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Riemann–Hilbert approach to the time-dependent generalized sine kernel

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Abstract

We derive the leading asymptotic behavior and build a new series representation for the Fredholm determinant of integrable integral operators appearing in the representation of the time and distance-dependent correlation functions of integrable models described by a six-vertex R-matrix. This series representation opens a systematic way for the computation of the long-time, long-distance asymptotic expansion for the correlation functions of the aforementioned integrable models *away* from their free fermion point. Our method builds on a Riemann–Hilbert based analysis.

1 Introduction

Highly structured determinants appear as a natural way for representing the correlation functions in integrable models that are equivalent to the so-called free fermions. It was already shown by Kaufman and Onsager that

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certain two-point functions of the two-dimensional (2D)-Ising model can be represented by Toeplitz determinants [30]. Then Montroll *et al.* [44] made this observation more systematic by expressing the so-called row-to-row two-point function of this model in terms of a Toeplitz determinant. It was observed by Lieb *et al.* [40] that such Toeplitz determinant-based representations also hold for the so-called XY model. Then, the systematic study of the correlation functions of the impenetrable Bose gas, the XY model or its isotrpoic version the XX model lead to the representation of various correlators in terms of Fredholm determinants (or their minors) of the so-called integrable operators [7, 8, 36, 39, 42, 47]. Such types of Fredholm determinants also appear in other branches of mathematical physics. For instance, the determinant of the so-called sine-kernel acting on an interval J is directly related to the gap probability (probability that in the bulk scaling limit a given matrix has no eigenvalues lying in J) in the Gaussian unitary ensemble [21]. Integrable integral operators [10] are operators of the type $I + V$ where the integral kernel V takes a very specific form. This fact allows for a relatively simple characterization of the resolvent kernel and often for a construction of a system of partial differential equations satisfied by the associated Fredholm determinant or minors thereof [14, 24, 26, 29, 49].

In all of the aforementioned examples, the integrable integral operators $I + V$ act on some curve $\mathscr{C}$ with a kernel $V(\lambda, \mu)$ depending, in an oscillatory way, on a parameter x. In the previous examples a lot of interesting information can be drawn out of the asymptotic behavior of $\det[I + V]$ for large values of x. For instance, when dealing with the correlation functions of integrable models, x plays the role of a spacial and/or temporal separation between the two operators entering in the correlation function. In such a case, computing the large-x asymptotic expansion of the associated Fredholm determinants, allows one to test the predictions of conformal field theories. The form of the asymptotic behavior of the pure sine kernel determinant $\log\det[I + S]$ was strongly argued in [6, 19] and then proven, to some extend, using operator methods [3, 20, 51]. Also, the discovery of non-linear differential equations of Painlevé V type for this determinant [29] allowed to compute many terms in the large-x asymptotic expansion of the associated correlation functions [29, 41, 42, 43]

However, a really systematic and efficient approach to the asymptotic analysis of various quantities related to integrable integral operators $I + V$ has been made possible thanks to the results obtained in [26]. There it was shown that the analysis of such operators can be reduced to a resolution of an associated Riemann–Hilbert problem (RHP). The jump contour in this RHP coincides with the one on which the integral operator acts and the jump matrix is built out of the functions entering in the description of the kernel. In this way, one deals with a RHP depending on x in an oscillatory way.

The asymptotic analysis of their solutions is possible thanks to the non-linear steepest descent method of Deift–Zhou [16, 17]. It is in this context that the full characterization of the leading asymptotic behavior of Fredholm determinants of kernels related to correlation functions in free-fermion equivalent models (the long-distance, long-time/long-distance at zero and also non-zero temperature) has been carried out in the series of papers [4, 9, 23, 25, 27, 28]. Also, it was shown that the value of Dyson's constant arising in the asymptotics of the pure sine kernel determinant can be obtained in the RHP-based framework as well [13, 38].

This paper deals with an extension of these analysis to the case of a Fredholm determinant of an integrable integral operator whose integral kernel has a more involved structure then in the aforementioned cases. We call our kernel the time-dependent generalized sine kernel. The Fredholm determinant we analyze arises in the representation of the zero temperature long-distance/long-time asymptotic behavior of two-point functions in a wide class of integrable models *away* from their free fermion point. In particular, its asymptotics expansion (and especially the new series representation that we obtain for it) plays a crucial role in the computation of the long-time/long-distance asymptotic behavior of these two-point functions. Therefore the results in this paper can be seen as a first step towards such an asymptotic analysis of two-point functions. Let us be slightly more specific.

In a wide class of algebraic Bethe Ansatz solvable models, one is able to compute the so-called form factors (matrix elements of local operators) and represent them as finite-size determinants [35, 46]. It has been shown in [36] that, for free fermion equivalent models, it is possible to build on these representations so as to explicitly sum-up the form factor expansion and compute the zero-temperature (and even the non-zero temperature) correlation functions of the model. In the limit of infinite lattice sizes, a two-point function is then represented by a Fredholm determinant (or its minors) of an integrable integral operator $I + V$ acting on some contour $\mathscr{C}$ determined by the properties of the model. For time and space translation invariant models, the kernel V depends on the distance separating the two operators as well as on the difference of time evolution between them. One can show that for general free-fermion type models, the integral operator $I + V$ associated with the form factor expansion of the time and space-dependent two-point functions acts on a finite subinterval $[\,-q\,;q\,]$ of $\mathbb{R}$ and its kernel V belongs to the class of kernels

$$V\left(\lambda,\mu\right) = 4\frac{\sin\left[\pi\nu\left(\lambda\right)\right]\sin\left[\pi\nu\left(\mu\right)\right]}{2i\pi\left(\lambda-\mu\right)}\left\{E\left(\lambda\right)e\left(\mu\right)-E\left(\mu\right)e\left(\lambda\right)\right\}. \qquad (1.1)$$

There ν is some function encoding the fine structure of the excitations above the ground state whereas e and E are oscillating factors. The function E is expressed in terms of e

$$E(\lambda) = ie(\lambda) \left\{ \fint_{\mathscr{C}_E} \frac{ds}{2\pi} \frac{e^{-2}(s)}{s-\lambda} + \frac{e^{-2}(\lambda)}{2} \cot[\pi\nu(\lambda)] \right\} . \tag{1.2}$$

The functions ν and e just as the integration curve $\mathscr{C}_E$ appearing in (1.2) depend on the specific model that one considers. We will give more precision about their properties in the core of the paper. We stress that, for free-fermion equivalent models, $\nu(\lambda)$ is some constant and e takes a simple form. It was in such a context that the asymptotic analysis of det $[I+V]$ has been carried out previously.

When considering integrable models that are away of their free-fermion point, as it has been shown in [34, 37], it is as well possible to build on the finite-size determinant representation for the form factors of local operators in integrable models out of their free fermion point so as to sum up the form factor series over the relevant sector of excited states. Even though Fredholm determinant based representations are no longer possible in models away from their free fermion point, the above procedure leads to a representation for the time and space-dependent two-point functions in terms of series of multiple integrals which take the generic form

$$\sum_{n\geq 0} \frac{1}{n!} \int_{-q}^{q} \prod_{a=1}^{n} \frac{d\lambda_a}{2\mathrm{i}\pi} \oint_{\mathscr{C}_{\boldsymbol{z}}} \prod_{a=1}^{n} \frac{dz_a}{2\mathrm{i}\pi} \oint_{\mathscr{C}_{\boldsymbol{y}}} \prod_{a=1}^{n} \frac{dy_a}{2\mathrm{i}\pi} \prod_{a=1}^{n} \frac{\mathrm{e}^{\mathrm{i}x[u(y_a)-u(\lambda_a)]}}{z_a-\lambda_a}$$
$$\cdot \det_n \left[\frac{1}{z_a-\lambda_b} \right] \cdot \mathcal{F}_n \begin{pmatrix} \lambda_1, \dots, \lambda_n \\ y_1, \dots, y_n \end{pmatrix} . \tag{1.3}$$

There, $\mathscr{C}_{\boldsymbol{z}}$ and $\mathscr{C}_{\boldsymbol{y}}$ are some contours in $\mathbb{C}$ surrounding $[-q\,;q]$, u is a holomorphic function in a neighborhood of $[-q\,;q] \cup \mathscr{C}_{\boldsymbol{y}}$ and $\mathcal{F}_n$ some function. In fact, an analogous type of series representation (1.3) have been first obtained, through other method, in [33], this in the case of the space-dependent functions only. That paper also proposed an approach to the extraction of the large-x asymptotic behavior out of the "pure-distance" analog of the series (1.3).

An important observation made in [34, 37] is that the intermediate computations can be shown to boil down to effective generalized free fermionic models. As such, they involve, again, the Fredholm determinants of operators $I+V$ with V given by (1.1). However, then, the functions ν and e become much more complex that at the free fermion point. In some sense, the approach of [34, 37] shows that kernels (1.1) appear as a natural basis of special functions allowing one to represent the correlation functions of a wide class of interacting (i.e., away from their free fermion point) integrable models as certain (infinite) linear combinations thereof. Therefore, the main motivation for our study of the time-dependent generalized sine kernel (1.1) is to obtain a convenient and effective representation — that we call the Natte series[1] — for the associated Fredholm determinant. The Natte series allows one to re-sum the aforementioned decomposition of the representation (1.3) as a linear combination of Fredholm determinants into some compact and explicit form. The latter provides one with a new type of series of multiple integrals representation for two-point functions. Moreover, it is built in such a way that one is able to *read-off* the asymptotic behavior of the correlators out of it. Therefore, the results established in this paper provide one of the fundamental tools that are necessary for carrying out the long-distance and large-time asymptotic analysis of the two-point functions in massless integrable models proposed in [34, 37].

This article contains two main results. We first derive the leading asymptotic behavior of the Fredholm determinant of $I + V$ understood as acting on $L^2([-q\,;q])$, with $q < +\infty$. This sets the ground for the second main result of the paper. Namely, we derive a new series representation — the Natte series — for the Fredholm determinant. This series is converging rather fast in the asymptotic $x \to +\infty$ regime. Its main advantage is to provide a direct (i.e., without the need to perform any additional analysis) approach to the asymptotic expansion of the determinant. As already stressed out, this series representation plays a crucial role in the computation of the large-time/long-distance asymptotic expansion of the two-point functions in integrable models corresponding to a six-vertex R-matrix. Also, the very form of the asymptotic expansion stemming from the Natte series proves several conjectures relative to the structure of the asymptotic expansions for certain particularizations (for specific values of ν, and e) of such Fredholm determinants [45, 48]. Also, upon specialization, it yields the general structure of the asymptotic expansion of the fifth Painleve transcendent associated to the pure sine kernel [14, 29]

This article is organized as follows. In Section 2, we outline the main assumptions that we rely upon throughout the article and give a discussion of the class of functions e that we deal with. After introducing several notations, we present the two main results of the paper. The remaining part is of technical nature. In Section 3, we present the RHP problem that is at

[1]The origin of this name issues from the so-called pig-tail (or braid) hairstyle that is called Natte in French. A braid is a specifically ordered reorganization of the loose hair-do style. Similarly, the Natte series reorganizes the Fredholm series in a very specific way, so that the resulting representation is perfectly fit for carrying out an asymptotic expansion.

the base of the asymptotic analysis of det $[I+V]$ and the construction of its Natte series. We also outline the chain of transformations corresponding to the implementation of the Deift–Zhou [17] steepest-descent method. In Section 5, we build the various local parametrices. This brings the original RHP into one that can be solved through a series expansion of the associated singular integral equation [5]. The latter naturally provides the large-x asymptotic expansion of the solution. We build on these results so as to derive the leading asymptotic expansion of the Fredholm determinant in Section 6. Finally, Section 7 is devoted to the construction of Natte series for the Fredholm determinant of $I+V$. In particular, we establish the main properties of such series. We then give a conclusion and discuss the further possible applications. In Appendix A, we recall all the properties of the special functions that we use in this article. In Appendix B, we gather some proofs relative to the structure of the large-x asymptotic expansions of certain matrix valued Neumann series representing the solution to a singular integral equation of interest to us. In appendix C, we establish bounds for certain matrices appearing in our analysis.

2 The main results and assumptions

In this article, we will focus on the case where the function e takes the form

$$e^{-1}(\lambda) = \mathrm{e}^{\mathrm{i}\frac{xu(\lambda)}{2}+\frac{g(\lambda)}{2}}. \tag{2.1}$$

$e(\lambda)$ is quickly oscillating in the $x \to +\infty$ limit and the function g entering in the definition of $e(\lambda)$ has been introduced so as to allow for some finite, x-independent oscillatory behavior of the function $e(\lambda)$. The principal value integral appearing in the definition of E (1.2) is carried out along a curve $\mathscr{C}_E$ which corresponds to a slight deformation of the real axis and is depicted in figure 1. Under the forthcoming hypothesis, such a contour allows to strengthen the convergence of the integral defining E at infinity (in the case of $\mathbb{R}$, the convergence would be the one of an oscillating non-absolutely integrable power-law whereas it is exponentially fast along $\mathscr{C}_E$).

2.1 The main assumptions

Throughout this paper, we make several assumptions on the function u, g and ν entering in the description of the integrable kernel (1.1).

- There exists an open neighborhood U of $\mathbb{R}$ such that u and g are simultaneously holomorphic on U.

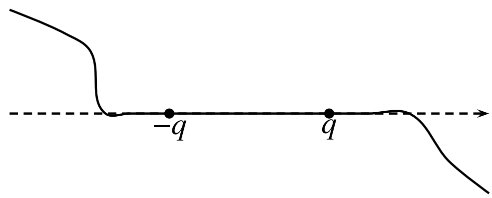

Figure 1: Contour $\mathscr{C}_E$ for the definition of E.

- The function g is bounded on U.
- The function u is real valued on $\mathbb{R}$ and has a unique saddle-point in U located at $\lambda_0 \in \mathbb{R}$. This saddle-point is a zero of u' with multiplicity one, that is to say $\exists\,!\,\lambda_0 \in \mathbb{R} : u'(\lambda_0) = 0$ and $u''(\lambda_0) < 0$.

  We also assume that the saddle-point lies away from the boundaries: $\lambda_0 \neq \pm q$.
- u is such that, given any $\eta > 0$, $\mathrm{e}^{\mathrm{i}\eta u(\lambda)}$ decays exponentially fast in λ when $\pm\Im(\lambda) > \delta > 0$ for any fixed $\delta > 0$, and $\Re(\lambda) \underset{\lambda\in U}{\to} \mp\infty$.
- The function ν is holomorphic on U and such that $\sin[\pi\nu(\lambda)]$ has no zeroes in some open neighborhood of $[\,-q\,;q\,]$ lying in U.
- The function ν has a "sufficiently" small real part at $\pm q$, i.e., $|\Re[\nu(\pm q)]| < 1/2$.

For technical reasons, one has to distinguish between two situations when the saddle point λ_0 is inside of $]\,-q\,;q\,[$ or outside. Following the tradition we refer to the first case ($-q < \lambda_0 < q$) as the time-like regime and to the second one ($|\lambda_0| > q$) as the space-like regime. Actually, in this article we will only consider the case where $\lambda_0 > -q$. Also, we do not treat the limiting case when $\lambda_0 = \pm q$ as this would require a significant modification of our approach.

2.2 The main result

We now gather the main results of this paper into two theorems.

Theorem 2.1. *Let $V(\lambda,\mu)$ be as in (1.1) and $I+V$ act on $L^2([\,-q\,;q\,])$. Then, under the assumption stated in Section 2.1, the leading $x \to +\infty$ asymptotic behavior of* $\det[I+V]$ *reads:*

$$\det_{[\,-q\,;q\,]}[I+V]\,\mathrm{e}^{-\int_{-q}^{q}[ixu'(\lambda)+g'(\lambda)]\nu(\lambda)d\lambda}$$
$$= B_x[\nu,u]\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right) + \frac{b_1[\nu,u,g]}{x^{\frac{3}{2}}}B_x[\nu,u]\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right)$$

$$+\,\mathrm{e}^{\mathrm{i}x[u(q)-u(-q)]+g(q)-g(-q)}B_x\left[\nu+1,u\right]\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right)$$
$$+\,\mathrm{e}^{\mathrm{i}x[u(-q)-u(q)]+g(-q)-g(q)}B_x\left[\nu-1,u\right]\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right). \tag{2.2}$$

The functional $B_x\left[\nu,u\right]$ takes the form

$$B_x\left[\nu,u\right]=\mathrm{e}^{C_1[\nu]}\frac{G^2\left(1+\nu\left(q\right)\right)G^2\left(1-\nu\left(-q\right)\right)}{\left[2qx\left(u'\left(q\right)+i0^+\right)\right]^{\nu^2(q)}\left[2qxu'\left(-q\right)\right]^{\nu^2(-q)}}$$
$$\times\left(2\pi\right)^{\nu(-q)-\nu(q)}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\left(\nu^2(q)-\nu^2(-q)\right)}. \tag{2.3}$$

It is expressed in terms of the Barnes G function [1] *and the auxiliary functional*

$$C_1\left[\nu\right]=\frac{1}{2}\int_{-q}^{q}d\lambda d\mu\frac{\nu'\left(\lambda\right)\nu\left(\mu\right)-\nu'\left(\mu\right)\nu\left(\lambda\right)}{\lambda-\mu}+\nu\left(q\right)\int_{-q}^{q}\frac{\nu\left(q\right)-\nu\left(\lambda\right)}{q-\lambda}d\lambda$$
$$+\,\nu\left(-q\right)\int_{-q}^{q}\frac{\nu\left(-q\right)-\nu\left(\lambda\right)}{q+\lambda}d\lambda. \tag{2.4}$$

The functional $b_1\left[\nu,u,g\right]$ takes different forms depending whether one is in the so-called space-like regime ($\lambda_0>q$) or in the time-like regime ($\lambda_0\in\,]-q\,;q\,[$):

$$b_1\left[\nu,u,g\right]=\frac{1}{\sqrt{-2\pi u''\left(\lambda_0\right)}}$$
$$\times\begin{cases}\dfrac{\nu\left(-q\right)}{u'\left(-q\right)\left(\lambda_0+q\right)^2}\dfrac{\mathcal{S}_0}{\mathcal{S}_-}-\dfrac{\nu\left(q\right)}{u'\left(q\right)\left(\lambda_0-q\right)^2}\dfrac{\mathcal{S}_0}{\mathcal{S}_+} & \textit{time-like}\\ \dfrac{\nu\left(-q\right)}{u'\left(-q\right)\left(\lambda_0+q\right)^2}\dfrac{\mathcal{S}_-}{\mathcal{S}_0}-\dfrac{\nu\left(q\right)}{u'\left(q\right)\left(\lambda_0-q\right)^2}\dfrac{\mathcal{S}_+}{\mathcal{S}_0} & \textit{space-like}\end{cases}. \tag{2.5}$$

There, we agree upon

$$\mathcal{S}_+=\left[2qxu'\left(q\right)+i0^+\right]^{2\nu(q)}e^2\left(q\right)\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(q)}-1\right)\frac{\Gamma\left(1-\nu\left(q\right)\right)}{\Gamma\left(1+\nu\left(q\right)\right)}$$
$$\times\exp\left\{-2\int_{-q}^{q}\frac{\nu\left(q\right)-\nu\left(\mu\right)}{q-\mu}d\mu\right\}, \tag{2.6}$$
$$\mathcal{S}_-=\frac{\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(-q)}-1\right)}{\left[2qxu'\left(-q\right)\right]^{2\nu(-q)}}e^2\left(-q\right)\frac{\Gamma\left(1+\nu\left(-q\right)\right)}{\Gamma\left(1-\nu\left(-q\right)\right)}$$
$$\times\exp\left\{2\int_{-q}^{q}\frac{\nu\left(-q\right)-\nu\left(\mu\right)}{q+\mu}d\mu\right\}, \tag{2.7}$$

$$\mathcal{S}_0 = e^2(\lambda_0)\, \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \left(\frac{\lambda_0 + q}{\lambda_0 - q - i0^+} \right)^{2\nu(\lambda_0)} \exp\left\{ -2 \int_{-q}^{q} \frac{\nu(\lambda_0) - \nu(\mu)}{\lambda_0 - \mu} d\mu \right\}$$
$$\times \begin{cases} \left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda_0)} - 1\right)^2 & \textit{time-like} \\ 1 & \textit{space-like} \end{cases}. \tag{2.8}$$

The proof of this theorem will be given in Section 6.2. It heavily relies on the asymptotic analysis of the RHP associated with V that will be carried out in Sections 3–5.

Above, the $i0^+$ regularization is important only in the time-like regime as then $u'(q) < 0$. It allows one for a non-ambiguous definition of the power-laws appearing above. In the space-like regime, the $i0^+$ regularization makes no difference.

A special limit of the kernel (1.1) can be related to the generalized sine kernel studied in [32]. Indeed, when the saddle point λ_0 is send to infinity, by deforming slightly the contours $\mathscr{C}_E$, the function E can be seen to be proportional to e^{-1}, up to corrections that are uniformly $\mathrm{O}(x^{-\infty})$ on $[-q\,;q]$. In particular, one has that the $x \to +\infty$ asymptotic expansion of the two Fredholm determinants coincide in this limit. This can be seen directly by inspection of our formulae, at least in what concerns the leading asymptotics.

A specific case of our kernel $u(\lambda) = \lambda - t\lambda^2/x$, $g = 0$, $q = 1$ and $\nu = cst$ has been studied in the literature in the context of its relation with the impenetrable Fermion gas [4]. Upon such a specialization, our results agree with the coefficients of the asymptotic expansion obtained in that paper.

The second main result obtained in this paper is the Natte series representation for the Fredholm determinant.

Theorem 2.2. *Under the assumptions stated in Section* 2.1*, the Fredholm determinant of the operator* $I + V$ *where the kernel* V *is given by* (1.1) *admits the below absolutely convergent Natte series expansion. In other words, there exists functionals* $\mathcal{H}_n\left[\nu, \mathrm{e}^g, \mathrm{e}^{ixu}\right]$ *such that*

$$\det[I+V][\nu,u,g] = \det[I+V]^{(0)}[\nu,u,g] \left\{ 1 + \sum_{n\geq 1} \mathcal{H}_n[\nu, \mathrm{e}^g, u] \right\}. \tag{2.9}$$

There

$$\det[I+V]^{(0)}[\nu,u,g] = B_x[\nu,u] \cdot \exp\left\{ \int_{-q}^{q} \left[ixu'(\lambda) + g'(\lambda)\right] \nu(\lambda)\, d\lambda \right\}. \tag{2.10}$$

A more detailed structure of the functionals $\mathcal{H}_N$ can be found in the core of the text, formulae (7.10)*. One has the following estimates for the functionals* $|\mathcal{H}_n[\nu, \mathrm{e}^g, x]| \leq [m(x)]^n$*, with* $m(x) = \mathrm{O}(x^{-w})$ *being n-independent and*

$$w = \frac{3}{4}\min\left(1/2, 1 - \widetilde{w} - 2\max_{\epsilon=\pm}|\Re\nu(\epsilon q)|\right) \quad \text{with}$$
$$\widetilde{w} = 2\sup\{|\Re[\nu(\lambda) - \nu(\epsilon q)]| : |\lambda - \epsilon q| = \delta, \epsilon = \pm\}, \tag{2.11}$$

where $\delta > 0$ is taken small enough. Hence, the series is convergent for x large enough.

The functionals $\mathcal{H}_n[\nu, \mathrm{e}^g, u]$ take the form

$$\mathcal{H}_n[\nu, \mathrm{e}^g, u] = \mathcal{H}_n^{(\infty)}[\nu, \mathrm{e}^g, u] + \sum_{m=-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]} \frac{\mathrm{e}^{\mathrm{i}xm[u(q)-u(-q)]}}{x^{2m[\nu(q)+\nu(-q)]}}\mathcal{H}_n^{(m)}[\nu, \mathrm{e}^g, u]$$
$$+\sum_{b=1}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{b}\sum_{m=b-\left[\frac{n}{2}\right]}^{\left[\frac{n}{2}\right]-b} \frac{\mathrm{e}^{\mathrm{i}xm[u(q)-u(-q)]}}{x^{2m[\nu(q)+\nu(-q)]}}$$
$$\cdot\, x^{\frac{b}{2}}\, \frac{\mathrm{e}^{\mathrm{i}x\boldsymbol{\eta}[bu(\lambda_0)-pu(q)-(b-p)u(-q)]}}{x^{2\boldsymbol{\eta}(b-p)\nu(-q)-2\boldsymbol{\eta}p\nu(q)}} \cdot \mathcal{H}_n^{(m,b,p)}[\nu, \mathrm{e}^g, u]\,. \tag{2.12}$$

Above, we agree upon $\boldsymbol{\eta} = 1$ in the space-like regime and $\boldsymbol{\eta} = -1$ in the time-like regime. There $\mathcal{H}_n^{(\infty)}[\nu, \mathrm{e}^g, x] = \mathrm{O}(x^{-\infty})$ *and the functionals* $\mathcal{H}_n^{(\ell)}[\nu, \mathrm{e}^g, u]$ *and* $\mathcal{H}_n^{(m,p,b)}[\nu, \mathrm{e}^g, u]$ *admit the asymptotic expansions*

$$\mathcal{H}_n^{(m)}[\nu, \mathrm{e}^g, u] \sim \sum_{r\geq 0}\mathcal{H}_{n;r}^{(m)}[\nu, \mathrm{e}^g, u]$$
$$\text{with} \quad \mathcal{H}_{n;r}^{(m)}[\nu, \mathrm{e}^g, u] = \mathrm{O}\left(\frac{(\log x)^{n+r-2m}}{x^{n+r}}\right),$$
$$\mathcal{H}_n^{(m,b,p)}[\nu, \mathrm{e}^g, u] \sim \sum_{r\geq 0}\mathcal{H}_{n;r}^{(m,b,p)}[\nu, \mathrm{e}^g, u]$$
$$\text{with} \quad \mathcal{H}_{n;r}^{(m,b,p)}[\nu, \mathrm{e}^g, u] = \mathrm{O}\left(\frac{(\log x)^{n+r-2(m+b)}}{x^{n+r}}\right). \tag{2.13}$$

This theorem, together with a more explicit expressions for the functionals $\mathcal{H}_n$, will be proven in Section 7. Here, we would however like to comment on the form of the asymptotic expansion. Indeed the above asymptotic expansion is not of the type usually encountered for higher transcendental functions. In fact, the large x-behavior of the functionals $\mathcal{H}_n\left(\nu, \mathrm{e}^g, u\right)$ and hence of the determinant $\det\left[I+V\right]$ contains a tower of different fractional powers of x, each appearing with its own oscillating pre-factor. Once that one has fixed a given phase factor and its associated fractional power of x, then the corresponding functional coefficients $\mathcal{H}_n^{(m)}\left[\nu, \mathrm{e}^g, u\right]$ or $\mathcal{H}_n^{(m,b,p)}\left[\nu, \mathrm{e}^g, u\right]$ admit an asymptotic expansion in the more-or-less standard sense. That is to say, each of their entries admits an asymptotic expansion into a series whose r^{th} term can be written as $P_{r+n}\left(\log x\right)/x^{n+r}$ with P_{r+n} being a polynomial of degree at most $r+n$. One of the consequences of such a structure is that an oscillating term that appears in a sense "farther" (large values of n) in the asymptotic series might be dominant in respect to a non-oscillating term present in the "lower" orders of the Natte series. This structure of the asymptotic expansion proves the conjectures raised in [45, 48] for certain specializations of this kernel. Also, upon specialization, it yields the general structure of the asymptotic expansion of the fifth Painlevé transcendent associated to the pure sine kernel [29].

