p belongs to, and let π_p the orthogonal projection on the affine subspace spanned by Δ_p. A smooth differential form ω on Δ^k is said to have *sitting instants* along the boundary if for any point p in Δ^k there is a neighbourhood V_p of p such that $\omega = \pi_p^*(\omega|_{\Delta_p})$ on V_p.

For any $U \in \mathrm{CartSp}$, a smooth differential form ω on $U \times \Delta^k$ is said to have sitting instants if for all points $u : * \to U$ the pullback along $(u, \mathrm{Id}) : \Delta^k \to U \times \Delta^k$ has sitting instants.

Smooth forms with sitting instants clearly form a sub-dg-algebra of all smooth forms. We shall write $\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)$ to denote this sub-dg-algebra.

Remark 4.2.2. The inclusion $\Omega^\bullet_{\mathrm{si}}(\Delta^k) \hookrightarrow \Omega^\bullet(\Delta^k)$ is a quasi-isomorphism. Indeed, by using bump functions with sitting instants one sees that the sheaf of differential forms with sitting instants is fine, and it is immediate to show that the stalkwise Poincaré lemma holds for this sheaf. Hence the usual hypercohomology argument applies. We thank Tom Goodwillie for having suggested a sheaf-theoretic proof of this result.

Remark 4.2.3. For a point p in the interior of the simplex Δ^k the sitting instants condition is clearly empty; this justifies the name "sitting instants along the boundary". Also note that the dimension of the normal direction to the boundary depends on the dimension of the boundary stratum: there is one perpendicular direction to a codimension-1 face, and there are k perpendicular directions to a vertex.

Definition 4.2.4. For a Cartesian space U, we denote by the symbol

$$\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} \subset \Omega^\bullet(U \times \Delta^k)$$

the sub-dg-algebra on forms that are *vertical* with respect to the projection $U \times \Delta^k \to U$.

Equivalently this is the completed tensor product,

$$\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} = C^\infty(U; \mathbb{R}) \hat{\otimes} \Omega^\bullet_{\mathrm{si}}(\Delta^k),$$

where $C^\infty(U; \mathbb{R})$ is regarded as a dg-algebra concentrated in degree zero.

Example 4.2.5.
- A 0-form (a smooth function) has sitting instants on Δ^1 if in a neighbourhood of the endpoints it is constant. A smooth function $f : U \times \Delta^1 \to \mathbb{R}$ is in $\Omega^0_{\mathrm{si}}(U \times \Delta^1)_{\mathrm{vert}}$ if for each $u \in U$ it is constant in a neighbourhood of the endpoints of Δ^1.
- A 1-form has sitting instants on Δ^1 if in a neighbourhood of the endpoints it vanishes.

- Let X be a smooth manifold and $\omega \in \Omega^\bullet(X)$ be a smooth form on X. Let $\phi : \Delta^k \to X$ be a smooth map with sitting instants in the ordinary sense: for every r-face of Δ^k there is a neighbourhood such that ϕ is perpendicularly constant on that neighbourhood. Then the pullback form $\phi^*\omega$ is a form with sitting instants on Δ^k.

Remark 4.2.6. The point of the definition of sitting instants, clearly reminiscent of the use of normal cylindrical collars in cobordism theory, is that, when gluing compatible forms on simplices along faces, the resulting differential form is smooth.

Proposition 4.2.7. *Let $\Lambda^k_i \subset \Delta^k$ be the ith horn of Δ^k, regarded naturally as a closed subset of $\mathbb{R}^{k-1}$. If $\{\omega_j \in \Omega^\bullet_{\mathrm{si}}(\Delta^{k-1})\}$ is a collection of smooth forms with sitting instants on the $(k-1)$-simplices of Λ^k_i that match on their coinciding faces, then there is a unique smooth form ω on Λ^k_i that restricts to ω_j on the jth face.*

Proof. By the condition that forms with sitting instants are constant perpendicular to their value on a face in a neighbourhood of any face it follows that if two agree on an adjacent face then all derivatives at that face of the unique form that extends both exist in all directions. Hence that unique form extending both is smooth. □

Definition 4.2.8. For $\mathfrak{g}$ an L_∞-algebra, the simplicial presheaf $\exp_\Delta(\mathfrak{g})$ on the site of Cartesian spaces is defined as

$$\exp_\Delta(\mathfrak{g}) : (U, [k]) \mapsto \mathrm{Hom}_{\mathrm{dgAlg}}\left(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}}\right).$$

Note that the construction of $\exp_\Delta(\mathfrak{g})$ is functorial in $\mathfrak{g}$: a morphism of L_∞-algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$, i.e., a dg-algebra morphism $\mathrm{CE}(\mathfrak{g}_2) \to \mathrm{CE}(\mathfrak{g}_1)$, induces a morphism of simplicial presheaves $\exp_\Delta(\mathfrak{g}_1) \to \exp_\Delta(\mathfrak{g}_2)$.

Remark 4.2.9. A k-simplex in $\exp_\Delta(\mathfrak{g})(U)$ may be thought of as a smooth family of *flat* $\mathfrak{g}$-valued forms on Δ^n, parameterized by U. We write $\exp_\Delta(\mathfrak{g})$ for this simplicial presheaf to indicate that it plays a role analogous to the formal exponentiation of a Lie algebra to a Lie group.

Proposition 4.2.10. *The simplicial presheaf $\exp_\Delta(\mathfrak{g})$ is a smooth ∞-groupoid in that it is fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$: it takes values in Kan complexes. We say that the smooth ∞-groupoid $\exp_\Delta(\mathfrak{g})$ integrates the L_∞-algebra $\mathfrak{g}$.*

Proof. Since our forms have sitting instants, this follows in direct analogy to the standard proof that the singular simplicial complex of any topological space is a Kan complex: we may use the standard retracts of simplices onto their horns to pullback forms from horns to simplices. The retraction maps are smooth except where they cross faces, but since the forms have sitting instants there, their smooth pullback exists nevertheless.

Let $\pi : \Delta^k \to \Lambda^k_i$ be the standard retraction map of a k-simplex on its ith horn. Since π is smooth away from the preimages of the faces, the commutative diagram

$$\begin{array}{ccc} U \times \Lambda^k_i & \xrightarrow{\mathrm{id}\times i} & U \times \Delta^k \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow {\scriptstyle \mathrm{id}\times\pi} \\ & & U \times \Lambda^k_i \end{array}$$

induces a commutative diagram of dgcas

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Lambda^k_i)_{\mathrm{vert}}) & \xleftarrow{(\mathrm{id}\times i)^* \circ -} & \mathrm{Hom}_{\mathrm{dgAlg}}\left(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}}\right) \; , \\ & {\scriptstyle \mathrm{id}} \nwarrow & \uparrow {\scriptstyle (\mathrm{id}\times\pi)^* \circ -} \\ & & \mathrm{Hom}_{\mathrm{dgAlg}}\left(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Lambda^n_i)_{\mathrm{vert}}\right) \end{array}$$

so that, in particular, the horn-filling map $\mathrm{Hom}_{\mathrm{dgAlg}}(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}} (U \times \Lambda^k_i)_{\mathrm{vert}}) \to \mathrm{Hom}_{\mathrm{dgAlg}}\left(\mathrm{CE}(\mathfrak{g}), \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}}\right)$ is surjective. ☐

Example 4.2.11. We may parameterize the 2-simplex as

$$\Delta^2 = \left\{(x, y) \in \mathbb{R}^2 | |x| \leq 1, \ 0 \leq y \leq 1 - |x|\right\}.$$

The retraction map $\Delta^2 \to \Lambda^2_1$ in this parameterization is

$$(x, y) \mapsto (x, 1 - |x|).$$

This is smooth away from $x = 0$. A 1-form with sitting instants on Λ^1_1 vanishes in a neighbourhood of $x = 0$, hence its pullback along this map exists and is smooth.

Typically one is interested not in $\exp_\Delta(\mathfrak{g})$ itself, but in a *truncation* thereof. For our purposes truncation is best modelled by the coskeleton operation.

Write $\Delta_{\leq n} \hookrightarrow \Delta$ for the full subcategory of the simplex category on the first n objects $[k]$, with $0 \leq k \leq n$. Write $\mathrm{sSet}_{\leq n}$ for the category of presheaves on $\Delta_{\leq n}$. By general abstract reasoning the canonical projection

$\mathrm{tr}_n : \mathrm{sSet} \to \mathrm{sSet}_{\leq n}$ has a left adjoint $\mathrm{sk}_n : \mathrm{sSet}_{\leq n} \to \mathrm{sSet}$ and a right adjoint $\mathrm{cosk}_n : \mathrm{sSet}_{\leq n} \to \mathrm{sSet}$.

$$(\mathrm{sk}_n \dashv \mathrm{tr}_n \dashv \mathrm{cosk}_n) : \mathrm{sSet} \underset{\mathrm{cosk}_n}{\overset{\mathrm{sk}_n}{\overleftarrow{\xrightarrow{\mathrm{tr}_n}}}} \mathrm{sSet}_{\leq n} .$$

The *coskeleton* operation on a simplicial set is the composite

$$\mathbf{cosk}_n := \mathrm{cosk}_n \circ \mathrm{tr}_n : \mathrm{sSet} \to \mathrm{sSet}.$$

Since $\mathbf{cosk}_n$ is a functor, it extends to an operation of simplicial presheaves, which we shall denote by the same symbol

$$\mathbf{cosk}_n : [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}] \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$$

For $X \in \mathrm{sSet}$ or $X \in [\mathrm{CartSp}^{\mathrm{op},\mathrm{sSet}}]$ we say $\mathbf{cosk}_n X$ is its *n-coskeleton.*

Remark 4.2.12. Using the adjunction relations, we have that the k-cells of $\mathbf{cosk}_n X$ are images of the n-truncation of $\Delta[k]$ in the n-truncation of X:

$$(\mathbf{cosk}_n X)_k = \mathrm{Hom}_{\mathrm{sSet}}\left(\Delta[k], \mathbf{cosk}_n X\right) = \mathrm{Hom}_{\mathrm{sSet}_{\leq n}}\left(\mathrm{tr}_n \Delta[k], \mathrm{tr}_n X\right).$$

A standard fact (e.g. [14, 23]) is

Proposition 4.2.13. *For X a Kan complex*

- *the simplicial homotopy groups π_k of $\mathbf{cosk}_n X$ vanishing in degree $k \geq n$;*
- *the canonical morphism $X \to \mathbf{cosk}_n X$ (the unit of the adjunction) is an isomorphism on all π_k in degree $k < n$;*
- *in fact, the sequence*

$$X \to \cdots \to \mathbf{cosk}_k X \to \mathbf{cosk}_{k-1} X \to \cdots \to \mathbf{cosk}_1 X \to \mathbf{cosk}_0 X \simeq *$$

is a model for the Postnikov tower of X.

Example 4.2.14. For $\mathcal{G}$ a groupoid and $N\mathcal{G}$ its simplicial nerve, the canonical morphism $N\mathcal{G} \to \mathbf{cosk}_2 N\mathcal{G}$ is an isomorphism.

Definition 4.2.15. We say a Kan complex or L_∞-groupoid X is an *n-groupoid* if the canonical morphism

$$X \to \mathbf{cosk}_{n+1} X$$

is an isomorphism. If this morphism is just a weak equivalence, we say X is an *n-type.*

We now spell out details of the Lie ∞-integration for:

(1) an ordinary Lie algebra;
(2) the *string Lie 2-algebra;*
(3) the line Lie n-algebras $b^{n-1}\mathbb{R}$.

The basic mechanism is is that discussed in [25], there for Banach ∞-groupoids. We present now analogous discussions for the context of smooth ∞-groupoids that we need for the differential refinement in 4.3 and then for the construction of the ∞-Chern–Weil homomorphism in 5

4.2.1 Ordinary Lie group

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then every smooth $\mathfrak{g}$-valued 1-form on the 1-simplex defines an element of G by parallel transport:

$$\mathrm{tra} : \Omega^1_{\mathrm{si}}([0,1], \mathfrak{g}) \to G$$
$$\omega \mapsto \mathcal{P}\mathrm{exp}\left(\int_{[0,1]} \omega\right),$$

where the right hand $\mathcal{P}\exp(\cdots)$ is notation defined to be the endpoint evaluation $f(1)$ of the unique solution $f : [0,1] \to G$ to the differential equation

$$df + r_{f_*}(\omega) = 0$$

with initial condition $f(0) = \mathrm{e}$, where $r_g : G \to G$ denotes the right action of $g \in G$ on G itself. In the special case that G is simply connected, there is a unique smooth path $\gamma : [0,1] \to G$ starting at the neutral element e such that ω equals the pullback $\gamma^*\theta$ of the Maurer–Cartan form on G. The value of the parallel transport is then the endpoint of this path in G.

More generally, this construction works in families and produces for every Cartesian space U, a parallel transport map

$$\mathrm{tra} : \Omega^1_{\mathrm{si}}(U \times [0,1], \mathfrak{g})_{\mathrm{vert}} \to C^\infty(U, G)$$

from smooth U-parameterized families of $\mathfrak{g}$-valued 1-forms on the interval to smooth functions from U to G. If we now consider a $\mathfrak{g}$-valued 1-form ω on the n-simplex instead, parallel transport along the sequence of edges $[0,1],\ [1,2], \ldots, [n-1,n]$ defines an element in G^{n+1}, and so we have an induced map $\Omega^1_{\mathrm{si}}(U \times \Delta^n, \mathfrak{g})_{\mathrm{vert}} \to \mathbf{B}G(U)_n$. This map, however is *not* in general a map of simplicial sets: the composition of parallel transport along $[0,1]$ and $[1,2]$ is in general not the same as the parallel transport along the edge $[0, 2]$ so parallel transport is not compatible with face maps. But precisely if the $\mathfrak{g}$-valued 1-form is flat does its parallel transport (over the contractible simplex) only depend on the endpoint of the path along which it is transported. Therefore, we have in particular the following

Proposition 4.2.16. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

- *Parallel transport along the edges of simplices induces a morphism of smooth ∞-groupoids*

$$\mathrm{tra} : \exp_\Delta(\mathfrak{g}) \to \mathbf{B}G.$$

- *When G is simply connected, there is a canonical bijection between smooth flat $\mathfrak{g}$-valued 1-forms A on Δ^n and smooth maps $\phi : \Delta^n \to G$ that send the 0-vertex to the neutral element. This bijection is given by $A = \phi^*\theta$, where θ is the Maurer–Cartan form of G.*

As for every morphism of Kan complexes, we can look at *coskeletal approximations* of parallel transport given by the morphism of coskeletal towers

$$\begin{array}{ccccccccc}
\exp(\mathfrak{g}) & \longrightarrow \cdots \longrightarrow & \mathbf{cosk}_{n+1}(\exp(\mathfrak{g})) & \longrightarrow & \mathbf{cosk}_n(\exp(\mathfrak{g})) & \longrightarrow \cdots \longrightarrow & * \\
\downarrow{\scriptstyle \mathrm{tra}} & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{B}G & \longrightarrow \cdots \longrightarrow & \mathbf{cosk}_{n+1}(\mathbf{B}G) & \longrightarrow & \mathbf{cosk}_n(\mathbf{B}G) & \longrightarrow \cdots \longrightarrow & *
\end{array}$$

Proposition 4.2.17. *If the Lie group G is $(k-1)$-connected, then the induced maps*

$$\mathbf{cosk}_n(\exp_\Delta(\mathfrak{g})) \to \mathbf{cosk}_n(\mathbf{B}G)$$

are acyclic fibrations in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{proj}$ for any $n \leq k$.

Proof. Recall that an acyclic fibration in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ is a morphism of simplicial presheaves that is objectwise an acyclic Kan fibration of simplicial sets. By standard simplicial homotopy theory [23], the latter are precisely the maps that have the left lifting property against all simplex boundary inclusions $\partial\Delta[p] \hookrightarrow \Delta[p]$.