The series representation (2.9) might appear abstract since there is no generic simple expression for the functionals $\mathcal{H}_n$, $n \geq 1$. However, the slightly more explicit (but also more cumbersome so that we did not present it a this point) characterization of the functionals $\mathcal{H}_n$, gives a thorough and explicit description of the way $\mathcal{H}_n$ acts on e^g. This characterization, together with the overall form of the Natte series (2.9), is enough to build a multidimensional deformation of (2.9) which describes a class of correlation functions appearing in integrable models *away* from their free fermion points. The very fact that the series representation one starts with has good properties from the point of view of an asymptotic analysis (for instance it immediately provides the leading asymptotics) leads to a multidimensional deformation which has basically the same good properties in respect to the asymptotic analysis, in the sense that it admits an expansion of the type (2.9), (2.12), (2.13). As a consequence, the long-time/long-distance asymptotic behavior of two-point functions in an *interacting* integrable model can be simply *read-off* by looking at the multidimensional series.

2.3 Notations

We now introduce several notations that we use throughout the article.

- $\mathcal{D}_{z_0,\delta} = \{z \in \mathbb{C} : |z - z_0| < \delta\}$ is the open disk of radius δ centered at z_0. $\partial\mathcal{D}_{z_0,\delta}$ stands for its canonically oriented boundary and $-\partial\mathcal{D}_{z_0,\delta}$ for the boundary equipped with the opposite orientation.
- σ_3, $\sigma^{\pm}$ and I_2 stand for the below matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.14}$$

- Given an oriented curve $\mathscr{C}$ in $\mathbb{C}$, $\Gamma(\mathscr{C})$ stands for a small counterclockwise loop around the curve $\mathscr{C}$. This loop is always chosen in such a way that the only potential singularities of the integrand inside of the loop are located on $\mathscr{C}$. For instance, if $\mathscr{C}$ consists of one point λ, then $\Gamma(\mathscr{C})$ can be taken as $\partial\mathcal{D}_{\lambda,\delta}$, for some $\delta > 0$ and small enough.
- When no confusion is possible, the variable dependence will be omitted, i.e., $u(\lambda) = u$, $g(\lambda) = g$, *etc.*
- log refers to the $]-\pi\,;\pi\,[$ determination of the logarithm, and it is this determination that is used for defining powers.
- Given a set U, $\overset{\circ}{U}$ refers to its interior and $\overline{U}$ to its closure.
- $\mathbb{H}_+$, resp. $\mathbb{H}_-$, stands for the upper $\{z \in \mathbb{C} : \Im(z) > 0\}$, resp. lower $\{z \in \mathbb{C} : \Im(z) < 0\}$, half-planes.
- Given matrix valued functions $M(\lambda)$, $N(\lambda)$, the relation $M(\lambda) = \mathrm{O}(N(\lambda))$ is to be understood entry-wise $M_{k\ell}(\lambda) = \mathrm{O}(N_{k\ell}(\lambda))$.
- Given an oriented curve $\mathscr{C}$, one defines its $+$ (resp. $-$) side as the one lying to the left (resp. right) when moving along the curve. Above and in the following, given any function or matrix function f, $f_{\pm}(\lambda)$ stands for the non-tangential limit of $f(z)$ when z approaches the point $\lambda \in \mathscr{C}$ from the $\pm$ side of the oriented curve $\mathscr{C}$.
- Given a piecewise smooth curve $\mathscr{C}$ and matrix M with entries in $L^p(\mathscr{C})$, $p = 1, 2, \infty$, we use the canonical matrix norms ($\dagger$ stands for Hermitian conjugation):

$$\|M\|_{L^\infty(\mathscr{C})} = \max_{i,j} \|M_{ij}\|_{L^\infty(\mathscr{C})}\,, \quad \|M\|_{L^2(\mathscr{C})} = \sqrt{\|\mathrm{tr}\,[M^\dagger M]\|_{L^1(\mathscr{C})}} \quad \text{and}$$
$$\|M\|_{L^1(\mathscr{C})} = \max_{ij} \|M_{ij}\|_{L^1(\mathscr{C})}\,. \tag{2.15}$$

- The distance between any two subsets A, B of $\mathbb{C}$ will be denoted by $\mathrm{d}(A, B) \equiv \inf\{|x - y| : x \in A, y \in B\}$.

2.4 Several remarks

It now seems to be a good place so as to gather several remarks in respect to our assumptions.

- The assumptions on the type of the saddle-point at λ_0 guarantee that there exists a local parametrization for $u(\lambda)$ around λ_0, $u(\lambda) - u(\lambda_0) = -\omega^2(\lambda)$ with $\omega(\lambda) = (\lambda - \lambda_0)\, h(\lambda)$, where $h(\lambda_0) \neq 0$ and h is holomorphic on $\mathcal{D}_{\lambda_0,\delta}$ for some $\delta > 0$.
- As it will become apparent from our asymptotic analysis, given functions u, ν, g satisfying to all the hypothesis, one has that $\det[I + V] \neq 0$ for x large enough.
- The assumption on the number of saddle-points and their order can be relaxed in principle. RHP with multiple saddle-points have been considered in [50]. This work was later extended to the case of less regular functions and higher order saddle-points in [18].
- The restriction on the real part of ν in the vicinity of $\pm q$ is of technical nature. It allows us to avoid the analysis related to the so-called ambiguous Fisher–Hartwig symbols. The method for dealing with such kinds of problems in the framework of RHPs has been proposed in [11, 12]. The cases where $\Re(\nu(\pm q)) \geq 1/2$ could in principle be treated along these techniques, but we chose not to venture into these technicalities.
- We have depicted the contour $\mathscr{C}_E$ appearing in principal value integral in (1.2) in figure 1. This contour $\mathscr{C}_E$ is chosen in such a way that the integral is converging exponentially fast at infinity. This avoids us unnecessary complications and corresponds to most, if not all, situations that can arise in interacting integrable models.
- In the case of kernels involved in the representation of the two-point functions in integrable models, the function u takes the form $u(\lambda) = p(\lambda) - t\varepsilon(\lambda)/x$. p corresponds to the momentum of excitations, whereas ε corresponds to their energy. The parameter t plays the role of the time-shift between the two operators and x that of their distance of separation. In general, one is interested in the large-distance/long-time behavior of the two-point function in the case where the ratio t/x is fixed. In such a limit, for many models of interest, the function u has a unique saddle-point on $\mathbb{R}$. This physical interpretation can be seen as a motivation for certain of our assumptions.
- It is not a problem to carry out the same analysis in the case where the contour $\mathscr{C}_E$ given in figure 1 is replaced by $\mathscr{C}_E^{(w)} = \mathscr{C}_E \cap \{z \in \mathbb{C} : |\Re(z)| \leq w\}$, with $w \in \mathbb{R}^+$ such that q, $-q$ and λ_0 belong to $]-w\,; w[$. Up to minor modifications due to such a truncation of the remote part of the contour, the results remain unchanged.

3 The initial RHP and some transformation

3.1 The RHP for χ

The kernel of any integrable integral operator can be recast in a form allowing one to give a convenient characterization of the kernel $R(\lambda, \mu)$ of the resolvent operator $I - R$ to $I + V$.

Namely, in the case of the kernel V given in (1.1) one sets

$$\begin{aligned} \left| \, E^R(\lambda) \right\rangle &= \frac{2 \sin [\pi \nu(\lambda)]}{i\pi} \begin{pmatrix} E(\lambda) \\ e(\lambda) \end{pmatrix}, \\ \left\langle E^L(\lambda) \, \right| &= \sin [\pi \nu(\lambda)] \left(-e(\lambda), E(\lambda) \right), \end{aligned} \tag{3.1}$$

so that the kernel V is expressed as the scalar product:

$$V(\lambda, \mu) = \frac{\langle E^L(\lambda) \mid E^R(\mu) \rangle}{\lambda - \mu}. \tag{3.2}$$

The resolvent $I - R$ of $I + V$ exists if $\det [I + V] \neq 0$. In that case, one defines $\left| \, F^R(\lambda) \right\rangle$ as the unique solution to the integral equation:

$$\begin{aligned} \left| \, F^R(\mu) \right\rangle + \int_{-q}^{q} V(\mu, \lambda) \left| \, F^R(\lambda) \right\rangle d\lambda &= \left| \, E^R(\mu) \right\rangle, \\ \left| \, F^R(\lambda) \right\rangle &= \frac{2 \sin [\pi \nu(\lambda)]}{i\pi} \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix} \end{aligned} \tag{3.3}$$

where the integration is to be understood entry-wise. $\left\langle F^L(\lambda) \, \right|$ corresponds to the solution of the integral equation where $\left| \, E^R(\lambda) \right\rangle$ has been replaced with $\left\langle E^L(\lambda) \, \right|$. It was shown in [26] that the resolvent kernel can be represented as:

$$R(\lambda, \mu) = \frac{\langle F^L(\lambda) \mid F^R(\mu) \rangle}{\lambda - \mu}. \tag{3.4}$$

It is well know since the results established in [14, 24, 26] that the study of many properties (construction of the resolvent, calculation of the Fredholm determinant, construction of a system of partial differential equations for the determinant) of the so-called integrable integral operators $I + V$ can be deduced from the solution of a certain RHP. In the case of the kernel of interest, this RHP reads

- χ is analytic on $\mathbb{C} \setminus [-q\,;q]$ and has continuous boundary values on $]-q\,;q\,[$;
- $\chi(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \log\left|\lambda^2 - q^2\right| \quad \text{for } \lambda \to \pm q$;
- $\chi(\lambda) = I_2 + \lambda^{-1}\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ uniformly in $\lambda \to \infty$;
- $\chi_+(\lambda)\, G_\chi(\lambda) = \chi_-(\lambda) \quad \text{for } \lambda \in\,]-q\,;q\,[\,.$

We remind that $\chi_\pm$ stands for the $\pm$ boundary value of χ from the $\pm$-side of its jump curve.

The jump matrix $G_\chi(\lambda)$ appearing in the formulation of this RHP reads

$$G_\chi(\lambda) = I_2 + 4\sin^2[\pi\nu(\lambda)] \begin{pmatrix} E(\lambda) \\ e(\lambda) \end{pmatrix} (-e(\lambda)\,,\, E(\lambda))\,. \tag{3.5}$$

The above RHP admits a solution as long as $\det[I+V] \neq 0$. Indeed, it has been shown in [26] that the matrix

$$\chi(\lambda) = I_2 - \int_{-q}^{q} \frac{\left|\, F^R(\mu)\,\right\rangle \left\langle\, E^L(\mu)\,\right|}{\mu - \lambda}\, d\mu,$$
$$\chi^{-1}(\lambda) = I_2 + \int_{-q}^{q} \frac{\left|\, E^R(\mu)\,\right\rangle \left\langle\, F^L(\mu)\,\right|}{\mu - \lambda}\, d\mu. \tag{3.6}$$

solves the above RHP. This solution is in fact unique, as can be seen by standard arguments [2]. It follows [26] readily from (3.6) that the solution $\chi(\lambda)$ allows one to construct $\left|\, F^R(\lambda)\,\right\rangle$ and $\left\langle\, F^L(\mu)\,\right|$:

$$\left|\, F^R(\lambda)\,\right\rangle = \chi(\lambda)\left|\, E^R(\lambda)\,\right\rangle, \qquad \left\langle\, F^L(\lambda)\,\right| = \left\langle\, E^L(\lambda)\,\right| \chi^{-1}(\lambda)\,. \tag{3.7}$$

3.2 Relation between χ and $\det[I+V]$

One can express partial derivatives of $\det[I+V]$ in respect to the various parameters entering in the definition of $V(\lambda,\mu)$ in terms of the solution $\chi(\lambda)$ to the above RHP. We will derive a set of such identities below. These will play an important role in our analysis.

Proposition 3.1. *Let $\eta \geq 0$ and $\Gamma(\mathscr{C}_E)$ be a loop in U enlacing counterclockwisely $\mathscr{C}_E$ and such that it goes to infinity in the regions where* $\mathrm{e}^{\mathrm{i}\eta u(z)}$,

$\eta > 0$ is decaying exponentially fast. Then,

$$\partial_x \log\det\left[I+V\right] = -i\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\mathrm{e}^{\mathrm{i}\eta u(z)}\mathrm{tr}[(\partial_z\chi)(z) \times(\sigma_3+2C\left[e^{-2}\right](z)\sigma^+)\chi^{-1}(z)]\right\}_{\eta=0^+}, \tag{3.8}$$

$$\partial_{\lambda_0} \log\det\left[I+V\right] = x\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\left[\partial_{\lambda_0}u(z)\right]\mathrm{e}^{\mathrm{i}\eta u(z)}\mathrm{tr}[(\partial_z\chi)(z) \times(\sigma_3+2C\left[e^{-2}\right](z)\sigma^+)\chi^{-1}(z)]\right\}_{\eta=0^+}. \tag{3.9}$$

There, $C[f]$ stands for the Cauchy transform on $\mathscr{C}_E$ and $C_{\pm}[G]$ for its $\pm$ boundary values on $\mathscr{C}_E$. One has more explicitly

$$C[G](\lambda) = \int_{\mathscr{C}_E}\frac{G(\mu)}{\mu-\lambda}\frac{d\mu}{2\mathrm{i}\pi}, \quad \text{and} \quad C_+[G](\lambda)-C_-[G](\lambda)=G(\lambda), \quad \text{for} \quad \lambda\in\mathscr{C}_E. \tag{3.10}$$

Proof. The proof goes along similar lines to [32]. It is straightforward that

$$\partial_x \log\det\left[I+V\right] = \int_{-q}^{q}\left[\partial_x V\cdot(I-R)\right](\lambda,\lambda)\,d\lambda. \tag{3.11}$$

In order to transform (3.11) into (3.8), one should start by writing a convenient representation for $\partial_x V(\lambda,\mu)$. One has that

$$\partial_x e(\lambda) = -\frac{i}{2}u(\lambda)e(\lambda),$$
$$\partial_x E(\lambda) = \frac{i}{2}u(\lambda)E(\lambda) - e(\lambda)\int_{\mathscr{C}_E}\frac{ds}{2\pi}\frac{u(s)-u(\lambda)}{s-\lambda}e^{-2}(s). \tag{3.12}$$

The last integral can be recast in a more convenient form

$$\int_{\mathscr{C}_E}\frac{d\mu}{2\pi}\frac{u(\mu)-u(\lambda)}{\mu-\lambda}e^{-2}(\mu) = \int_{\mathscr{C}_E}\frac{d\mu}{2\pi}\oint_{\Gamma(\{\lambda,\mu\})}\frac{dz}{2\mathrm{i}\pi}\frac{u(z)}{(z-\lambda)(z-\mu)}e^{-2}(\mu)$$

$$= -\mathrm{i}\frac{\partial}{\partial\eta}\left\{\int_{\mathscr{C}_E}\frac{d\mu}{2\pi}\oint_{\Gamma(\{\lambda,\mu\})}\frac{d}{2\mathrm{i}\pi}\frac{\mathrm{e}^{\mathrm{i}\eta u(z)}}{(z-\lambda)(z-\mu)}e^{-2}(\mu)\right\}_{\eta=0^+}$$

$$= -\mathrm{i}\frac{\partial}{\partial\eta}\left\{\int_{\mathscr{C}_E}\frac{d\mu}{2\pi}\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{2\mathrm{i}\pi}\frac{\mathrm{e}^{\mathrm{i}\eta u(z)}}{(z-\lambda)(z-\mu)}e^{-2}(\mu)\right\}_{\eta=0^+}$$

$$= -\mathrm{i}\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{2\mathrm{i}\pi}\frac{\mathrm{e}^{\mathrm{i}\eta u(z)}}{(z-\lambda)}\int_{\mathscr{C}_E}\frac{d\mu}{2\pi}\frac{e^{-2}(\mu)}{z-\mu}\right\}_{\eta=0^+}$$

$$= \mathrm{i}\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{2\pi}\frac{\mathrm{e}^{\mathrm{i}\eta u(z)}C\left[e^{-2}\right](z)}{(z-\lambda)}\right\}_{\eta=0^+}. \tag{3.13}$$

We first have replaced the ratio of differences by a contour integral on $\Gamma(\{\lambda,\mu\})$. Here $\Gamma(\{\lambda,\mu\})$ consists of two small loops around the points λ and μ. In order to manipulate convergent integrals, we then wrote the integral as an η-derivative. The derivative symbol could then be taken out of the integral. Next we deformed the contour of integration from a compact one $\Gamma(\{\lambda,\mu\})$ into $\Gamma(\mathscr{C}_E)$. Such a replacement is allowed as $\mathscr{C}_E$ is chosen precisely in such a way so as to make $\mathrm{e}^{\mathrm{i}\eta u(\lambda)}$, $\eta>0$, decay exponentially on a small neighborhood of $\mathscr{C}_E$ where one can draw $\Gamma(\mathscr{C}_E)$. Such a choice of contours allows us to satisfy to the hypothesis of Fubini's theorem and hence permute the orders of integration. Also, we stress that one should compute the η-derivative only once that all integrals have been computed. Indeed, for generic choices of functions u, permuting the η-derivation with the λ-integration in the last line of (3.13), leads to an *apriori* divergent integral.

Once that this differential identity is established, one readily convinces oneself that

$$\partial_x V(\lambda,\mu) = \mathrm{i}\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{\mathrm{e}^{\mathrm{i}\eta u(z)}}{(z-\lambda)(z-\mu)}\right.$$
$$\left.\times\langle E^L(\lambda)\mid\left(\sigma_3+2C\left[e^{-2}\right](z)\,\sigma^+\right)\mid E^R(\mu)\rangle\frac{dz}{4\pi}\right\}_{\eta=0^+}. \tag{3.14}$$

Denoting $S(z)=\sigma_3+2C\left[e^{-2}\right](z)\,\sigma^+$, using the representation (3.4) of the resolvent R and the fact that $\langle F^L(\lambda)\mid F^R(\mu)\rangle = \mathrm{tr}\left[\big|\,F^R(\mu)\,\big\rangle\big\langle\,F^L(\lambda)\,\big|\right]$, we get

$$\partial_x \log\det\left[I+V\right] = \mathrm{i}\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\,\mathrm{e}^{\mathrm{i}\eta u(z)} \times \int_{-q}^{q} d\lambda\,\frac{\left\langle E^L(\lambda)\,\middle|\,S(z)\,\middle|\,E^R(\lambda)\right\rangle}{(z-\lambda)^2}\right\}_{\eta=0^+} - \mathrm{i}\frac{\partial}{\partial\eta}\mathrm{tr}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\,\mathrm{e}^{\mathrm{i}\eta u(z)}\int_{-q}^{q} d\lambda d\mu\,\left|\,F^R(\lambda)\right\rangle\left\langle E^L(\lambda)\,\right| \times\left(\frac{1}{\lambda-z}-\frac{1}{\lambda-\mu}\right)S(z)\,\frac{\left|\,E^R(\mu)\right\rangle\left\langle F^L(\mu)\,\right|}{(\mu-z)^2}\right\}_{\eta=0^+}. \tag{3.15}$$

Using the integral expressions (3.6) for χ and χ^{-1}, we obtain

$$\begin{aligned}\partial_x \log\det\left[I+V\right] &= \mathrm{i}\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\,\mathrm{e}^{\mathrm{i}\eta u(z)}\int_{-q}^{q} d\lambda\,\frac{\left\langle E^L(\lambda)\,\middle|\,S(z)\,\middle|\,E^R(\lambda)\right\rangle}{(z-\lambda)^2}\right. \\ &\quad - \mathrm{i}\frac{\partial}{\partial\eta}\mathrm{tr}\left[\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\,\mathrm{e}^{\mathrm{i}\eta u(z)}\int_{-q}^{q} d\mu\,(\chi(\mu)-\chi(z)) \right. \\ &\quad \left.\left.\times\; S(z)\,\frac{\left|\,E^R(\mu)\right\rangle\left\langle F^L(\mu)\,\right|}{(\mu-z)^2}\right]\right\}_{\eta=0^+} \\ &= -\mathrm{i}\frac{\partial}{\partial\eta}\left\{\oint_{\Gamma(\mathscr{C}_E)}\frac{dz}{4\pi}\,\mathrm{e}^{\mathrm{i}\eta u(z)}\mathrm{tr}\left\{\partial_z\chi(z)\,S(z)\,\chi^{-1}(z)\right\}\right\}_{\eta=0^+},\end{aligned} \tag{3.16}$$

where we used (3.7). The proof of identity (3.9) goes along very similar lines. □

4 The first set of transformations on the RHP

We now perform several transformations on the original RHP. We first simplify the form of the oscillating functions E appearing in the formulation of the RHP. This step in carried in the spirit of [28]. Then, we map this new RHP into one whose jump matrix can be written as the identity plus some purely off-diagonal matrix. Finally, we apply the non-linear steepest descent method by deforming the contour so as to obtain jump matrices that are a $\mathrm{O}(x^{-\infty})$ uniformly away from $\pm q$ and λ_0. These last steps are a standard implementation of the Deift–Zhou steepest descent method [16, 17].

4.1 Simplification of the function E

In order to replace the complicated function E by e^{-1}, we perform the substitution

$$\chi(\lambda) = \widetilde{\chi}(\lambda)\left(I_2 + \sigma^+ C\left[e^{-2}\right](\lambda)\right), \tag{4.1}$$

where C is the rational Cauchy transform with support on $\mathscr{C}_E$ defined in (3.10).

It is readily checked that this new matrix $\widetilde{\chi}$ is the unique solution to the RHP

- $\widetilde{\chi}$ is analytic on $\mathbb{C} \setminus \mathscr{C}_E$ and has continuous boundary values on $\mathscr{C}_E \setminus \{\pm q\}$;
- $\widetilde{\chi}(\lambda) = I_2 + \lambda^{-1}\mathrm{O}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, uniformly in $\lambda \to \infty$;
- $\widetilde{\chi}(\lambda) = \mathrm{O}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\log\left|\lambda^2 - q^2\right| \quad$ for $\lambda \to \pm q$;
- $\widetilde{\chi}_+(\lambda)\, G_{\widetilde{\chi}}(\lambda) = \widetilde{\chi}_-(\lambda) \quad$ for $\lambda \in \mathscr{C}_E$.

The jump matrix for $\widetilde{\chi}$ takes two different forms

$$G_{\widetilde{\chi}}(\lambda) = \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)} & 0 \\ \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} e^2(\lambda)\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)} - 1\right)^2 & \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} \end{pmatrix}, \quad \text{for } \lambda \in \left]-q\,;q\right[\tag{4.2}$$

and

$$G_{\widetilde{\chi}}(\lambda) = \begin{pmatrix} 1 & e^{-2}(\lambda) \\ 0 & 1 \end{pmatrix}, \quad \text{for } \lambda \in \mathscr{C}_E \setminus [-q\,;q]. \tag{4.3}$$

The existence and uniqueness of solutions for the RHP for $\widetilde{\chi}$ ensures that there is a one-to-one correspondence between χ and $\widetilde{\chi}$.

4.2 Uniformization of the jump matrices

We now carry out the second substitution that will yield an RHP with upper or lower diagonal jump matrices whose diagonal is the identity. For this purpose, we define

$$\alpha(\lambda) = \kappa(\lambda) \left(\frac{\lambda+q}{\lambda-q}\right)^{\nu(\lambda)}, \quad \text{where} \quad \log \kappa(\lambda) = -\int_{-q}^{q} \frac{\nu(\lambda)-\nu(\mu)}{\lambda-\mu} d\mu. \tag{4.4}$$

The function $\alpha(\lambda)$ is holomorphic on $\mathbb{C} \setminus [-q\,;q]$, $\alpha(\lambda) \underset{\lambda\to\infty}{\longrightarrow} 1$ and satisfies to the jump condition

$$\alpha_+(\lambda)\, \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} = \alpha_-(\lambda), \quad \text{for} \quad \lambda \in \,]-q\,;q\,[\,. \tag{4.5}$$

Then we set

$$\Xi(\lambda) = \widetilde{\chi}(\lambda)\, \alpha^{\sigma_3}(\lambda)\,. \tag{4.6}$$

The matrix $\Xi(\lambda)$ is the unique solution to the RHP:

- Ξ is analytic on $\mathbb{C} \setminus \mathscr{C}_E$ and has continuous boundary values on $\mathscr{C}_E \setminus \{\pm q\}$;
- $\Xi(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\lambda-q)^{-\sigma_3\nu(q)} (\lambda+q)^{\sigma_3\nu(-q)} \log\left|\lambda^2-q^2\right|$ for $\lambda \to \pm q$;
- $\Xi(\lambda) = I_2 + \lambda^{-1}\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ uniformly in $\lambda \to \infty$;
- $\Xi_+(\lambda)\, G_\Xi(\lambda) = \Xi_-(\lambda)$ for $\lambda \in \mathscr{C}_E$.

The new jump matrix $G_\Xi(\lambda)$ reads

$$G_\Xi = \begin{pmatrix} 1 & \alpha^{-2}e^{-2} \\ 0 & 1 \end{pmatrix} \quad \lambda \in \mathscr{C}_E \setminus [-q\,;q\,] \quad \text{and}$$
$$G_\Xi = \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{2\mathrm{i}\pi\nu}\alpha_+\alpha_- e^2\left(\mathrm{e}^{-2\mathrm{i}\pi\nu}-1\right)^2 & 1 \end{pmatrix} \quad \lambda \in \,]-q\,;q\,[\,.$$

Again, there is a one-to-one correspondence between Ξ and χ.

4.3 Deformation of the contour

We now perform the third substitution that will result in a change of the shape of jump contour. Due to the fact that $e^{\pm 1}(\lambda)$ are exponentially small in x in appropriate regions of the complex plane, we will end up with an RHP for an unknown matrix Υ whose jump matrices are $I_2 + \mathrm{O}(x^{-\infty})$ and this for λ uniformly away from the points $\pm q$ and λ_0.

4.3.1 The time-like regime

We first introduce three auxiliary matrices $M(\lambda)$ and $N^{(L/R)}(\lambda)$

$$M(\lambda) = \begin{pmatrix} 1 & \alpha^{-2}(\lambda)\, e^{-2}(\lambda) \\ 0 & 1 \end{pmatrix} = I_2 + P(\lambda)\,\sigma^{+}, \tag{4.7}$$

$$\begin{aligned} N^{(L)}(\lambda) &= \begin{pmatrix} 1 & 0 \\ \alpha_-^2(\lambda)\, e^2(\lambda) \left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)} - 1\right)^2 & 1 \end{pmatrix} = I_2 + Q^{(L)}(\lambda)\,\sigma^{-}, \\ N^{(R)}(\lambda) &= \begin{pmatrix} 1 & 0 \\ \alpha_+^2(\lambda)\, \mathrm{e}^{4\mathrm{i}\pi\nu(\lambda)} e^2(\lambda) \left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)} - 1\right)^2 & 1 \end{pmatrix} = I_2 + Q^{(R)}(\lambda)\,\sigma^{-}. \end{aligned} \tag{4.8}$$

Note that although the matrices $N^{(R/L)}(\lambda)$ have different expressions, they coincide on U due to the jump conditions for $\alpha(\lambda)$. It is clear from its very definition that $N^{(L)}(\lambda)$, resp. $N^{(R)}(\lambda)$, has an analytic continuation to some neighborhood of $[-q\,;q]$ in the lower, resp. upper, half-plane. Also, the matrix $M(\lambda)$ has an analytic continuation to $U \setminus [-q\,;q]$ starting from $\mathscr{C}_E \setminus [-q\,;q]$.