Notice that for $n = 0$ and $n = 1$ the statement is trivial. For $n \geq 2$ we have an isomorphism $\mathbf{B}G \to \mathbf{cosk}_n\mathbf{B}G$. Hence we need to prove that for

$2 \leq n \leq k$ we have for all $U \in \mathrm{CartSp}$ lifts σ in diagrams of the form

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \exp_\Delta(\mathfrak{g}) \\ {\scriptstyle i}\downarrow & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle \mathrm{tra}} \\ \Delta[n] & \longrightarrow & \mathbf{B}G \end{array}$$

By parallel transport and using the Yoneda lemma, the outer diagram is equivalently given by a map $U \times \partial\Delta^p \to G$ that is smooth with sitting instants on each face Δ^{p-1}. By Proposition 4.2.7 this may be thought of as a smooth map $U \times S^{p-1} \to G$. The lift σ then corresponds to a smooth map with sitting instants $\sigma : U \times \Delta^n \to G$ extending this, hence to a smooth map $\sigma : U \times D^p \to G$ that in a neighbourhood of S^{p-1} is constant in the direction perpendicular to that boundary.

By the connectivity assumption on G there is a continuous map with these properties. By the Steenrod–Wockel-approximation theorem [49], this delayed homotopy on a smooth function is itself continuously homotopic to a smooth such function. This smooth enhancement of the continuous extension is a lift σ. □

For $n = 1$ the Kan complex $\mathbf{cosk}_1(\mathbf{B}G)$ is equivalent to the point. For $n = 2$ we have an isomorphism $\mathbf{B}G \to \mathbf{cosk}_2\mathbf{B}G$ (since $\mathbf{B}G$ is the nerve of a Lie groupoid) and so the proposition asserts that for simply connected Lie groups $\mathbf{cosk}_2 \exp_\Delta(\mathfrak{g})$ is equivalent to $\mathbf{B}G$.

Corollary 4.2.18. *If G is a compact connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, then the natural morphism $\exp_\Delta(\mathfrak{g}) \to \mathbf{B}G$ induces an acyclic fibration $\mathbf{cosk}_3\,(\exp_\Delta(\mathfrak{g})) \to \mathbf{B}G$ in the global model structure.*

Proof. Since a compact connected and simply connected Lie group is automatically 2-connected, we have an induced acyclic fibration $\mathbf{cosk}_3$ $(\exp(\mathfrak{g})) \to \mathbf{cosk}_3(\mathbf{B}G)$. Now notice that $\mathbf{B}G$ is 2-coskeletal, i.e, its coskeletal tower stabilizes at $\mathbf{cosk}_2(\mathbf{B}G) = \mathbf{B}G$. □

4.2.2 Line n-group

Definition 4.2.19. For $n \geq 1$ write $b^{n-1}\mathbb{R}$ for the *line Lie n-algebra*: the L_∞-algebra characterized by the fact that its Chevalley–Eilenberg algebra is generated from a single generator c in degree n and has trivial differential $\mathrm{CE}(b^{n-1}\mathbb{R}) = (\wedge^\bullet\langle c\rangle, d = 0)$.

Proposition 4.2.20. *Fiber integration over simplices induces an equivalence*

$$\int_{\Delta^\bullet} : \exp_\Delta(b^{n-1}\mathbb{R}) \xrightarrow{\simeq} \mathbf{B}^n\mathbb{R}.$$

Proof. By the DK correspondence we only need to show that integration along the simplices is a chain map from the normalized chain complex of $\exp_\Delta(b^{n-1}\mathbb{R})$ to $C^\infty(-)[n]$. The normalized chain complex $N_\bullet(\exp_\Delta(b^{n-1}\mathbb{R}))$ has in degree $-k$ the abelian group $C^\infty(-)\hat{\otimes}\Omega^n_{cl}(\Delta^k)$, and the differential

$$\partial : N^{-k}\left(\exp_\Delta(b^{n-1}\mathbb{R})\right) \to N^{-k+1}\left(\exp_\Delta(b^{n-1}\mathbb{R})\right)$$

maps a differential form ω to the alternating sum of its restrictions on the faces of the simplex. If ω is an element in $C^\infty(-) \otimes \Omega^n_{cl}(\Delta^k)$, integration of ω on Δ^k is zero unless $k = n$, which shows that integration along the simplex maps $N^\bullet\left(\exp_\Delta(b^{n-1}\mathbb{R})\right)$ to $C^\infty(-)[n]$. Showing that this map is actually a map of chain complexes is trivial in all degrees but for $k = n+1$; in this degree, checking that integration along simplices is a chain map amounts to checking that for a closed n-form ω on the $(n+1)$-simplex, the integral of ω on the boundary of Δ^{n+1} vanishes, and this is obvious by Stokes theorem. □

Remark 4.2.21. For $n = 1$, the morphism $\exp_\Delta(\mathbb{R}) \to \mathbf{B}\mathbb{R}$ coincides with the morphism described in Proposition 4.2.16, for $G = \mathbb{R}$.

4.2.3 Smooth string 2-group

Definition 4.2.22. Let $\mathfrak{string} := \mathfrak{so}_{\mu_3}$ be the extension of the Lie algebra $\mathfrak{so}$ classified by its 3-cocycle $\mu_3 = \frac{1}{2}\langle -, [-,-]\rangle$ according to Definition 4.1.23. This is called the *string Lie 2-algebra.* Let

$$\mathbf{B}\mathrm{String} := \mathbf{cosk}_3 \exp_\Delta(\mathfrak{so}_{\mu_3})$$

be its Lie integration. We call this the delooping of the smooth *string 2-group.*

The Banach-space ∞-groupoid version of this Lie integration is discussed in [25].

Remark 4.2.23. The 7-cocycle μ_7 on $\mathfrak{so}$ is still, in the evident way, a cocycle on $\mathfrak{so}_{\mu_3}$

$$\mu_7 : \mathfrak{so}_3 \to b^6\mathbb{R}.$$

Proposition 4.2.24. *We have an ∞-pullback of smooth ∞-groupoids*

$$\begin{array}{ccc} \mathbf{B}\mathrm{String} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3 U(1) \end{array}$$

presented by the ordinary pullback of simplicial presheaves

$$\begin{array}{ccc} \mathbf{B}\widetilde{\mathrm{String}} & \longrightarrow & \mathbf{E}\mathbf{B}^2(\mathbb{Z} \to \mathbb{R}) \\ \downarrow & & \downarrow \\ \mathbf{cosk}_3 \exp_\Delta(\mathfrak{so}) & \xrightarrow{\exp_\Delta(\mu_3)} & \mathbf{B}^3(\mathbb{Z} \to \mathbb{R}) \\ \downarrow \wr & & \\ \mathbf{B}\mathrm{Spin} & & \end{array} ,$$

where $\mathbf{B}\mathrm{String} \xrightarrow{\simeq} \mathbf{B}\widetilde{\mathrm{String}}$ *is induced by integrating the 2-form over simplices.*

Remark 4.2.25. In terms of Definition 3.2.19, $\mathbf{B}\mathrm{String}$ is the smooth $\mathbf{B}^2U(1)$-principal 3-bundle over $\mathbf{B}\mathrm{Spin}$ classified by the smooth refinement of the first fractional Pontryagin class.

Proof. Since all of $\mathbf{cosk}_3 \exp_\Delta(\mathfrak{so})$, $\mathbf{B}^3(\mathbb{Z} \to \mathbb{R})$ and $\mathbf{E}\mathbf{B}^2(\mathbb{Z} \to \mathbb{R})$ are fibrant in $[\mathrm{CartSp}^{op}, \mathrm{sSet}]_{\mathrm{proj}}$ and since $\mathbf{E}\mathbf{B}^2U(1) \to \mathbf{B}^3(\mathbb{Z} \to \mathbb{R})$ is a fibration (being the image under DK of a surjection of complexes of sheaves), we have by standard facts about homotopy pullbacks that the ordinary pullback is a homotopy pullback in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$. By [34] this presents the ∞-pullback of ∞-presheaves on CartSp. And since ∞-stackification is left exact, this is also presents the ∞-pullback of ∞-sheaves.

This ordinary pullback manifestly has 2-cells given by 2-simplices in G labelled by elements in $U(1)$ and 3-cells being 3-simplices in G such that the labels of their faces differ by $\int_{\Delta^3 \to G} \mu \,\mathrm{mod}\mathbb{Z}$. This is the definition of $\mathbf{B}\mathrm{String}$. That $\exp_\Delta(\mu_3)$ indeed presents a smooth refinement of the second fractional Pontryagin class as indicated is shown below. □

Proposition 4.2.26. *There is a zigzag of equivalences*

$$\mathbf{cosk}_3 \exp(\mathfrak{so}_{\mu_3}) \simeq \cdots \simeq \mathbf{B}\left(\hat{\Omega}\mathrm{Spin} \to P\mathrm{Spin}\right)$$

in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$, *of the Lie integration, Proposition* 4.2.24, *of* $\mathfrak{so}_{\mu_3}$ *with the strict 2-group, Definition* 3.2.10 *coming from the crossed module*

($\hat{\Omega}\mathrm{Spin} \to P\mathrm{Spin}$) *of Fréchet Lie groups, discussed in [8], consisting of the centrally extended loop group and the path group of* Spin.

This is proven in Section 5.2 of [41].

Proposition 4.2.27. *The object* $\mathbf{B}\mathrm{String} = \mathbf{cosk}_3\left(\exp(\mathfrak{so}_{\mu_3})\right)$ *is fibrant in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$.

Proof. Observe first that both object are fibrant in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ (the Lie integration by Proposition 4.2.10, the delooped strict 2-group by observation 3.2.11). The claim then follows with Proposition 4.2.26 and Proposition 3.2.12, which imply that for $C(\{U_i\}) \to \mathbb{R}^n$ the Čech nerve of a good open cover, hence a cofibrant resolutions, there is a homotopy equivalence

$$\begin{aligned}&[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U}), \mathbf{cosk}_3 \exp(\mathfrak{so}_{\mu_3})\right)\\ &\quad \simeq [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\Big(\check{C}(\mathcal{U}), \mathbf{B}(\hat{\Omega}\mathrm{Spin} \to P\mathrm{Spin})\Big).\end{aligned}$$ □

Corollary 4.2.28. *A Spin-principal bundle* $P \to X$ *can be lifted to a String-principal bundle precisely if it trivializes* $\frac{1}{2}\mathbf{p}_1$*, i.e., if the induced mophism* $\mathbf{H}(X, \mathbf{B}Spin) \to \mathbf{H}\left(X, \mathbf{B}^3 U(1)\right)$ *is homotopically trivial. The choice of such a lifting is called a String structure on the Spin-bundle.*

We discuss string structures and their twisted versions further in 6.

4.3 Principal ∞-bundles with connection

For an ordinary Lie group G with Lie algebra $\mathfrak{g}$, we have met in Section 3.2.1 the smooth groupoids $\mathbf{B}G$, $\mathbf{B}G_{\mathrm{conn}}$ and $\mathbf{B}G_{\mathrm{diff}}$ arising from G, and in 4.2 the smooth ∞-groupoid $\exp_\Delta(\mathfrak{g})$ coming from $\mathfrak{g}$, and have shown that they are related by a diagram

$$\begin{array}{ccccc} & & & & \exp_\Delta(\mathfrak{g}) \\ & & & & \downarrow \\ \mathbf{B}G_{\mathrm{conn}} & \hookrightarrow & \mathbf{B}G_{\mathrm{diff}} & \xrightarrow{\sim} & \mathbf{B}G \end{array}$$

and that $\mathbf{B}G_{\mathrm{conn}}$ is the moduli stack of G-principal bundles with connection. Now we discuss such *differential refinements* $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ and $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ that complete the above diagram for any integrated smooth ∞-group $\exp_\Delta$ $(\mathfrak{g})$. Where a truncation of $\exp_\Delta(\mathfrak{g})$ is the object that classifies G-principal

∞-bundles, the corresponding truncation of $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ classifies *principal ∞-bundles with connection.* Between $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ and $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$, we will also meet the *Chern–Weil ∞-groupoid* $\exp_\Delta(\mathfrak{g})_{\mathrm{CW}}$ which is the natural ambient for ∞-Chern–Weil theory to live in.

For the following, let $\mathfrak{g}$ be any Lie ∞-algebra.

Definition 4.3.1. The differential refinement $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ of $\exp_\Delta(\mathfrak{g})$ is the simplicial presheaf on the site of Cartesian spaces given by the assignment

$$(U,[k]) \mapsto \left\{ \begin{array}{ccc} \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) \end{array} \right\},$$

where on the right we have the set of commuting diagrams in dgcAlg as indicated.

Remark 4.3.2. This means that a k-cell in $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ over $U \in \mathrm{CartSp}$ is a $\mathfrak{g}$-valued form A on $U \times \Delta^k$ that satisfies the condition that its curvature forms F_A vanish when restricted in all arguments to vectors on the simplex. This is the analog of the *first Ehresmann condition* on a connection form on an ordinary principal bundle: the form A on the trivial simplex bundle $U \times \Delta^k \to U$ is flat along the fibres.

Proposition 4.3.3. *The evident morphism of simplicial presheaves*

$$\exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \xrightarrow{\sim} \exp_\Delta(\mathfrak{g})$$

is an acyclic fibration of smooth ∞-groupoids in the global model structure.

Proof. We need to check that, for all $U \in \mathrm{CartSp}$ and $[k] \in \Delta$ and for all diagrams

$$\begin{array}{ccc} \partial\Delta[k] & \xrightarrow{A|_\partial} & \exp_\Delta(\mathfrak{g})_{\mathrm{diff}}(U) \\ \downarrow & \overset{A}{\nearrow} & \downarrow \\ \Delta[k] & \xrightarrow{A_{\mathrm{vert}}} & \exp_\Delta(\mathfrak{g})(U) \end{array}$$

we have a lift as indicated by the dashed morphism. For that we need to extend the composite

$$\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g}) \stackrel{A_{\mathrm{vert}}}{\to} \Omega^\bullet_{\mathrm{si}}(U\times\Delta^n)_{\mathrm{vert}}$$

to an element in $\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \otimes \mathfrak{g}$ with fixed boundary value A_∂ in $\Omega^\bullet_{\mathrm{si}}(U \times \partial\Delta^k) \otimes \mathfrak{g}$. To see that this is indeed possible, use the decomposition

$$\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) = \Omega^\bullet_{\mathrm{si}}(U \times \Delta^n)_{\mathrm{vert}} \oplus \Big(\Omega^{>0}(U)\hat{\otimes}\Omega^\bullet_{\mathrm{si}}(\Delta^k)\Big)$$

to write $A_\partial = A_{\mathrm{vert}}|_{\partial\Delta^k} + A_\partial^{>0}$. Extend $A_\partial^{>0}$ to an element $A^{>0}$ in $\Omega^{>0}(U) \hat{\otimes} \Omega^\bullet_{\mathrm{si}}(\Delta^k)$. This is a trivial extension problem: any smooth differential form on the boundary of an k-simplex can be extended to a smooth differential form on the whole simplex. Then the degree 1 element $A_{\mathrm{vert}} + A^{>0}$ is a solution to our original extension problem. □

Remark 4.3.4. This means that $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ is a certain resolution of $\exp_\Delta(\mathfrak{g})$. In the full abstract theory [41], the reason for its existence is that it serves to model the canonical curvature characteristic map $\mathbf{B}G \to \flat_{\mathrm{dR}}\mathbf{B}^n U(1)$ in the ∞-topos of smooth ∞-groupoids by a truncation of the zigzag $\exp_\Delta(\mathfrak{g}) \stackrel{\sim}{\leftarrow} \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \to \exp_\Delta(b^{n-1}\mathbb{R})$ of simplicial presheaves. By the nature of acyclic fibrations, we have that for every $\exp_\Delta(\mathfrak{g})$-cocycle $X \xleftarrow{\sim_{\mathrm{loc}}} \check{C}(\mathcal{U}) \xrightarrow{g} \exp_\Delta(\mathfrak{g})$ there is a lift g_{diff} to an $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$-cocycle

$$\begin{array}{ccc}
QX & \xrightarrow{g_{\mathrm{diff}}} & \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \;. \\
\downarrow{\wr} & & \downarrow{\wr} \\
\check{C}(\mathcal{U}) & \xrightarrow{g} & \exp_\Delta(\mathfrak{g}) \\
\downarrow{\wr_{\mathrm{loc}}} & & \\
X & &
\end{array}$$

For the abstract machinery of ∞-theory to work, it is only the existence of this lift that matters. However, in practice it is useful to make certain nice choices of lifts. In particular, when X is a paracompact smooth manifold, there is always a choice of lift with the property that the corresponding curvature characteristic forms are globally defined forms on X, instead of more general (though equivalent) cocycles in total Čech–de Rham cohomology. Moreover, in this case the local connection forms can be chosen to have Δ-horizontal curvature. Lifts with this special property are *genuine ∞-connections* on the ∞-bundles classified by g. The following definitions formalize this. But it is important to note that genuine ∞-connections are but a certain choice of gauge among all differential lifts. Notably when the base X is not a manifold but for instance a non-trivial orbifold, then genuine ∞-connections will in general not even exist, whereas the differential lifts always do exist, and always support the ∞-Chern–Weil homomorphism.

Proposition 4.3.5. *If $\mathfrak{g}$ is the Lie algebra of a Lie group G, then there is a natural commutative diagram*

$$\begin{array}{ccc} \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{\sim} & \exp_\Delta(\mathfrak{g}) \\ \downarrow & & \downarrow \\ \mathbf{B}G_{\mathrm{diff}} & \xrightarrow{\sim} & \mathbf{B}G \end{array}$$

In particular, if G is $(k-1)$-connected, with $k \geq 2$, then the induced morphism $\mathrm{cosk}_k\,(\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}) \to \mathbf{B}G_{\mathrm{diff}}$ is an acyclic fibration in the global model structure.

Proof. We have seen in Remark 3.2.7 that there is a natural isomorphism $\mathbf{B}G_{\mathrm{diff}} \cong \mathbf{B}G \times \mathrm{Codisc}(\Omega^1(-;\mathfrak{g})$, so in order to give the morphism $\exp_\Delta$ $(\mathfrak{g})_{\mathrm{diff}} \to \mathbf{B}G_{\mathrm{diff}}$ making the above diagram commute we only need to give a natural morphism $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \to \mathrm{Codisc}(\Omega^1(-;\mathfrak{g})$; this is evaluation of the connection form A on the vertices of the simplex.

Assume now G is $(k-1)$-connected, with $k \geq 2$. Then, by Propositions 4.2.17 and 4.3.3, both $\mathrm{cosk}_k(\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}) \to \mathrm{cosk}_k(\exp_\Delta(\mathfrak{g}))$ and cosk_k $(\exp_\Delta(\mathfrak{g})) \to \mathbf{B}G$ are acyclic fibrations. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{cosk}_k\,(\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}) & \xrightarrow{\sim} & \mathrm{cosk}_k\,(\exp_\Delta(\mathfrak{g})) \\ \downarrow & & \downarrow\wr \\ \mathbf{B}G_{\mathrm{diff}} & \xrightarrow{\sim} & \mathbf{B}G, \end{array}$$

so, by the "two out of three" rule, also $\mathrm{cosk}_k\,(\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}) \to \mathbf{B}G_{\mathrm{diff}}$ is an acyclic fibration. □

Definition 4.3.6. The simplicial presheaf $\exp_\Delta(\mathfrak{g})_{\mathrm{CW}} \subset \exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ is the sub-presheaf of $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ on those k-cells $\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) \xleftarrow{A} \mathrm{W}(\mathfrak{g})$ that make also the bottom square in the diagram

$$\begin{array}{ccc} \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & CE(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \xleftarrow{A} & W(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) \end{array}$$

commute.

Remark 4.3.7. A k-cell in $\exp(\mathfrak{g})_{\mathrm{CW}}$ parameterized by a Cartesian space U is a $\mathfrak{g}$-valued differential form A on the total space $U \times \Delta^k$ such that

(1) its restriction to the fibre Δ^k of $U \times \Delta^k \to U$ is flat, and indeed equal to the canonical $\mathfrak{g}$-valued form there as encoded by the cocycle A_{vert} (which, recall, is the datum in $\exp(\mathfrak{g})$ that determines the G-bundle itself); this we may think of as the *first Ehresmann condition* on a connection;
(2) all its curvature characteristic forms $\langle F_A \rangle$ descend to the base space U of $U \times \Delta^k \to U$; this we may think of as a slightly weakened version of the *second Ehresmann condition* on a connection: this is the main *consequence* of the second Ehresmann condition.

These are the structures that have been considered in [43, 44].

Proposition 4.3.8. $\exp_\Delta(\mathfrak{g})_{\mathrm{CW}}$ *is fibrant in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. As in the proof of Proposition 4.2.10 we find horn fillers σ by pullback along the standard retracts, which are smooth away from the loci where our forms have sitting instants.

$$
\begin{array}{ccccc}
\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} & \longleftarrow & \Omega^\bullet_{\mathrm{si}}(U \times \Lambda^k_i) & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\
\uparrow & & \uparrow & & \uparrow \\
\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \longleftarrow & \Omega^\bullet_{\mathrm{si}}(U \times \Lambda^k_i) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\
\uparrow & & \uparrow & & \uparrow \\
\Omega^\bullet(U) & \longleftarrow & \Omega^\bullet(U) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g})
\end{array}
\qquad : \sigma
$$

□

We say that $\exp(\mathfrak{g}_\mu)_{\mathrm{CW}}$ is the *Chern–Weil ∞-groupoid of* $\mathfrak{g}$.

Definition 4.3.9. Write $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ for the simplicial sub-presheaf of $\exp_\Delta$ $(\mathfrak{g})_{\mathrm{diff}}$ given in degree k by those $\mathfrak{g}$-valued forms satisfying the following further *horizontality* condition:

- for all vertical (i.e., tangent to the simplex) vector fields v on $U \times \Delta^k$, we have

$$\iota_v F_A = 0.$$

Remark 4.3.10. This extra condition is the direct analog of the *second Ehresmann condition.* For ordinary Lie algebras we have discussed this form of the second Ehresmann condition in Section 2.2.

Remark 4.3.11. If we decompose differential forms on the products $U \times \Delta^k$ as

$$\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) = \bigoplus_{p,q \in \mathbb{N}} \Omega^p(U) \,\hat{\otimes}\, \Omega^q_{\mathrm{si}}(\Delta^k)$$

then

(1) k-simplices in $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ are those connection forms $A : W(\mathfrak{g}) \to \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)$ whose curvature form has only the (p,q)-components with $p > 0$;
(2) k-simplices in $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ are those k-simplices in $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ whose curvature is furthermore constrained to have precisely only the $(p,0)$-components, with $p > 0$.

Proposition 4.3.12. *We have a sequence of inclusions of simplicial pre-sheaves*

$$\exp_\Delta(\mathfrak{g})_{\mathrm{conn}} \hookrightarrow \exp_\Delta(\mathfrak{g})_{\mathrm{CW}} \hookrightarrow \exp_\Delta(\mathfrak{g})_{\mathrm{diff}}.$$

Proof. Let $\langle - \rangle$ be an invariant polynomial on $\mathfrak{g}$, and A a k-cell of $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$. Since $d_{W(\mathfrak{g})}\langle - \rangle = 0$, we have $d\langle F_A \rangle = 0$, and since $\iota_v F_A = 0$ we also have $\iota_v \langle F_A \rangle = 0$ for v tangent to the k-simplex. Therefore by Cartan's formula also the Lie derivatives $\mathcal{L}_v \langle F_A \rangle$ are zero. This implies that the curvature characteristic forms on $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ descend to U and hence that A defines a k-cell in $\exp_\Delta(\mathfrak{g})_{\mathrm{CW}}$. ☐

4.3.1 Examples

We consider the special case of the above general construction again for the special case that $\mathfrak{g}$ is an ordinary Lie algebra and for $\mathfrak{g}$ of the form $b^{n-1}\mathbb{R}$.

Proposition 4.3.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, for any $k \in \mathbb{N}$ there is a pullback diagram*

$$\begin{array}{ccc} \mathbf{cosk}_k \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} & \longrightarrow & \mathbf{cosk}_k \exp_\Delta(\mathfrak{g}) \\ \downarrow & & \downarrow \\ \mathbf{B}G_{\mathrm{conn}} & \longrightarrow & \mathbf{B}G \end{array}$$

in the category of simplicial presheaves.

Proof. The result is trivial for $n = 0$. For $n = 1$ we have to show that given two 1-forms $A_0, A_1 \in \Omega^1(U, \mathfrak{g})$, a gauge transformation $g : U \to G$ between them, and any lift $\lambda(u,t)dt$ of g to a 1-form in $\Omega^1(U \times \Delta^1, \mathfrak{g})$, there exists a unique 1-form $A \in \Omega^1(U \times \Delta^1, \mathfrak{g})$ whose vertical part is λ, whose curvature is of type (2,0), and such that

$$A\big|_{U\times\{0\}} = A_0; \quad A\big|_{U\times\{1\}} = A_1.$$

We may decompose such A into its vertical and horizontal components

$$A = \lambda\, dt + A_U,$$

where $\lambda \in C^\infty(U \times \Delta^1)$ and A_U in the image of $\Omega^1(U, \mathfrak{g})$. Then the horizontality condition $\iota_{\partial_t} F_A = 0$ on A is the differential equation

$$\frac{\partial}{\partial t} A_U = d_U \lambda + [A_U, \lambda].$$

For the given initial condition $A_U(t=0) = A_0$ this has a unique solution, given by

$$A_U(t) = g(t)^{-1} A_0 g(t) + g(t)^{-1} d_t g(t),$$

where $g(t) \in G$ is the parallel transport for the connection $\lambda\, dt$ along the path $[0,t]$ in the 1-simplex Δ^1. Evaluating at $t = 1$, and using $g(1) = g$, we find

$$A(1) = g^{-1} A_0 g + g^{-1} dg = A_1,$$

as required.

These computations carry on without substantial modification to higher simplices: using that $\lambda\, dt$ is required to be flat along the simplex, it follows the value of A_U at any point in the simplex is determined by a differential equation as above, for parallel transport along *any* path from the 0-vertex to that point. Accordingly we find unique lifts A, which concludes the proof. □

Corollary 4.3.14. *If G is a compact simply connected Lie group, there is a weak equivalence* $\mathbf{cosk}_3 \exp(\mathfrak{g})_{\mathrm{conn}} \xrightarrow{\sim} \mathbf{B}G_{\mathrm{conn}}$ *in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Proof. By Proposition 4.2.17 we have that $\mathbf{cosk}_3 \exp \Delta(\mathfrak{g}) \to \mathbf{B}G$ is an acyclic fibration in the global model structure. Since these are preserved under pullback, the claim follows by the above proposition. □

Proposition 4.3.15. *Integration along simplices gives a morphism of smooth ∞-groupoids*

$$\int_{\Delta^\bullet}^{\mathrm{diff}} : \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{diff}} \to \mathbf{B}^n\mathbb{R}_{\mathrm{diff}}.$$

Proof. By means of the DK correspondence we only need to show that integration along simplices is a morphism of complexes from the normalized chain complex of $\exp_\Delta(b^{n-1}\mathbb{R})$ to the cone

$$\begin{array}{ccccccccc} C^\infty(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(U) \\ \oplus & \nearrow_{\mathrm{Id}} & \oplus & \nearrow & \oplus & & \oplus & \nearrow_{\mathrm{Id}} & \\ \Omega^1(U) & \xrightarrow[d]{} & \Omega^2(U) & \longrightarrow & \cdots & \longrightarrow & \Omega^n(U) & \longrightarrow & 0 \end{array} \tag{3}$$

The normalized chain complex $N^\bullet\left(\exp_\Delta(b^{n-1}\mathbb{R})\right)$ has in degree $-k$ the subspace of $\Omega^n(U \times \Delta^k)$ consisting of those n-forms whose $(0,n)$-component $\omega_{0,n}$ lies in $C^\infty(U)\,\hat{\otimes}\,\Omega^n_{\mathrm{cl}}(\Delta^k)$; the differential

$$\partial : N^{-k}\left(\exp_\Delta(b^{n-1}\mathbb{R})\right) \to N^{-k+1}\left(\exp_\Delta(b^{n-1}\mathbb{R})\right)$$

maps an n-form ω on $U \times \Delta^k$ to the alternate sum of its restrictions to the faces of $U \times \partial\Delta^k$. For $k \neq 0$, let $\int_{\Delta^\bullet}^{\mathrm{diff}}$ be the map

$$\begin{aligned} \int_{\Delta^k}^{\mathrm{diff}} : N^{-k}\left(\exp_\Delta(b^{n-1}\mathbb{R})\right) &\to \Omega^{n-k}(-) \oplus \Omega^{n-k+1}(-) \\ \omega &\mapsto \left(\int_{\Delta^k} \omega, \int_{\Delta^k} d_{dR}\omega\right), \end{aligned}$$

and, for $k = 0$ let $\int_{\Delta^0}^{\mathrm{diff}}$ be the identity

$$\int_{\Delta^0}^{\mathrm{diff}} = \mathrm{id} : N^0\left(\exp_\Delta(b^{n-1}\mathbb{R})\right) \to \Omega^n(-).$$

The map $\int_{\Delta^\bullet}^{\mathrm{diff}}$ actually takes its values in the cone (3). Indeed, if $k > n+1$, then both the integral of ω and of $d_{dR}\omega$ are zero by dimensional reasons; for $k = n+1$, the only possibly non-trivial contribution to the integral over Δ^{n+1} comes from $d_{\Delta^{n+1}}\omega_{0,n}$, which is zero by hypothesis (where we have written $d_{dR} = d_{\Delta^k} + d_U$ for the decomposition of the de Rham differential associated with the product structure of $U \times \Delta^k$).

The fact that $\int_{\Delta^\bullet}^{\mathrm{diff}}$ is a chain map immediately follows by the Stokes formula:

$$\int_{\Delta^{k-1}} \partial\omega = \int_{\partial\Delta^k} \omega = \int_{\Delta^k} d_{\Delta^k}\omega$$

and by the identity $d_{dR} = d_{\Delta^k} + d_U$. □

Corollary 4.3.16. *Integration along simplices induces a morphism of smooth ∞-groupoids*

$$\int_{\Delta^\bullet}^{\mathrm{conn}} : \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{conn}} \to \mathbf{B}^n\mathbb{R}_{\mathrm{conn}}.$$

Proof. By Proposition 3.2.26, we only need to check that the image of the composition $\mathrm{curv} \circ \int_{\Delta^\bullet}^{\mathrm{diff}}$ lies in the subcomplex $(0 \to 0 \to \cdots \to \Omega^{n+1}_{\mathrm{cl}}(-))$ of $\flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}$, and this is trivial since by definition of $\exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{conn}}$ the curvature of ω, i.e., the de Rham differential $d_{dR}\omega$, is 0 along the simplex. □

5 ∞-Chern–Weil homomorphism

With the constructions that we have introduced in the previous sections, there is an evident Lie integration of a cocycle $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ on a L_∞-algebra $\mathfrak{g}$ to a morphism $\exp_\Delta(\mathfrak{g}) \to \exp_\Delta(b^{n-1}\mathbb{R})$ that truncates to a characteristic map $\mathbf{B}G \to \mathbf{B}^n\mathbb{R}/\Gamma$. Moreover, this has an evident lift to a morphism $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \to \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{diff}}$ between the differential refinements. Truncations of this we shall now identify with the Chern–Weil homomorphism and its higher analogues.

5.1 Characteristic maps by ∞-Lie integration

We have seen in Section 4 how L_∞-algebras $\mathfrak{g}$, $b^{n-1}\mathbb{R}$ integrate to smooth ∞-groupoids $\exp_\Delta(\mathfrak{g})$, $\exp_\Delta(b^{n-1}\mathbb{R})$ and their differential refinements $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$, $\exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{diff}}$ as well as to various truncations and quotients of these. We remarked at the end of 4.1 that a degree n cocycle μ on $\mathfrak{g}$ may equivalently be thought of as a morphism $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$, i.e., as a dg-algebra morphism $\mu : \mathrm{CE}(b^{n-1}\mathbb{R}) \to \mathrm{CE}(\mathfrak{g})$.

Definition 5.1.1. Given an L_∞-algebra cocycle $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ as in Section 4.1, define a morphism of simplicial presheaves

$$\exp_\Delta(\mu) : \exp_\Delta(\mathfrak{g}) \to \exp_\Delta(b^{n-1}\mathbb{R})$$

by componentwise composition with μ:

$$\exp_\Delta(\mu)_k : \left(\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} \stackrel{A_{\mathrm{vert}}}{\leftarrow} \mathrm{CE}(\mathfrak{g})\right) \mapsto \left(\Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} \stackrel{A_{\mathrm{vert}}}{\leftarrow} \mathrm{CE}(\mathfrak{g}) \stackrel{\mu}{\leftarrow} \mathrm{CE}(b^{n-1}\mathbb{R}) : \mu(A_{\mathrm{vert}})\right).$$

Write $\mathbf{B}^n\mathbb{R}/\mu$ for the pushout

$$\begin{array}{ccccc} \exp_\Delta(\mathfrak{g}) & \xrightarrow{\exp_\Delta(\mu)} & \exp_\Delta(b^{n-1}\mathbb{R}) & \xrightarrow[\sim]{\int_{\Delta^\bullet}} & \mathbf{B}^n\mathbb{R} \\ \downarrow & & & & \downarrow \\ \mathbf{cosk}_n \exp_\Delta(\mathfrak{g}) & & \longrightarrow & & \mathbf{B}^n\mathbb{R}/\mu \end{array} .$$

By slight abuse of notation, we shall denote also the bottom morphism by $\exp_\Delta(\mu)$ and refer to it as the *Lie integration of the cocycle* μ.