The functions P and $Q^{(L)}$ and $Q^{(R)}$ have the local parameterizations around $\pm q$

$$\begin{aligned} P(\lambda) &= \mathrm{e}^{\mathrm{i}\zeta_{-q}} \left[\zeta_{-q}\right]^{-2\nu(\lambda)} \frac{\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} - 1}{C^{(L)}(\lambda)} \quad \text{for } \lambda \in \partial\mathcal{D}_{-q,\delta}, \\ P(\lambda) &= \mathrm{e}^{-\mathrm{i}\zeta_q} \left[\zeta_q\right]^{2\nu(\lambda)} \frac{\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} - 1}{C^{(R)}(\lambda)}, \quad \text{for } \lambda \in \partial\mathcal{D}_{q,\delta}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} Q^{(L)}(\lambda) &= C^{(L)}(\lambda)\, \mathrm{e}^{-\mathrm{i}\zeta_{-q}} \left[\zeta_{-q}\right]^{2\nu(\lambda)} \left(\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} - 1\right) \quad \text{and} \\ Q^{(R)}(\lambda) &= C^{(R)}(\lambda)\, \mathrm{e}^{\mathrm{i}\zeta_q} \left[\zeta_q\right]^{-2\nu(\lambda)} \left(\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} - 1\right). \end{aligned} \tag{4.10}$$

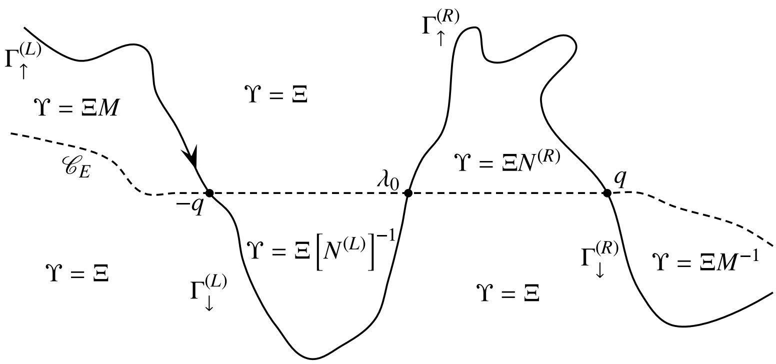

Figure 2: Contour $\Sigma_\Upsilon = \Gamma_{\uparrow}^{(L)} \cup \Gamma_{\downarrow}^{(L)} \cup \Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}$ appearing in the RHP for Υ (time-like regime).

There $\zeta_{-q} = x\left(u\left(\lambda\right) - u\left(-q\right)\right)$ and $\zeta_q = x\left(u\left(q\right) - u\left(\lambda\right)\right)$ and we have set

$$C^{(L)}\left(\lambda\right) = -\frac{\kappa^2\left(\lambda\right)\mathrm{e}^{-g(\lambda)-\mathrm{i}xu(-q)}}{\left[x\left(q-\lambda\right)\right]^{2\nu(\lambda)}}\left(\frac{\lambda+q}{u\left(\lambda\right)-u\left(-q\right)}\right)^{2\nu(\lambda)}\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1\right),$$

$$C^{(R)}\left(\lambda\right) = \frac{-\kappa^2\left(\lambda\right)}{\mathrm{e}^{g(\lambda)+\mathrm{i}xu(q)}}\left(\frac{u\left(\lambda\right)-u\left(q\right)}{\lambda-q}+\mathrm{i}0^+\right)^{2\nu(\lambda)} \times\left[x\left(\lambda+q\right)\right]^{2\nu(\lambda)}\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1\right). \tag{4.11}$$

We now define a piecewise analytic matrix Υ according to figure 2. We will be more specific about the choice of the contours $\Gamma_{\uparrow/\downarrow}^{(L/R)}$ around the points $\pm q$ and λ_0 when we will be constructing the local parametrices. Here, we only precise that the jump contour for Υ remains in U and that all jump curves are chosen so that, for a fixed $z \in \Gamma_{\uparrow}^{(L)} \cup \Gamma_{\uparrow}^{(R)} \setminus \{\pm q, \lambda_0\}$ (resp. $z \in \Gamma_{\downarrow}^{(L)} \cup \Gamma_{\downarrow}^{(R)}$), $\mathrm{e}^{\mathrm{i}xu(z)}$ (resp. $\mathrm{e}^{-\mathrm{i}xu(z)}$) is exponentially small in x. The matrix Υ is discontinuous across the curve $\Sigma_\Upsilon = \Gamma_{\uparrow}^{(L)} \cup \Gamma_{\downarrow}^{(L)} \cup \Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}$. One readily checks that the matrix Υ is the unique solution of the below RHP (and hence there is a one-to-one correspondence between χ and Υ):

- Υ is analytic on $\mathbb{C} \setminus \Sigma_\Upsilon$ and has continuous boundary values on $\Sigma_\Upsilon \setminus \{\pm q\}$;
- $\Upsilon(\lambda) = \mathrm{O}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}(\lambda-q)^{-\sigma_3\nu(q)}(\lambda+q)^{\sigma_3\nu(-q)}\log\left|\lambda^2-q^2\right|$ for $\lambda \to \pm q$;
- $\Upsilon(\lambda) = I_2 + \lambda^{-1}\mathrm{O}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ uniformly in $\lambda \to \infty$;
- $\Upsilon_+(\lambda)\, G_\Upsilon(\lambda) = \Upsilon_-(\lambda)$ for $\lambda \in \Sigma_\Upsilon \setminus \{\pm q, \lambda_0\}$.

With

$$G_{\Upsilon}(\lambda)=M(\lambda)\ \text{on}\ \ \Gamma_{\uparrow}^{(L)}\cup\Gamma_{\downarrow}^{(R)},\quad G_{\Upsilon}(\lambda)=N^{(L)}(\lambda)\ \text{on}\ \ \Gamma_{\downarrow}^{(L)}\ \ \text{and}$$
$$G_{\Upsilon}(\lambda)=N^{(R)}(\lambda)\ \text{on}\ \Gamma_{\uparrow}^{(R)}. \tag{4.12}$$

4.3.2 The space-like regime

We introduce two matrices M and N

$$M=\begin{pmatrix}1 & \alpha^{-2}e^{-2}\\ 0 & 1\end{pmatrix}=I_2+P(\lambda)\,\sigma^{+}\quad\text{and}$$
$$N=\begin{pmatrix}1 & 0\\ \alpha_-^2e^2\left(\mathrm{e}^{-2\mathrm{i}\pi\nu}-1\right)^2 & 1\end{pmatrix}=I_2+Q(\lambda)\,\sigma^{-}. \tag{4.13}$$

The matrix $M(\lambda)$ has an analytic continuation to $U\setminus[-q\,;q\,]$ starting from $\mathscr{C}_E\setminus[-q\,;q\,]$. The matrix $N(\lambda)$ has an analytic continuation to $U\cap\mathbb{H}_-$.

This allows to write convenient local parameterizations around $\pm q$ for P and Q:

$$P(\lambda)=\mathrm{e}^{\mathrm{i}\zeta_{-q}}\left[\zeta_{-q}\right]^{-2\nu(\lambda)}\frac{\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)}-1}{C^{(L)}(\lambda)}=-\mathrm{e}^{\mathrm{i}\zeta_q}\left[\zeta_q\right]^{2\nu(\lambda)}\frac{\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1}{C^{(R)}(\lambda)}\quad\text{with}$$
$$\begin{cases}\zeta_{-q}=x\left(u(\lambda)-u(-q)\right),\\ \zeta_q=x\left(u(\lambda)-u(q)\right).\end{cases} \tag{4.14}$$

Similarly,

$$\begin{aligned}Q(\lambda)&=C^{(L)}(\lambda)\ \mathrm{e}^{-\mathrm{i}\zeta_{-q}}\left[\zeta_{-q}\right]^{2\nu(\lambda)}(\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)}-1)\\ &=-C^{(R)}(\lambda)\,\mathrm{e}^{-\mathrm{i}\zeta_q}\left[\zeta_q\right]^{-2\nu(\lambda)}(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1).\end{aligned} \tag{4.15}$$

Here, we bring to the reader's attention the difference of signs in the definition of ζ_q in the time-like and space-like regimes. Also, the functions $C^{(L/R)}(\lambda)$ have been defined in (4.11). The sole difference is that, in the space-like regime, the $+\mathrm{i}0^+$ regularization plays no role.

The matrix $\Upsilon(\lambda)$ defined in figure 3 is the unique solution to exactly the same RHP as formulated for the time-like case but with the contours being defined in figure 3 and the jump matrix being now given by

$$G_{\Upsilon}(\lambda)=M(\lambda)\ \text{on}\ \ \Gamma_{\uparrow}^{(L)}\cup\Gamma_{\uparrow}^{(R)}\cup\Gamma_{\downarrow}^{(R)}\quad\text{and}\quad G_{\Upsilon}(\lambda)=N(\lambda)\ \text{on}\ \ \Gamma_{\downarrow}^{(L)}. \tag{4.16}$$

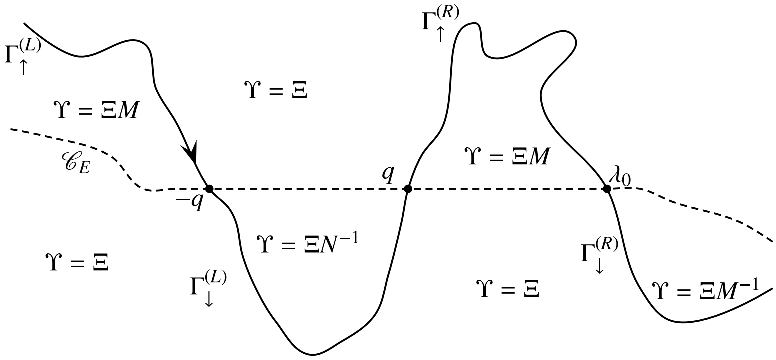

Figure 3: Contour $\Sigma_\Upsilon = \Gamma_{\uparrow}^{(L)} \cup \Gamma_{\downarrow}^{(L)} \cup \Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}$ appearing in the RHP for Υ (space-like regime).

5 The local parametrices

We now build the parametrices around $\pm q$ and λ_0. These will allows us to put the RHP for Υ (and hence the one for χ) in correspondence with an RHP that has its jump matrices close to the identity, uniformly on its whole jump contour (in the case of $\Upsilon(\lambda)$ the jump matrices are close to the identity only uniformly away from the points λ_0 and $\pm q$). The role of the parametrices is to mimic the complicated local behavior of the solution χ near the stationary point λ_0 and the endpoints $\pm q$. Once again, due to slight differences between the two regimes, we treat the space-like and the time-like regimes separately.

5.1 The time-like regime

We recall that for the time-like regime, the functions $P(\lambda)$ and $Q^{(L/R)}(\lambda)$ appearing in the jump matrices are given respectively by (4.9) and (4.10) with $C^{(L/R)}(\lambda)$ given by (4.11).

5.1.1 The parametrix around λ_0

It follows from the assumptions gathered in Section 2.1, that the function u admits a local parameterization around λ_0, i.e., there exists $\delta > 0$ such that $\overline{\mathcal{D}}_{\lambda_0,\delta} \subset U$ and a holomorphic function h on some open neighborhood of $\overline{\mathcal{D}}_{\lambda_0,\delta}$ such that $u(\lambda) - u(\lambda_0) = -\omega^2(\lambda)$ with $\omega(\lambda) = (\lambda - \lambda_0)\, h(\lambda)$, and $h\left(\overline{\mathcal{D}}_{\lambda_0,\delta} \cap \mathbb{H}_\pm\right) \subset \mathbb{H}_\pm$.

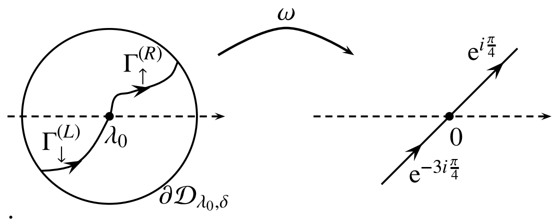

Figure 4: Contours appearing in the local RHP around λ_0 in the time-like case.

The curves $\Gamma_{\downarrow/\uparrow}^{(L/R)}$ in $\overline{\mathcal{D}}_{\lambda_0,\delta}$ are defined according to figure 4.

The parametrix $\mathcal{P}_0$ around λ_0 reads

$$\mathcal{P}_0(\lambda) = I_2 - \overline{b}_{21}(\lambda)\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \frac{\omega(\lambda)\sqrt{\pi x}}{2i\pi} \Psi\left(1, \frac{3}{2}; \mathrm{i}x\omega^2(\lambda)\right) \sigma^- . \tag{5.1}$$

There, $\Psi(a, b; z)$ is Tricomi's confluent hypergeometric function whose definition is recalled in appendix A. The function $\overline{b}_{21}$ is defined piecewise:

$$\overline{b}_{21}(\lambda) = \alpha^2(\lambda)\, \mathrm{e}^{-\mathrm{i}xu(\lambda_0) - g(\lambda)} (\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)} - 1)^2, \quad \text{for } \lambda \in \mathbb{H}^- \cap \mathcal{D}_{\lambda_0,\delta}, \tag{5.2}$$
$$\overline{b}_{21}(\lambda) = \alpha^2(\lambda)\, \mathrm{e}^{4\mathrm{i}\pi\nu(\lambda)} \mathrm{e}^{-\mathrm{i}xu(\lambda_0) - g(\lambda)} (\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)} - 1)^2, \quad \text{for } \lambda \in \mathbb{H}^+ \cap \mathcal{D}_{\lambda_0,\delta}. \tag{5.3}$$

It is holomorphic on $\mathcal{D}_{\lambda_0,\delta}$ due to the jump condition satisfied by α (4.5).

The paramertix $\mathcal{P}_0$ solves the RHP:

- $\mathcal{P}_0$ is analytic in $\mathcal{D}_{\lambda_0,\delta} \setminus \left\{\Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(L)}\right\} \cap \mathcal{D}_{\lambda_0,\delta}$ with continuous boundary values on $\left\{\Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(L)}\right\} \cap \mathcal{D}_{\lambda_0,\delta}$;
- $\mathcal{P}_0 = I_2 + \dfrac{1}{\sqrt{x}} \mathrm{O}(\sigma^-)$ uniformly in $\lambda \in \partial\mathcal{D}_{\lambda_0,\delta}$;
- $[\mathcal{P}_0]_+(\lambda)\left(I_2 + \overline{b}_{21}(\lambda)\, \mathrm{e}^{\mathrm{i}x\omega^2(\lambda)}\sigma^-\right) = [\mathcal{P}_0]_-(\lambda)$.

The first two points in the formulation of the RHP for $\mathcal{P}_0$ are obvious. The validity of the jump conditions can be checked with the help of identity (A.1).

5.1.2 The parametrix at $-q$

The parametrices for the local RHPs at $\pm q$ are well known. They have already appeared in a series of works [4, 11, 32] and can be constructed from the differential equation method [22]. Here, we recall their form.

The parametrix $\mathcal{P}_{-q}$ around $-q$ reads

$$\mathcal{P}_{-q}(\lambda) = \Psi(\lambda)\, L(\lambda) \left[x\left(u(\lambda) - u(-q)\right)\right]^{\nu(\lambda)\sigma_3} \mathrm{e}^{-\frac{\mathrm{i}\pi\nu(\lambda)}{2}}, \tag{5.4}$$

$$\Psi(\lambda) = \begin{pmatrix} \Psi(\nu(\lambda), 1; -\mathrm{i}x[u(\lambda) - u(-q)]) & \mathrm{i}b_{12}(\lambda)\,\Psi(1-\nu(\lambda), 1; \mathrm{i}x[u(\lambda) - u(-q)]) \\ -\mathrm{i}b_{21}(\lambda)\,\Psi(1+\nu(\lambda), 1; -\mathrm{i}x[u(\lambda) - u(-q)]) & \Psi(-\nu(\lambda), 1; ix[u(\lambda) - u(-q)]) \end{pmatrix}, \tag{5.5}$$

$$\begin{aligned} b_{12}(\lambda) &= -\mathrm{i}\frac{\sin[\pi\nu(\lambda)]}{\pi C^{(L)}(\lambda)}\Gamma^2(1-\nu(\lambda)) \\ b_{21}(\lambda) &= -\mathrm{i}\frac{\pi C^{(L)}(\lambda)}{\sin[\pi\nu(\lambda)]\,\Gamma^2(-\nu(\lambda))} \end{aligned}, \quad \text{so that} \quad b_{12}(\lambda)\, b_{21}(\lambda) = -\nu^2(\lambda). \tag{5.6}$$

$C^{(L)}(\lambda)$ is given by (4.11) and

$$L(\lambda) = \begin{cases} I_2 & -\pi/2 < \arg[u(\lambda) - u(-q)] < \pi/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} \end{pmatrix} & \pi/2 < \arg[u(\lambda) - u(-q)] < \pi, \\ \begin{pmatrix} \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg[u(\lambda) - u(-q)] < -\pi/2. \end{cases} \tag{5.7}$$

$\mathcal{P}_{-q}$ is an exact solution of the RHP:

- $\mathcal{P}_{-q}$ is analytic on $\mathcal{D}_{-q,\delta'} \setminus \left\{\Gamma_{\uparrow}^{(L)} \cup \Gamma_{\downarrow}^{(L)}\right\}$ with continuous boundary values on $\left\{\Gamma_{\uparrow}^{(L)} \cup \Gamma_{\downarrow}^{(L)}\right\} \setminus \{-q\}$;
- $\mathcal{P}_{-q}(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\lambda + q)^{\sigma_3\nu(-q)} \log|\lambda + q|\,, \lambda \longrightarrow -q;$
- $\mathcal{P}_{-q}(\lambda) = I_2 + \frac{1}{x^{1-\rho_\delta}}\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, uniformly in $\lambda \in \partial\mathcal{D}_{-q,\delta'}$;
- $\begin{cases} [\mathcal{P}_{-q}]_+(\lambda)\, M(\lambda) = [\mathcal{P}_{-q}]_-(\lambda) & \text{for } \lambda \in \Gamma_{\uparrow}^{(L)} \cap \mathcal{D}_{-q,\delta'}; \\ [\mathcal{P}_{-q}]_+(\lambda)\, N^{(L)}(\lambda) = [\mathcal{P}_{-q}]_-(\lambda) & \text{for } \lambda \in \Gamma_{\downarrow}^{(L)} \cap \mathcal{D}_{-q,\delta'}. \end{cases}$

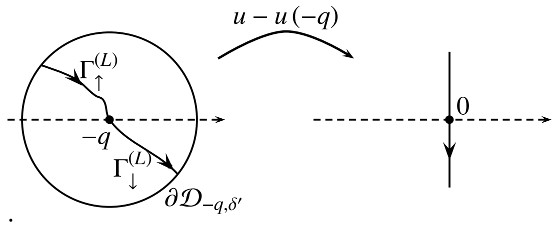

Figure 5: Contours for the local RHP around $-q$ in the time-like case.

Here, we have set

$$\rho_{\delta'} = 2\sup\left\{\left|\Re\left(\nu\left(\lambda\right)\right) : \lambda \in \partial\mathcal{D}_{\pm q,\delta'} \cup \partial\mathcal{D}_{-q,\delta'}\right|\right\} < 1. \tag{5.8}$$

The fact that $\rho_{\delta'} < 1$ for sufficiently small δ' is a consequence of the assumptions that $|\Re\left(\nu\left(\pm q\right)\right)| < 1/2$. The canonically oriented contour $\partial\mathcal{D}_{-q,\delta'}$ together with the definition of the contours $\Gamma_{\uparrow/\downarrow}^{(L)}$ is depicted in figure 5. $\delta' > 0$ is chosen in such a way that $\overline{\mathcal{D}}_{\pm q,\delta'} \subset \overset{\circ}{U}$, $\overline{\mathcal{D}}_{\pm q,\delta'} \cap \overline{\mathcal{D}}_{\lambda_0,\delta} = \emptyset$ and $\overline{\mathcal{D}}_{q,\delta'} \cap \overline{\mathcal{D}}_{-q,\delta'} = \emptyset$. Playing with the δ entering in the definition of the parametrix $\mathcal{P}_0$, one can tune it in such a way that $\delta' = \delta$. We shall assume such a choice in the following.

5.1.3 The parametrix at q

The parametrix $\mathcal{P}_q$ around q reads

$$\mathcal{P}_q\left(\lambda\right) = \Psi\left(\lambda\right) L\left(\lambda\right) \left[x\left(u\left(q\right) - u\left(\lambda\right)\right)\right]^{-\nu(\lambda)\sigma_3} \mathrm{e}^{-\frac{\mathrm{i}\pi\nu(\lambda)}{2}}. \tag{5.9}$$

Here,

$$\Psi\left(\lambda\right) = \begin{pmatrix} \Psi\left(-\nu\left(\lambda\right), 1; -\mathrm{i}x\left[u\left(\lambda\right) - u\left(q\right)\right]\right) & i\,\widetilde{b}_{12}\left(\lambda\right)\,\Psi\left(1+\nu\left(\lambda\right), 1; \mathrm{i}x\left[u\left(\lambda\right) - u\left(q\right)\right]\right) \\ -i\,\widetilde{b}_{21}\left(\lambda\right)\,\Psi\left(1-\nu\left(\lambda\right), 1; -\mathrm{i}x\left[u\left(\lambda\right) - u\left(q\right)\right]\right) & \Psi\left(\nu\left(\lambda\right), 1; ix\left[u\left(\lambda\right) - u\left(q\right)\right]\right) \end{pmatrix}, \tag{5.10}$$

$$\begin{aligned} \widetilde{b}_{12}\left(\lambda\right) &= i\frac{\pi\left[C^{(R)}\left(\lambda\right)\right]^{-1}}{\Gamma^2\left(-\nu\left(\lambda\right)\right)\sin\left[\pi\nu\left(\lambda\right)\right]} \\ \widetilde{b}_{21}\left(\lambda\right) &= i\pi^{-1}\Gamma^2\left(1-\nu\left(\lambda\right)\right)C^{(R)}\left(\lambda\right)\sin\left[\pi\nu\left(\lambda\right)\right] \end{aligned}, \quad \widetilde{b}_{12}\left(\lambda\right)\widetilde{b}_{21}\left(\lambda\right) = -\nu^2\left(\lambda\right). \tag{5.11}$$

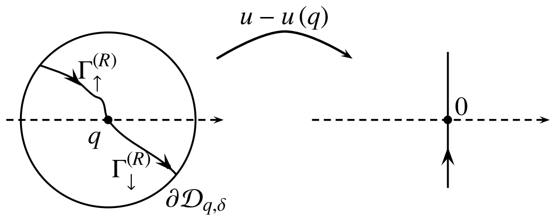

Figure 6: Contours for the local RHP around q in the time-like case.

$C^{(R)}(\lambda)$ is given by (4.11) and

$$L(\lambda)=\begin{cases} I_2 & -\pi/2<\arg\left[u(q)-u(\lambda)\right]<\pi/2, \\ \begin{pmatrix} \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} & \pi/2<\arg\left[u(q)-u(\lambda)\right]<\pi, \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} \end{pmatrix} & -\pi<\arg\left[u(q)-u(\lambda)\right]<-\pi/2. \end{cases} \tag{5.12}$$

$\mathcal{P}_q$ is an exact solution of the RHP:

- $\mathcal{P}_q$ is analytic on $\mathcal{D}_{q,\delta}\setminus\left\{\Gamma_{\uparrow}^{(R)}\cup\Gamma_{\downarrow}^{(R)}\right\}\cap\mathcal{D}_{q,\delta}$ and has continuous boundary values on $\left\{\Gamma_{\uparrow}^{(R)}\cup\Gamma_{\downarrow}^{(R)}\right\}\cap\mathcal{D}_{q,\delta}\setminus\{q\}$;
- $\mathcal{P}_q(\lambda)=\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}(\lambda-q)^{-\sigma_3\nu(q)}\log|\lambda-q|\,,\lambda\longrightarrow q$;
- $\mathcal{P}_q(\lambda)=I_2+\dfrac{1}{x^{1-\rho_\delta}}\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, uniformly in $\lambda\in\partial\mathcal{D}_{q,\delta}$;
- $\begin{cases} \left[\mathcal{P}_q\right]_+(\lambda)\,N^{(R)}(\lambda)=\left[\mathcal{P}_q\right]_-(\lambda) \quad \text{for } \lambda\in\Gamma_{\uparrow}^{(R)}\cap\mathcal{D}_{q,\delta}\setminus\{q\}\,, \\ \left[\mathcal{P}_q\right]_+(\lambda)\,M(\lambda)=\left[\mathcal{P}_q\right]_-(\lambda) \quad \text{for } \lambda\in\Gamma_{\downarrow}^{(R)}\cap\mathcal{D}_{q,\delta}\setminus\{q\}\,. \end{cases}$

The canonically oriented contour $\partial\mathcal{D}_{q,\delta}$ together with the definition of the curves $\Gamma_{\uparrow/\downarrow}^{(R)}$ in the vicinity of q is depicted in figure 6. Note the change of orientation of the jump curve due to $u'(q)<0$. Also ρ_δ is as given in (5.8).

5.1.4 Asymptotically analysable RHP for Π

We now define a piecewise analytic matrix Π in terms of Υ and the parametrices according to figure 7. In particular one has $\Pi=\Upsilon$ everywhere outside of the disks. The matrix Π has its jump matrices uniformly close to the identity matrix in respect to the $x\to+\infty$ limit. Hence, it can be computed perturbatively in x by the use [17] of Neumann series expansion

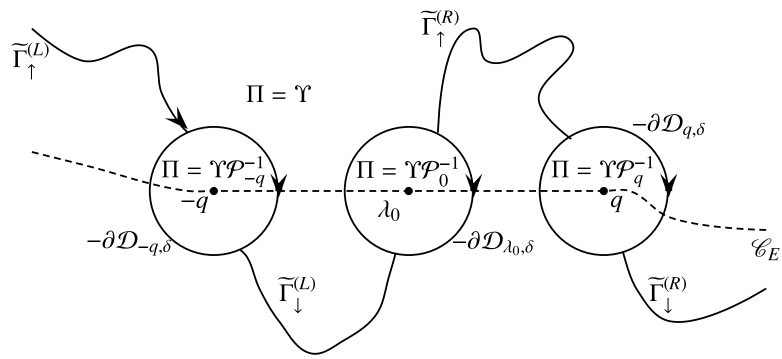

Figure 7: Contour Σ_Π appearing in the RHP for Π, time-like regime.

for the solution of the singular integral equation equivalent to the RHP for Π. This matrix Π is the unique solution to the RHP:

- Π is analytic on $\mathbb{C} \setminus \Sigma_\Pi$ and has continuous boundary values on Σ_Π;
- $\Pi(\lambda) = I_2 + \lambda^{-1}\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, uniformly in $\lambda \to \infty$;
- $\Pi_+(\lambda)\, G_\Pi(\lambda) = \Pi_-(\lambda) \quad$ for $\lambda \in \Sigma_\Pi$.