Remark 5.1.2. The object $\mathbf{B}^n\mathbb{R}/\mu$ is typically equivalent to the n-fold delooping $\mathbf{B}^n(\Lambda_\mu \to \mathbb{R})$ of the real modulo a lattice $\Lambda_\mu \subset \mathbb{R}$ of periods of μ, as discussed below. Moreover, as discussed in Section 4, we will be considering weak equivalences $\mathbf{cosk}_n \exp_\Delta(\mathfrak{g}) \stackrel{\sim}{\to} \mathbf{B}G$. Therefore $\exp_\Delta(\mu)$ defines a characteristic morphism of smooth ∞-groupoids $\mathbf{B}G \to \mathbf{B}^n(\Lambda_\mu \to \mathbb{R})$, presented by the span of morphisms of simplicial presheaves

$$\begin{array}{ccc} \mathbf{cosk}_n \exp_\Delta(\mathfrak{g}) & \xrightarrow{\exp_\Delta(\mu)} & \mathbf{B}^n\mathbb{R}/\mu \\ \downarrow \wr & & \\ \mathbf{B}G & & \end{array} .$$

Proposition 5.1.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mu : \mathfrak{g} \to b^{n-1}\mathbb{R}$ a degree n Lie algebra cocycle. Then there is a smallest subgroup Λ_μ of $(\mathbb{R}, +)$ such that we have a commuting diagram*

$$\begin{array}{ccccc} \exp_\Delta(\mathfrak{g}) & \xrightarrow{\exp_\Delta(\mu)} & \exp_\Delta(b^{n-1}\mathbb{R}) & \xrightarrow[\sim]{\int_{\Delta^\bullet}} & \mathbf{B}^n\mathbb{R} \\ \downarrow & & & & \downarrow \\ \mathbf{cosk}_n \exp_\Delta(\mathfrak{g}) & & \longrightarrow & & \mathbf{B}^n(\Lambda_\mu \to \mathbb{R}) \end{array} .$$

Proof. We exhibit the commuting diagram naturally over each Cartesian space U. The vertical map $\mathbf{B}^n\mathbb{R}(U) \to \mathbf{B}^n(\Lambda_\mu \to \mathbb{R})(U)$ is the obvious quotient map of simplicial abelian groups. Since $\mathbf{B}^n(\Lambda_\mu \to \mathbb{R})$ is $(n-1)$-connected and $\mathbf{cosk}_n \exp_\Delta(\mathfrak{g})$ is n-coskeletal, it is sufficient to define the horizontal map $\mathbf{cosk}_n \exp_\Delta(\mathfrak{g}) \to \mathbf{B}^n(\Lambda_\mu \to \mathbb{R})$ on n-cells. For the diagram to commute, the bottom morphism must send a form $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{si}}(U \times \Delta^n, \mathfrak{g})$ to the image of $\int_{\Delta^n} \mu(A_{\mathrm{vert}}) \in \mathbb{R}$ under the quotient map. For this assignment to constitute a morphism of simplicial sets, it must be true that for all $A_{\mathrm{vert}} \in \Omega^1_{\mathrm{si}}(U \times \partial\Delta^{n-1}, \mathfrak{g})$ the integral $\int_{\partial\Delta^{n+1}} A_{\mathrm{vert}} \in \mathbb{R}$ lands in $\Lambda_\mu \subset \mathbb{R}$.

Recall that we may identify flat $\mathfrak{g}$-valued forms on $\partial\Delta^{n+1}$ with based smooth maps $\partial\Delta^{n+1} \to G$. We observe that $\int_{\partial\Delta^{n+1}} A_{\mathrm{vert}}$ only depends on the homotopy class of such a map: if we have two homotopic n-spheres A_{vert} and A'_{vert} then by the arguments as in the proof of Proposition 4.2.17, using [49], there is a smooth homotopy interpolating between them, hence a corresponding extension of $\hat{A}_{\mathrm{vert}}$. Since this is closed, the fibre integrals of A_{vert} and A'_{vert} coincide.

Therefore we have a group homomorphism $\int_{\partial\Delta^{n+1}} : \pi_n(G, e_G) \to \mathbb{R}$. Take Λ_μ to be the subgroup of $\mathbb{R}$ generated by its image. This is the minimal subgroup of $\mathbb{R}$ for which we have a commutative diagram as stated. □

Remark 5.1.4. If G is compact and simply connected, then its homotopy groups are finitely generated and so is Λ_μ.

Example 5.1.5. Let G be a compact, simple and simply connected Lie group and μ_3 the canonical 3-cocycle on its semisimple Lie algebra, normalized such that its left-invariant extension to a differential 3-form on G represents a generator of $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$ in de Rham cohomology. In this case we have $\Lambda_{\mu_3} \simeq \mathbb{Z}$ and the diagram of morphisms discussed above is

$$\begin{array}{ccc}
\exp_\Delta(\mathfrak{g}) & \xrightarrow{\int_{\Delta^\bullet} \exp_\Delta(\mu)} & \mathbf{B}^3\mathbb{R} \\
\downarrow & & \downarrow \\
\mathbf{cosk}_3(\exp_\Delta(\mathfrak{g})) & \longrightarrow & \mathbf{B}^3(\mathbb{Z} \to \mathbb{R}) \\
\downarrow \wr & & \downarrow \wr \\
\mathbf{B}G & & \mathbf{B}^3 U(1)
\end{array}$$

This presents a morphism of smooth ∞-groupoids $\mathbf{B}G \to \mathbf{B}^3U(1)$. Let $X \to \mathbf{B}G$ be a morphism of smooth ∞-groupoids presented by a Čech-cocycle $X \xleftarrow{\simeq_{\mathrm{loc}}} \check{C}(\mathcal{U}) \to \mathbf{B}G$ as in Section 3. Then the composite $X \to \mathbf{B}G \to \mathbf{B}^3U(1)$ is a cocycle for a $\mathbf{B}^2U(1)$-principal 3-bundle presented by a span of simplicial presheaves

$$\begin{array}{ccccc} QX & \xrightarrow{\hat{g}} & \mathbf{cosk}_3 \exp_\Delta(\mathfrak{g}) & \xrightarrow{\exp_\Delta(\mu)} & \mathbf{B}^3(\mathbb{Z} \to \mathbb{R}) \,. \\ \uparrow\!\!\downarrow \wr & & \downarrow \wr & & \\ \check{C}(\mathcal{U}) & \longrightarrow & \mathbf{B}G & & \\ \downarrow \wr_{\mathrm{loc}} & & & & \\ X & & & & \end{array}$$

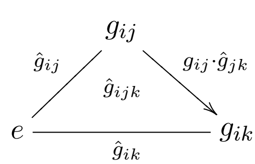
Here the acyclic fibration $QX \to \check{C}(\mathcal{U})$ is the pullback of the acyclic fibration $\mathbf{cosk}_3 \exp_\Delta(\mathfrak{g}) \to \mathbf{B}G$ from Proposition 4.2.17 and $\check{C}(\mathcal{U}) \to QX$ is any choice of section, guaranteed to exist, uniquely up to homotopy, since $\check{C}(\mathcal{U})$ is cofibrant according to Proposition 2.

This span composite encodes a morphism of 3-groupoids of Čech-cocycles

$$\mathbf{c}_\mu : \check{C}(\mathcal{U}, \mathbf{B}G) \to \check{C}(\mathcal{U}, \mathbf{B}^3(\mathbb{Z} \to \mathbb{R}))$$

given as follows

(1) it reads in a Čech-cocycle (g_{ij}) for a G-principal bundle;
(2) it forms a lift $\hat{g}$ of this Čech-cocycle of the following form:
   - over double intersections we have that $\hat{g}_{ij} : (U_i \cap U_j) \times \Delta^1 \to G$ is a smooth family of based paths in G, with $\hat{g}_{ij}(1) = g_{ij}$;
   - over triple intersections we have that $\hat{g}_{ijk} : (U_i \cap U_j \cap U_k) \times \Delta^2 \to G$ is a smooth family of 2-simplices in G with boundaries labelled by the based paths on double overlaps:



   - on quadruple intersections we have that $\hat{g}_{ijkl} : (U_i \cap U_j \cap U_k \cap U_l) : \Delta^3 \to G$ is a smooth family of 3-simplices in G, cobounding the union of the 2-simplices corresponding to the triple intersections.

(1) The morphism $\exp_\Delta(\mu) : \mathbf{cosk}_3 \exp_\Delta(\mathfrak{g}) \to \exp_\Delta(b^2\mathbb{R})$ takes these smooth families of 3-simplices and integrates over them the 3-form $\mu_3(\theta \wedge \theta \wedge \theta)$ to obtain the Čech-cocycle

$$\left(\int_{\Delta^3} \hat{g}^*_{ijkl}\, \mu(\theta \wedge \theta \wedge \theta) \quad \mod \mathbb{Z}) \in \check{C}(\mathcal{U}, \mathbf{B}^3 U(1) \right).$$

Note that $\mu_3(\theta \wedge \theta \wedge \theta)$ is the canonical 3-form representative of a generator of $H^3(G, \mathbb{Z})$.

In total the composite of spans therefore encodes a map that takes a Čech-cocycle (g_{ij}) for a G-principal bundle to a degree 3 Čech-cocycle with values in $U(1)$.

Remark 5.1.6. The map of Čech cocycles obtained in the above example from a composite of spans of simplicial presheaves is seen to be the special case of the construction considered in [5] that is discussed in Section 4 there, where an explicit Čech cocycle for the second Chern class of a principal $SU(n)$-bundle is described. See [2] for the analogous treatment of the first Pontryagin class of a principal $SO(n)$-bundle and also [3, 4].

Proposition 5.1.7. *For* $G = \mathrm{Spin}$ *the morphism* $\mathbf{c}_{\mu_3}$ *from Example* 5.1.5 *is a smooth refinement of the first fractional Pontryagin class*

$$\exp_\Delta(\mu_3) = \frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \xleftarrow{\simeq} \mathbf{cosk_3} \exp_\Delta(\mathfrak{g}) \to \mathbf{B}^3 U(1)$$

in that postcomposition with this characteristic map induces the morphism

$$\frac{1}{2}p_1 : H^1(X, \mathrm{Spin}) \to H^4(X, \mathbb{Z}).$$

Proof. Using the identification from Example 5.1.5 of the composite of spans with the construction in [5] this follows from the main theorem there. The strategy there is to refine to a secondary characteristic class with values in Deligne cocycles that provide the differential refinement of $H^4(X, \mathbb{Z})$. The proof is completed by showing that the curvature 4-form of the refining Deligne cocycle is the correct de Rham image of $\frac{1}{2}p_1$. □

Below we shall rederive this theorem as a special case of the more general ∞-Chern–Weil homomorphism. We now turn to an example that genuinely lives in higher Lie theory and involves higher principal bundles.

Proposition 5.1.8. *The canonical projection*

$$\mathbf{cosk}_7 \exp_\Delta(\mathfrak{so}_{\mu_3}) \to \mathbf{B}\mathrm{String}$$

is an acyclic fibration in the global model structure.

Proof. The 3-cells in $\mathbf{B}$String are pairs consisting of 3-cells in $\exp_\Delta(\mathfrak{so})$, together with labels on their boundary, subject to a condition that guarantees that the boundary of a 4-cell in String never wraps a 3-cycle in Spin. Namely, a morphism $\partial\Delta^4 \to \mathbf{B}$String is naturally identified with a smooth map $\phi : S^3 \to$ Spin equipped with a 2-form $\omega \in \Omega^2(S^3)$ such that $d\omega = \phi^*\mu_3(\theta\wedge\theta\wedge\theta)$. But since $\mu_3(\theta\wedge\theta\wedge\theta)$ is the image in de Rham cohomology of the generator of $H^3(\mathrm{Spin},\mathbb{Z}) \simeq \mathbb{Z}$ this means that such ϕ must represent the trivial element in $\pi_3(\mathrm{Spin})$.

Using this, the proof of the claim proceeds verbatim as that of Proposition 4.2.17, using that the next non-vanishing homotopy group of Spin after π_3 is π_7 and that the generator of $H^8(B\mathrm{String},\mathbb{Z})$ is $\frac{1}{6}p_2$. ☐

Remark 5.1.9. Therefore the Lie integration of the 7-cocycle

$$\begin{array}{ccc} \mathbf{cosk}_7 \exp_\Delta(\mathfrak{so}_{\mu_3}) & \xrightarrow{\exp_\Delta(\mu_7)} & \mathbf{B}^7(\mathbb{Z}\to\mathbb{R}) \\ \downarrow \wr & & \\ \mathbf{B}\mathrm{String} & & \end{array}$$

presents a characteristic map $\mathbf{B}\mathrm{String} \to \mathbf{B}^7U(1)$.

Proposition 5.1.10. *The Lie integration of $\mu_7 : \mathfrak{so}_{\mu_3} \to b^6\mathbb{R}$ is a smooth refinement*

$$\frac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{String} \to \mathbf{B}^7U(1).$$

of the second fractional Pontryagin class [42] in that postcomposition with this morphism represents the top horizontal morphism in

$$\begin{array}{ccc} H^1(X,\mathrm{String}) & \xrightarrow{\frac{1}{6}p_2} & H^8(X,\mathbb{Z}) \\ \downarrow & & \downarrow \cdot 6 \\ H^1(X,\mathrm{Spin}) & \xrightarrow{p_2} & H^8(X,\mathbb{Z}) \end{array}.$$

Proof. As in the above case, we shall prove this below by refining to a morphism of differential cocycles and showing that the corresponding curvature 8-form represents the fractional Pontryagin class in de Rham cohomology. ☐

5.2 Differential characteristic maps by ∞-Lie integration

We wish to lift the integration in Section 5.1 of Lie ∞-algebra cocycles from $\exp_\Delta(\mathfrak{g})$ to its differential refinement $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ in order to obtain differential characteristic maps with coefficients in differential cocycles such that postcomposition with these is the ∞-Chern–Weil homomorphism. We had obtained $\exp_\Delta(\mu)$ essentially by postcomposition of the k-cells in $\exp_\Delta(\mathfrak{g})$ with the cocycle $\mathfrak{g} \xrightarrow{\mu} b^{n-1}\mathbb{R}$. Since the k-cells in $\exp_\Delta(\mathfrak{g})_{\mathrm{diff}}$ are diagrams, we need to extend the morphism μ accordingly to a diagram. We had discussed in Section 4.1 how transgressive cocycles extend to a diagram

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{W}(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(b^{n-1}\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{inv}(\mathfrak{g}) & \xleftarrow{\langle - \rangle} & \mathrm{inv}(b^{n-1}\mathbb{R}) \end{array},$$

where $\langle - \rangle$ is an invariant polynomial in transgression with μ and cs is a Chern–Simons element witnessing that transgression.