The jump matrix $G_\Pi(\lambda)$ for $\Pi(\lambda)$ reads

$$G_\Pi(\lambda) = G_\Upsilon(\lambda) \;\text{ on } \widetilde{\Gamma} = \widetilde{\Gamma}_{\uparrow}^{(L)} \cup \widetilde{\Gamma}_{\downarrow}^{(L)} \cup \widetilde{\Gamma}_{\downarrow}^{(R)} \cup \widetilde{\Gamma}_{\uparrow}^{(R)} \quad \text{and}$$
$$G_\Pi(\lambda) = \begin{cases} \mathcal{P}_{\pm q}^{-1}(\lambda) & \text{on } -\partial\mathcal{D}_{\pm q,\delta}, \\ \mathcal{P}_{0}^{-1}(\lambda) & \text{on } -\partial\mathcal{D}_{\lambda_0,\delta}. \end{cases} \tag{5.13}$$

5.2 Asymptotic expansion for the algebraically small jump matrices

Note that the jump matrices along $\widetilde{\Gamma}$ are exponentially close to I_2 in x and this in the $L^1(\Sigma_\Pi) \cap L^2(\Sigma_\Pi) \cap L^\infty(\Sigma_\Pi)$ sense. Only the jump matrices on the disks are algebraically in x close to the identity matrix. The latter jump matrices have the below asymptotic expansion into inverse powers of x, valid uniformly on the boundary of their respective domains of definition ($\partial\mathcal{D}_{\pm q,\delta}$ or $\partial\mathcal{D}_{\lambda_0,\delta}$):

$$\mathcal{P}_{-q}^{-1}(s) \simeq I_2 + \sum_{n\geq 0} \frac{V^{(-;n)}(s)}{(n+1)!\,[x(s+q)]^{n+1}},$$

$$\mathcal{P}_q^{-1}(s) \simeq I_2 + \sum_{n\geq 0} \frac{V^{(+;n)}(s)}{(n+1)!\,[x(s-q)]^{n+1}},$$
$$\mathcal{P}_0^{-1}(s) \simeq I_2 + \sigma^- \sum_{n\geq 0} \frac{d^{(n)}(s)}{x^{n+\frac{1}{2}}(s-\lambda_0)^{2n+1}}. \tag{5.14}$$

where

$$V^{(-;n)}(s) = (-i)^{n+1}\left(\frac{s+q}{u(s)-u(-q)}\right)^{n+1} \times \begin{pmatrix} (-1)^{n+1}(-\nu)_{n+1}^2 & i(n+1)\,b_{12}(-1)^{n+1}(1-\nu)_n^2 \\ -i(n+1)\,b_{21}(1+\nu)_n^2 & (\nu)_{n+1}^2 \end{pmatrix}, \tag{5.15}$$

$$V^{(+;n)}(s) = (-i)^{n+1}\left(\frac{s-q}{u(s)-u(q)}\right)^{n+1} \times \begin{pmatrix} (-1)^{n+1}(\nu)_{n+1}^2 & i(n+1)\,\widetilde{b}_{12}(-1)^{n+1}(1+\nu)_n^2 \\ -i(n+1)\,\widetilde{b}_{21}(1-\nu)_n^2 & (-\nu)_{n+1}^2 \end{pmatrix}, \tag{5.16}$$

$$d^{(n)}(s) = -\frac{i^n\Gamma(1/2+n)}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{h^{2n+1}(s)}\overline{b}_{21}(s). \tag{5.17}$$

We remind that $\omega(\lambda) = (\lambda-\lambda_0)\,h(\lambda)$ and we have used the conditions $\det[\mathcal{P}_{\pm q}] = 1 = \det[\mathcal{P}_0]$ so as to invert the parametrices and then infer their asymptotic expansion from the one of CHF (A.2). Also, we have not made explicit that b_{ij}, $\widetilde{b}_{ij}$ and ν are functions of s.

5.3 The space-like regime

The discussion of the space-like regime resembles, up to minor subtelties, to the previous one. Therefore, we make it as short as possible.

5.3.1 The parametrix around λ_0

The parametrix $\mathcal{P}_0$ on $\mathcal{D}_{\lambda_0,\delta}$ for the local RHP around λ_0 reads

$$\mathcal{P}_0(\lambda) = I_2 - \overline{b}_{12}(\lambda)\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\frac{\omega(\lambda)\sqrt{\pi x}}{2i\pi}\Psi\left(1,\frac{3}{2};-ix\omega^2(\lambda)\right)\sigma^+ \quad \text{with}$$
$$\overline{b}_{12}(\lambda) = \alpha^{-2}(\lambda)\,\mathrm{e}^{\mathrm{i}xu(\lambda_0)+g(\lambda)}. \tag{5.18}$$

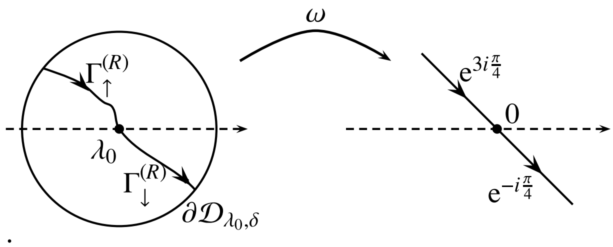

Figure 8: Contours for the local RHP around λ_0 in the space-like case.

$\overline{b}_{12}$ is holomorphic on $\mathcal{D}_{\lambda_0,\delta}$. The parametrix $\mathcal{P}_0$ is a solution to the RHP

- $\mathcal{P}_0$ is analytic in $\mathcal{D}_{\lambda_0,\delta} \setminus \left\{\Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}\right\} \cap \mathcal{D}_{\lambda_0,\delta}$ and has continuous boundary values on $\left\{\Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}\right\} \cap \mathcal{D}_{\lambda_0,\delta}$;
- $\mathcal{P}_0 = I_2 + \dfrac{1}{\sqrt{x}}\mathrm{O}\left(\sigma^+\right)$ uniformly in $\lambda \in \partial D_{\lambda_0,\delta}$;
- $\left[\mathcal{P}_0\right]_+(\lambda)\left(I_2 + \overline{b}_{12}(\lambda)\,\mathrm{e}^{-\mathrm{i}x\omega^2(\lambda)}\sigma^+\right) = \left[\mathcal{P}_0\right]_-(\lambda)$.

The jump curve for the parametrix $\mathcal{P}_0$ is depicted in figure 8.

5.3.2 The parametrix around $-q$

This parametrix $\mathcal{P}_{-q}$ is exactly the same as in the time-like regime. Hence, we do not present it here.

5.3.3 The parametrix around q

The parametrix $\mathcal{P}_q$ around q reads

$$\mathcal{P}_q(\lambda) = \Psi(\lambda)\,L(\lambda)\left[x\left(u(\lambda) - u(q)\right)\right]^{-\nu(\lambda)\sigma_3}\,\mathrm{e}^{\frac{\mathrm{i}\pi\nu(\lambda)}{2}}. \tag{5.19}$$

Here,

$$\Psi(\lambda) = \begin{pmatrix} \Psi\left(-\nu(\lambda), 1; -\mathrm{i}x\left[u(\lambda) - u(q)\right]\right) & i\,\widetilde{b}_{12}(\lambda)\;\Psi\left(1+\nu(\lambda), 1; \mathrm{i}x\left[u(\lambda) - u(q)\right]\right) \\ -i\,\widetilde{b}_{21}(\lambda)\;\Psi\left(1-\nu(\lambda), 1; -\mathrm{i}x\left[u(\lambda) - u(q)\right]\right) & \Psi\left(\nu(\lambda), 1; ix\left[u(\lambda) - u(q)\right]\right) \end{pmatrix}, \tag{5.20}$$

$$\left.\begin{aligned} \widetilde{b}_{12}(\lambda) &= \mathrm{i}\frac{\Gamma^2\left(1+\nu(\lambda)\right)}{\pi C^{(R)}(\lambda)}\sin\left[\pi\nu(\lambda)\right] \\ \widetilde{b}_{21}(\lambda) &= \frac{\mathrm{i}\pi C^{(R)}(\lambda)}{\Gamma^2\left(\nu(\lambda)\right)\sin\left[\pi\nu(\lambda)\right]} \end{aligned}\right., \quad \widetilde{b}_{12}(\lambda)\,\widetilde{b}_{21}(\lambda) = -\nu^2(\lambda). \tag{5.21}$$

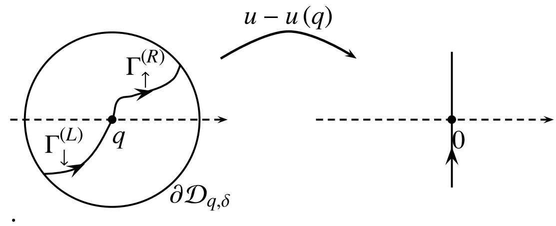

Figure 9: Contours for the parametrix around q in the space-like regime.

$C^{(R)}$ is given by (4.11) and

$$L(\lambda) = \begin{cases} I_2 & -\pi/2 < \arg[u(\lambda) - u(q)] < \pi/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-2i\pi\nu(\lambda)} \end{pmatrix} & \pi/2 < \arg[u(\lambda) - u(q)] < \pi, \\ \begin{pmatrix} \mathrm{e}^{-2i\pi\nu(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg[u(\lambda) - u(q)] < -\pi/2. \end{cases} \tag{5.22}$$

$\mathcal{P}_q$ is an exact solution of the RHP:

- $\mathcal{P}_q$ is analytic on $\mathcal{D}_{q,\delta} \setminus \left\{\Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}\right\} \cap \mathcal{D}_{q,\delta}$ with continuous boundary values on $\left\{\Gamma_{\uparrow}^{(R)} \cup \Gamma_{\downarrow}^{(R)}\right\} \cap \mathcal{D}_{q,\delta} \setminus \{q\}$;
- $\mathcal{P}_q(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\lambda - q)^{-\sigma_3\nu(q)} \log|\lambda - q|, \lambda \longrightarrow q$;
- $\mathcal{P}_q(\lambda) = I_2 + \frac{1}{x^{1-\rho_\delta}}\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, uniformly in $\lambda \in \partial\mathcal{D}_{q,\delta}$;
- $\begin{cases} [\mathcal{P}_q]_+(\lambda) M(\lambda) = [\mathcal{P}_q]_-(\lambda) & \text{for } \lambda \in \Gamma_{\uparrow}^{(R)} \cap \mathcal{D}_{q,\delta}, \\ [\mathcal{P}_q]_+(\lambda) N(\lambda) = [\mathcal{P}_q]_-(\lambda) & \text{for } \lambda \in \Gamma_{\uparrow}^{(R)} \cap \mathcal{D}_{q,\delta}. \end{cases}$

The canonically oriented contour $\partial\mathcal{D}_{q,\delta}$ as well as the definition of the jump curves $\Gamma_{\downarrow/\uparrow}^{(L/R)}$ is depicted in figure 9. Finally, ρ_δ has been defined in (5.8).

5.3.4 The RHP for Π

The matrix Π is defined according to figure 10 and is the unique solution to the RHP formulated in exactly the same way as for the time-like regime. The difference consists in the precise form of the contours due to the fact that in the space-like regime $\lambda_0 > q$.

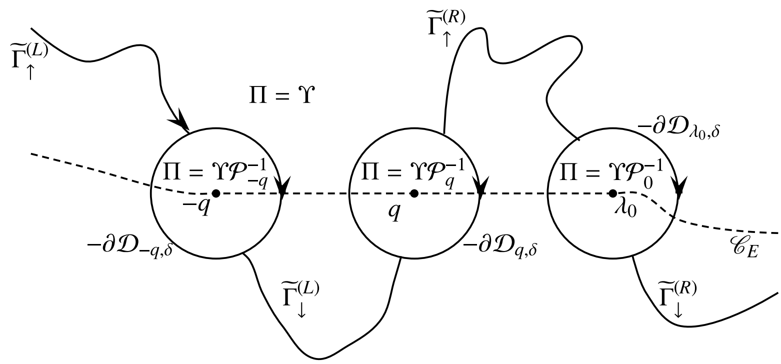

Figure 10: Contour Σ_Π appearing in the RHP for Π, space-like regime.

5.4 The asymptotic expansion for the parametrices

The jump matrices $\mathcal{P}^{-1}_{\pm q}$ have the same asymptotic expansion as in the time-like regime (5.14) with the sole exception that the coefficients $\widetilde{b}_{12}$, $\widetilde{b}_{21}$ entering in the definition of $V^{(+;n)}$ (5.16) are now given by (5.21). The matrix $\mathcal{P}^{-1}_0$ has the below asymptotic expansion

$$\mathcal{P}^{-1}_0(s) \simeq I_2 + \sigma^+ \sum_{n\geq 0} \frac{d^{(n)}(s)}{x^{n+\frac{1}{2}}(s-\lambda_0)^{2n+1}} \quad \text{with}$$
$$d^{(n)}(s) = \overline{b}_{12}(s)(-i)^n \frac{\Gamma(1/2+n)}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{h^{2n+1}(s)}. \qquad (5.23)$$

6 Asymptotic expansion of the Fredholm determinant

Starting from now on, we will treat both regimes (space and time-like) simultaneously.

6.1 The asymptotic expansion for Π

In this subsection we present two ways of writing down the asymptotic expansion for the matrix Π. The first, given in Proposition 6.1, traces back all the different fractional powers of x and oscillating terms that appear in the asymptotic expansion of Π. It also provides one with a sharp and quite optimal control of the remainders. The second one, given in Proposition 6.2, is considerably less explicit and, by far, does not provide optimal estimates

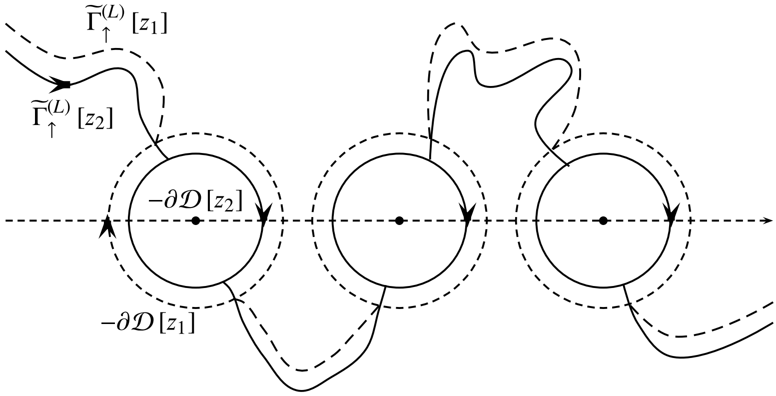

Figure 11: The inslotted contour for $N = 2$. The integration over z_1 runs through the dotted contour whereas the one over z_2 runs through the full one. $\partial\mathcal{D}[z_i]$ refers to the three disks over which the variable z_i is integrated.

for the remainders. However, it is easier to implement from the computational point of view, especially when one is interested in calculating only a couple of terms in the asymptotics. One can then build on the first asymptotic expansion so as to on the one hand argue for a sharper form of the estimates for the remainders and on the other hand identify which among the computed terms are relevant and which are not. We start this section by presenting the Neumann series expansion for Π.

Definition 6.1. Let Σ_Π be the jump contour for the matrix Π. We define the contour $\Sigma_\Pi^{(N)}$ as being the inslotted version of the N-fold Carthesian product $\Sigma_\Pi \times \cdots \times \Sigma_\Pi$. Namely it is obtained from $\Sigma_\Pi \times \cdots \times \Sigma_\Pi$ by putting the contour for z_{k+1} slightly shifted to the right from the contour for z_k. We have depicted the inslotted contour for $N = 2$ in figure 11.

Let pr_k stands for the projection on the k^{th} factor of an N-fold Carthesian product, i.e., given $\boldsymbol{z} = (z_1, \ldots, z_N)$ one has $\mathrm{pr}_k(\boldsymbol{z}) = z_k$. The contour $\Sigma_\Pi^{(N)}$ thus defines N curves $\Sigma_\Pi[z_k] \equiv \mathrm{pr}_k(\Sigma_\Pi^{(N)})$, $k = 1, \ldots, N$. Each of these can be interpreted as giving rise to the jump contour for the RHP problem associated with the matrix Π. In the following whenever Δ, resp. ∇, is integrated along $\Sigma_\Pi[z_k]$, it should be understood as originating from the jump matrix $I_2 + \Delta$, resp. $I_2 + \nabla$, appearing in the RHP for Π, resp. Π^{-1}, when the latter is formulated on the jump contour $\Sigma_\Pi[z_k]$.

Lemma 6.1. *Let $I + \Delta$ be the jump matrix for Π and $\nabla = \mathrm{Comat}(\Delta)^t$ be the transpose of the adjugate matrix to Δ. Then, for x-large enough, the* *matrices* Π *and* Π^{-1} *admit the below uniformly convergent Neumann series*

$$\Pi(\lambda) = I_2 + \sum_{N\geq 1} \int_{\Sigma_\Pi^{(N)}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \frac{\Delta(z_N)\ldots\Delta(z_1)}{(\lambda - z_1)\prod_{s=1}^{N-1}(z_s - z_{s+1})}, \tag{6.1}$$

$$\Pi^{-1}(\lambda) = I_2 + \sum_{N\geq 1} \int_{\Sigma_\Pi^{(N)}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \frac{\nabla(z_1)\ldots\nabla(z_N)}{(\lambda - z_1)\prod_{s=1}^{N-1}(z_s - z_{s+1})}. \tag{6.2}$$

The convergence holds in $L^\infty(O)$ *sense for* $\lambda \in O$*, with* O *any subset of* $\mathbb{C}$ *such that* $\mathrm{d}(O, \Sigma_\Pi) > 0$*. Also, it holds for* $\lambda_\pm \in \Sigma_\Pi$ *in the* $L^2(\Sigma_\Pi)$ *sense. Finally, the matrices* Δ *and* ∇ *that are integrated along the inslotted contour* $\Sigma_\Pi^{(N)}$ *should be understood according to definition* 6.1*.*

Proof. We define two linear operators on 2×2 $L^2(\Sigma_\Pi)$-valued matrices

$$\begin{aligned} \mathcal{C}_{\Sigma_\Pi}^{\Delta}[M](\lambda) &= \int_{\Sigma_\Pi} \frac{ds}{2\mathrm{i}\pi(\lambda_+ - s)} M(s)\Delta(s) \quad \text{and} \\ {}^{\boldsymbol{t}}\mathcal{C}_{\Sigma_\Pi}^{\nabla}[M](\lambda) &= \int_{\Sigma_\Pi} \frac{ds}{2\mathrm{i}\pi(\lambda_+ - s)} \nabla(s) M(s). \end{aligned} \tag{6.3}$$

Using that for sufficiently regular, not necessarily bounded, contours Σ_Π, the $\pm$ limits of the Cauchy transform with support on Σ_Π are continuous operators on $L^2(\Sigma_\Pi)$ with norm $c(\Sigma_\Pi)$ [31], it is easy to see that that the two above operators are also continuous[2] on the space $\mathscr{M}_2\left(L^2(\Sigma_\Pi)\right)$ of 2×2 matrices with $L^2(\Sigma_\Pi)$ entries:

$$\left\|\mathcal{C}_{\Sigma_\Pi}^{\Delta}[M]\right\|_{L^2(\Sigma_\Pi)} \leq 2c(\Sigma_\Pi) \|\Delta\|_{L^\infty(\Sigma_\Pi)} \|M\|_{L^2(\Sigma_\Pi)} \tag{6.4}$$

$$\left\|{}^{\boldsymbol{t}}\mathcal{C}_{\Sigma_\Pi}^{\nabla}[M]\right\|_{L^2(\Sigma_\Pi)} \leq 2c(\Sigma_\Pi) \|\Delta\|_{L^\infty(\Sigma_\Pi)} \|M\|_{L^2(\Sigma_\Pi)} \tag{6.5}$$

There we made use of the fact that ∇ is the transpose of the adjugate matrix to Δ so that $\|\Delta\|_{L^\infty(\Sigma_\Pi)} = \|\nabla\|_{L^\infty(\Sigma_\Pi)}$ and $\|\Delta\|_{L^2(\Sigma_\Pi)} = \|\nabla\|_{L^2(\Sigma_\Pi)}$.

It is a standard fact [5] that there is a one-to-one correspondence between the solution to the RHP for Π (or Π^{-1}) and the unique solution to the

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[2] By interchanging the roles of Δ and M, it is easy to see that $\mathcal{C}_{\Sigma_\Pi}^{\Delta}$ is continuous on $\mathscr{M}_2(L^\infty(\Sigma_\Pi))$ since $\Delta \in L^2(\Sigma_\Pi)$.

singular integral equations

$$\Pi_+ - \mathcal{C}^{\Delta}_{\Sigma_\Pi}\left[\Pi_+\right] = I_2 \qquad \text{and} \qquad \Pi_+^{-1} - {}^{\boldsymbol{t}}\,\mathcal{C}^{\nabla}_{\Sigma_\Pi}\left[\Pi_+^{-1}\right] = I_2. \tag{6.6}$$

Indeed, provided that Π_+ is known, the matrix Π (or Π^{-1}) admits the below integral representation for λ away from Σ_Π

$$\Pi(\lambda) = I_2 + \int_{\Sigma_\Pi} \frac{ds}{2i\pi(\lambda - s)} \Pi_+(s)\,\Delta(s) \quad \text{and}$$
$$\Pi^{-1}(\lambda) = I_2 + \int_{\Sigma_\Pi} \frac{ds}{2i\pi(\lambda - s)} \nabla(s)\,\Pi_+^{-1}(s)\,. \tag{6.7}$$

The estimates for the jump matrices on the boundary of the discs $\partial\mathcal{D}_{\pm q,\delta}$ and $\partial\mathcal{D}_{\lambda_0,\delta}$ and the specific choice for the shape of the contour Σ_Π at infinity lead to $\|\Delta\|_{L^2(\Sigma_\Pi)} + \|\Delta\|_{L^\infty(\Sigma_\Pi)} = \mathrm{O}\left(x^{-w}\right)$ with $w = \min\left(1/2, 1-\rho_\delta\right) > 1$ and ρ_δ defined in (5.8). This implies that for x-large enough the operators $I - \mathcal{C}^{\Delta}_{\Sigma_\Pi}$ and $I - {}^{\boldsymbol{t}}\,\mathcal{C}^{\nabla}_{\Sigma_\Pi}$ are invertible and that their inverse can be computed by a Neumann series expansion converging in $L^2(\Sigma_\Pi)$:

$$\begin{aligned}\Pi_+(\lambda) &= I_2 + \sum_{N\geq 1} \left\{\mathcal{C}^{\Delta}_{\Sigma_\Pi}\right\}^N [I_2](\lambda) \\ &= I_2 + \sum_{N\geq 1} \int_{\Sigma_\Pi} \frac{d^N z}{(2\mathrm{i}\pi)^N} \frac{\Delta(z_N)\ldots\Delta(z_1)}{(\lambda_+ - z_1)\prod_{s=1}^{N-1}\left([z_s]_+ - z_{s+1}\right)}.\end{aligned} \tag{6.8}$$

$$\begin{aligned}\Pi_+^{-1}(\lambda) &= I_2 + \sum_{N\geq 1} \left\{{}^{\boldsymbol{t}}\mathcal{C}^{\nabla}_{\Sigma_\Pi}\right\}^N [I_2](\lambda) \\ &= I_2 + \sum_{N\geq 1} \int_{\Sigma_\Pi} \frac{d^N z}{(2\mathrm{i}\pi)^N} \frac{\nabla(z_1)\ldots\nabla(z_N)}{(\lambda_+ - z_1)\prod_{s=1}^{N-1}\left([z_s]_+ - z_{s+1}\right)}.\end{aligned} \tag{6.9}$$

Where $\left\{\mathcal{C}^{\Delta}_{\Sigma_\Pi}\right\}^N = \mathcal{C}^{\Delta}_{\Sigma_\Pi} \circ \cdots \circ \mathcal{C}^{\Delta}_{\Sigma_\Pi}$ stands for the composition of N operators $\mathcal{C}^{\Delta}_{\Sigma_\Pi}$. In (6.8)–(6.9) the integration runs across the Carthesian product of N copies of Σ_Π: $\Sigma_\Pi \times \cdots \times \Sigma_\Pi$.

The fact that $\Pi^{\pm 1}(\lambda)$ admits a uniformly convergent Neumann series for λ belonging to any open set O at finite distance from Σ_Π follows from the $L^2(\Sigma_\Pi)$ convergence of the series (6.8)–(6.9), the fact that $\Delta \in \mathscr{M}_2\left(L^2(\Sigma_\Pi) \cap L^1(\Sigma_\Pi)\right)$, and that $\mathrm{d}(O, \Sigma_\Pi) > 0$.

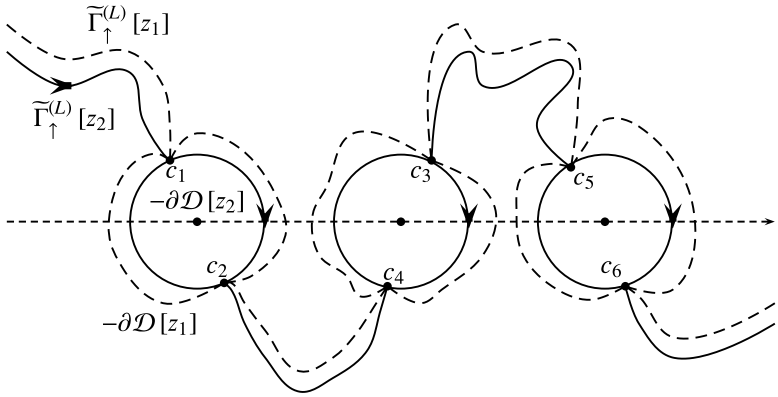

Figure 12: Construction of the inslotted contour.

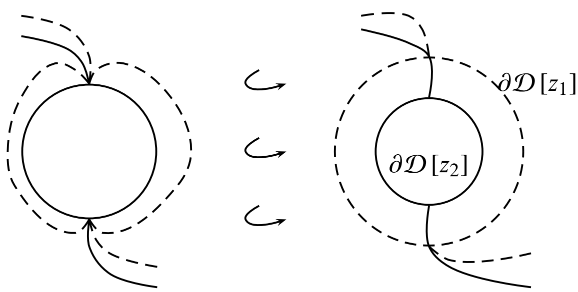

Figure 13: Deformation of the circles.

Finally, it is easy to check that one gets the expression for $\Pi(\lambda)$ (resp. $\Pi^{-1}(\lambda)$) on $\mathbb{C}\setminus\Sigma_\Pi$ by replacing the $+$ type regularization λ_+ of λ in (6.8) (resp. (6.9)) by $\lambda\in\mathbb{C}\setminus\Sigma_\Pi$.

The N^{th} summand of the Neumann series for $\Pi_+^{\pm 1}$ can be expressed in a regularized form by deforming the original contour $\Sigma_\Pi\times\cdots\times\Sigma_\Pi$ to the inslotted one $\Sigma_\Pi^{(N)}$. The latter manipulation is possible due to the properties of the locally analytic matrices $\Delta(z)$ and $\nabla(z)$. It allows one to get rid of the $+$ regularization in the integrals.

The construction of the inslotted contour $\Sigma_\Pi^{(N)}$ is depicted in figures 12 and 13. Initially, the integral is performed with the use of the $+$ boundary value of z_1 on the integration contour for z_2. Hence, away from the points of triple intersections c_i, we can deform the integration contour for z_1 to the $+$ side of the integration contour for z_2. One ends up with a contour as depicted in figure 12. There, the dotted lines correspond to the integration contour for z_1 whereas the full lines give the integration contour for z_2. One then proceeds inductively in this way up to z_N. As λ is assumed to lie uniformly away from the original contour Σ_Π, there is no problem to deform the integration contour for z_1 in the vicinity of Σ_Π as the pole at $z_1 = \lambda$ is lying "far" away.

It remains to threat the integration on the intersection points of the disks $\partial\mathcal{D}[z_j]$ with the curves $\widetilde{\Gamma}^{(L/R)}_{\uparrow/\downarrow}[z_j]$. We first reduce the most interior disc (the one over which z_N is integrated and then the procedure is repeated by induction) to smaller a one. The jump matrices Δ have different analytic continuations from the right and left of the points c_i (this corresponds to the discontinuity lines of $\mathcal{P}_0$ and $\mathcal{P}_\pm$). Taking this difference into account produces the small extensions of the contours $\Gamma^{(L/R)}_{\uparrow/\downarrow}[z_N]$ as depicted on the right part of figure 13 together with smaller discs $\partial\mathcal{D}[z_N]$. It is in this way that the matrix Δ integrated over $\Sigma_\Pi[z_N]$ is identified with the one stemming from the jump matrix for Π when the latter is defined as in (5.13) but with jumps on $\Sigma_\Pi[z_N]$ (what corresponds to slight deformations of the curves $\widetilde{\Gamma}^{(L/R)}_{\uparrow/\downarrow}$). $\square$

Proposition 6.1. *The matrix* Π *admits the series expansion*

$$\Pi(\lambda) = I_2 + \sum_{N\geq 1}^{\infty} \frac{\Pi_N(\lambda)}{x^N}, \tag{6.10}$$

that is valid uniformly away from Σ_Π *and also on the boundary* Σ_Π *in the sense of* $L^2(\Sigma_\Pi)$ *boundary values. The coefficients* Π_N *of this expansion take the form*

$$\begin{aligned}\Pi_N(\lambda) = A_N(\lambda) &+ \sum_{m=-\left[\frac{N}{2}\right]}^{[N/2]} \frac{\mathrm{e}^{\mathrm{i}xm[u(q)-u(-q)]}}{x^{2m[\nu(q)+\nu(-q)]}} \Pi_N^{(m)}(\lambda)\\ &+ \sum_{b=1}^{[N/2]} \sum_{p=0}^{b} \sum_{m=b-\left[\frac{N}{2}\right]}^{\left[\frac{N}{2}\right]-b} \frac{\mathrm{e}^{\mathrm{i}xm[u(q)-u(-q)]}}{x^{2m[\nu(q)+\nu(-q)]}}\\ &\cdot x^{\frac{b}{2}} \frac{\mathrm{e}^{\mathrm{i}x\boldsymbol{\eta}[bu(\lambda_0)-pu(q)+(p-b)u(-q)]}}{x^{2\boldsymbol{\eta}(b-p)\nu(-q)-2p\boldsymbol{\eta}\nu(q)}} \Pi_N^{(m,b,p)}(\lambda),\end{aligned} \tag{6.11}$$

and one should set $\boldsymbol{\eta} = 1$ *in the space-like regime and* $\boldsymbol{\eta} = -1$ *in the time-like.*

The matrix $A_N(\lambda)$ *contains only exponentially small corrections, i.e.,* $[A_N]_{ij}(\lambda) = \mathrm{O}(x^{-\infty})$ *with a* O *that is uniform for* λ*-uniformly away from* Σ_Π.

The matrices $\Pi_N^{(m)}(\lambda)$ *and* $\Pi_N^{(m,b,p)}(\lambda)$ *admit the asymptotic expansion*

$$\begin{aligned}
&\Pi_N^{(m)}(\lambda) = \sum_{r\geq 0} \Pi_{N;r}^{(m)}(\lambda) \quad \text{with} \\
&\Pi_{N;r}^{(m)}(\lambda) = \mathrm{O}\left(\frac{(\log x)^{N+r-\delta_{m,0}-2m}}{x^r} M \right), \qquad (6.12) \\
&\Pi_N^{(m,\,b,\,p)}(\lambda) = \sum_{r\geq 0} \Pi_{N;r}^{(m,\,b,\,p)}(\lambda) \quad \text{with} \\
&\Pi_{N;r}^{(m,\,b,\,p)}(\lambda) = \mathrm{O}\left(\frac{(\log x)^{N+r-2(m+b)}}{x^r} M \right), \qquad (6.13)
\end{aligned}$$

The estimates hold for λ *uniformly away from* Σ_Π.

The matrix M *appearing in the various* O *estimates takes the form:*

$$M = \begin{pmatrix} 1 & \widetilde{m}_+ \dfrac{x^{2\nu(q)}}{\mathrm{e}^{\mathrm{i}xu(q)}} + \widetilde{m}_- \dfrac{\mathrm{e}^{-\mathrm{i}xu(-q)}}{x^{2\nu(-q)}} + \widetilde{m}_0 \dfrac{\sqrt{x}}{\mathrm{e}^{\mathrm{i}xu(\lambda_0)}} \\ m_+ \dfrac{\mathrm{e}^{\mathrm{i}xu(q)}}{x^{2\nu(q)}} + m_- x^{2\nu(-q)} \mathrm{e}^{\mathrm{i}xu(-q)} + m_0 \sqrt{x} \mathrm{e}^{\mathrm{i}xu(\lambda_0)} & 1 \end{pmatrix} \qquad (6.14)$$

There $m_\pm$, m_0, $\widetilde{m}_\pm$ *and* $\widetilde{m}_0$ *are* x*-independent coefficients. Moreover, necessarily,* $m_0 = 0$ *in the time-like regime and* $\widetilde{m}_0 = 0$ *in the space-like one.*

We postpone the proof of this asymptotic expansion to Appendix B as it is rather cumbersome and long. However, at this point, we would like to make several comments in respect to the form of the asymptotic expansion.

The above asymptotic expansion is in a form very similar to the one of the functionals $\mathcal{H}_n[\nu, \mathrm{e}^g, u]$ given in Theorem 2.2. In fact, the large-x behavior of the matrix Π_N contains various fractional powers of x, each appearing with its own oscillating pre-factor. Once that one has fixed a given phase factor and fractional power of x, then the corresponding matrix coefficients $\Pi_N^{(m)}$ or $\Pi_N^{(m,\,b,\,p)}$ admit an asymptotic expansion in the more-or-less standard sense. That is to say, each of their entries admits an asymptotic expansion into a series whose n^{th} term can be written as $P_{N+n}(\log x)/x^n$ with P_{N+n} being a polynomial of degree at most $N+n$. One of the consequences of such a structure is that an oscillating term present in $x^{-n}\Pi_n(\lambda)$ may be in fact dominant in respect to, say, a non-oscillating term present in $x^{-n'}\Pi_{n'}(\lambda)$ where $n' < n$.

We would also like to point out that the asymptotic expansions of Π_N and Π_{N+1} share many oscillating terms at equal frequencies (e.g., $\mathrm{e}^{\mathrm{i}x[u(q)-u(-q)]}$ is present in Π_N and Π_{N+1} for any $N \geq 2$). However, those issued from Π_{N+1} have an additional dumping pre-factor $\log x \cdot x^{-1}$ in respect to the same ones issued from Π_N. Finally, there may also appear additional oscillatory terms $\mathrm{e}^{\pm \mathrm{i}xu(z)}$, $z = \pm q$ or λ_0 (and their associated fractional powers of x) in the off-diagonal parts of $\Pi_N^{(m)}$ and $\Pi_N^{(m,b,p)}$, cf. (6.14).

There is also another way of writing down the asymptotic expansion of $\Pi(\lambda)$. Although it is more compact, it is also less explicit and provides one with weaker estimates for the remainders.

Proposition 6.2. *The matrix Π admits the asymptotic expansion*

$$\Pi(\lambda) = I_2 + \sum_{n\geq 0}^{N} \frac{\Pi^{(n)}(\lambda)}{x^{\frac{1+n}{2}}} + \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x^{-(N+1)w} \quad \text{with}$$
$$w = \min\left(\frac{1}{2}, 1-\rho_\delta\right), \tag{6.15}$$

that is valid uniformly away from Σ_Π.

For λ belonging to any connected component of ∞ in $\mathbb{C}\setminus\Sigma_\Pi$, the first few terms appearing in this expansion read

$$\Pi^{(0)}(\lambda) = -\frac{d^{(0)}(\lambda_0)}{\lambda-\lambda_0}\sigma, \qquad \Pi^{(1)}(\lambda) = -\sum_{\epsilon=\pm}\frac{V^{(\epsilon;0)}(\epsilon q)}{\lambda-\epsilon q}, \tag{6.16}$$

$$\Pi^{(2)}(\lambda) = \sum_{\epsilon=\pm}\frac{d^{(0)}(\lambda_0)}{\lambda_0-\epsilon q}\left\{\frac{V^{(\epsilon;0)}(\epsilon q)\,\sigma}{\lambda-\lambda_0} - \frac{\sigma V^{(\epsilon;0)}(\epsilon q)}{\lambda-\epsilon q}\right\}$$
$$- \frac{\sigma}{2}\frac{\partial^2}{\partial s^2}\left\{\frac{d^{(1)}(s)}{\lambda-s}\right\}_{s=\lambda_0}. \tag{6.17}$$

The expression for $\Pi^{(3)}$ is a bit more involved.

$$\Pi^{(3)}(\lambda) = \frac{\left[d^{(0)}(\lambda_0)\right]^2}{\lambda-\lambda_0}\sum_{\epsilon=\pm}\frac{\sigma V^{(\epsilon;0)}(\epsilon q)\,\sigma}{(\lambda_0-\epsilon q)^2} + \sum_{\epsilon=\pm}\epsilon\frac{V^{(-\epsilon;0)}(-\epsilon q)\,V^{(\epsilon;0)}(\epsilon q)}{2q(\lambda-\epsilon q)}$$
$$-\frac{1}{2}\sum_{\epsilon=\pm}\frac{\partial}{\partial s}\left\{\frac{V^{(\epsilon;1)}(s) - 2V^{(\epsilon;0)}(\epsilon q)\,V^{(\epsilon;0)}(s)}{\lambda-s}\right\}_{s=\epsilon q}. \tag{6.18}$$

There $\sigma = \sigma^{+}$ in the space-like regime and $\sigma = \sigma^{-}$ in the time-like regime.

This form of the asymptotic expansion is the closest, in spirit, to the one appearing in the literature, *cf eg* [15]. However, it does not represent a "well-ordered" asymptotic expansion in the sense that each matrix $\Pi^{(n)}$ depends on the various fractional powers of x and oscillating corrections. Some terms present in the entries of $\Pi^{(p)}$ are dominant in respect to the ones present $\Pi^{(\ell)}$, $\ell < p$. Moreover, the expansion (6.15) does not provide one with a precise identification of these terms. This form is however very convenient from the computational point of view, and having explicit expressions for the matrices $\Pi^{(\ell)}$ easily allows one to identify the various matrices entering in the "well-ordered" asymptotic expansion (6.10)–(6.11).

Proof. The unique solution Π_{+} to the singular integral equation (6.6) equivalent to the uniquely solvable RHP for Π provides an integral representation for Π on $\mathbb{C} \setminus \Sigma_{\Pi}$. Namely,

$$\Pi(\lambda) = I_2 + \int_{\Sigma_{\Pi}} \frac{ds}{2i\pi(\lambda - s)} \Pi_{+}(s)\,\Delta(s), \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \Sigma_{\Pi}. \tag{6.19}$$

The only places where the jump matrix for Π is not exponentially close to the identity are the three boundaries of the discs $-\partial\mathcal{D}_{\pm q,\delta}$ and $-\partial\mathcal{D}_{\lambda_0,\delta}$. There one has

$$\Pi_{+}\mathcal{P}_{\pm q}^{-1} = \Pi_{-} \quad \text{on} \quad -\partial\mathcal{D}_{\pm q,\delta} \quad \text{and} \quad \Pi_{+}\mathcal{P}_{0}^{-1} = \Pi_{-} \quad \text{on} \quad -\partial\mathcal{D}_{\lambda_0,\delta}. \tag{6.20}$$

Note that the minus sign refers to the clockwise orientation of the boundary of the discs in figures 10 and 7.

By using the estimate

$$\mathcal{N}(\Delta) = \|\Delta\|_{L^1(\Sigma_{\Pi})} + \|\Delta\|_{L^{\infty}(\Sigma_{\Pi})} = \mathrm{O}\left(x^{-w}\right) \quad \text{with} \quad w = \min\left(\frac{1}{2}, 1 - \rho_{\delta}\right) \tag{6.21}$$

and equation (6.6), one shows by standard methods (see e.g. [15]) the existence of the asymptotic expansion (6.15) for $\Pi(\lambda)$. This expansion is valid uniformly away from the jump curve Σ_{Π}.

We would like to stress that for computing the coefficients of the asymptotic expansion, we can drop the integration contours other then the boundaries of the disks $\partial\mathcal{D}_{\pm q/\lambda_0;\delta}$. Indeed as $(\Pi_{+} - I_2) \in L^2(\Sigma_{\Pi})$, cf. (6.1), and

$\|\Delta\|_{L^2\cap L^1(\widetilde{\Gamma})} = \mathrm{O}\left(x^{-\infty}\right)$, it is clear that the integration along $\widetilde{\Gamma}$ in (6.6) can only produce exponentially small corrections. Then, it cannot contribute to the asymptotic expansion (6.15). As a consequence, the matrix coefficients $\Pi^{(n)}$ in (6.15) can be computed by plugging[3] the asymptotic series into the integral equation (6.19), dropping there all the exponentially small corrections (stemming from the integration along $\widetilde{\Gamma}$) and replacing the jump matrices $\mathcal{P}_0^{-1}$, $\mathcal{P}_{\pm q}^{-1}$ by their asymptotic expansions which are valid uniformly on the boundaries of the three discs. This leads to the formal (in the sense that valid order by order in x) equation,

$$\Pi_+(\lambda) \simeq I_2 - \frac{1}{2\mathrm{i}\pi}\int_{\partial\mathcal{D}_{\lambda_0,\delta}} \frac{ds}{\lambda_+ - s}\sum_{n\geq 0}\frac{\Pi_+(s)\, d^{(n)}(s)\,\sigma}{(s-\lambda_0)^{2n+1}\, x^{n+\frac{1}{2}}}$$
$$- \frac{1}{2\mathrm{i}\pi}\sum_{\epsilon=\pm}\int_{\partial\mathcal{D}_{\epsilon q,\delta}} \frac{ds}{\lambda_+ - s}\sum_{n\geq 0}\frac{\Pi_+(s)\, V^{(\epsilon;n)}(s)}{(n+1)!\,(s-\epsilon q)^{n+1}\, x^{n+1}}. \qquad (6.22)$$

It now remains to equate the coefficients of equal inverse powers in x. This yields sets of recurrence relations between the various terms appearing in the asymptotic expansion for Π. A straightforward residue computation leads to the result for $\Pi^{(n)}$, $n = 0, \dots, 3$, for λ belonging to any connected component of ∞ in $\mathbb{C}\setminus\Sigma_\Pi$. □

6.2 Proof of the leading asymptotics of the determinant

We now prove Theorem 2.1. We divde the proof into three part. First, we obtain a modified version of the integral representation (3.8) for $\partial_x \log\det[I+V]$ that will be more suited for our further computations. Then, we use this integral representation so as to compute the first few x-dependent terms in the asymptotics. Finally, we fix the constant, x-independent part of the asymtptotics.

• Modification of the integral representation

The first few terms of the asymptotic expansion of $\det[I+V]$ can be obtained by using the identity (3.8) between the x-derivative of $\log\det[I+V]$ and the RHP data χ, together with the asymptotic expansion for Π. As a

[3]It is possible to insert the asymptotic expansion, which a priori is valid only uniformly away from Σ_Π in (6.15) in as much as one slightly deforms the contour Σ_Π in the $+$ direction what is allowed in virtue of the analytic properties of Π_+ and the local analyticity of Δ.

starting remark, we observe that one can always choose the contour $\Gamma\left(\mathscr{C}_E\right)$ appearing in (3.8) in such a way that it only passes in the region where

$$\chi\left(\lambda\right)=\Pi\left(\lambda\right)\alpha^{-\sigma_3}\left(\lambda\right)\left(I_2+C\left[e^{-2}\right]\left(\lambda\right)\sigma^+\right). \tag{6.23}$$

Then, plugging this exact expression for χ into the trace appearing in (3.8), one gets that

$$\begin{aligned}&\operatorname{tr}\left\{\partial_\lambda\chi\left(\lambda\right)\left[\sigma_3+2C\left[e^{-2}\right]\left(\lambda\right)\sigma^+\right]\chi^{-1}\left(\lambda\right)\right\}\\&\quad=\operatorname{tr}\left[\partial_\lambda\Pi\left(\lambda\right)\sigma_3\Pi^{-1}\left(\lambda\right)\right]-2\partial_\lambda\left(\log\alpha\right)\left(\lambda\right).\end{aligned} \tag{6.24}$$

It remarkable, but also important from the computational point of view, that the matrix allowing one to simplify the complicated functions $E\left(\lambda\right)$ (1.2) appearing in the formulation of the initial RHP, does not play a direct a role in the computation of the asymptotics of the determinant. In particular, one does not have to deal with integrations on $\Gamma\left(\mathscr{C}_E\right)$ of Cauchy transforms $C\left[e^{-2}\right]\left(\lambda\right)$. Inserting (6.24) into (3.8), one obtains that the contribution of $-2\partial_\lambda\left(\log\alpha\right)\left(\lambda\right)$ can be separated from the rest, so that

$$\begin{aligned}&\partial_x\log\det\left[I+V\right]\left[\nu,u,g\right]\\&\quad=a_{-1}-i\frac{\partial}{\partial\eta}\left\{\int_{\Gamma\left(\mathscr{C}_E\right)}\frac{d\lambda}{4\pi}\,\mathrm{e}^{\mathrm{i}\eta u(\lambda)}\operatorname{tr}\left[\partial_\lambda\Pi\left(\lambda\right)\sigma_3\Pi^{-1}\left(\lambda\right)\right]\right\}_{\eta=0^+},\end{aligned} \tag{6.25}$$

where we have set

$$a_{-1}=\int_{-q}^{q}\frac{d\lambda}{2\pi}\,u'\left(\lambda\right)\log\left(\frac{\alpha_-\left(\lambda\right)}{\alpha_+\left(\lambda\right)}\right)=i\int_{-q}^{q}u'\left(\lambda\right)\nu\left(\lambda\right)d\lambda. \tag{6.26}$$

Note that one cannot exchange the η-derivation and the λ-integration symbols in (6.25) yet. To be able to do so, we deform the most exterior parts of the contour $\Gamma\left(\mathscr{C}_E\right)$ in (6.25) to $\widetilde{\Gamma}^{(L)}_{\uparrow}$ and $\widetilde{\Gamma}^{(R)}_{\downarrow}$, cf. figure 14. We denote $\gamma^{(0)}$ the resulting interior loop. The integrand along $\gamma^{(0)}$ remains unchanged. However, when integrating along the contours $\widetilde{\Gamma}^{(L/R)}_{\uparrow/\downarrow}$, one should replace $\operatorname{tr}\left[\partial_\lambda\Pi\left(\lambda\right)\sigma_3\Pi^{-1}\left(\lambda\right)\right]$ by the difference between the two boundary values:

$$\operatorname{tr}\left[\partial_\lambda\Pi_-\left(\lambda\right)\sigma_3\Pi_-^{-1}\left(\lambda\right)\right]-\operatorname{tr}\left[\partial_\lambda\Pi_+\left(\lambda\right)\sigma_3\Pi_+^{-1}\left(\lambda\right)\right]. \tag{6.27}$$

We remind that not only the $\pm$ boundary values themselves, but also theses of Π's derivatives do exist on $\widetilde{\Gamma}^{(L/R)}_{\uparrow/\downarrow}$. This is a consequence of the fact the jump matrix $I_2+\Delta$ for Π on $\widetilde{\Gamma}^{(L/R)}_{\uparrow/\downarrow}$ admits an analytic continuation to a

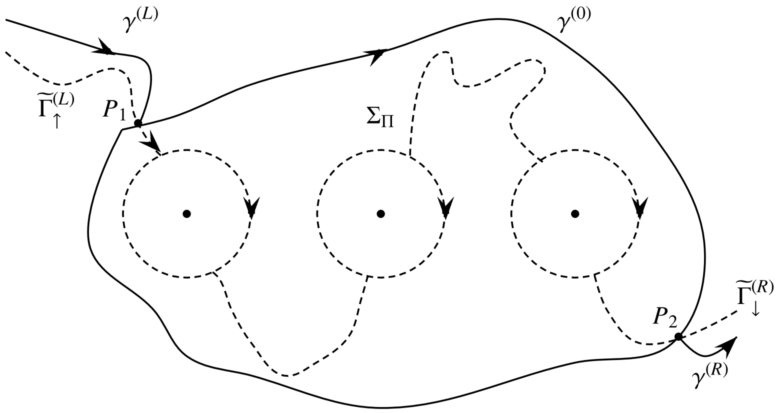

Figure 14: Contour $\gamma = \gamma^{(L)} \cup \gamma^{(0)} \cup \gamma^{(R)}$. The contour Σ_Π is depicted in dotted lines.

neighborhood of these curves. This fact allows one for a local deformation of the jump contour Σ_Π, meaning that Π_+ (resp. Π_-) admits an analytic continuation to some neighborhood of Σ_Π located on its $-$ (resp. $+$) side.

The difference in (6.27) can be estimated with the help of the jump condition for Π along $\widetilde{\Gamma}_{\uparrow/\downarrow}^{(L/R)}$: $\Pi_+ M = \Pi_-$ where M is given by (4.7):

$$\begin{aligned}
&\operatorname{tr}\left[\partial_\lambda \Pi_-(\lambda)\,\sigma_3 \Pi_-^{-1}(\lambda)\right] \\
&\quad= \operatorname{tr}\left[\left\{\left[\partial_\lambda \Pi_+(\lambda)\right] M(\lambda) + \Pi_+(\lambda)\,\partial_\lambda M(\lambda)\right\} \sigma_3 M^{-1}(\lambda)\,\Pi_+^{-1}(\lambda)\right] \\
&\quad= \operatorname{tr}\left[\Pi_+^{-1}\left(\partial_\lambda \Pi_+(\lambda)\right) M^2(\lambda)\,\sigma_3\right] + \operatorname{tr}\left[\partial_\lambda M(\lambda)\,\sigma_3 M^{-1}(\lambda)\right]. \qquad (6.28)
\end{aligned}$$

Using that $M = I_2 + P\sigma^+$, with P being defined in (4.9), we obtain the jump formula

$$\begin{aligned}
&\operatorname{tr}\left[\partial_\lambda \Pi_-(\lambda)\,\sigma_3 \Pi_-^{-1}(\lambda)\right] - \operatorname{tr}\left[\partial_\lambda \Pi_+(\lambda)\,\sigma_3 \Pi_+^{-1}(\lambda)\right] \\
&\quad= 2\alpha^{-2}(\lambda)\,e^{-2}(\lambda)\operatorname{tr}\left[\partial_\lambda \Pi_+(\lambda)\,\sigma^+ \Pi_+^{-1}(\lambda)\right]. \qquad (6.29)
\end{aligned}$$

Using, once again, the jump condition on $\widetilde{\Gamma}_{\uparrow/\downarrow}^{(L/R)}$, we see that $\operatorname{tr}[\partial_\lambda \Pi(\lambda)\,\sigma^+ \Pi^{-1}(\lambda)]$ has no discontinuity across those parts of $\widetilde{\Gamma}_{\uparrow/\downarrow}^{(L/R)}$ that we focus on. It can thus be extended to a holomorphic function in some neighborhood of this curve. Thence, we can deform the contours of integration $\widetilde{\Gamma}_{\uparrow/\downarrow}^{(L/R)}$ to $\gamma^{(L/R)}$ as depicted in figure 14. Once that this has been done, there is no problem anymore to exchange the η-derivation with the λ-integration. Indeed, $\operatorname{tr}[\partial_\lambda \Pi(\lambda)\,\sigma^+ \Pi(\lambda)]$ is bounded when $\Re(\lambda) \to \pm\infty$ along $\gamma^{(L/R)}$, and the function G given by

$$G(\lambda) = u(\lambda)\left\{\mathbf{1}_{\gamma^{(0)}}(\lambda) + 2\alpha^{-2}(\lambda)\, e^{-2}(\lambda)\, \mathbf{1}_{\gamma^{(L)}\cup\gamma^{(R)}}(\lambda)\right\} \tag{6.30}$$

is integrable. Here, $\mathbf{1}_A$ stands for the characteristic function of the set A. Once that the η-derivative is computed, we get the below integral representation

$$\partial_x \log\det[I+V][\nu,u,g] = a_{-1} + \oint_\gamma \frac{d\lambda}{4\pi} G(\lambda)\,\mathrm{tr}\left[\partial_\lambda \Pi(\lambda)\,\sigma(\lambda)\,\Pi^{-1}(\lambda)\right]. \tag{6.31}$$

The final contour γ is depicted in figure 14 and the matrix-valued function reads $\sigma(\lambda) = \sigma_3 \mathbf{1}_{\gamma^{(0)}}(\lambda) + \sigma^+ \mathbf{1}_{\gamma^{(L)}\cup\gamma^{(R)}}(\lambda)$.

- **Extracting the first few x-dependent terms**

$\mathrm{tr}\left[\partial_\lambda \Pi(\lambda)\,\sigma^+ \Pi^{-1}(\lambda)\right]$ is bounded on $\gamma^{(L)}\cup\gamma^{(R)}$ and $\|G\|_{L^1\left(\gamma^{(R)}\cup\gamma^{(L)}\right)} = \mathrm{O}(x^{-\infty})$. Hence, we can drop the part of integration over $\gamma^{(L)}\cup\gamma^{(R)}$ when computing the asymptotic expansion of $\log\det[I+V]$. It thus remains to treat the integration along $\gamma^{(0)}$.

As follows from Proposition 6.1, Π has a uniform asymptotic expansion on $\gamma^{(0)}$ given by (6.10). In order to obtain the leading asymptotic expansion for the x-derivative of the determinant, it is readily seen that it is enough to plug in the more compact expansion (6.15) to the desired order and then drop all the terms that are irrelevant. This is simpler from the point of view of computations and justified *a posteriori* by the form of the well-ordered asymptotic expansion (6.10). Therefore, we get

$$\begin{aligned}
&\mathrm{tr}[\partial_\lambda \Pi(\lambda)\,\sigma_3 \Pi^{-1}(\lambda)] \\
&\quad = \frac{1}{x}\mathrm{tr}\{[\Pi^{(1)}]'(\lambda)\,\sigma_3\} + \frac{1}{x^{\frac{3}{2}}}\mathrm{tr}\{[\Pi^{(2)}]'(\lambda) - \Pi^{(0)}(\lambda)\,[\Pi^{(1)}]'(\lambda) \\
&\qquad - \Pi^{(1)}(\lambda)\,[\Pi^{(0)}]'(\lambda)\}\sigma_3 \\
&\qquad + \frac{1}{x^2}\mathrm{tr}\{[\Pi^{(3)}]'(\lambda) - \Pi^{(1)}(\lambda)\,[\Pi^{(1)}]'(\lambda)\}\sigma_3 \\
&\qquad + \mathrm{o}\left(\frac{\mathrm{e}^{\pm ix[u(q)-u(-q)]}}{x^{\pm 2[\nu(q)+2\nu(-q)]+2}}, \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{\eta} x[u(\lambda_0)-u(\pm q)]}}{x^{\frac{3}{2}\mp 2\nu(\pm q)}}, \frac{(\log x)}{x^2}\right),
\end{aligned} \tag{6.32}$$

uniformly on $\gamma^{(0)}$. There the o refers to sub-leading terms that have been ignored. It distinguishes between the various oscillating and non-oscillating corrections that have been ignored. Also one should set $\boldsymbol{\eta}=1$ in the space-like regime and $\boldsymbol{\eta}=-1$ in the time-like regime. Note that due to the compactness of $\gamma^{(0)}$, the order of the o-remainder is preserved by the integration along $\gamma^{(0)}$.