Definition 5.2.1. Define the morphism of simplicial presheaves

$$\exp_\Delta(\mathrm{cs})_{\mathrm{diff}} : \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} \to \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{diff}}$$

degreewise by pasting composition with this diagram:

$$\exp_\Delta(cs)_k : \left(\begin{array}{ccc} \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \end{array} \right)$$

$$\mapsto \left(\begin{array}{cccccl} \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathbb{R}) & : \mu(A_{\mathrm{vert}}) \\ \uparrow & & \uparrow & & \uparrow & \\ \Omega^\bullet_{\mathrm{si}}(U \times \Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(b^{n-1}\mathbb{R}) & : \mathrm{cs}(A) \end{array} \right)$$

Write $(\mathbf{B}^n\mathbb{R}/\mathrm{cs})_{\mathrm{diff}}$ for the pushout

$$\begin{array}{ccccc} \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} & \xrightarrow{\exp_\Delta(\mathrm{cs})} & \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{diff}} & \underset{\sim}{\xrightarrow{\int_{\Delta^\bullet}}} & \mathbf{B}^n\mathbb{R}_{\mathrm{diff}} \\ \downarrow & & & & \downarrow \\ \mathbf{cosk}_n \exp_\Delta(\mathfrak{g})_{\mathrm{diff}} & & \longrightarrow & & (\mathbf{B}^n\mathbb{R}/\mathrm{cs})_{\mathrm{diff}} \end{array} .$$

Remark 5.2.2. This induces a corresponding morphism on the Chern–Weil subobjects

$$\exp_\Delta(\mathrm{cs})_{\mathrm{CW}} : \exp_\Delta(\mathfrak{g})_{\mathrm{CW}} \to \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{CW}}$$

degreewise by pasting composition with the full transgression diagram

$$\exp_\Delta(cs)_k : \left(\begin{array}{ccc} \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ \Omega^\bullet(U) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) \end{array}\right)$$

$$\mapsto \left(\begin{array}{ccccccc} \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k)_{\mathrm{vert}} & \xleftarrow{A_{\mathrm{vert}}} & \mathrm{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \mathrm{CE}(b^{n-1}\mathbb{R}) & & : \mu(A_{\mathrm{vert}}) \\ \uparrow & & \uparrow & & \uparrow & & \\ \Omega^\bullet_{\mathrm{si}}(U\times\Delta^k) & \xleftarrow{A} & \mathrm{W}(\mathfrak{g}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(b^{n-1}\mathbb{R}) & & : \mathrm{cs}(A) \\ \uparrow & & \uparrow & & \uparrow & & \\ \Omega^\bullet(U) & \xleftarrow{F_A} & \mathrm{inv}(\mathfrak{g}) & \xleftarrow{\langle - \rangle} & \mathrm{inv}(b^{n-1}\mathbb{R}) & & : \langle F_A \rangle \end{array}\right)$$

Moreover, this restricts further to a morphism of the genuine ∞-connection subobjects

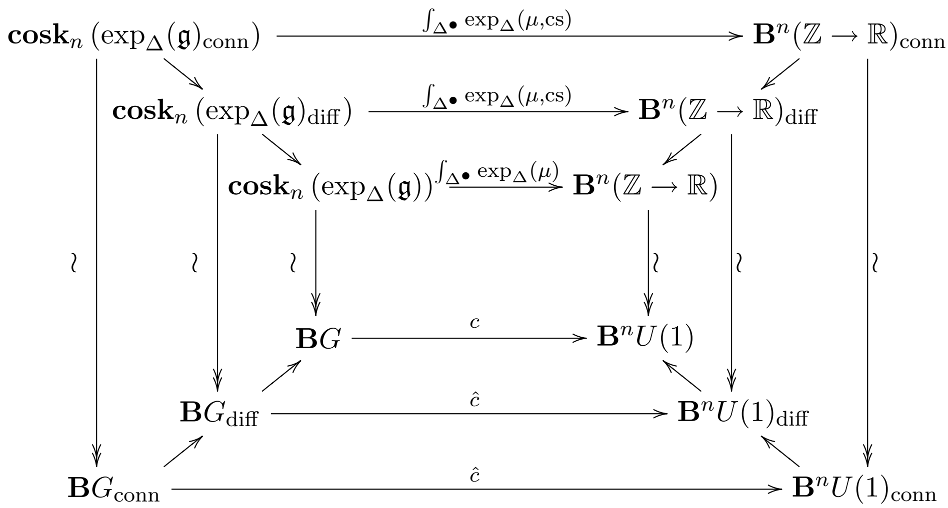
$$\exp_\Delta(\mathrm{cs})_{\mathrm{conn}} : \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} \to \exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{conn}}.$$

Indeed, the commutativity of the lower part of the diagram encodes the classical equation

$$d\mathrm{cs}(A) = \langle F_A \rangle$$

stating that the curvature of the connection $\mathrm{cs}(A)$ is the horizontal differential form $\langle F_A \rangle$ in $\Omega(U)$. This shows that the image of $\exp_\Delta(\mathrm{cs})_{\mathrm{CW}}$ is actually contained in $\exp_\Delta(b^{n-1}\mathbb{R})_{\mathrm{conn}}$, and so the restriction to $\exp_\Delta(\mathfrak{g})_{\mathrm{conn}}$ defines a morphism between the genuine ∞-connection subobjects.

Remark 5.2.3. In the typical application — see the examples discussed below — we have that $(\mathbf{B}^n\mathbb{R}/\mathrm{cs})_{\mathrm{diff}}$ is $\mathbf{B}^n(\Lambda_\mu \to \mathbb{R})_{\mathrm{diff}}$ and usually even $\mathbf{B}^n(\mathbb{Z} \to \mathbb{R})_{\mathrm{diff}}$. The above constructions then yield a sequence of spans in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$:



Here we have

- the innermost diagram presents the morphism of smooth ∞-groupoids $\mathbf{c}_\mu : \mathbf{B}G \to \mathbf{B}^n U(1)$ that is the characteristic map obtained by Lie integration from μ. Postcomposition with this is the morphism

$$\mathbf{c}_\mu : \mathbf{H}(X, \mathbf{B}G) \to \mathbf{H}\left(X, \mathbf{B}^n U(1)\right)$$

  that sends G-principal ∞-bundles to the corresponding circle n-bundles. In cohomology/on equivalence classes, this is the ordinary characteristic class

$$c_\mu : H^1(X, G) \to H^{n+1}(X, \mathbb{Z}).$$

- The middle diagram is the differential refinement of the innermost diagram. By itself this is weakly equivalent to the innermost diagram and hence presents the same characteristic map $\mathbf{c}_\mu$. But the middle diagram does support also the projection

$$\mathbf{B}G \xleftarrow{\sim} \mathbf{B}G_{\mathrm{diff}} \to \mathbf{B}^n(\mathbb{Z} \to \mathbb{R})_{\mathrm{diff}} \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}\mathbb{R}$$

  onto the curvature characteristic classes. This is the simple version of the ∞-Chern–Weil homomorphism that takes values in de Rham cohomology $H_{dR}^{n+1}(X) = \pi_0 \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R})$

$$\mathbf{H}(X, \mathbf{B}G) \to \mathbf{H}(X, \flat_{\mathrm{dR}} \mathbf{B}^{n+1}\mathbb{R}).$$

- The outermost diagram restrict the innermost diagram to differential refinements that are genuine ∞-connections. These map to genuine ∞-connections on circle n-bundles and hence support the map to secondary characteristic classes

$$\mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}}) \to \mathbf{H}\left(X, \mathbf{B}^n U(1)_{\mathrm{conn}}\right).$$

5.3 Examples

We spell out two classes of examples of the construction of the ∞-Chern–Weil homomorphism:

The Chern–Simons circle 3-bundle with connection. In Example 5.1.5 we had considered the canonical 3-cocycle $\mu_3 \in \mathrm{CE}(\mathfrak{g})$ on the semisimple Lie algebra $\mathfrak{g}$ of a compact, simple and simply connected Lie group G and discussed how its Lie integration produces a map from Čech-cocycles for G-principal bundles to Čech-cocycles for circle 3-bundles. This map turned out to coincide with that given in [5]. We now consider its differential refinement.

From Example 4.1.22 we have a Chern–Simons element cs_3 for μ_3 whose invariant polynomial is the Killing form $\langle -, - \rangle$ on $\mathfrak{g}$. By Definition 5.2.1 this induces a differential Lie integration $\exp_\Delta(\mathrm{cs})$ of μ.

As a consequence of all the discussion so far, we now simply read off the following corollary.

Corollary 5.3.1. *Let*

$$\begin{array}{ccc} \mathbf{cosk}_3 \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} & \xrightarrow{\int_{\Delta^\bullet} \exp_\Delta(\mathrm{cs}_3)} & \mathbf{B}^3(\mathbb{Z} \to \mathbb{R})_{\mathrm{conn}} \\ \downarrow{\wr} & & \\ \mathbf{B}G_{\mathrm{conn}} & & \end{array}$$

be the span of simplicial presheaves obtained from the Lie integration of the differential refinement of the cocycle from Example 5.1.5*. Composition with* *this span*

$$\begin{array}{ccccc}
QX & \xrightarrow{(\hat{g},\hat{\nabla})} & \mathbf{cosk}_3 \exp_\Delta(\mathfrak{g})_{\mathrm{conn}} & \xrightarrow{\int_{\Delta^\bullet} \exp_\Delta(\mathrm{cs}_3)} & \mathbf{B}^3(\mathbb{Z} \to \mathbb{R})_{\mathrm{conn}} \\
\downarrow \wr \; \uparrow & & \downarrow \wr & & \\
\check{C}(\mathcal{U}) & \xrightarrow{(g,\nabla)} & \mathbf{B}G_{\mathrm{conn}} & & \\
\downarrow \simeq_{\mathrm{loc}} & & & & \\
X & & & &
\end{array}$$

(where $QX \to \check{C}(\mathcal{U})$ is the pullback acyclic fibration and $\check{C}(\mathcal{U}) \to QX$ any choice of section from the cofibrant $\check{C}(\mathcal{U})$ through this acyclic fibration) produces a map from Čech cocycles for smooth G-principal bundles with connection to degree 4 Čech–Deligne cocycles

$$\hat{\mathbf{c}}_{\mathrm{cs}} : \check{C}(\mathcal{U}, \mathbf{B}G_{\mathrm{conn}}) \to \check{C}(\mathcal{U}, \mathbf{B}^3 U(1)_{\mathrm{conn}})$$

on a paracompact smooth manifold X as follows:

- *the input is a set of transition functions and local connection data (g_{ij}, A_i) on a differentiably good open cover $\{U_i \to X\}$ as in Section 2.2; (notice that there is a G-principal bundle $P \to X$ with Ehresmann connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ and local sections $\{\sigma_i : U_i \to P|_{U_i}\}$ such that $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}} g_{ij}$ and $A_i = \sigma_i^* A$)*
- *the span composition produces a lift of this data:*
  - *— on double intersections a smooth family $\hat{g}_{ij} : (U_i \cap U_j) \times \Delta^1 \to G$ of based paths in G, together with a 1-form $A_{ij} := \hat{g}_{ij}^* A_i \in \Omega^1(U_{ij} \times \Delta^1, \mathfrak{g})$;*
  - *— on triple intersections a smooth family $\hat{g}_{ijk} : (U_i \cap U_j \cap U_k) \times \Delta^2 \to G$ of based 2-simplices in G, together with a 1-form $A_{ijk} := \hat{g}_{ijk}^* A_i \in \Omega^1(U_{ijk} \times \Delta^1, \mathfrak{g})$;*
  - *— on quadruple intersections a smooth family $\hat{g}_{ijkl} : (U_i \cap U_j \cap U_k \cap U_l) \times \Delta^3 \to G$ of based 2-simplices in G, together with a 1-form $A_{ijkl} := \hat{g}_{ijkl}^* A_i \in \Omega^1(U_{ijkl} \times \Delta^1, \mathfrak{g})$;*
- *this lifted cocycle data is sent to the Čech–Deligne cocycle*

$$\begin{aligned}
&\left(\mathrm{cs}(A_i), \int_{\Delta^1} \mathrm{cs}(\hat{A}_{ij}), \int_{\Delta^2} \mathrm{cs}(\hat{A}_{ijk}), \int_{\Delta^3} \mu(\hat{A}_{ijkl})\right) \\
&\quad = \left(\mathrm{cs}(A_i), \int_{\Delta^1} \hat{g}_{ij}^* \mathrm{cs}(A), \int_{\Delta^2} \hat{g}_{ijk}^* \mathrm{cs}(A), \int_{\Delta^3} \hat{g}_{ijkl}^* \mu(A)\right),
\end{aligned}$$

where cs(A) *is the Chern–Simons 3-form obtained by evaluating a* $\mathfrak{g}$*-valued 1-form in the chosen Chern–Simons element* cs.

Proof. That we obtain Čech–Deligne data as indicated is a straightforward matter of inserting the definitions of the various morphisms. That the data indeed satisfies the Čech-cocycle condition follows from the very fact that by construction these are components of a morphism $\check{C}(\mathcal{U}) \to \mathbf{B}^3(\mathbb{Z} \to \mathbb{R})_{\mathrm{conn}}$, as discussed in Section 3. The curvature 4-form of the resulting Čech–Deligne cocycle is (up to a scalar factor) the Pontryagin form $\langle F_A \wedge F_A \rangle$. By the general properties of Deligne cohomology this represents in de Rham cohomology the integral class in $H^4(X, \mathbb{Z})$ of the cocycle, so that we find that this is a multiple of the class of the G-bundle $P \to X$ corresponding to the Killing form invariant polynomial.

In the case that $G = \mathrm{Spin}$, we have that $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$. By Proposition 5.1.3 it follows that the above construction produces a generator of this cohomology group: there cannot be a natural number ≥ 2 by which this $\mathbb{R}/\mathbb{Z}$-cocycle is divisible, since that would mean that $\mu_3(\theta \wedge \theta \wedge \theta)$ had a period greater than 1 around the generator of $\pi_3(G)$, which by construction it does not. But this generator is the fractional Pontryagin class $\frac{1}{2}p_1$ (see the review in [42] for instance). □

Definition 5.3.2. We write

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

in $\mathbf{H}$ for the morphism of smooth ∞-groupoids given by the above corollary and call this the *differential first fractional Pontryagin* map.

Remark 5.3.3. The Čech–Deligne cocycles produced by the span composition in the above corollary are again those considered in Section 4 of [5]. We may regard the above corollary as explaining the deeper origin of that construction. But the full impact of the construction in the above corollary is that it applies more generally in cases where standard Chern–Weil theory is not applicable, as discussed in the introduction. We now turn to the first non-trivial example for the ∞-Chern–Weil homomorphism beyond the traditional Chern–Weil homomorphism.

The Chern–Simons circle 7-bundle with connection. Recall from Proposition 5.1.10 the integration of the 7-cocycle μ_7 on the String 2-group. We find a Chern–Simons element $\mathrm{cs}_7 \in W(\mathfrak{so}_{\mu_3})$ and use this to obtain the differential refinement of this characteristic map.

Corollary 5.3.4. *Let*

$$\begin{array}{ccc} \mathbf{cosk}_7 \exp_\Delta(\mathfrak{so}_{\mu_3})_{\mathrm{conn}} & \xrightarrow{\int_{\Delta^\bullet} \exp_\Delta(\mathrm{cs}_7)} & \mathbf{B}^7(\mathbb{Z} \to \mathbb{R})_{\mathrm{conn}} \\ \downarrow \wr & & \\ \mathbf{B}\mathrm{String}_{\mathrm{conn}} & & \end{array}$$

be the span of simplicial presheaves obtained from the Lie integration of the differential refinement of the cocycle from Proposition 5.1.10*. Composition with this span*

$$\begin{array}{ccccc} QX & \xrightarrow{(\hat{g}, \hat{\nabla})} & \mathbf{cosk}_7 \exp_\Delta(\mathfrak{so}_{\mu_3})_{\mathrm{conn}} & \xrightarrow{\int_{\Delta^\bullet} \exp_\Delta(\mathrm{cs}_7)} & \mathbf{B}^7(\mathbb{Z} \to \mathbb{R})_{\mathrm{conn}} \\ \uparrow\downarrow \wr & & \downarrow \wr & & \\ \check{C}(\mathcal{U}) & \xrightarrow{(g, \nabla)} & \mathbf{B}\mathrm{String}_{\mathrm{conn}} & & \\ \downarrow \simeq_{\mathrm{loc}} & & & & \\ X & & & & \end{array}$$

(where $QX \to \check{C}(\mathcal{U})$ is the pullback acyclic fibration and $\check{C}(\mathcal{U}) \to QX$ any choice of section from the cofibrant $\check{C}(\mathcal{U})$ through this acyclic fibration) produces a map from Čech cocycles for smooth principal String *2-bundles with connection to degree 8 Čech–Deligne cocycles*

$$\hat{\mathbf{c}}_{\mathrm{cs}_7} : \check{C}(\mathcal{U}, \mathbf{B}\mathrm{String}_{\mathrm{conn}}) \to \check{C}(\mathcal{U}, \mathbf{B}^7 U(1)_{\mathrm{conn}})$$

on a paracompact smooth manifold X. For $P \to X$ a principal Spin *bundle with* String *structure, i.e., with a trivialization of $\frac{1}{2}p_1(P)$, the integral part of $\hat{\mathbf{c}}_{\mathrm{cs}_7}(P)$ is the second fractional Pontryagin class $\frac{1}{6}p_2(P)$.*

Proof. As above.