Note that, in (6.32), we have been able to simplify certain products by exploiting that, regardless of the time or space-like regimes $[\Pi^{(0)}]^2=0$ and that traces of matrices proportional to σ (with $\sigma=\sigma^{\pm}$ depending on the space to time-like regime) vanish (*eg* $\Pi^{(0)}\Pi^{(1)}\left[\Pi^{(0)}\right]'\propto\sigma$).

We now insert the explicit form of the first few matrix coefficients appearing in the expansion of Π and then integrate the expansion (6.32) along $\gamma^{(0)}$ with the appropriate weight. At the end of the day, by using the precise estimates provided by the expansion (6.10), we get

$$\begin{aligned}\partial_x \log\det\left[I+V\right] = a_{-1} + \frac{a_0}{x} + \frac{a_1}{x^{\frac{3}{2}}}\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right) + \frac{a_2^{\mathrm{osc}}}{x^2}\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right)\\ + \frac{a_2^{\mathrm{no}}}{x^2}\left(1+\mathrm{O}\left(\frac{\log x}{x}\right)\right) + \mathrm{O}\left(\frac{a_1}{x^{w+\frac{3}{2}}}\right) + \mathrm{O}\left(\frac{a_2^{\mathrm{osc}}}{x^{w+2}}\right).\end{aligned}$$

Above the last O corresponds to higher order oscillating correction with bigger phases than those involved in the definition of a_1 and a_2^{osc}. The term responsible for the logarithmic contribution to the determinant coincides with the one appearing in the time-independent case (the so-called generalized sine kernel) considered in [32]:

$$a_0 = -\left(\nu^2\left(q\right)+\nu^2\left(-q\right)\right). \tag{6.33}$$

The first λ_0-dependent term is an oscillating correction

$$a_1 = \frac{i}{2}\sum_{\epsilon=\pm} d^{(0)}(\lambda_0)\,\frac{u\left(\lambda_0\right)-u\left(\epsilon q\right)}{\left(\lambda_0-\epsilon q\right)^2}\mathrm{tr}\{V^{(\epsilon;0)}\left(\epsilon q\right)\left[\sigma_3,\sigma\right]\}. \tag{6.34}$$

a_2^{osc} contains the oscillating term coming from the boundaries $\pm q$:

$$a_2^{\mathrm{osc}} = -\frac{u\left(q\right)-u\left(-q\right)}{2i\left(2q\right)^2}\mathrm{tr}\left\{\left[V^{(+;0)}(q)\,,V^{(-;0)}(-q)\right]\sigma_3\right\}. \tag{6.35}$$

Finally, a_2^{no} corresponds to the first non-oscillating corrections issued from the endpoints $\pm q$:

$$\begin{aligned} a_2^{\mathrm{no}} = \frac{i}{4} \sum_{\epsilon=\pm} u''(\epsilon q)\, \mathrm{tr}\left\{ V^{(\epsilon;1)}(\epsilon q) - \left[V^{(\epsilon;0)}(\epsilon q) \right]^2 \right\} \sigma_3 \\ + u'(\epsilon q)\, \mathrm{tr}\left\{ \left[V^{(\epsilon;1)} \right]'(\epsilon q) - 2V^{(\epsilon;0)}(\epsilon q) \left[V^{(\epsilon;0)} \right]'(\epsilon q) \right\} \sigma_3 . \end{aligned} \tag{6.36}$$

Here, we precise that $u' = \partial_\lambda u$, $u'' = \partial_\lambda^2 u$ and $\left[V^{(\epsilon,a)} \right]' = \partial_\lambda V^{(\epsilon,a)}$.

It now remains to insert the explicit expressions for $V^{(\pm,k)}$ as well as $d^{(n)}$ so as to obtain the expressions for the coefficients a_k, $k = 1, 2$.

We get that, independently of the time-like or space-like regime,

$$\frac{a_2^{\mathrm{osc}}}{x^2} = i \left[u\left(q\right) - u\left(-q\right) \right] \cdot \frac{\nu\left(q\right) \nu\left(-q\right)}{u'(q)\, u'(-q)\, \left(2qx\right)^2} \left(\frac{\mathcal{S}_-}{\mathcal{S}_+} - \frac{\mathcal{S}_+}{\mathcal{S}_-} \right) . \tag{6.37}$$

As for a_1, we have

$$\begin{aligned} \frac{a_1}{x^{\frac{3}{2}}} = \frac{1}{2\sqrt{\pi} h\left(\lambda_0\right) x^{\frac{3}{2}}} \\ \times \left\{ i\nu\left(-q\right) \cdot \frac{\left[u\left(-q\right) - u\left(\lambda_0\right) \right]}{u'(-q)\left(\lambda_0 + q\right)^2} \frac{\mathcal{S}_0}{\mathcal{S}_-} - i\nu\left(q\right) \cdot \frac{\left[u\left(q\right) - u\left(\lambda_0\right) \right]}{u'\left(q\right)\left(\lambda_0 - q\right)^2} \frac{\mathcal{S}_0}{\mathcal{S}_+} \right\} \end{aligned} \tag{6.38}$$

in the time-like regime, and

$$\begin{aligned} \frac{a_1}{x^{\frac{3}{2}}} = \frac{1}{2\sqrt{\pi} h\left(\lambda_0\right) x^{\frac{3}{2}}} \\ \times \left\{ i\nu\left(-q\right) \cdot \frac{\left[u\left(\lambda_0\right) - u\left(-q\right) \right]}{u'(-q)\left(\lambda_0 + q\right)^2} \frac{\mathcal{S}_-}{\mathcal{S}_0} - i\nu\left(q\right) \cdot \frac{\left[u\left(\lambda_0\right) - u\left(q\right) \right]}{u'(q)\left(\lambda_0 - q\right)^2} \frac{\mathcal{S}_+}{\mathcal{S}_0} \right\} \end{aligned} \tag{6.39}$$

in the space-like one. We remind that $\mathcal{S}_\pm$ and $\mathcal{S}_0$ have been defined in (2.7) and (2.8)

- **The constant term**

The x-derivative cannot fix the constant in x part of the leading asymptotics. We use the λ_0-derivative identity so as to fix the λ_0-dependent part of this constant. Then, in the space-like regime one obtains the λ_0-independent part of the constant term by sending $\lambda_0 \to \infty$ (the asymptotic expansion is uniform in λ_0 lying uniformly away to the right from q). In such a limit, the determinant can be related, up to $\mathrm{O}\left(x^{-\infty}\right)$ corrections, to the generalized sine kernel determinant studied in [32]. In this way, we are able to fix the constant in this regime. In the time-like regime, in order to fully fix the constant, one has also to compute the q-derivative of the determinant asymptotically.

We already know from the above analysis that $\log\det\left[I+V\right] = xa_{-1} + \log xa_0 + C\left[\nu,u,g\right] + \mathrm{o}\left(1\right)$. Using (3.9), we get

$$\begin{aligned}\partial_{\lambda_0} C\left[\nu,u,g\right] &= \oint_{\gamma(0)} \frac{dz}{4\pi}\left(\partial_{\lambda_0} u\right)(z) \sum_{\epsilon=\pm} \frac{\mathrm{tr}\left[V^{(\epsilon;0)}\left(\epsilon q\right)\sigma_3\right]}{\left(z-\epsilon q\right)^2} \\ &= -\partial_{\lambda_0}\left\{\nu^2\left(q\right)\log\left|u'\left(q\right)\right| + \nu^2\left(-q\right)\log u'\left(-q\right)\right\}. \qquad (6.40)\end{aligned}$$

The absolute value has been chosen so as to treat the space-like and time-like regimes simultaneously.

In the space-like regime, the asymptotics are uniform in λ_0, as long as λ_0 remains uniformly away from q. Hence, one can set $\lambda_0 = \infty$ in the asymptotics so as to fix the constant term. When $\lambda_0 = +\infty$, the function u has no saddle-point, a straightforward computation shows that $V\left(\lambda,\mu\right) = V_{GSK}\left(\lambda,\mu\right) + \mathrm{O}\left(x^{-\infty}\right)$, with

$$V_{GSK}\left(\lambda,\mu\right) = -\left\{1-\mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)}\right\}^{\frac{1}{2}}\left\{1-\mathrm{e}^{2\mathrm{i}\pi\nu(\mu)}\right\}^{\frac{1}{2}} \cdot \frac{\widetilde{e}^{-1}(\lambda)\,\widetilde{e}\left(\mu\right) - \widetilde{e}^{-1}(\mu)\,\widetilde{e}\left(\lambda\right)}{2\mathrm{i}\pi\left(\lambda-\mu\right)}$$

$$\text{with} \quad \widetilde{e}\left(\lambda\right) = e\left(\lambda\right)\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1\right)^{\frac{1}{2}}.$$

Moreover, the big O symbol is uniform on $\left[-q\,;q\right]$. This means that

$$\det_{\left[-q\,;q\right]}\left[I+V\right] = \det_{\left[-q\,;q\right]}\left[I+V_{GSK}\right]\cdot\left(1+\mathrm{O}\left(x^{-\infty}\right)\right). \qquad (6.41)$$

This last identity stems from the fact that the resolvent of a generalized sine kernel is polynomially bounded in x, and this uniformly on $\left[-q\,;q\right]$, *cf* [32]. Using the $x \to +\infty$ asymptotic behavior of $\det_{\left[-q\,;q\right]}\left[I+V_{GSK}\right]$ obtained in [32], we get that

$$\begin{aligned}C\left[\nu,u,g\right] &= -\nu^2\left(q\right)\log\left[2q\left(u'(q)+i0^+\right)\right] - \nu^2\left(-q\right)\log\left[2qu'\left(-q\right)\right] \\ &\quad + \log G\left(1,\nu\left(q\right)\right)G\left(1,\nu\left(-q\right)\right) + C_1\left[\nu\right] \\ &\quad + \int_{-q}^{q} d\lambda g'\left(\lambda\right)\nu\left(\lambda\right) - \int_{-q}^{q} d\lambda\nu\left(\lambda\right)\log'\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1\right). \qquad (6.42)\end{aligned}$$

The functional C_1 has been defined in (2.4) and we agree upon the shorthand notation $G(1,z) = G(1+z)\,G(1-z)$ for the product of two Barnes functions. Note that the $i0^{+}$ regularization only matters in the time-like regime where $u'(q) < 0$. Of course, for the moment we have only proven the value of the constant term in the space-like regime. To see that the constant term is indeed given by $C[\nu, u, g]$ (6.42) in the time-like regime as well, we apply the so-called q derivative method [32]. Namely, starting from the identity

$$\partial_q \log \det [I+V] = R(q,q) + R(-q,-q)\,, \tag{6.43}$$

one replaces the resolvent R by its leading in x part corresponding to sending $\Pi = I_2$ in the reconstruction formula for $\left| F^R(\lambda) \right\rangle$ in terms of χ. The leading resolvent around $\pm q$ is then expressed in terms of CHF with the use of identities (A.6). Then, following word for word the steps described in [32] one obtains that, in the time-like regime, $\partial_q C[\nu, u, g]$ is indeed given by the partial q-derivative of (6.42). This fixes the λ_0 and q-dependent part of the constant term in this regime. As the remaining λ_0 and q-independent part has to be the same in both regimes, the constant term is fully fixed.

The form of the asymptotic expansion given in Theorem 2.1 follows once upon applying the identity

$$\begin{aligned} &\mathrm{e}^{-\int_{-q}^{q} \log'\left(\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}-1\right)\nu(\lambda)d\lambda} G(1,\nu(q))\, G(1,\nu(-q)) \\ &\quad = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\left(\nu^2(q)-\nu^2(-q)\right)} (2\pi)^{\nu(-q)-\nu(q)}\, G^2(1+\nu(q))\, G^2(1-\nu(-q))\,. \end{aligned} \tag{6.44}$$

This identity is a direct consequence of (A.8). □

7 Natte Series for the determinant

In this section, we derive a new series representation, that we call the Natte series, for $\det[I+V]$. Just as a Fredholm series is well adapted for computing the determinant of the operator $I+V$ perturbatively when the kernel V is small, the Natte series is built in such a way that it is immediately fit for an asymptotic analysis of the determinant. The form and existence of the series is closely related to the fact that the asymptotic behavior of this determinant can be obtained by an application of the Deift–Zhou steepest-descent method.

Let $I+\Delta$ be the jump matrix for Π. Then, according to Sections 3 and 5, Δ has an asymptotic expansion that is valid uniformly on the contour Σ_{Π}:

$$\Delta(z) \simeq \sum_{n\geq 0} \frac{\Delta^{(n)}(z;x)}{x^{n+1}}$$

$$\text{where} \quad \begin{cases} \Delta^{(n)}(z;x) & = & \sqrt{x}\dfrac{d^{(n)}(z)}{(z-\lambda_0)^{2n+1}}\sigma & \text{for } z\in\partial\mathcal{D}_{\lambda_0,\delta} \\ \Delta^{(n)}(z;x) & = & \dfrac{V^{(\pm;n)}(z)}{(n+1)!\,(z\mp q)^{n+1}} & \text{for } z\in\partial\mathcal{D}_{\pm q,\delta} \end{cases} \tag{7.1}$$

and everywhere else $\Delta^{(n)}(z;x)=0$. In other words $\Delta(z;x)$ is a $\mathrm{O}(x^{-\infty})$ everywhere else on the contour. Moreover, one can convince oneself that this $\mathrm{O}(x^{-\infty})$ holds in the $L^1\cap L^\infty(\mathscr{C})$ sense, for any curve $\mathscr{C}$ that is lying sufficiently close to $\mathscr{C}_E$. Finally, we remind that $\sigma=\sigma^+$ in the space-like regime and $\sigma=\sigma^-$ in the time-like regime.

7.1 The leading Natte series

We start the derivation of the Natte series by providing a convenient integral representation for $\log\det[I+V]$.

Lemma 7.1. *Let V be the kernel defined in* (1.1) *and $\Pi(\lambda)\equiv\Pi(\lambda;x)$ be the unique solution to the associated RHP. Then, the logarithm of the Fredholm determinant admits the below representation*

$$\log\det[I+V][\nu,u,g] = \log\det[I+V]^{(0)}[\nu,u,g] + \log\det[I+V]^{(\mathrm{sub})}[\nu,u,g] \tag{7.2}$$

where

$$\log\det[I+V]^{(0)}[\nu,u,g] = ix\int_{-q}^{q} u'(\lambda)\,\nu(\lambda)\,d\lambda - \big(\nu^2(q)+\nu^2(-q)\big)\log x + C[\nu,u,g] \tag{7.3}$$

and $C[\nu,u,g]$ has been defined in (6.42)*. Also*

$$\log\det[I+V]^{(\mathrm{sub})}[\nu,u,g] = \int_{+\infty}^{x} dx' \oint_{\gamma} \frac{d\lambda}{4\pi} G(\lambda) \Big\{ \mathrm{tr}[\partial_\lambda \Pi(\lambda;x')\sigma(\lambda)\Pi^{-1}(\lambda;x')] + \frac{1}{x'}\int_{\partial\mathcal{D}} \frac{dz}{2i\pi} \frac{\mathrm{tr}\left[\Delta^{(1)}(z;x')\,\sigma_3\right]}{(\lambda-z)^2} \Big\}. \tag{7.4}$$

The contour γ *is as defined in figure* 14 *and* $\partial\mathcal{D} = -\partial\mathcal{D}_{q,\delta} \cup -\partial\mathcal{D}_{-q,\delta} \cup -\partial\mathcal{D}_{\lambda_0,\delta}$. *The function* G *has been defined in* (6.30), $\Delta^{(1)}$ *in* (7.1) *and we remind that* $\sigma(\lambda) = \sigma_3 \mathbf{1}_{\gamma^{(0)}}(\lambda) + \sigma^{+}\mathbf{1}_{\gamma^{(R)}\cup\gamma^{(L)}}(\lambda)$.

The convergence of this integral representation is part of the conclusion of the lemma.

Proof. The formula for the x-derivative of the determinant (3.8) is the starting point of the proof. By re-ordering the terms we get, exactly as in the proof of Theorem 2.1,

$$\partial_x \log\det[I+V][\nu,u,g] = \partial_x \log\det[I+V]^{(0)}[\nu,u,g] + \mathcal{R}, \tag{7.5}$$

in which

$$\mathcal{R} = -\mathrm{i}\frac{\partial}{\partial\eta}\left[\int_{\Gamma(\mathscr{C}_E)} \frac{d\lambda}{4\pi} \mathrm{e}^{\mathrm{i}\eta u(\lambda)} \left\{ \mathrm{tr}\left[\partial_\lambda \Pi(\lambda;x)\,\sigma_3 \Pi^{-1}(\lambda;x)\right] \right.\right.$$
$$\left.\left. + \int_{\partial\mathcal{D}} \frac{dz}{2\mathrm{i}\pi} \frac{\mathrm{tr}\left[\Delta^{(1)}(z;x)\,\sigma_3\right]}{x(\lambda-z)^2} \right\}\right]_{\eta=0^+}. \tag{7.6}$$

Here the matrix $\Delta^{(1)}$ appears in (7.6) as its contribution has already been taken into account in $\partial_x \log\det[I+V]^{(0)}$. It had thus to be subtracted.

Now, performing exactly the same steps as in the proof of Theorem 2.1, we recast the integral in such a way that the η-derivative can be moved inside of the integration symbol. Note that the operation of squeezing the contour $\Gamma(\mathscr{C}_E)$ in (7.6) to γ does not affect the term coming from $\Delta^{(1)}$ as it is holomorphic outside of $\partial\mathcal{D}$. Once that the η-derivative has been computed, the result follows by an x-integration. This integration is licit as, due to the presence of $\Delta^{(1)}$, the η-differentiated integrand behaves as $\mathrm{O}\left(\log x/x^2\right)$, for $x \to +\infty$, in what concerns the non-oscillating contributions and as $\mathrm{e}^{\mathrm{i}xv}x^{-w}$ for the oscillating terms. Here, v and w are constants such that $v \in \mathbb{R}$ and $\Re(w) > 0$. The oscillating contributions are thus also integrable, at least in the Riemann-sense. □

We are now in position to derive the logarithmic Natte series representation for $\log\det[I+V]$.

Theorem 7.1. *There exists a sequence of functionals $\mathcal{F}_N\left[\nu, u, \mathrm{e}^g\right](x)$, such that*

$$\log\det\left[I+V\right]\left[\nu,u,g\right]=\log\det\left[I+V\right]^{(0)}\left[\nu,u,g\right]+\sum_{N\geq 1}\mathcal{F}_N\left[\nu,u,\mathrm{e}^g\right](x)\,. \tag{7.7}$$

There exists a positive N-independent constant $m(x)$ such that $\left|\mathcal{F}_N\left[\nu,u,\mathrm{e}^g\right](x)\right|\leq\left[m(x)\right]^N$. $m(x)$ is such that $m(x)=\mathrm{O}\left(x^{-w}\right)$ where, for $\delta>0$ but small enough

$$w=\frac{3}{4}\min\Bigl(1/2,1-\widetilde{w}-2\max_{\epsilon=\pm}\left|\Re\left[\nu\left(\epsilon q\right)\right]\right|\Bigr)\quad\text{and}$$
$$\widetilde{w}=2\max_{\epsilon=\pm}\left\{\sup_{\partial\mathcal{D}_{\epsilon q,\delta}}\left|\Re\left[\nu-\nu\left(\epsilon q\right)\right]\right|\right\}. \tag{7.8}$$

The functionals $\mathcal{F}_N\left[\nu,u,\mathrm{e}^g\right](x)$ admit the integral representation

$$\mathcal{F}_N\left[\nu,u,\mathrm{e}^g\right](x)=\sum_{r=1}^{N}\sum_{\substack{\Sigma\epsilon_k=0\\ \epsilon_k\in\{\pm1,0\}}}\sum_{\tau=\downarrow,\uparrow}\int_{+\infty}^{x}dx'\oint_{\gamma^\tau}\frac{d\lambda}{4\pi}\int_{\left\{\Sigma_{\Pi}^{\tau}\right\}^{(r,N)}}\frac{d^N z}{\left(2\mathrm{i}\pi\right)^N}$$
$$\cdot H_{N,r}\Bigl(\lambda,\{z_j\}_{j=1}^N,x';\{\epsilon_j\}_{j=1}^N\Bigr)\left[\nu,u\right]\cdot\prod_{p=1}^{N}\mathrm{e}^{\epsilon_p g(z_p)} \tag{7.9}$$

in terms of the auxiliary functionals

$$H_{N,r}\Bigl(\lambda,\{z_j\}_{j=1}^N,x;\{\epsilon_j\}_{j=1}^N\Bigr)\left[\nu,u\right]$$
$$=\frac{-G(\lambda)\,D_{N,r}\Bigl(\lambda,\{z_j\}_{j=1}^N,x;\{\epsilon_j\}_{j=1}^N\Bigr)\left[\nu,u\right]}{\left(\lambda-z_1\right)^2\left(\lambda-z_{r+1}\right)\prod\limits_{p=1}^{r-1}\left(z_p-z_{p+1}\right)\prod\limits_{p=r+1}^{N}\left(z_p-z_{p+1}\right)}. \tag{7.10}$$

The first summation in (7.9) runs through all possible choices of the variables $\epsilon_k\in\{\pm1,0\}$ subject to the constraint $\sum\epsilon_k=0$. Then, one sums over integrals running over the upper/lower part $\gamma^{\uparrow/\downarrow}$ of the contour γ and also over the associated inslotted contour $\left\{\Sigma_N^{\uparrow/\downarrow}\right\}^{(r,N)}$.

For $N \geq 2$, the functionals D_N are defined as the functionals of ν and u that appear in the expansion of

$$\operatorname{tr}\left[\Delta\left(z_r\right) \ldots \Delta\left(z_1\right) \sigma\left(\lambda\right) \nabla\left(z_{r+1}\right) \ldots \nabla\left(z_N\right)\right]$$
$$= \sum_{\substack{\Sigma \epsilon_k = 0 \\ \epsilon_k \in \{\pm 1, 0\}}} D_{N,r}\left(\lambda, \{z_j\}_{j=1}^{N}, x; \{\epsilon_j\}_{j=1}^{N}\right)[\nu, u] \ \mathrm{e}^{\sum\limits_{p=1}^{N} \epsilon_p g(z_p)}, \qquad (7.11)$$

into different powers of $\mathrm{e}^{g(z_\ell)}$. *For $N = 1$ one has*

$$D_1\left(\lambda, z_1, x\right) = \operatorname{tr}\left[\left(\Delta\left(z_1\right) - x^{-1}\Delta^{(1)}(z_1)\right)\sigma_3\right] \mathbf{1}_{\gamma^{(0)}}(\lambda) \,. \qquad (7.12)$$

Above, ∇ is the adjugate matrix to Δ: $\nabla = \mathrm{Comat}\left[\Delta\right]^t$, so that $I_2 + \nabla$ corresponds to the jump matrix for Π_+^{-1}. Finally, $\gamma^{\uparrow/\downarrow}$ denotes that part of the curve γ which lies above/below of Σ_Π. Let P_1, P_2 stand for the two intersection points between γ and Σ_Π, cf. figure 14. Then, $\Sigma_\Pi^{\uparrow/\downarrow}$ is the contour equal everywhere to Σ_Π except in a small vicinity of the points P_k, where it avoids these points by below/above. Then $\left\{\Sigma_\Pi^{\uparrow/\downarrow}\right\}^{(r,N)}$ is realized as the Carthesian product of two inslotted contours of length r and $N - r$: $\left\{\Sigma_\Pi^{\uparrow/\downarrow}\right\}^{(r,N)} = \left\{\Sigma_\Pi^{\uparrow/\downarrow}\right\}^{(r)} \times \left\{\Sigma_\Pi^{\uparrow/\downarrow}\right\}^{(N-r)}$.

Starting from the definition 6.1 of the matrices Δ and ∇ on inslotted contours $\Sigma_\Pi^{(N)}$, one defines the matrices Δ and ∇ on $\Sigma_\Pi^{\uparrow/\downarrow}\left[z_k\right]$ as the analytic continuations of Δ from $\Sigma_\Pi\left[z_k\right]$.

Proof. The functional $\mathcal{F}_N\left[\nu, u, g\right]$ will be constructed by merging the integral representation (7.4) with the Neumann series for Π (6.1) and Π^{-1} (6.2). These series converge uniformly in λ (and every finite-order λ-derivative) on every open set O such that $\mathrm{d}(O, \Sigma_\Pi) > 0$. However, in (7.4), one integrates $\operatorname{tr}\left[\partial_\lambda \Pi\left(\lambda\right) \sigma\left(\lambda\right) \Pi^{-1}(\lambda)\right]$ with a weight along γ, where the contour γ is depicted in figure 14. The latter contour intersects Σ_Π. Hence, it contains points that are not uniformly away from Σ_Π. However, we have already argued that $\operatorname{tr}\left[\partial_\lambda \Pi_\pm\left(\lambda\right) \sigma\left(\lambda\right) \Pi_\pm^{-1}(\lambda)\right]$ can be analytically continued to a small neighborhood of Σ_Π located to the right/left of Σ_Π. Such an analytic continuation can be also performed on the level of the Neumann series for $\Pi^{\pm 1}$.