This completes the proof of theorem 1.0.1. □

Definition 5.3.5. We write

$$\frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{conn}}$$

in $\mathbf{H}$ for the morphism of smooth ∞-groupoids presented by the above construction, and speak of the *differential second fractional Pontryagin* map.

Remark 5.3.6. Notice how the fractional differential class $\frac{1}{6}\hat{\mathbf{p}}_2$ comes out as compared to the construction in [5], where a Čech cocycle representing $-2p_2$ is obtained . There, in order to be able to fill the simplices in the 7-coskeleton one works with chains in the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(q)$ and *multiplies* these with the cardinalities of the torsion homology groups in order to ensure that they they become chain boundaries that may be filled.

On the other hand, in the construction above the lift to the Čech cocycle of a String 2-bundle ensures that all the simplices of the cocycle in $\mathrm{Spin}(n)$ can already be filled genuinely, without passing to multiples. Therefore the cocycle constructed here is a fraction of the cocycle constructed there by these integer factors.

6 Homotopy fibres of Chern–Weil: twisted differential structures

Above we have shown how to construct refined secondary characteristic maps as morphisms of smooth ∞-groupoids of differential cocycles. This homotopical refinement of secondary characteristic classes gives access to their *homotopy fibres*. Here we discuss general properties of these and indicate how the resulting *twisted differential structures* have applications in string physics.

Some of the computations necessary for the following go beyond the scope of this paper and will not be spelled out. Details on these can be found in Section 5.2 of [41].

In Section 6.1 below we consider some basic concepts of obstruction theory in order to set the scene for the its differential refinement further below in Section 6.2. Before we get to that, it may be worthwhile to note the following subtlety.

There are two different roles played by topological spaces in the homotopy theory of higher bundles:

(1) they serve as a model for *discrete* ∞-groupoids via the standard Quillen equivalence

$$\mathrm{Top} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{sSet} \simeq \infty\mathrm{Grpd},$$

where the ∞-groupoids on the right are "discrete" in direct generalization to the sense in which a *discrete group* is discrete,

(2) and they also serve to model actual geometric structure in the sense of "continuous cohesion", that for instance distinguishes a non-discrete topological group from the underlying discrete group.

Therefore a *topological group* and more generally a *simplicial topological group* is a model for something that pairs these two aspects of topological spaces.

To make this precise, let Top be a small category of suitably nice topological spaces and continuous maps between them, equipped with the standard Grothendieck topology of open covers. Then we can consider the the ∞-topos of ∞-sheaves over Top, presented by simplicial presheaves over Top, and this is the context that contains *topological ∞-groupoids* in direct analogy to the smooth ∞-groupoids that we considered in the bulk of the paper.

$$\infty\mathrm{Grpd} \simeq \mathrm{Sh}_\infty(*) \simeq (\mathrm{sSet})^{\mathrm{op}}$$
$$\mathrm{Top}\infty\mathrm{Grpd} \simeq \mathrm{Sh}_\infty(\mathrm{Top}) \simeq ([\mathrm{Top}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{loc}})^{\mathrm{op}}$$
$$\mathbf{H} := \mathrm{Smooth}\infty\mathrm{Grpd} \simeq \mathrm{Sh}_\infty(\mathrm{CartSp}) \simeq ([\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{loc}})^{\mathrm{op}}$$

We have geometric realization functors (see Section 4 in [41])

$$\Pi : \mathrm{Top}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd}$$

and

$$\Pi : \mathrm{Smooth}\infty\mathrm{Grpd} \to \infty\mathrm{Grpd},$$

which on objects represented by simplicial topological spaces are given by the traditional geometric realization operation. For G a topological group or topological ∞-group, we write $\mathbf{B}G$ for its delooping in Top∞Grpd. Under geometric realization this becomes the standard classifying space $BG := \Pi(\mathbf{B}G)$, which, while naturally presented by a topological space, is really to be regarded as a presentation for a discrete ∞-groupoid.

6.1 Topological and smooth c-structures

An important fact about the geometric realization of topological ∞-groupoids is Milnor's theorem [37]:

Theorem 6.1.1. *For every connected ∞-groupoid (for instance presented by a connected homotopy type modelled on a topological space) there is a topological group such that its topological delooping groupoid $\mathbf{B}G$ has a geometric realization weakly equivalent to it.*

This has the following simple, but important consequence. Let G be a topological group and consider some characteristic map $c : BG \to K(\mathbb{Z}, n+1)$, representing a characteristic class $[c] \in H^{n+1}(BG, \mathbb{Z})$. Then consider the homotopy fibre

$$\begin{array}{ccc} BG^c & \longrightarrow & * \\ \downarrow & & \downarrow \\ BG & \xrightarrow{c} & K(\mathbb{Z}, n+1) \end{array}$$

formed in ∞Grpd. While this homotopy pullback takes place in discrete ∞-groupoids, Milnor's theorem ensures that there is in fact a topological group G^c such that BG^c is indeed its classifying space.

For $X \in \mathrm{Top}_{\mathrm{sm}}$, the set of homotopy classes $[X, BG]$ is in natural bijection with equivalence classes of G-principal topological bundles $P \to X$. One says that P *has c-structure* if it is in the image of $[X, BG^c] \to [X, BG]$.

Remark 6.1.2. By the defining universal property of homotopy fibres, the datum of a (equivalence class of a) principal G^c-bundle over X is equivalent to the datum of a principal G-bundle P over X whose characteristic class $[c(P)]$ vanishes.

Example 6.1.3. Classical examples of this construction are $O^{w_1} = SO$ and $U^{c_1} = SU$. Indeed is well known that the structure group of an O-bundle can be reduced to SO if and only if its first Stiefel–Withney class vanishes. More precisely, an principal SO-bundle can be seen as a principal O-bundle with a trivializiation of the associated orientation $\mathbb{Z}/2\mathbb{Z}$-bundle. Similarly, an SU-bundle is a U bundles with a trivialization of the associated determinant bundle, and such a trivialization exists if and only if the first Chern class of the given $U(n)$-bundle vanishes.

A more advanced example is the one described in Section 5: $\mathrm{Spin}^{\frac{1}{2}p_1} = \mathrm{String}$, i.e., String-bundles are Spin-bundles with a trivialization of the associated 2-gerbe.

For a more refined description of c-structures, we need to consider not just the *set* of equivalence classes of bundles, but the full cocycle ∞-groupoids: whose objects are such bundles, whose morphisms are equivalences between such bundles, whose 2-morphisms are equivalences between such equivalences, and so on. But for this purposes it matters whether we form homotopy fibres in *discrete* or in *topological* ∞-groupoids. We shall be interested in homotopy fibres of topological ∞-groupoids.

Definition 6.1.4. Let G be a simplicial topological group and $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ a classifying map. Write $\mathbf{B}G^c$ for the homotopy fibre

$$\begin{array}{ccc} \mathbf{B}G^c & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}^n U(1) \end{array}$$

of topological ∞-groupoids. Then for X a paracompact topological space, we say that the ∞-groupoid

$$\mathbf{c}\mathrm{Struc}(X) := \mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}G^c)$$

is the *∞-groupoid of topological $\mathbf{c}$-structures* on X.

Analogously, for G a smooth ∞-group and $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ a morphism of smooth ∞-groupoids as in Section 3, we write $\mathbf{B}G^c$ for its homotopy fibre in $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ and says that

$$\mathbf{c}\mathrm{Struc}(X) := \mathbf{H}(X, \mathbf{B}G^c)$$

is the *∞-groupoid of smooth $\mathbf{c}$-structures* on X.

Among the first non-trivial examples for these notions is the following

Definition 6.1.5. Let

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3 U(1)$$

be the smooth refinement of the first fractional Pontryagin class, from Corollary 5.3.1. We write

$$\mathbf{B}\mathrm{String} := \mathbf{B}\mathrm{Spin}^{\frac{1}{2}\mathbf{p}_1},$$

and call String the *smooth String 2-group.*

By Proposition 4.2.24 the smooth 2-groupoid $\mathbf{B}$String is presented by the simplicial presheaf $\mathbf{cosk}_3 \exp(\mathfrak{so}_{\mu_3})$.

Proposition 6.1.6. *Under geometric realization the delooping of the smooth String 2-group yields the classifying space of the topological string group*

$$\Pi \mathbf{B}\mathrm{String} \simeq B\mathrm{String}.$$

Moreover, in cohomology smooth $\frac{1}{2}\mathbf{p}_1$-structures on a manifold X are equivalent to ordinary String structures, hence $\frac{1}{2}p_1$-structures.

Proof. The first statement is proven in Section 5.2 of [41]. The second statement follows with Proposition 4.2.26 from Proposition 4.1 in [39]. □

6.2 Twisted differential c-structures

By the universal property of the homotopy pullback, the ∞-*groupoid of topological* $\mathbf{c}$-*structures* on X, Definition 6.1.4, can be equivalently described as the homotopy pullback

$$\begin{array}{ccc} \mathbf{c}\mathrm{Struc}(X) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}G) & \xrightarrow{\mathbf{c}} & \mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}^n U(1)) \end{array}$$

of ∞-groupoids of cocycles over X, where the right vertical morphism picks any cocycle representing the trivial class. From this point of view, there is no reason to restrict one's attention to the fibre of

$$\mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}G) \xrightarrow{\mathbf{c}} \mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}^n U(1))$$

over the distinguished point in $\mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}^n U(1)$ corresponding to the trivial $\mathbf{B}^n U(1)$-bundle over X. Rather, it is more natural and convenient to look at all homotopy fibres at once, i.e., to consider all possible (isomorphism classes of) $\mathbf{B}^n U(1)$-bundles over X.

Definition 6.2.1. For $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$ a characteristic map in either $\mathbf{H} = \mathrm{Top}\infty\mathrm{Grpd}$ or $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$, and for X a paracompact topological space or paracompact smooth manifold, respectively, let $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ be the ∞-groupoid defined by the homotopy pullback

$$\begin{array}{ccc} \mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & H^{n+1}(X;\mathbb{Z}), \\ {\scriptstyle \chi}\downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}G) & \xrightarrow{\mathbf{c}} & \mathbf{H}(X, \mathbf{B}^n U(1)) \end{array}$$

where the right vertical morphism from the cohomology set into the cocycle n-groupoid picks one basepoint in each connected component, i.e., picks a representative $U(1)$-$(n-1)$-gerbe for each degree $n+1$ integral cohomology class.

We call $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ the ∞-*groupoid of (topological or smooth) twisted* $\mathbf{c}$-*structures.* For $\tau \in \mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ we say $\mathrm{tw}(\tau) \in H^{n+1}(X;\mathbb{Z})$ is its *twist* and $\chi(\tau) \in \mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}G)$ is the (topological or smooth) *underlying G-principal ∞-bundle* of τ, or that τ is a $\mathrm{tw}(\tau)$*-twisted lift of* $\chi(\tau)$.

For $[\omega] \in H^{n+1}(X; \mathbb{Z})$ a cohomology class, $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}=[\omega]}(X)$ is the full sub-∞-groupoid of $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ on those twisted structures with twist $[\omega]$.

The following list basic properties of $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ that follow directly on general abstract grounds.

Proposition 6.2.2. (1) *The definition of* $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ *is independent, up to equivalence, of the choice of the right vertical morphism. Indeed, all choices of such are (non-canonically) equivalent as ∞-functors.*
(2) *For $\mathbf{B}G$ a topological k-groupoid for $k \leq n-1$, the ∞-groupoid* $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X)$ *is an $(n-1)$-groupoid.*
(3) *The following pasting diagram of homotopy pullbacks shows how* $\mathbf{c}\mathrm{Struc}_{\mathrm{tw}=[\omega]}(X)$ *can be equivalently seen as the homotopy fibre of* $\mathrm{Top}\infty\mathrm{Grpd}(X, \mathbf{B}G) \xrightarrow{\mathbf{c}} \mathrm{Top}\infty\mathrm{Grpd}\left(X, \mathbf{B}^n U(1)\right)$ *over a representative $U(1)$-$(n-1)$-gerbe for the cohomology class $[\omega]$:*

$$\begin{array}{ccc} \mathbf{c}\mathrm{Struc}_{\mathrm{tw}=[\omega]}(X) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle [\omega]} \\ \mathbf{c}\mathrm{Struc}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & H^{n+1}(X;\mathbb{Z}) \\ {\scriptstyle \chi}\downarrow & & \downarrow \\ \mathrm{Top}\infty\mathrm{Grpd}(X,\mathbf{B}G) & \xrightarrow{\mathbf{c}} & \mathrm{Top}\infty\mathrm{Grpd}(X,\mathbf{B}^n U(1)) \end{array} ,$$

In particular one has

$$\mathbf{c}\mathrm{Struc}_{\mathrm{tw}=0}(X) \cong \mathbf{c}\mathrm{Struc}(X).$$

We consider the following two examples, being the direct differential refinement of those of Definition 6.2:

Definition 6.2.3. For $\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^3 U(1)$ the smooth first fractional Pontryagin class from Proposition 5.1.7, we call

$$\frac{1}{2}\mathbf{p}_1\mathrm{Struc}_{\mathrm{tw}}(X)$$

the 2-groupoid of *smooth twisted String structures* on X. For $\frac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{Spin} \to \mathbf{B}^7 U(1)$ the smooth second fractional Pontryagin class from

Proposition 5.1.10, we call

$$\frac{1}{6}\mathbf{p}_2\mathrm{Struc}_{\mathrm{tw}}(X)$$

the 6-groupoid of *twisted differential fivebrane structures* on X.

The terminology here arises from the applications in string theory that originally motivated these constructions, as described in [42].

In order to explicitly compute simplicial sets modelling ∞-groupoids of smooth twisted $\mathbf{c}$-structures, the usual recipe for computing homotopy fibres applies: it is sufficient to present the smooth cocycle $\mathbf{c}$ by a fibration of simplicial presheaves and then form an ordinary pullback of simplicial presheaves. We shall discuss now how to obtain such fibrations by Lie integration of factorizations of the L_∞-cocycles $\mu_3 : \mathfrak{so} \to b^2\mathbb{R}$ and $\mu_7 : \mathfrak{so}_{\mu_3} \to b^6\mathbb{R}$. These factorizations at the L_∞-algebra level are due to [44]. A detailed proof that their Lie integration produces the desired fibration can be found in Section 5.2 of [41].

Definition 6.2.4. Let $\mathfrak{string} := \mathfrak{so}_{\mu_3}$ be the string Lie 2-algebra from Definition 4.2.22, and let $(b\mathbb{R} \to \mathfrak{string})$ be the Lie 3-algebra defined by the fact that its Chevalley–Eilenberg algebra is that of $\mathfrak{so}$ with two additional generators, b in degree 2 and c in degree 3, and with the differential extended to these as

$$d_{\mathrm{CE}}b = c - \mu_3,$$
$$d_{\mathrm{CE}}c = 0.$$

There is an evident sequence of morphisms of L_∞-algebras

$$\mathfrak{so} \to (b\mathbb{R} \to \mathfrak{string}) \to b^2\mathbb{R}$$

factoring the 3-cocycle $\mu_3 : \mathfrak{so} \to b^2\mathbb{R}$.

Proposition 6.2.5. *The Lie integration, according to Definition* 4.2.8*, of this sequence of* L_∞*-algebra morphisms is a factorization*

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{cosk}_3 \exp(\mathfrak{so}) \xrightarrow{\sim} \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{string}) \twoheadrightarrow \mathbf{B}^3U(1)$$

of the smooth refinement of the first fractional Pontryagin class from Proposition 5.1.7 *into a weak equivalence followed by a fibration in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

Corollary 6.2.6. *The 2-groupoid of twisted string structures on a smooth manifold X is presented by the ordinary fibres of*

$$[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U}), \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{so}_{\mu_3})\right) \\ \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U})\right), \quad \mathbf{B}^3 U(1)).$$

We spell out the explicit presentation for $\frac{1}{2}\mathbf{p}_1\mathrm{Struc}_{\mathrm{tw}}(X)$ further below, after passing to the following differential refinement.

Recall that when an L_∞-algebra cocycle $\mu : \mathfrak{g} \to b^n\mathbb{R}$ can be transgressed to an invariant polynomial by a Chern–Simons element, as in Section 5.2, then the smooth characteristic map $\mathbf{c} = \exp(\mu)$ refines to a *differential characteristic map*

$$\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}},$$

where

$$\mathbf{B}G_{\mathrm{conn}} := \mathbf{cosk}_{n+1} \exp_\Delta(\mathfrak{g})_{\mathrm{conn}}.$$

In terms of this there is a straightforward refinement of 6.2.1:

Definition 6.2.7. For X a smooth manifold, let $\hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X)$ be the ∞-groupoid defined by the homotopy pullback

$$\begin{array}{ccc} \hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X) & \xrightarrow{\mathrm{tw}} & \hat{H}^{n+1}_{\mathrm{diff}}(X;\mathbb{Z}) \\ \chi\downarrow & & \downarrow \\ \mathbf{H}(X, \mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{H}\left(X, \mathbf{B}^n U(1)_{\mathrm{conn}}\right) \end{array},$$

where the right vertical morphism from the cohomology set into the cocycle n-groupoid picks one basepoint in each connected component.

We call $\hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}}(X)$ the ∞-groupoid of *twisted differential* $\hat{\mathbf{c}}$*-structures* on X.

Such twisted differential structures enjoy the analogous properties listed in Proposition 6.2.2. In particular, also for differential refinements one has a natural interpretation of untwisted $\hat{\mathbf{c}}$-structures: the component of $\hat{\mathbf{c}}\mathrm{Struc}(X)$ over the 0-twist is the ∞-groupoid of $\hat{G}$-∞-connections

$$\hat{\mathbf{c}}\mathrm{Struc}_{\mathrm{tw}=0}(X) \simeq \mathrm{Smooth}\infty\mathrm{Grpd}(X, \mathbf{B}\hat{G}_{\mathrm{conn}}),$$

where $\mathbf{B}^{n-2}U(1) \to \hat{G} \to G$ is the extension of ∞-groups classified by $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^n U(1)$. This is shown in detail in Section 5.2 of [41].

6.3 Examples

We consider the following two examples:

Definition 6.3.1. For $\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ the differential first fractional Pontryagin class from Definition 5.3.2 and $\frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7U(1)_{\mathrm{conn}}$ the differential second fractional Pontryagin class from Definition 5.3.5, we call

$$\frac{1}{2}\hat{\mathbf{p}}_1\mathrm{Struc}_{\mathrm{tw}}(X),$$

the 2-groupoid of *twisted differential String structures* on X and

$$\frac{1}{6}\hat{\mathbf{p}}_2\mathrm{Struc}_{\mathrm{tw}}(X),$$

the 6-groupoid of *twisted differential Fivebrane structures* on X.

We indicate now explicit constructions of these higher groupoids of twisted structures.

Twisted differential string-structures. The factorization

$$\frac{1}{2}\mathbf{p}_1 : \mathbf{cosk}_3 \exp(\mathfrak{so}) \xrightarrow{\simeq} \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{string}) \longrightarrow\!\!\!\!\to \mathbf{B}^3U(1)$$

of the smooth first fractional Pontryagin class from Proposition 6.2.5 has a differential refinement, from which we can compute the 2-groupoid of twisted differential string structures by an ordinary pullback of simplicial sets. This is achieved by factoring the commutative diagram

$$\begin{array}{ccc} \mathrm{CE}(\mathfrak{so}) & \xleftarrow{\mu} & \mathrm{CE}(b^2\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{W}(\mathfrak{so}) & \xleftarrow{\mathrm{cs}} & \mathrm{W}(b^2\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{inv}(\mathfrak{so}) & \xleftarrow{\langle - \rangle} & \mathrm{inv}(b^2\mathbb{R}) \end{array}$$

as a commutative diagram

$$\begin{array}{ccccc}
\mathrm{CE}(\mathfrak{so}) & \xleftarrow{\sim} & \mathrm{CE}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{CE}(b^2\mathbb{R}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{W}(\mathfrak{so}) & \xleftarrow{\sim} & \tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{W}(b^2\mathbb{R}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{so}) & \xleftarrow{=} & \mathrm{inv}(b\mathbb{R} \to \mathfrak{string}) & \longleftarrow & \mathrm{inv}(b^2\mathbb{R})
\end{array}$$

as in [44]. In the above diagram, the Weil algebra $\mathrm{W}(b\mathbb{R} \to \mathfrak{string})$ is replaced by the modified Weil algebra $\tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{string})$ presented by

$$\begin{aligned}
&dt^a = -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c + r^a, \\
&db = c - \mathrm{cs}_3 + h, \\
&dc = g, \\
&dr^a = -C^a{}_{bc}t^b \wedge r^c, \\
&dh = \langle -, - \rangle - g, \\
&dg = 0.
\end{aligned}$$

Here $\{t^a\}$ are the coordinates on $\mathfrak{so}$ relative to a basis $\{\mathrm{e}_a\}$, $C^a{}_{bc}$ are the structure constants of the Lie brackets of $\mathfrak{so}$ with respect to this basis, b and c are the additional generators of the Chevalley–Eilenberg algebra $\mathrm{CE}(b\mathbb{R} \to \mathfrak{string})$, the generators r^a, h, g are the images of of t^a, b, c via the shift isomorphism, and cs_3 is a Chern–Simons element transgressing the cocycle μ_3 to the Killing form $\langle -, - \rangle$. The modified Weil algebra $\tilde{\mathrm{W}}(b\mathbb{R} \to \mathfrak{string})$ is isomorphic (via a distinguished isomorphism) to the Weil algebra $\mathrm{W}(b\mathbb{R} \to \mathfrak{string})$ as a dgca, but the isomorphism between the two does not preserves the graded subspaces of polynomials in the shifted generators. In particular, the modified algebra takes care of realizing the horizontal homotopy between $\langle -, - \rangle$ and g as a polynomial in the shifted generators, see the third item in Example 4.1.22. Since the notion of curvature forms depends on the splitting of the generators of the Weil algebra into shifted and unshifted generators (see Remark 4.1.9), the modified Weil algebra will lead to a modified version of $\exp(b\mathbb{R} \to \mathfrak{string})_{\mathrm{conn}}$, which we will denote by $\exp(b\mathbb{R} \to \mathfrak{string})_{\widehat{\mathrm{conn}}}$. This is a resolution of $\exp(\mathfrak{so})_{\mathrm{conn}}$ that is naturally adapted to the computation of the homotopy fibre of $\frac{1}{2}\mathbf{p}_1$. As we will show below, it is precisely this resolution that is the relevant one for applications to the Green–Schwarz mechanism.

Proposition 6.3.2. *Lie integration of the above diagram of differential L_∞-algebra cocycles provides a factorization*

$$\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{cosk}_3 \exp(\mathfrak{so})_{\mathrm{conn}} \xrightarrow{\sim} \mathbf{cosk}_3 \exp(b\mathbb{R} \to \mathfrak{string})_{\widehat{\mathrm{conn}}} \longrightarrow\!\!\!\!\rightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

of the differential first fractional Pontryagin class from Definition 5.3.2 *into a weak equivalence followed by a fibration in* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$.

A detailed proof can be found in Section 5.2 of [41].

Corollary 6.3.3. *The 2-groupoid of twisted differential string structures on a smooth manifold X with respect to a differentiably good open cover $\mathcal{U} = \{U_i \to X\}$ is presented by the ordinary fibres of the morphism of simplicial sets*

$$\begin{aligned}&[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U}), \mathbf{cosk}_3 \exp_\Delta(b\mathbb{R} \to \mathfrak{string})_{\widehat{\mathrm{conn}}}\right)\\ &\quad \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]\left(\check{C}(\mathcal{U}), \mathbf{B}^3 U(1)_{\mathrm{conn}}\right).\end{aligned}$$

A k-simplex for $k \leq 3$ in the simplicial set of local differential forms data describing a differential twisted string structure consists, for any k-fold intersection $U_I := U_{i_0,\ldots,i_k}$ in the cover $\mathcal{U}$, of a triple $(\omega, B, C)_I$ of connection data such the corresponding curvature data $(F_\omega, H, \mathcal{G})_I$ are horizontal. Here

$$\omega_I \in \Omega^1_{\mathrm{si}}(U_I \times \Delta^k; \mathfrak{so}), \quad B_I \in \Omega^2_{\mathrm{si}}(U_I \times \Delta^k; \mathbb{R}), \quad C_I \in \Omega^3_{\mathrm{si}}(U_I \times \Delta^k; \mathbb{R})$$

and

$$F_{\omega_I} = d\omega_I + \frac{1}{2}[\omega_I, \omega_I], \quad H_I = dB_I + \mathrm{cs}(\omega_I) - C_I, \quad \mathcal{G}_I = dC_I.$$

Remark 6.3.4. The curvature forms of a twisted string structure obey the *Bianchi identities*

$$dF_{\omega_I} = -[\omega_I, F_{\omega_I}], \quad dH_I = \langle F_{\omega_I} \wedge F_{\omega_I} \rangle - \mathcal{G}_I, \quad d\mathcal{G}_I = 0.$$

Twisted differential String structures and the Green–Schwarz mechanism. The above is the local differential form data governing what in string theory is called the *Green–Schwarz mechanism.* We briefly indicate what this means and how it is formalized by the notion of twisted differential String structures (for background and references on the physics story see for instance [42]).

The standard action functionals of higher-dimensional supergravity theories are generically *anomalous* in that instead of being functions on the space of field configurations, they are just sections of a line bundle over these spaces. In order to get a well defined action principle as input for a path-integral quantization to obtain the corresponding quantum field theories, one needs to prescribe in addition the data of a *quantum integrand.* This is a choice of trivialization of these line bundles, together with a choice of flat connection. For this to be possible, the line bundle has to be trivializable and flat in the first place. Its failure to be tivializable — its Chern class — is called the *global anomaly*, and its failure to be flat — its curvature 2-form — is called its local anomaly.

But moreover, the line bundle in question is the tensor product of two different line bundles with connection. One is a Pfaffian line bundle induced from the fermionic degrees of freedom of the theory, the other is a line bundle induced from the higher form fields of the theory in the presence of higher *electric and magnetic charge.* The Pfaffian line bundle is fixed by the requirement of supersymmetry, but there is freedom in choosing the background higher electric and magnetic charge. Choosing these appropriately such as to ensure that the tensor product of the two anomaly line bundles produces a flat trivializable line bundle is called an *anomaly cancellation* by a *Green–Schwarz mechanism.*

Concretely, the higher gauge background field of ten-dimensional heterotic supergravity is the Kalb–Ramond field, which in the absence of *fivebrane magnetic charge* is modelled by a circle 2-bundle (a bundle gerbe) with connection and curvature 3-form $H \in \Omega^3(X)$, satisfying the higher *Maxwell equation*

$$dH = 0.$$

In order to cancel the relevant quantum anomaly it turns out that a magnetic background charge density is to be added to the system, whose differential form representative is the difference $j_{\mathrm{mag}} := \langle F_{\nabla_{\mathrm{Spin}}} \wedge F_{\nabla_{\mathrm{Spin}}} \rangle - \langle F_{\nabla_{\mathrm{SU}}} \wedge F_{\nabla_{\mathrm{SU}}} \rangle$ between the Pontryagin forms of the Spin-tangent bundle and of a given SU-gauge bundle (here we leave normalization constants implicit in the definition of the invariant polynomials $\langle -, - \rangle$). This modifies the above Maxwell equation locally, on a patch $U_i \subseteq X$ to

$$dH_i = \langle F_{\omega_i} \wedge F_{\omega_i} \rangle - \langle F_{A_i} \wedge F_{A_i} \rangle.$$

Comparing with Proposition 6.3.3 we see that, while such H_i is no longer be the local curvature 3-forms of a circle 2-bundle (2-gerbe), they are that of a *twisted circle 3-bundle* – a Čech–Deligne 2-cochain that trivializes the difference of the two Chern–Simons Čech–Deligne 3-cocycles — that is part of the data of a twisted differential string-structure with $\mathcal{G}_i = \langle F_{A_i} \wedge F_{A_i} \rangle$. Note that the above differential form equation exhibits a de Rham homotopy between the two Pontryagin forms. This is the local differential aspect of the very definition of a twisted differential string-structure: a homotopy from the Chern–Simons circle 3-bundle of the Spin-tangent bundle to a given twisting circle 3-bundle, which here is itself a Chern–Simons 3-bundle, coming from an SU-bundle.

This anomaly cancellation has been known in the physics literature since the seminal article [31]. Recently, Bunke [9] has given a rigorous proof in the special case that underlying topological class of the twisting gauge bundle is trivial. This proof used the model of twisted differential string structures with topologically trivial twist given in [48]. This model is constructed in terms of bundle 2-gerbes and does not exhibit the homotopy pullback property of Definition 6.2.7 explicitly. However, the author shows that his model satisfies the properties 6.2.2 satisfied by the abstract homotopy pullback.

Twisted differential fivebrane structures. The construction of an explicit Kan complex model for the 6-groupoid of twisted differential fivebrane structures proceeds in close analogy to the above discussion for twisted differential string structures, by adding throughout one more layer of generators in the CE-algebra.

Definition 6.3.5. Write

$$\mathfrak{fivebrane} := (\mathfrak{so}_{\mu_3})_{\mu_7}$$

for the L_∞-algebra extension of the $\mathfrak{string}$ Lie 2-algebra (Definition 4.2.22) by the 7-cocycle $\mu_7 : \mathfrak{so}_{\mu_3} \to b^6\mathbb{R}$ (remark 4.2.23) according to Proposition 4.1.23. Following [43], we call this the *fivebrane Lie 6-algebra.*

Remark 6.3.6. The Chevelley–Eilenberg algebra $\mathrm{CE}(\mathfrak{fivebrane})$ is given by

$$\begin{aligned}
dt^a &= -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c, \\
db_2 &= -\mu_3 := -\frac{1}{2}\langle -, [-,-]\rangle, \\
db_6 &= -\mu_7 := -\frac{1}{8}\langle -, [-,-], [-,-], [-,-]\rangle
\end{aligned}$$

for $\{t^a\}$ and b_2 generators of degree 1 and 2, respectively, as for the string Lie 2-algebra, and b_6 a new generator in degree 6.

Definition 6.3.7. Let $(b^5\mathbb{R} \to \mathfrak{fivebrane})$ be the Lie 7-algebra defined by having CE-algebra given by

$$\begin{aligned} dt^a &= -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c, \\ db_2 &= c_3 - \mu_3, \\ db_6 &= c_7 - \mu_7, \\ dc_3 &= 0, \\ dc_7 &= 0. \end{aligned}$$

Proposition 6.3.8. *In the evident factorization*

$$\mu_7 : \ \mathrm{CE}(\mathfrak{string}) \xleftarrow{\sim} \mathrm{CE}(b^5 \to \mathfrak{fivebrane}) \longleftarrow \mathrm{CE}(b^6\mathbb{R})$$

of the 7-cocycle μ_7, the first morphism is a quasi-isomorphism.