In order to have a Neumann series representation for Π or Π^{-1} that is uniformly convergent in $\lambda \in \gamma^{\uparrow/\downarrow} \setminus \{P_1, P_2\}$, we use the local analyticity of the jump matrices for Π so as to deform the original jump contour Σ_Π appearing in the RHP for Π and Π^{-1} (6.1)–(6.2) into the contour $\Sigma_\Pi^{\uparrow/\downarrow}$:

$$\Pi(\lambda) = I_2 + \sum_{N\geq 1} \int_{\Sigma_\Pi^{\uparrow/\downarrow}} \frac{dz}{2i\pi(\lambda - z)} \left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^{\uparrow/\downarrow}} \right\}^{N-1} [I_2](z)\, \Delta(z) \quad \text{and}$$

$$\Pi^{-1}(\lambda) = I_2 + \sum_{N\geq 1} \int_{\Sigma_\Pi^{\uparrow/\downarrow}} \frac{dy}{2i\pi(\lambda - y)} \nabla(y) \left\{ {}^{\boldsymbol{t}}\mathcal{C}^{\nabla}_{\Sigma_\Pi^{\uparrow/\downarrow}} \right\}^{N-1} [I_2](y)\,.$$

According to these formulae Π, Π^{-1} are holomorphic on some small vicinity of $\gamma^{\uparrow/\downarrow}$. However, we do stress that these analytic continuation from above and below P_k differ at P_k. Hence, for $\lambda \in \gamma^{\uparrow/\downarrow} \setminus \{P_1, P_2\}$, we get

$$\operatorname{tr}\left[\partial_\lambda \Pi(\lambda)\, \sigma(\lambda)\, \Pi^{-1}(\lambda)\right] = \sum_{N\geq 1} f_N(\lambda, x) \tag{7.13}$$

with

$$\begin{aligned} f_N(\lambda, x) = & \sum_{r=1}^{N-1} \int_{\Sigma_\Pi^{\uparrow/\downarrow}} \frac{-dz dy}{(2\mathrm{i}\pi)^2 (\lambda - z)^2 (\lambda - y)} \\ & \times \operatorname{tr}\left[\left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^{\uparrow/\downarrow}} \right\}^{r-1} [I_2](z)\, \Delta(z, x)\, \sigma(\lambda)\, \nabla(y, x) \left\{ {}^{\boldsymbol{t}}\mathcal{C}^{\nabla}_{\Sigma_\Pi^{\uparrow/\downarrow}} \right\}^{N-r-1} [I_2](y) \right] \\ & + \int_{\Sigma_\Pi^{\uparrow/\downarrow}} \frac{-dz}{2i\pi(\lambda - z)^2} \cdot \operatorname{tr}\left[\left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^{\uparrow/\downarrow}} \right\}^{N-1} [I_2](z)\, \Delta(z, x)\, \sigma(\lambda) \right]. \end{aligned} \tag{7.14}$$

Above, we have insisted on the dependence on x of the matrices Δ and ∇. The representation (7.13) allows us to define the functional $\mathcal{F}_N[\nu, u, \mathrm{e}^g]$:

$$\mathcal{F}_1[\nu, u, \mathrm{e}^g] = \int_{+\infty}^{x} dx' \oint_\gamma \left\{ f_1(\lambda, x') + \frac{1}{x'} \int_{\partial\mathcal{D}} \frac{dz}{2\mathrm{i}\pi} \frac{\operatorname{tr}\left[\Delta^{(1)}(z; x')\, \sigma_3\right]}{(\lambda - z)^2} \right\} G(\lambda) \frac{d\lambda}{4\pi} \tag{7.15}$$

and, for $N \geq 2$,

$$\mathcal{F}_N[\nu, u, \mathrm{e}^g] = \int_{+\infty}^{x} dx' \oint_\gamma f_N(\lambda, x')\, G(\lambda, x') \frac{d\lambda}{4\pi}. \tag{7.16}$$

We remind that G is given by (6.30) and above, we have explicitly insisted on its x-dependence.

In the following we justify that (7.15)–(7.16) are well defined and that one can exchange the integrals over γ and $[\, x\, ; +\infty\, [$ in (7.4) with the summation

(7.13). Then, we provide explicit bounds for $\mathcal{F}_N\left[\nu, u, \mathrm{e}^{g}\right]$ and finally outline the steps leading to the derivation of the representation (7.9) for $\mathcal{F}_N$.

Exchange of symbols

Building on the identities:

$$|\mathrm{tr}\,(AB)| \le \max\nolimits_{j,k}\left(|B_{jk}|\right) \sum_{j,k} |A_{jk}| \quad \text{and}$$
$$\sum_{j,k} \left\| (AB)_{jk} \right\|_{L^1(\mathscr{C})} \le \sum_{j,k,\ell} \|A_{j\ell}\|_{L^2(\mathscr{C})} \|B_{\ell k}\|_{L^2(\mathscr{C})} \le 4\, \|A\|_{L^2(\mathscr{C})} \|B\|_{L^2(\mathscr{C})}\,,$$

and after some algebra one obtains that

$$\|f_N\|_{L^\infty(\gamma)} \le 4 \max_{\substack{\tau \in \{\uparrow,\downarrow\}\\ k=2,3}} \{\mathrm{d}^{-k}(\gamma^\tau, \Sigma_\Pi^\tau)\}$$
$$\times \sum_{\tau=\uparrow,\downarrow} \sum_{r=1}^{N-1} \left\{ \left\| \left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^\tau} \right\}^{r-1} [I_2] \frac{\Delta}{2\pi} \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \left\| \frac{\nabla}{2\pi} \left\{ {}^{\boldsymbol{t}}\mathcal{C}^{\nabla}_{\Sigma_\Pi^\tau} \right\}^{N-1-r} [I_2] \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \right.$$
$$\left. + \left\| \left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^\tau} \right\}^{N-1} [I_2] \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \left\| \frac{\Delta}{2\pi} \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \right\}.$$

In the intermediate calculation we have used $\|\sigma(\lambda)\|_{L^\infty(\gamma)} = 1$. By using the estimates (6.4)–(6.5), one gets that for $\tau = \uparrow$ or $\downarrow$

$$\left\| \left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^\tau} \right\}^{r} [I_2] \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \le \left\{ 2c\left(\Sigma_\Pi^\tau\right) \|\Delta\|_{L^\infty\left(\Sigma_\Pi^\tau\right)} \right\}^{r-1} \left\| \mathcal{C}^{I_2}_{\Sigma_\Pi^\tau} [\Delta] \right\|_{L^2\left(\Sigma_\Pi^\tau\right)}$$
$$\le \left\{ 2c\left(\Sigma_\Pi^\tau\right) \right\}^{r} \|\Delta\|^{r-1}_{L^\infty\left(\Sigma_\Pi^\tau\right)} \|\Delta\|_{L^2\left(\Sigma_\Pi^\tau\right)}$$

and

$$\left\| \left\{ {}^{\boldsymbol{t}}\mathcal{C}^{\nabla}_{\Sigma_\Pi^\tau} \right\}^{r} [I_2] \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \le \left\{ 2c\left(\Sigma_\Pi^\tau\right) \right\}^{r} \|\Delta\|^{r-1}_{L^\infty\left(\Sigma_\Pi^\tau\right)} \|\Delta\|_{L^2\left(\Sigma_\Pi^\tau\right)}.$$

Also, one has

$$\left\| \left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^\tau} \right\}^{r} [I_2]\, \Delta \right\|_{L^2\left(\Sigma_\Pi^\tau\right)} \le 2\, \|\Delta\|_{L^\infty\left(\Sigma_\Pi^\tau\right)} \left\| \left\{ \mathcal{C}^{\Delta}_{\Sigma_\Pi^\tau} \right\}^{r} [I_2] \right\|_{L^2\left(\Sigma_\Pi^\tau\right)}. \tag{7.17}$$

The estimates and asymptotic expansions of Δ ensure that there exists an x-independent constant C_2 such that

$$\max_{\tau\in\{\uparrow,\downarrow\}}\left\{\|\Delta\|_{L^\infty(\Sigma_\Pi^\tau)}+\|\Delta\|_{L^2(\Sigma_\Pi^\tau)}\right\}\leq\frac{C_2}{c\left(\Sigma_\Pi\right)}x^{-w}$$
$$\text{where } c\left(\Sigma_\Pi\right)=\max_{\tau\in\{\uparrow,\downarrow\}}c\left(\Sigma_\Pi^\tau\right).$$

Hence,

$$\|f_N\|_{L^\infty(\gamma)}\leq\frac{2N}{\pi c(\Sigma_\Pi)}\max_{\substack{\tau\in\{\uparrow,\downarrow\}\\ k=2,3}}\left\{\mathrm{d}^{-k}(\gamma^\tau,\Sigma_\Pi^\tau)\right\}\left(\frac{2C_2}{x^w}\right)^N\left(\pi^{-1}+c\left(\Sigma_\Pi\right)\right)$$
$$\text{with } w=\frac{3}{4}\min(1/2,1-\widetilde{w}-2\max_\pm|\Re\nu\left(\pm q\right)|)\,. \tag{7.18}$$

It follows that for x large enough and for $N\geq N_0$ (with $N_0w>1$) $(\lambda,x')\mapsto\sum_{p=N_0}^N G\left(\lambda,x'\right)f_p\left(\lambda,x'\right)$ is bounded on $\gamma\times\,]\,x\,;+\infty\,[$ by an integrable function. The terms corresponding to $p=1,\dots,N_0-1$ are also integrable as will be shown below. Hence, by the dominated convergence theorem one can exchange the summation and the integration symbols leading to (7.7). It now remains to provide sharper estimates for each summand.

Sharper estimates for $\mathcal{F}_N\left[\nu,u,\mathrm{e}^g\right]$

It follows from Proposition C.1 applied to the jump contour $\Sigma_\Pi^{\uparrow/\downarrow}$, that for $\lambda\in\gamma^{\uparrow/\downarrow}$ one has the representation

$$\Pi_N\left(\lambda\right)=A_N\left(\lambda\right)+\sum_{b=0}^{[N/2]}\sum_{p=0}^{b}\sum_{m=b-[N/2]}^{[N/2]-b}\left(\frac{\mathfrak{e}\left(q;x\right)}{\mathfrak{e}\left(-q;x\right)}\right)^{m-\boldsymbol{\eta}p}\left(\frac{\mathfrak{e}\left(\lambda_0;x\right)}{\mathfrak{e}\left(-q;x\right)}\right)^{\boldsymbol{\eta}b}$$
$$\times\sum_{\epsilon\in\{\pm1,0\}}\left[\mathfrak{e}\left(v_\epsilon;x\right)\right]^{\frac{\sigma_3}{2}}\Pi_{N;\epsilon}^{(m,b,p)}\left(\lambda\right)\left[\mathfrak{e}\left(v_\epsilon;x\right)\right]^{-\frac{\sigma_3}{2}}$$

where $\mathfrak{e}\left(q,x\right)=\mathrm{e}^{\mathrm{i}xu(q)}x^{-2\nu(q)}$, $\mathfrak{e}\left(-q,x\right)=\mathrm{e}^{\mathrm{i}xu(-q)}x^{2\nu(-q)}$ and $\mathfrak{e}\left(\lambda_0,x\right)=\mathrm{e}^{\mathrm{i}xu(\lambda_0)}x^{-\frac{\boldsymbol{\eta}}{2}}$. Also, we remind that $\boldsymbol{\eta}=1$ in the space-like regime and $\boldsymbol{\eta}=-1$ in the time-like and we agree upon $v_0=\lambda_0$ and $v_{\pm q}=\pm q$. The matrix $A_N(\lambda)$ contains exponentially small corrections in x and the remaining part represents the algebraically small ones.

This representation ensures that for $\lambda \in \gamma$

$$f_N(\lambda, x) = \frac{a_N(\lambda, x)}{x^N} + \frac{1}{x^N} \sum_{b=0}^{[N/2]+1} \sum_{p=0}^{b} \sum_{m=b-[N/2]-1}^{[N/2]+1-b} \left(\frac{\mathfrak{e}(q, x)}{\mathfrak{e}(-q, x)} \right)^{m-\boldsymbol{\eta} p}$$
$$\times \left(\frac{\mathfrak{e}(\lambda_0, x)}{\mathfrak{e}(-q, x)} \right)^{\boldsymbol{\eta} b} c_N^{(m,b,p)}(\lambda, x). \tag{7.19}$$

The functions $c_N^{(m,b,p)}(\lambda, x)$ and $a_N(\lambda, x)$ can be expressed as traces involving appropriate combinations of the matrices A_N and $\Pi_{N;\epsilon}^{(m,b,p)}$. We have included all the exponentially small corrections stemming from the A_j's, $j = 1, \ldots, N$ into $a_N(\lambda, x)$.

It follows from the properties of $A_j(\lambda)$ and $\Pi_{j;\epsilon}^{(m,b,p)}(\lambda)$, $j = 1, \ldots, N$ that these functions are smooth in $\lambda \in \gamma$ and x. Moreover, by using the estimates for the L^∞ norms of the aforementioned matrices A_N and $\Pi_{N;\epsilon}^{(m,b,p)}$ (C.4), after some algebra one shows that, for x-large enough, given any $k \in \mathbb{N}$ there exists an N-independent constant $C > 0$ such that

$$|a_N(\lambda, x)| \leq \frac{C^N}{x^k} \quad \text{and}$$
$$\left| c_N^{(m,b,p)}(\lambda, x) \right| \leq C^N x^{N\widetilde{w}}, \quad \text{uniformly in } \lambda \in \gamma^{\uparrow/\downarrow}. \tag{7.20}$$

These estimates remain unchanged when considering first-order partial derivatives in respect to x of these functions. Hence for all integers m, b, p of interest the function

$$(\lambda, y) \mapsto \phi_{m,b,p}(\lambda, y) = y^{-N} \left(\frac{\mathfrak{e}(q, y)}{\mathfrak{e}(-q, y)} \right)^{m-\boldsymbol{\eta} p}$$
$$\times \left(\frac{\mathfrak{e}(\lambda_0, y)}{\mathfrak{e}(-q, y)} \right)^{\boldsymbol{\eta} b} c_N^{(m,b,p)}(\lambda, y)\, G(\lambda; y) \tag{7.21}$$

is Riemann-integrable on $\gamma \times \,]\, x\,; +\infty\, [$. Suppose that m, b or p is non-zero. Then, for $N \geq 2$ an integration by parts leads to the estimate

$$\left| \int_x^{+\infty} dy \phi_{m,b,p}(\lambda, y) \right| \leq \frac{[\widetilde{C}]^N}{x^{Nw}} |G(\lambda; x)|, \tag{7.22}$$

with w being defined as in (7.18). When $m=b=p=0$ one simply deals with a non-oscillating integral. In that case,

$$\left|\int_x^{+\infty} dy\phi_{m,b,p}(\lambda,y)\right| \le \frac{[\widetilde{C}]^N}{x^{N(1-\widetilde{w})-1}}\left|G(\lambda;x)\right|. \tag{7.23}$$

There are two cases of interest to consider. If $w=3/8$, then since $\widetilde{w}=\mathrm{O}(\delta)$, taking δ sufficiently small we get that $Nw \le N(1-\widetilde{w})-1$. It remains to treat the case when, for all $\delta>0$ small enough $w<3/8$. In other words, $1-\widetilde{w}-2\max_{\pm}|\Re\nu(\pm q)|<1/2$. Therefore

$$\frac{1}{4}-\frac{\widetilde{w}}{2} \le \max_{\pm}|\Re\nu(\pm q)|. \tag{7.24}$$

Thus, taking δ small enough, so that $\widetilde{w}\le 1/10$ one gets $\max_{\pm}|\Re\nu(\pm q)| \ge 1/5$. Hence, for $N\ge 2$

$$\left|\int_x^{+\infty} dy\phi_{m,b,p}(\lambda,y)\right| \le \frac{[\widetilde{C}]^N}{x^{Nw}}\left|G(\lambda;x)\right|. \tag{7.25}$$

Thus, once upon the integration over $\lambda\in\gamma^{\uparrow}\cup\gamma^{\downarrow}=\gamma$, we get that there exists a constant $m(x)=\mathrm{O}(x^{-w})$ such that $|\mathcal{F}_N[\nu,u,g](x)|\le[m(x)]^N$ for $N\ge 2$. The fact that $|\mathcal{F}_1[\nu,u,g](x)|\le m(x)$ follows from a direct calculation based on the representation (7.15) and the first few terms of the asymptotic expansion of the matrix Δ.

Justification of (7.9)

We conclude this proof by explaining how one can obtain a slightly more convenient representation for each individual functional $\mathcal{F}_N[\nu,u,g](x)$. Starting from the Neumann series representations (6.8) and (6.9), it follows that for $\lambda\in\gamma^{\uparrow/\downarrow}$

$$\begin{aligned}
&\operatorname{tr}\left[\partial_\lambda\Pi(\lambda)\sigma(\lambda)\Pi^{-1}(\lambda)\right]\\
&\quad= -\sum_{N\ge1}\sum_{r=1}^{N}\int_{\left\{\Sigma_{\Pi}^{\uparrow/\downarrow}\right\}^{(r,N)}}\frac{d^N z}{(2\mathrm{i}\pi)^N}\\
&\qquad\times\frac{\operatorname{tr}\left[\Delta(z_r)\ldots\Delta(z_1)\sigma(\lambda)\nabla(z_{r+1})\ldots\nabla(z_N)\right]}{(\lambda-z_1)^2(\lambda-z_{r+1})\prod\limits_{p=1}^{r-1}(z_p-z_{p+1})\prod\limits_{p=r+1}^{N-1}(z_p-z_{p+1})}.
\end{aligned} \tag{7.26}$$

Above, $\left\{\Sigma_{\Pi}^{\uparrow/\downarrow}\right\}^{(r,N)}$ is the Cartesian product of two inslotted contours $\left\{\Sigma_{\Pi}^{\uparrow/\downarrow}\right\}^{(r)} \times \left\{\Sigma_{\Pi}^{\uparrow/\downarrow}\right\}^{(N-r)}$. To obtain (7.26) it is enough to multiply out the two series for $\partial_{\lambda}\Pi(\lambda)$, $\Pi^{-1}(\lambda)$ and then take the trace. One can convice oneself that the matrices Δ and ∇ are such that

$$\Delta(z) = \mathrm{e}^{\frac{g(z)\sigma_3}{2}} \Delta_{|g=0}(z)\, \mathrm{e}^{-\frac{g(z)\sigma_3}{2}} \quad \text{and} \quad \nabla(z) = \mathrm{e}^{\frac{g(z)\sigma_3}{2}} \nabla_{|g=0}(z)\, \mathrm{e}^{-\frac{g(z)\sigma_3}{2}}. \tag{7.27}$$

This means that there exists functionals $D_{N,r}\Big(\lambda, \{z_j\}_{j=1}^{N}, x; \{\epsilon_j\}_{j=1}^{N}\Big)[\nu, u]$ such that

$$\begin{aligned}
&\operatorname{tr}\left[\Delta(z_r)\ldots\Delta(z_1)\,\sigma(\lambda)\,\nabla(z_{r+1})\ldots\nabla(z_N)\right] \\
&\quad = \sum_{\substack{\Sigma\epsilon_k=0 \\ \epsilon_k\in\{\pm1,0\}}} D_{N,r}\Big(\lambda, \{z_j\}_{j=1}^{N}, x; \{\epsilon_j\}_{j=1}^{N}\Big)[\nu, u] \exp\left\{\sum_{p=1}^{N} \epsilon_p g(z_p)\right\}.
\end{aligned} \tag{7.28}$$

Above, the sum is taken over all possible choices of N integers $\epsilon_k \in \{\pm1, 0\}$, such that $\sum_{k=1}^{N}\epsilon_k = 0$. We replace the trace in (7.26) by (7.28) and then insert the result into the integral representation for $\log\det\left[I+V\right]^{\mathrm{sub}}$. One can exchange the γ and x'-integrals with the summation over N in virtue of the previous discussion. One can also pull-out the finite sum over $\epsilon_k \in \{\pm1, 0\}$ out of the integrals. Indeed, given any choice of $\{\epsilon_k\}$, the function $(\lambda, x') \mapsto G(\lambda, x')\, D_{N,r}$ is Riemann–integrable along $\gamma \times \,]\,x\,; +\infty\,[$. This stems from the fact that $D_{N,r}$ is bounded in λ, and as follows from the previous discussion, is at least Riemann-integrable in x as an oscillatory integral. Moreover, it is clear that by harping on the steps that allow one to prove the expansion for Π_N given in Proposition 6.1, one can just as well prove a similar type of expansion for $\mathcal{F}_N[\nu, u, \mathrm{e}^{g}]$. □

Lemma 7.2. *Under the assumptions of section* 2.1*, the Fredholm determinant of $I+V$, with V being given by* (1.1)*, admits the below, absolutely convergent for x-large enough, Natte series representation:*

$$\begin{aligned}
\det\left[I+V\right][\nu, u, g] = \det\left[I+V\right]^{(0)}[\nu, u, g]\Bigg\{1 \\
+ \sum_{n\geq1}\sum_{\boldsymbol{k}\in\mathcal{K}_n}\sum_{\{\epsilon_{\boldsymbol{t}}\}\in\mathcal{E}(\boldsymbol{k})} H_n\Big(x, \{k\}, \{\epsilon_{\boldsymbol{t}}\}_{\boldsymbol{t}\in J_{\{k\}}}\Big)\Big[\nu, u, \Pi_{\boldsymbol{t}\in J_{\{k\}}}\mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})}\Big]\Bigg\}.
\end{aligned} \tag{7.29}$$

The n^{th} term of this series is a $\mathrm{O}([m(x)]^n)$, with $m(x) = \mathrm{O}(x^{-w})$ w being given as in (7.8) and the O *being n independent. The functional H_n appearing above is a linear functional in respect to the function $\prod_{\boldsymbol{t}\in J_{\{k\}}} \mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})}$ of the n-variables $z_{\boldsymbol{t}}$. It produces a weighted integration of this function over curves lying in some small neighborhood of the real axis:*

$$\begin{aligned} & H_n\left(x, \{k\}, \{\epsilon_{\boldsymbol{t}}\}_{\boldsymbol{t}\in J_{\{k\}}}\right)\left[\nu, u, \prod_{\boldsymbol{t}\in J_{\{k\}}} \mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})}\right] \\ & \quad = \sum_{\substack{r_{\boldsymbol{d}}=1 \\ \boldsymbol{d}\in D_{\{k\}}}}^{\boldsymbol{d}_1} \sum_{\tau_{\boldsymbol{d}}=\uparrow/\downarrow} \prod_{\boldsymbol{d}\in D_{\{k\}}} \int_{+\infty}^{x} dx_{\boldsymbol{d}} \oint_{\gamma^{\tau_{\boldsymbol{d}}}} \frac{d\lambda_{\boldsymbol{d}}}{4\pi} \int_{\left\{\Sigma_{\Pi}^{\tau_{\boldsymbol{d}}}\right\}^{(r_{\boldsymbol{d}},\boldsymbol{d}_1)}} \frac{d^{\boldsymbol{d}_1} z_{\boldsymbol{d},j}}{(2\mathrm{i}\pi)^{\boldsymbol{d}_1}} \prod_{j=1}^{n} \frac{1}{k_j!} \\ & \quad \cdot \prod_{\boldsymbol{d}\in D_{\{k\}}} H_{\boldsymbol{d}_1, r_{\boldsymbol{d}}}\left(\lambda_{\boldsymbol{d}}, \{z_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1}, x_{\boldsymbol{d}}; \{\epsilon_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1}\right)[\nu, u] \cdot \prod_{\boldsymbol{t}\in J_{\{k\}}} \mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})}. \end{aligned} \tag{7.30}$$

In (7.29), *the sum is carried out over all the possible choices of n-uples of integers $\boldsymbol{k} = (k_1, \ldots, k_n)$ belonging to*

$$\mathcal{K}_n = \left\{\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n : \sum_{s=1}^{n} s k_s = n\right\} \tag{7.31}$$

Each such n-uple of integers defines a set of triplets

$$J_{\{k\}} = \Big\{ (s,p,j)\,, s \in [\![\, 1\,;\, n \,]\!], p \in [\![\, 1\,;\, k_s \,]\!], j \in [\![\, 1\,;\, s \,]\!] \Big\} \tag{7.32}$$

and a set of doublets

$$D_{\{k\}} = \Big\{ (s,p)\,, s \in [\![\, 1\,;\, n \,]\!], p \in [\![\, 1\,;\, k_s \,]\!] \Big\}. \tag{7.33}$$

A triplet (s,p,j) belonging to $J_{\{k\}}$ is denoted by $\boldsymbol{t}$ and a doublet (s,p) belonging to $D_{\{k\}}$ is denoted by $\boldsymbol{d} = (s,p)$. The notation $\boldsymbol{d}_1$ stands for the first coordinate of $\boldsymbol{d}$, i.e., if $\boldsymbol{d} = (s,p)$, then $s = \boldsymbol{d}_1$. Once that a choice of $\boldsymbol{k}$ is made, one sums over all the possible elements of

$$\mathcal{E}(\boldsymbol{k}) = \left\{ \{\epsilon_{\boldsymbol{t}}\}_{\boldsymbol{t}\in J_{\{k\}}} : \epsilon_{\boldsymbol{t}} \in \{\pm 1, 0\} \quad \text{and} \quad \sum_{j=1}^{\boldsymbol{d}_1} \epsilon_{\boldsymbol{d},j} = 0 \quad \text{for all } \boldsymbol{d} \in D_{\{k\}} \right\}. \tag{7.34}$$

The sums and integrations in (7.30) *are also ordered by the sets of triplets* $J_{\{k\}}$ *and doublets* $D_{\{k\}}$. *One first starts to sum up over* $r_{\boldsymbol{d}}$, *where* $\boldsymbol{d}$ *runs through* $D_{\{k\}}$. *There are* $\#D_{\{k\}}$ *such sums in total, corresponding to* $\boldsymbol{d}$ *running through the set* $D_{\{k\}}$. *Finally, for each* $\boldsymbol{d} \in D_{\{k\}}$, *there is one integral over the corresponding* $x_{\boldsymbol{d}}$, *one over the corresponding* $\lambda_{\boldsymbol{d}}$ *and* $\boldsymbol{d}_1$ *integrals over the subordinate set of* z*-variables* $\{z_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1}$.

Proof. We define an auxiliary function

$$A(\gamma) = \sum_{N\geq 1} \gamma^N \mathcal{F}_N[\nu, u, g](x), \tag{7.35}$$

which is holomorphic in γ on the open disc of radius $m^{-1}(x)$, with $m(x) = \mathrm{O}(x^{-w})$ and w given by (7.8), as follows from the estimates on the growth of $\mathcal{F}_N$ with x. This means that $\mathrm{e}^{A(\gamma)}$ is holomorphic on the same disc. The radius of convergence of its Taylor series around $\gamma = 0$ has its lower bound given by $m^{-1}(x)$.

The series for the Fredholm determinant is obtained by using the Faa-di-Bruno formula so as to compute the nth derivative of $\mathrm{e}^{A(\gamma)}$ at $\gamma = 0$

$$\begin{aligned} \frac{1}{n!}\frac{d^n}{d\gamma^n}\mathrm{e}^{A(\gamma)}\bigg|_{\gamma=0} &= \sum_{\substack{\{k\} \\ \Sigma s k_s = n}} \mathrm{e}^{A(0)} \prod_{j=1}^{n} \left\{ \frac{1}{k_j!}\left(\frac{A^{(j)}(0)}{j!}\right)^{k_j} \right\} \\ &\Rightarrow \left| \frac{1}{n!}\frac{d^n}{d\gamma^n}\mathrm{e}^{A(\gamma)}\bigg|_{\gamma=0} \right| \leq \frac{[4m(x)]^n}{n!}\frac{d^n}{d\gamma^n}\exp\left\{\frac{1}{2-\gamma}\right\}\bigg|_{\gamma=0}. \end{aligned} \tag{7.36}$$

Where we used that $|\mathcal{F}_N[\nu, u, g]| \leq (m(x))^N$, with $m(x) = \mathrm{O}(x^{-w})$. The last estimates allow one to see explicitly that the Taylor series at $\gamma = 0$ for $\mathrm{e}^{A(\lambda)}$ has a radius of convergence that scales as $m^{-1}(x)$. It is in particular convergent at $\gamma = 1$ leading to

$$\begin{aligned} \det[I+V][\nu, u, g] &= \det[I+V]^{(0)}[\nu, u, g] \\ &\times \left\{ 1 + \sum_{n\geq 1}\sum_{\Sigma s k_s = n}\prod_{j=1}^{n}(k_j!)^{-1}\cdot\prod_{j=1}^{n}[\mathcal{F}_j[\nu, u, g](x)]^{k_j} \right\}. \end{aligned} \tag{7.37}$$

In this language of doublets and triplets, the expression for the product in (7.37) reads

$$\prod_{j=1}^{n}\left(\mathcal{F}_j\left[\nu,u,g\right](x)\right)^{k_j}$$
$$= \sum_{\substack{r_{\boldsymbol{d}}=1\\ \boldsymbol{d}\in D_{\{k\}}}}^{\boldsymbol{d}_1} \sum_{\substack{\sum_{j=1}^{\boldsymbol{d}_1}\epsilon_{\boldsymbol{d},j}=0\\ \epsilon_{\boldsymbol{d},j}\in\{\pm1,0\}}} \sum_{\tau_{\boldsymbol{d}}=\uparrow/\downarrow} \prod_{\boldsymbol{d}\in D_{\{k\}}} \int_{+\infty}^{x} dx_{\boldsymbol{d}} \oint_{\gamma^{\tau_{\boldsymbol{d}}}} \frac{d\lambda_{\boldsymbol{d}}}{4\pi} \int_{\left\{\Sigma_{\Pi}^{\tau_{\boldsymbol{d}}}\right\}^{(r_{\boldsymbol{d},\boldsymbol{d}_1})}} \prod_{j=1}^{\boldsymbol{d}_1} \frac{dz_{\boldsymbol{d},j}}{(2\mathrm{i}\pi)}$$
$$\times \prod_{\boldsymbol{d}\in D_{\{k\}}} H_{\boldsymbol{d}_1,r_{\boldsymbol{d}}}\Big(\lambda_{\boldsymbol{d}},\{z_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1},x_{\boldsymbol{d}};\{\epsilon_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1}\Big)\left[\nu,u\right]\exp\left\{\sum_{\boldsymbol{t}\in J_{\{k\}}}\epsilon_{\boldsymbol{t}}g\left(z_{\boldsymbol{t}}\right)\right\}. \tag{7.38}$$

The result follows. □

The expression (7.30) for the functionals involved in the Natte series is more explicit then as it was given in Theorem 2.2.