As before, it is convenient to lift this factorization to the differential refinement by using a slightly modified Weil algebra to collect horizontal generators

$$\tilde{W}(b^5 \to \mathfrak{fivebrane}) \simeq W(b^5 \to \mathfrak{fivebrane})$$

given by

$$\begin{aligned} dt^a &= -\frac{1}{2}C^a{}_{bc}t^b \wedge t^c + r^a, \\ db_2 &= c_3 - \mathrm{cs}_3 + h_3, \\ db_6 &= c_7 - \mathrm{cs}_7 + h_7, \\ dc_3 &= g_4, \\ dc_7 &= g_8, \\ dh_3 &= \langle -,- \rangle - g_4, \\ dh_7 &= \langle -,-,-,- \rangle - g_8, \end{aligned}$$

where $\langle -,-,-,- \rangle$ is the second Pontryagin polynomial for $\mathfrak{so}$, to obtain a factorization

$$
\begin{array}{ccccc}
\mathrm{CE}(\mathfrak{string}) & \stackrel{\sim}{\longleftarrow} & \mathrm{CE}(b^5\mathbb{R} \to \mathfrak{fivebrane}) & \longleftarrow & \mathrm{CE}(b^6\mathbb{R}) \; . \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{W}(\mathfrak{string}) & \stackrel{\sim}{\longleftarrow} & \tilde{\mathrm{W}}(b^5\mathbb{R} \to \mathfrak{fivebrane}) & \longleftarrow & \mathrm{W}(b^6\mathbb{R}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{inv}(\mathfrak{string}) & \stackrel{=}{\longleftarrow} & \mathrm{inv}(b^5\mathbb{R} \to \mathfrak{fivebrane}) & \longleftarrow & \mathrm{inv}(b^6\mathbb{R})
\end{array}
$$

This is the second of the big diagrams in [44]. Using this and following through the same steps as for twisted differential string-structures above, one finds that the 6-groupoid of twisted differential fivebrane structures over some X with respect to a diffrentiably good open cover $\mathcal{U}$ has k-cells for $k \leq 7$ given by differential form data

$$
\begin{gathered}
\omega_I \in \Omega^1_{\mathrm{si}}(U_I \times \Delta^k; \mathfrak{so}), \quad (B_2)_I \in \Omega^2_{\mathrm{si}}(U_I \times \Delta^k; \mathbb{R}), (B_6)_I \in \Omega^6_{\mathrm{si}}(U_I \times \Delta^k; \mathbb{R}), \\
(C_3)_I \in \Omega^3_{\mathrm{si}}(U_I \times \Delta^k; \mathbb{R}), \quad (C_7)_I \in \Omega^7_{\mathrm{si}}(U_I \times \Delta^k; \mathbb{R})
\end{gathered}
$$

with horizontal curvature forms

$$
\begin{gathered}
F_{\omega_I} = d\omega_I + \frac{1}{2}[\omega_I, \omega_I], \\
(H_3)_I = d(B_2)_I + \mathrm{cs}_3(\omega_I) - (C_3)_I, \quad (H_7)_I = d(B_6)_I + \mathrm{cs}_7(\omega_I) - (C_7)_I, \\
(\mathcal{G}_4)_I = d(C_4)_I, \quad (\mathcal{G}_8)_I = d(C_8)_I.
\end{gathered}
$$

And *Bianchi identities*

$$
\begin{gathered}
dF_{\omega_I} = -[\omega_I, F_{\omega_I}], \\
d(H_3)_I = \langle F_{\omega_I} \wedge F_{\omega_I} \rangle - (\mathcal{G}_4)_I, \quad d(H_7)_I = \langle F_{\omega_I} \wedge F_{\omega_I} \wedge F_{\omega_I} \wedge F_{\omega_I} \rangle - (\mathcal{G}_8)_I, \\
d(\mathcal{G}_4)_I = 0, \quad d(\mathcal{G}_8)_I = 0
\end{gathered}
$$

Twisted differential fivebrane structures and the dual Green–Schwarz mechanism. On a ten-dimensional smooth manifold X a (twisted) circle 2-bundle with local connection form $\{(B_2)_I\}$ and (local) curvature forms $\{(H_3)_I\}$ is the electric/magnetic dual of a (twisted) circle 6-bundle with local connection 6-forms $\{(B_2)_I\}$ and (local) curvature forms $\{(H_7)_I\}$. It is expected (see the references in [42]) that there is a magnetic dual quantum heterotic string theory where the string — electrically charged under B_2 — is replaced by the fundamental fivebrane — magnetically charged under B_6. While the understanding of the six-dimensional fivebrane sigma model is rudimentary, its fermionic worldvolume quantum anomaly can and has been computed and the corresponding anomaly cancelling Green–Schwarz mechanism has been written down (all reviewed in [42]). If X does have differential string structure then its local differential expression is the relation

$$d(H_7)_I = \langle F_{\omega_I} \wedge F_{\omega_I} \wedge F_{\omega_I} \wedge F_{\omega_I} \rangle - \langle F_{A_I} \wedge F_{A_I} \wedge F_{A_I} \wedge F_{A_I} \rangle$$

for some normalization of invariant polynomials, where the second term is the curvature characteristic form of the next higher Chern class of the background SU-principal gauge bundle. Comparing with the above formula, we find that this is indeed modelled by twisted differential fivebrane structures.

Appendix A ∞-Stacks over the site of Cartesian spaces

Here, we give a formal description of simplicial presheaves over the site of Cartesian spaces and prove several statements mentioned in Section 3.

Definition A.1. For X a d-dimensional paracompact smooth manifold, a *differentiably good open cover* is an open cover $\mathcal{U} = \{U_i \to X\}_{i \in I}$ such that for all $n \in \mathbb{N}$ every n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ is either empty or *diffeomorphic* to $\mathbb{R}^d$.

Note that this is asking a little more than that the intersections are contractible, as for ordinary good open covers.

Proposition A.1. *Differentiably good open covers always exist.*

Proof. By Greene [24], every paracompact manifold admits a Riemannian metric with positive convexity radius $r_{\mathrm{conv}} \in \mathbb{R}$. Choose such a metric and choose an open cover consisting for each point $p \in X$ of the geodesically convex open subset $U_p := B_p(r_{\mathrm{conv}})$ given by the geodesic r_{conv}-ball at p. Since the injectivity radius of any metric is at least $2r_{\mathrm{conv}}$ [1] it follows from the minimality of the geodesics in a geodesically convex region that inside every finite non-empty intersection $U_{p_1} \cap \cdots \cap U_{p_n}$ the geodesic flow around any point u is of radius less than or equal the injectivity radius and is therefore a diffeomorphism onto its image. Moreover, the preimage of the intersection region under the geometric flow is a star-shaped region in the tangent space $T_u X$: the intersection of geodesically convex regions is itself geodesically convex, so that for any $v \in T_u X$ with $\exp(v) \in U_{p_1} \cap \cdots \cap U_{p_n}$

the whole geodesic segment $t \mapsto \exp(tv)$ for $t \in [0,1]$ is also in the region. So we have that every finite non-empty intersection of the U_p is diffeomorphic to a star-shaped region in a Euclidean space. It is then a folk theorem that every star-shaped region is diffeomorphic to an $\mathbb{R}^n$; an explicit proof of this fact is in theorem 237 of [17]. □

Recall the following notions [29].

Definition A.2. A *coverage* on a small category $\mathcal{C}$ is for each object $U \in \mathcal{C}$ a choice of collections of morphisms $\mathcal{U} = \{U_i \to U\}$ — called *covering families* — such that whenever $\mathcal{U}$ is a covering family and $V \to U$ any morphism in $\mathcal{C}$ there exists a covering family $\{V_j \to V\}$ such that all diagrams

$$\begin{array}{ccc} V_j & \xrightarrow{\exists} & U_i \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

exist as indicated. The *covering sieve* corresponding to a covering family $\mathcal{U}$ is the colimit

$$S(\mathcal{U}) = \lim_{\to\, [k]\in\Delta} \check{C}(\mathcal{U})_k \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

of the Čech-nerve, formed after Yoneda embedding in the category of presheaves on $\mathcal{C}$.

Definition A.3. A *site* is a small category $\mathcal{C}$ equipped with a coverage. A *sheaf* on a site is a presheaf $A : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ such that for each covering sieve $S(\mathcal{U}) \to U$ the morphism

$$A(U) \simeq [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](U, A) \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](S(\mathcal{U}), A)$$

is an isomorphism.

Remark A.1. Often this is formulated in terms of Grothendieck topologies instead of coverages. But every coverage induces a unique Gorthendieck topology such that the corresponding notions of sheaf coincide. An advantage of using coverages is that there are fewer morphisms to check the sheaf condition against.

In the language of left exact reflective localizations: the coverage sieve projections of a covering family form a small set such that localizing the presheaf category at this set produces the category of sheaves. This localization however inverts more morphisms than just the coverage sieves. This saturated class of inverted morphisms contains also the sieve projections of the corresponding Grothendieck topology.

Below we use this for obtaining the ∞-stacks/∞-sheaves by left Bousfield localization just at a coverage.

Corollary A.1. *Differentiably good open covers form a coverage on the category* CartSp.

Proof. The pullback of a differentiably good open cover always exists in the category of manifolds, where it is an open cover. By the above, this may always be refined again by a differentiably good open cover. $\square$

Definition A.4. We consider CartSp as a site by equipping it with this differentiably-good-open-cover coverage.

Definition A.5. Write $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ for the global projective model category structure on simplicial presheaves whose weak equivalences and fibrations are objectwise those of simplicial sets. Write $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ for the left Bousfield localization of $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj}}$ at at the set of coverage Čech-nerve projections $\check{C}(\mathcal{U}) \to U$. This is a simplicial model category with respect to the canonical simplicial enrichment of simplicial presheaves, see [15]. For X, A two objects, we write $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](X, A) \in \mathrm{sSet}$ for the simplicial hom-complex of morphisms between them.

Proposition A.2. *In* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *the Čech-nerve* $\check{C}(\mathcal{U}) \to X$ *of a differentiably good open cover over a paracompact smooth manifold* X *is a cofibrant resolution of* X.

Proof. By assumption $\check{C}(\mathcal{U})$ is degreewise a coproduct of representables (this is what the definition of *differentiably good open cover* formulates). Clearly its degeneracies split off as a direct summand in each degree (the summand of intersections $U_{i_0} \cap \cdots U_{i_n}$ where at least one index repeats). With this it follows from corollary 9.4 in [15] that $\check{C}(\mathcal{U})$ is cofibrant in the global projective model structure. Since left Bousfield localization keeps the cofibrations unchanged, it follows that it is also cofibrant in the local structure. That the projection $\check{C}(U) \to X$ is a weak equivalence in the local structure follows by using our theorem A.1 below in Proposition A.4 of [13]. $\square$

Corollary A.2. *The fibrant objects of* $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{proj,loc}}$ *are precisely those simplicial presheaves* A *that are objectwise Kan complexes and such that for all differentiably good open covers* $\mathcal{U}$ *of a Cartesian space* U *the* *induced morphism*

$$A(U) \xrightarrow{\simeq} [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](U, A) \to [\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}](\check{C}(\mathcal{U}), A)$$

is a weak equivalence of Kan complexes.

This is the descent condition or ∞*-sheaf/*∞*-stack condition on* A*.*

Proof. By standard facts about left Bousfield localizations we have that the fibrant objects are the degreewise fibrant object such that the morphisms

$$\mathbb{R}\mathrm{Hom}(U, A) \to \mathbb{R}\mathrm{Hom}(\check{C}(\mathcal{U}), A)$$

are weak equivalences of Kan complexes, where $\mathbb{R}\mathrm{Hom}$ denotes the right derived simplicial hom-complex in the global projective model structure. Since every representable U is cofibrant and since $\check{C}(\mathcal{U})$ is cofibrant by the above proposition, these hom-complexes are equivalent to the hom-complexes in $[\mathrm{CartSp}^{\mathrm{op}}, \mathrm{sSet}]$ as indicated. □

Finally we establish the equivalence of the localization at a coverage that we are using to the localization at the corresponding Grothendieck topology, which is the one commonly found discussed in the literature.

Theorem A.1. *Let* $\mathcal{C}$ *be any small category equipped with a coverage given by covering families* $\{U_i \to U\}$*.*

Then the ∞*-topos presented by the left Bousfield localization of* $[\mathcal{C}^{\mathrm{op}}, \mathrm{CartSp}]_{\mathrm{proj}}$ *at the coverage covering families is equivalent to that presented by the left Bousfield localization at the covers for the corresponding Grothendieck topology.*

We prove this for the *injective* model structure on simplicial presheaves. The result then follows since that is Quillen equivalent to the projective one and so presents the same ∞-topos.

Write $S(\mathcal{U}) \to j(U)$ for the sieve corresponding to a covering family, regarded as a subfunctor of the representable functor $j(U)$ (the Yoneda embedding of U), which we both regard as simplicially discrete objects in $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]$. Write $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,cov}}$ for the left Bousfield localization of the injective model structure at the morphisms $S(\mathcal{U}) \to j(U)$ corresponding to covering families.

Lemma A.1. *A subfunctor inclusion* $\tilde{S} \hookrightarrow j(U)$ *corresponding to a sieve that contains a covering sieve* $S(\mathcal{U})$ *is a weak equivalence in* $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,cov}}$

Proof. Let J be the set of morphisms in the bigger sieve that are not in the smaller sieve. By assumption we can find for each $j \in J$ a covering family $\{V_{j,k} \to V_j\}$ such that for all j, i the diagrams

$$\begin{array}{ccc} V_{j,k} & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ V_j & \xrightarrow{f} & U \end{array}$$

commute. Consider then the commuting diagram

$$\begin{array}{ccc} \coprod_j S(\{V_{j,k}\}) & \hookrightarrow & S(\{U_i\} \cup \{V_{j,i}\})\,. \\ \downarrow \wr & & \downarrow \\ \coprod_j j(V_j) & \longrightarrow & S(\{U_i\} \cup \{V_j\}) = \tilde{S} \end{array}$$

Observe that this is a pushout in $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]$, that the top morphism is a cofibration in $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}$ and hence in $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,cov}}$, that the left morphism a weak equivalence in the local structure and that by general properties of left Bousfield localization the localization is left proper. Therefore the pushout morphism $S(\{U_i\} \cup \{V_{j,k}\}) \to S(\{U_i\} \cup \{V_j\}) = \tilde{S}$ is a weak equivalence.

Then observe that from the horizontal morphisms of the above commuting diagrams that defined the covers $\{V_{j,k} \to V_j\}$ we have an induced morphism $S(\{U_i\} \cup \{V_{j,k}\}) \to S(\{U_i\})$ that exhibit $S(\{U_i\})$ as a retract

$$\begin{array}{ccccc} S(\{U_i\}) & \longrightarrow & S(\{U_i\} \cup \{V_{j,k}\}) & \longrightarrow & S(\{U_i\})\,. \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{S} & \xrightarrow{=} & \tilde{S} & \xrightarrow{=} & \tilde{S} \end{array}$$

By closure of weak equivalences under retracts, this shows that the inclusion $S(\{U_i\}) \to \tilde{S}$ is a weak equivalence. By 2-out-of-3 this finally means that $\tilde{S} \hookrightarrow j(U)$ is a weak equivalence. □

Corollary A.3. *For $S(\{U_i\}) \to j(U)$ a covering sieve, its pullback f^*S $(\{U_i\}) \to j(V)$ in $[\mathcal{C}, \mathrm{sSet}]$ along any morphism $j(f) : j(V) \to j(U)$*

$$\begin{array}{ccc} f^*S(\{U_i\}) & \longrightarrow & S(\{U_i\}) \\ \downarrow & & \downarrow \\ j(V) & \xrightarrow{j(f)} & j(U) \end{array}$$

is also a weak equivalence.

Lemma A. 2. *If $S(\{U_i\}) \to j(U)$ is the sieve of a covering family and $\tilde{S} \hookrightarrow j(U)$ is any sieve such that for every $f_i : U_i \to U$ the pullback $f^*\tilde{S}$ is a weak equivalence, then $\tilde{S} \to j(U)$ becomes an isomorphism in the homotopy category.*

Proof. First note that if $f_i^*\tilde{S}$ is a weak equivalence for every i, then the pullback of $\tilde{S}$ to any element of the sieve $S(\{U_i\})$ is a weak equivalence. Use the Yoneda lemma to write

$$S(\{U_i\}) \simeq \varinjlim_{V \to U_i \to U} j(V).$$

Then consider these objects in the ∞-category of ∞-presheaves that is presented by $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj}}$ [34]. Since that has universal colimits we have the pullback square

$$\begin{array}{ccccc}
i^* \varinjlim j(V) & \xrightarrow{\sim} & \varinjlim f_V^* \tilde{S} & \longrightarrow & \tilde{S} \\
 & & \downarrow & & \downarrow{\scriptstyle i} \\
S(\{U_i\}) & \xrightarrow{\sim} & \varinjlim\limits_{f_V : V \to U_i \to U} j(V) & \xrightarrow{(f_V)} & j(U)
\end{array}$$

and the left vertical morphism is a colimit over morphisms that are weak equivalences in $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,loc}}$. By the general properties of reflective sub-∞-categories this means that the total left vertical morphism becomes an isomorphism in the homotopy category of $[\mathcal{C}^{\mathrm{op}}, \mathrm{sSet}]_{\mathrm{inj,cov}}$. Also the bottom morphism is an isomorphism there, and hence the right vertical one is. □

Proof of the theorem. The two lemmas show that all morphisms $S(\{V_j\}) \to j(V)$ for covering sieves of the Grothendieck topology that is generated by the coverage are also weak equivalences in the left Bousfield localization just at the coverage sieves. It follows that this coincides with the localization at the full Grothendieck topology. □

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