7.2 Proof of Theorem 2.2

The first part of Theorem 2.2, i.e., the very form of the expansion (2.9) is a consequence of Lemma 7.2. The latter provides moreover a more explicit form for the functionals $\mathcal{H}_n\left[\nu,\mathrm{e}^g,u\right]$.

The well-ordered asymptotic expansion in x for each functional $\mathcal{H}_n[\nu,\mathrm{e}^g,u]$, as given in (2.12), is a direct consequence of the existence of a similar representation for $\mathcal{F}_N\left[\nu,u,g\right]$ together the correspondence (7.37) between $\mathcal{F}_N\left[\nu,u,g\right]$ and $\det\left[I+V\right]$. Finally, the existence of a representation for $\mathcal{F}_N\left[\nu,u,g\right]$ in the spirit of (2.12) can be readily obtained by inserting the well-ordered series representation for Π and Π^{-1} (we remind that $\Pi^{-1}={}^t\mathrm{Comat}\left(\Pi\right)$ since $\det\left[\Pi\right]=1$) given in Proposition 6.1 into the integral representation for $\mathcal{F}_N$, (7.14), (7.15), (7.16). □

7.3 Higher order Natte series

The higher order Natte series is a generalization of the Natte series derived in the previous sub-sections. It gives a direct access to part of the asymptotic expansion without having to compute the effective form of the functionals $D_{N,r}$ and then carry out the contour integrals. Indeed, even if it is possible in principle to compute explicitly, order-by-order the functionals $D_{N,r}$ and thus H_n, this task becomes very quickly monstrously cumbersome. In order to get the corrections, it is more desirable to apply the procedure below (or its obvious extension to higher order asymptotics) if one wants to access to the higher order correction then those contained in $\det\left[I+V\right]^{(0)}\left[\nu,u,g\right]$.

Proposition 7.1. *The Fredholm determinant of $I+V$ admits the below convergent Natte series representation:*

$$\det\left[I+V\right]\left[\nu,u,g\right]=\det\left[I+V\right]^{(0)}\left[\nu,u,g\right]$$
$$\times\exp\left\{\int_{+\infty}^{x}dx'\left[x'\right]^{-\frac{3}{2}}a_1(x')+\left[x'\right]^{-2}\left[a_2^{\mathrm{osc}}(x')+a_2^{\mathrm{no}}(x')\right]\right\}$$
$$\times\left\{1+\sum_{n\geq1}\sum_{\boldsymbol{k}\in\mathcal{K}_n}\sum_{\{\epsilon_{\boldsymbol{t}}\}\in\mathcal{E}(\boldsymbol{k})}\widetilde{H}_n\left(x,\{k\},\{\epsilon_{\boldsymbol{t}}\}\right)\left[\nu,u,\Pi_{\boldsymbol{t}\in J_{\{k\}}}\mathrm{e}^{\epsilon_{\boldsymbol{t}}g(z_{\boldsymbol{t}})}\right]\right\}. \tag{7.39}$$

There $\widetilde{H}_n$ is defined as in (7.30), (7.10), *but with the minor difference that the functionals D_N in* (7.10) *should be replaced by the functionals $\widetilde{D}_N$ as given in* (7.41). *Also, a_1, a_2^{osc}, a_2^{nosc} are given by* (6.36), (6.37), (6.38), (6.39). *Note that here we have explicitly insisted on their dependence on the large-parameter x'. The fundamental difference between the higher order Natte series and the one discussed previously is that for x large enough and for an n-independent* O*:*

$$\left|H_n\left(x,\{k\},\{\epsilon_{\boldsymbol{t}}\}\right)\left[\nu,u,\prod_{\boldsymbol{t}\in J_{\{k\}}}\mathrm{e}^{\epsilon_{\boldsymbol{t}}g(z_{\boldsymbol{t}})}\right]\right|$$
$$=\min\left\{\mathrm{O}\left(\frac{1}{x^{nw}}\right),\ \mathrm{O}\left(\frac{\log^2 x}{x^3},\frac{a_1\log x}{x^{\frac{5}{2}}},\frac{a_2^{\mathrm{osc}}\log x}{x^3},\frac{a_1}{x^{\frac{3}{2}+w}},\frac{a_2^{\mathrm{osc}}}{x^{2+w}}\right)\right\}.$$

The constant w is as defined in (7.8).

Proof. One starts by performing the decomposition

$$\log\det\left[I+V\right]\left[\nu,u,g\right]$$
$$=\log\det\left[I+V\right]^{(0)}\left[\nu,u,g\right]+\int_{+\infty}^{x}dy\left(\frac{a_1(y)}{y^{\frac{3}{2}}}+\frac{a_2^{\mathrm{osc}}(y)+a_2^{\mathrm{no}}(y)}{y^2}\right)$$
$$+\log\det\left[I+V\right]^{(\mathrm{sub2})}\left[\nu,u,g\right].$$

There, $\log\det\left[I+V\right]^{(\mathrm{sub2})}[\nu,u,g]$ corresponds to that part of $\log\det\left[I+V\right]^{(\mathrm{sub})}[\nu,u,g]$ (7.4) where all terms that give rise to the integral involving a_1 and $a_2^{\mathrm{osc/no}}$ have been subtracted. Namely,

$$\begin{aligned}
\log\det\left[I+V\right]^{(\mathrm{sub2})}[\nu,u,g] &= \int_{+\infty}^{x} dx' \oint_{\gamma} \frac{d\lambda}{4\pi} G(\lambda) \\
&\times \Bigg\{ \mathrm{tr}\left[\partial_\lambda \Pi(\lambda)\,\sigma(\lambda)\,\Pi^{-1}(\lambda)\right] - \frac{\mathbf{1}_{\gamma^{(0)}}(\lambda)}{y} \mathrm{tr}\left\{\left[\Pi^{(1)}\right]'(\lambda)\,\sigma_3\right\} \\
&- \frac{\mathbf{1}_{\gamma^{(0)}}(\lambda)}{y^{\frac{3}{2}}} \mathrm{tr}\left\{\left[\Pi^{(2)}\right]'(\lambda) - \Pi^{(0)}(\lambda)\left[\Pi^{(1)}\right]'(\lambda) - \Pi^{(1)}(\lambda)\left[\Pi^{(0)}\right]'(\lambda)\right\}\sigma_3 \\
&- \frac{\mathbf{1}_{\gamma^{(0)}}(\lambda)}{y^{2}} \mathrm{tr}\left\{\left[\Pi^{(3)}\right]'(\lambda) - \Pi^{(1)}(\lambda)\left[\Pi^{(1)}\right]'(\lambda)\right\}\sigma_3 \Bigg\} . \qquad (7.40)
\end{aligned}$$

We stress that the variable of integration x' corresponds to the large parameter (denoted by x before) that enters in the formulation of the RHP for Π and on which Π depends implicitly.

The expansion of $\log\det\left[I+V\right]^{(\mathrm{sub})}[\nu,u,g]$ goes along the same lines as before, with the minor difference that the functionals $D_{N,r}$ are defined slightly differently. Indeed one has to subtract from the Neumann series like expansion for $\mathrm{tr}\left[\partial_\lambda\Pi(\lambda)\,\sigma(\lambda)\,\Pi^{-1}(\lambda)\right]$ all the subleading contributions that appear in (7.40). For this purpose, we define

$$\begin{aligned}
&\mathrm{tr}\left[\Delta(z_r)\ldots\Delta(z_1)\,\sigma(\lambda)\,\nabla(z_{r+1})\ldots\nabla(z_N)\right] - R_{N,r}(\{z\},x) \\
&\quad = \sum_{\substack{\Sigma\epsilon_k=0 \\ \epsilon_k\in\{\pm1,0\}}} \widetilde{D}_{N,r}\left(\lambda,\{z_j\}_1^N, x; \{\epsilon_j\}_{j=1}^N\right)[\nu,u]\, \mathrm{e}^{\sum\limits_{p=1}^{N}\epsilon_p g(z_p)} . \qquad (7.41)
\end{aligned}$$

In order to define $R_{N,r}(\{z\},x)$ we represent the asymptotic expansion of $\Delta(z)$ slightly differently then in (7.1).

$$\Delta(z) \simeq \sum_{p\geq1} x^{-\frac{p}{2}} M^{(p)}(z,x) \quad \text{with}$$

$$\begin{cases}
M^{(2p+1)}(z;x) = \dfrac{d^{(p)}(z)}{(z-\lambda_0)^{2p+1}} \cdot \mathbf{1}_{\mathcal{D}_{\lambda_0,2\delta}\setminus\mathcal{D}_{\lambda_0,\delta'}}(z) \\[2ex]
M^{(2p)}(z;x) = \sum\limits_{\epsilon=\pm} \dfrac{V^{(\epsilon;p-1)}(z)}{p!\,(z-\epsilon q)^{p-1}} \cdot \mathbf{1}_{\mathcal{D}_{\epsilon q,2\delta}\setminus\mathcal{D}_{\epsilon q,\delta'}}(z) .
\end{cases} \qquad (7.42)$$

There we took δ small enough and $\delta > \delta' > 0$. In terms of such matrices one has

$$\begin{aligned} R_{N,r}\left(\{z\},x\right) &= \delta_{N,1}\delta_{r,N}\left[\sum_{p=1}^{4} x^{-\frac{p}{2}}\mathrm{tr}\left[M^{(p)}(z_1,x)\,\sigma_3\right]\right] \\ &\quad + \frac{\delta_{N,3}\delta_{r,3}}{x^2}\mathrm{tr}\left[M^{(1)}(z_3,x)\,M^{(2)}(z_2,x)\,M^{(1)}(z_1,x)\,\sigma_3\right] \\ &\quad + \sum_{r=1}^{2}\delta_{N,2}\left(-1\right)^r\left[\sum_{p,p'=1}^{2} x^{-\frac{p+p'}{2}}\mathrm{tr}\left[M^{(p)}(z_1,x)\,\sigma_3 M^{(p')}(z_2,x)\right]\right]. \end{aligned}$$

Finally, defining $\widetilde{H}_n$ as in (7.30), (7.10), but with the minor difference that the functionals D_N in (7.10) should be replaced by the functionals $\widetilde{D}_N$ given in (7.41). One gets the desired representation. $\square$

A similar Natte series can be obtained for other quantities that are also related with the correlation functions in integrable models.

Proposition 7.2. *Let F_1 be as defined in* (3.3)*, then the below Fredholm minor admits a Natte series representation:*

$$\begin{aligned} &\left\{\int_{\mathscr{C}_E}\frac{d\lambda}{2\pi}e^{-2}(\lambda) + \int_{-q}^{q}\frac{d\lambda}{2\pi}4\sin^2\left[\pi\nu\left(\lambda\right)\right]F_1\left(\lambda\right)E\left(\lambda\right)\right\}\det\left[I+V\right]\left[\nu,u,g\right] \\ &\quad = \det\left[I+V\right]^{(0)}\left[\nu,u,g\right]\left\{\frac{\mathcal{S}_0^{-1}\mathbf{1}_{]\,q\,;+\infty\,[}\left(\lambda_0\right)}{\sqrt{-2\pi x u''(\lambda_0)}} + \frac{\nu\left(q\right)}{xu'\left(q\right)}\mathcal{S}_+^{-1} - \frac{\nu\left(-q\right)}{xu'\left(-q\right)}\mathcal{S}_-^{-1}\right. \\ &\qquad \left. + \sum_{n\geq 1}\sum_{\boldsymbol{k}\in\widetilde{\mathcal{K}}_n}\sum_{\{\epsilon_{\boldsymbol{t}}\}\in\widetilde{\mathcal{E}}(\boldsymbol{k})}\widetilde{H}_n^{(+)}\left(x,\boldsymbol{k},\{\epsilon_{\boldsymbol{t}}\}\right)\left[\nu,u,\Pi_{\boldsymbol{t}\in J_{\{k\}}}\mathrm{e}^{\epsilon_{\boldsymbol{t}}g(z_{\boldsymbol{t}})}\right]\right\}. \end{aligned} \tag{7.43}$$

There the summation runs through all the possible choice of integers $k_1,\ldots,k_{N+1}$ belonging to

$$\mathcal{K}_n = \left\{\boldsymbol{k} = (k_1,\ldots,k_{n+1})\ :\ k_s\in\mathbb{N},\ s=1,\ldots,n \quad \text{and} \right. \\ \left. k_{n+1}\in\mathbb{N}^* \quad k_{n+1}+\sum_{s=1}^{n} sk_s = n\right\}. \tag{7.44}$$

The remaining summation run through sets that are labelled by doublets and triplets belonging to

$$J_{\{k\}} = \Big\{ \, (s,p,j) \, , s \in [\![\, 1 \, ; \, n+1 \,]\!] \, , \; p \in [\![\, 1 \, ; \, k_s \,]\!] \, , \; j \in [\![\, 1 \, ; \, s - \delta_{s,n+1} n \,]\!] \Big\}$$
$$D_{\{k\}} = \Big\{ \, (s,p,j) \, , s \in [\![\, 1 \, ; \, n \,]\!] \, , \; p \in [\![\, 1 \, ; \, k_s \,]\!] \Big\}.$$

Indeed, then

$$\widetilde{\mathscr{E}}\,(\boldsymbol{k}) = \Bigg\{ \{\epsilon_{\boldsymbol{t}}\}_{\boldsymbol{t}\in J_{\{k\}}} \; : \; \epsilon_{\boldsymbol{t}} \in \{\pm 1, 0\} \, , \; \sum_{j=1}^{\boldsymbol{d}_1} \epsilon_{\boldsymbol{d},j} = 0 \;\; \text{with} \;\; \boldsymbol{d} \in D_{\{k\}} \;\; \text{and} \;\; \sum_{p=1}^{k_{n+1}} \epsilon_{k_{n+1},p,1} = 1 \Bigg\}. \tag{7.45}$$

Finally, the functionals $\widetilde{H}_n^{(+)}\,(x, \{k\}\,, \{\epsilon_{\boldsymbol{t}}\})$ *read*

$$\widetilde{H}_n^{(+)}\,(x, \{k\}\,, \{\epsilon_{\boldsymbol{t}}\}) \Big[\nu, u, \prod_{\boldsymbol{t}\in J_{\{k\}}} \mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})} \Big]$$
$$= \int_{\Sigma_\Pi^{(n)}} \prod_{p=1}^{n} \frac{dz_{n+1,p,1}}{2\mathrm{i}\pi} \widetilde{C}_n \Big(\{z_{n+1,p,1}\}_{p=1}^n, \{\epsilon_{n+1,p,1}\}_{p=1}^n, x \Big) \, [\nu, u]$$
$$\times \prod_{p=1}^{n} \mathrm{e}^{\epsilon_{n+1,p,1} g(z_{n+1,p,1})}.$$

when $k_{n+1} = n$ *and in all other cases,*

$$\widetilde{H}_n^{(+)} \Big(x, \{k\}\,, \{\epsilon_{\boldsymbol{t}}\}_{\boldsymbol{t}\in J_{\{k\}}} \Big) \Big[\nu, u, \prod_{\boldsymbol{t}\in J_{\{k\}}} \mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})} \Big]$$
$$= \sum_{\substack{r_{\boldsymbol{d}}=1 \\ \boldsymbol{d}\in D_{\{k\}}}}^{\boldsymbol{d}_1} \; \sum_{\tau_{\boldsymbol{d}}=\uparrow/\downarrow} \; \prod_{\boldsymbol{d}\in D_{\{k\}}} \; \int_{+\infty}^{x} dx_{\boldsymbol{d}} \oint_{\gamma^{\tau_{\boldsymbol{d}}}} \frac{d\lambda_{\boldsymbol{d}}}{4\pi} \int_{\{\Sigma_\Pi^{\tau_{\boldsymbol{d}}}\}^{(r_{\boldsymbol{d}}, \boldsymbol{d}_1)}} \frac{d^{\boldsymbol{d}_1} z_{\boldsymbol{d},j}}{(2\mathrm{i}\pi)^{\boldsymbol{d}_1}}$$
$$\times \int_{\Sigma_\Pi^{(k_{n+1})}} \prod_{p=1}^{k_{n+1}} \frac{dz_{k_{n+1},p,1}}{2\mathrm{i}\pi} \times \prod_{j=1}^{n} \frac{1}{k_j!}$$

$$\times\ C_{k_{n+1}}\left(\{z_{k_{n+1},p,1}\}_{p=1}^{k_{n+1}}, x, \{\epsilon_{k_{n+1},p,1}\}_{p=1}^{k_{n+1}}\right)[\nu, u]$$
$$\cdot \prod_{\boldsymbol{d}\in D_{\{k\}}} H_{\boldsymbol{d}_1, r_{\boldsymbol{d}}}\left(\lambda_{\boldsymbol{d}}, \{z_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1}, x_{\boldsymbol{d}}; \{\epsilon_{\boldsymbol{d},j}\}_{j=1}^{\boldsymbol{d}_1}\right)[\nu, u] \cdot \prod_{\boldsymbol{t}\in J_{\{k\}}} \mathrm{e}^{\epsilon_{\boldsymbol{t}} g(z_{\boldsymbol{t}})}.$$

Note that we have used above the notation introduced in Lemma 7.2. Also we have set

$$i\frac{\mathrm{tr}\left[\Delta\left(z_N\right)\ldots\Delta\left(z_1\right)\sigma^-\right] - \delta_{N,1}\sum_{p=1}^{2} x^{-\frac{p}{2}}\mathrm{tr}\left[M^{(p)}(z_1, x)\,\sigma^-\right]}{\prod_{p=1}^{N-1}\left(z_p - z_{p-1}\right)}$$
$$= \sum_{\Sigma\epsilon_s=1} \widetilde{C}_N\left(\{z_j\}_1^N, x, \{\epsilon_j\}_1^N\right)[\nu, u]\prod_{p=1}^{N}\mathrm{e}^{\epsilon_p g(z_p)}$$

$$i\mathrm{tr}\left[\Delta\left(z_N\right)\ldots\Delta\left(z_1\right)\sigma^-\right]\cdot\prod_{p=1}^{N-1}\left(z_p - z_{p-1}\right)^{-1}$$
$$= \sum_{\Sigma\epsilon_s=1} C_N\left(\{z_j\}_1^N, x, \{\epsilon_j\}_1^N\right)[\nu, u]\prod_{p=1}^{N}\mathrm{e}^{\epsilon_p g(z_p)}.$$

The sums in the two equations above run over all choices of the variables ϵ_s, $s = 1, \ldots, N$ with $\epsilon_s \in \{\pm 1, 0\}$ and $\sum_{s=1}^{N}\epsilon_s = 1$.

Proof. By using the integral representation for χ (3.6), one readily gets that

$$i\left[\chi_\infty\right]_{12} \equiv \lim_{\lambda\to+\infty}\mathrm{tr}\left[\lambda\chi\left(\lambda\right)\sigma^-\right] = \int_{-q}^{q}\frac{d\lambda}{2\pi}4\sin^2\left[\pi\nu\left(\lambda\right)\right]F_1\left(\lambda\right)E\left(\lambda\right). \quad (7.46)$$

Also, it is easy to convince oneself that

$$i\left[\Pi_\infty\right]_{12} = i\left[\chi_\infty\right]_{12} + \int_{\mathscr{C}_E} e^{-2}(\lambda)\,\frac{d\lambda}{2\pi}. \quad (7.47)$$

The claim follows by expanding $[\Pi_\infty]_{12}$ into a higher order Natte series (where the first few terms of the asymptotics have been taken into account) and then taking the product of this series with the Natte series for the Fredholm determinant. The details are left to the reader. □

It follows from the leading asymptotics given in (7.43) that, in the case of the time-like regime, the saddle-point λ_0 does not contribute to the leading order. It can however be checked that it does eventually contribute. Its contribution is a $\mathrm{O}\Big(x^{-5/2}, x^{-5/2}\big(x^{-\nu(q)}+x^{-\nu(-q)}\big)^2\Big)$.

8 Conclusion

In this paper we have obtained the first few terms in the leading $x \to +\infty$ asymptotics of the Fredholm determinant of a class of integrable integral operator that provide a starting point for the analysis of the large-distance/long-time asymptotic behavior in integrable models away from their free fermion point. Also, we have derived a new series representation for the Fredholm determinant, the so-called Natte series. This series is well adapted for an asymptotic analysis of the Fredholm determinant and can thus be thought of as being an analog of the Mellin–Barnes integral representation for hypergeometric functions. In two subsequent paper, the Natte series will appear as a central tool in computing the large-distance/long-time asymptotic behavior of the correlation functions in the non-linear Schrödinger model away from its free Fermion point [34, 37]. As a byproduct of our analysis, we have been able to bring a little more order to the structure of the asymptotic expansion of Fredholm determinants of operators belonging to the class of the generalized sine kernel. It would be interesting to extend/push forward the form of the full asymptotic expansion of the determinant given in Theorem 2.2, in particular by providing a closed form (i.e., the explicit values of coefficients/functionals), at least in the case of some particular kernel such as the sine kernel.

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Appendix A. Several Properties of CHF

One can check that for $z \in \mathbb{R}^+$

$$\Psi\left(1, \frac{3}{2}; -\mathrm{e}^{\mathrm{i}0^+} z\right) - \Psi\left(1, \frac{3}{2}; -\mathrm{e}^{-\mathrm{i}0^+} z\right) = 2i\sqrt{\frac{\pi}{z}}\mathrm{e}^{-z}. \tag{A.1}$$

$\Psi(a,c;z)$ has an asymptotic expansion at $z \to \infty$ given by

$$\Psi(a,c;z) = \sum_{n=0}^{M} (-1)^n \frac{(a)_n (a-c+1)_n}{n!} z^{-a-n} + \mathrm{O}\left(z^{-M-a}\right), \quad \text{for } -\frac{3\pi}{2} < \arg(z) < \frac{3\pi}{2}. \tag{A.2}$$

It also satisfies to the monodromy properties

$$\Psi(a,1;ze^{2\mathrm{i}\pi}) = \Psi(a,1;z)e^{-2i\pi a} + \frac{2\pi i e^{-i\pi a+z}}{\Gamma^2(a)}\Psi(1-a,1;-z), \quad \Im(z) < 0, \tag{A.3}$$

$$\Psi(a,1;ze^{-2i\pi}) = \Psi(a,1;z)e^{2i\pi a} - \frac{2\pi i e^{i\pi a+z}}{\Gamma^2(a)}\Psi(1-a,1;-z), \quad \Im(z) > 0. \tag{A.4}$$

Tricomi's CHF can be expressed in terms of Humbert's CHF

$$\Psi(a,c;z) = \Gamma\begin{pmatrix} 1-c \\ a-c+1 \end{pmatrix} \Phi(a,c;z) + \Gamma\begin{pmatrix} c-1 \\ a \end{pmatrix} z^{1-c}\Phi(a-c+1, 2-c; z). \tag{A.5}$$

There exists a similar formula expression Tricomi's CHF $\Psi(a,c;z)$ in terms of Humbert's one

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(c-a)}\mathrm{e}^{\mathrm{i}\epsilon a\pi}\Psi(a,c;z) + \frac{\Gamma(c)}{\Gamma(a)}\mathrm{e}^{\mathrm{i}\epsilon\pi(a-c)+z}\Psi(c-a,c;-z), \tag{A.6}$$

where $\epsilon = \mathrm{sgn}(\Im(z))$, and

$$\Phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}. \tag{A.7}$$

The Barnes' G function satisfies to the reflection property

$$G(1-z) = \frac{G(1+z)}{(2\pi)^z} \exp\left\{\int_0^z \pi x \cot[\pi x]\; dx\right\}, \tag{A.8}$$

which holds for $\Re(z) < 1$ in the usual sense (and also everywhere else by analytic continuation).

Appendix B. Proof of the asymptotic expansion for Π

B.1 Two lemmas

We first need a technical lemma

Lemma B.1. *Let the matrices Δ_j take the form*

$$\Delta_j = \left[\mathfrak{e}_j\right]^{\frac{\sigma_3}{2}} \widetilde{\Delta}_j \left[\mathfrak{e}_j\right]^{-\frac{\sigma_3}{2}} = \begin{pmatrix} a_j & b_j \mathfrak{e}_j \\ c_j \mathfrak{e}_j^{-1} & d_j \end{pmatrix}, \tag{B.1}$$

where the entries a_j, b_j, c_j, d_j do not depend on $\mathfrak{e}_j$. Then

$$\Delta_N \dots \Delta_1 = \sum_{a=0}^{\left[\frac{N}{2}\right]} \sum_{\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;N}} \begin{pmatrix} A_{\boldsymbol{j}^{(a)}} & \mathfrak{e}_N B_{\boldsymbol{j}^{(a)}} \\ \mathfrak{e}_N^{-1} C_{\boldsymbol{j}^{(a)}} & D_{\boldsymbol{j}^{(a)}} \end{pmatrix} \cdot \left(\frac{\mathfrak{e}_{j_2} \dots \mathfrak{e}_{j_{2a}}}{\mathfrak{e}_{j_1} \dots \mathfrak{e}_{j_{2a-1}}} \right)^{\sigma_3}. \tag{B.2}$$

Above, the sum runs through all choices of $2a$-uples of integers $\boldsymbol{j}^{(a)} = (j_1, \dots, j_{2a})$ with $\boldsymbol{j}^{(a)}$ belonging to

$$\mathcal{B}_{a;N} = \left\{ (j_1, \dots, j_{2a}) \in [\mathbb{N}^*]^{2a} \; : \; 1 \leq j_1 < \dots < j_{2a} \leq N \right\}. \tag{B.3}$$

The entries of each matrix appearing in the sum are linear polynomials the entries of the matrices $\widetilde{\Delta}_p$.

This lemma allow us to trace back all the dependence on the fractional power of x in the products $\Delta(z_N) \dots \Delta(z_1)$ of the non-trivial parts of the jump matrices for Π.

Proof. The result clearly holds for $N = 1$ as then, the only possibility is to take $a = 0$.

We prove the induction hypothesis for the 11 entry. It goes similarly for all the others. By applying the induction hypothesis to $\Delta_N \dots \Delta_1$ and then explicitly multiplying out with Δ_{N+1}, we get that

$$\begin{aligned} \left[\Delta_{N+1} \dots \Delta_1\right]_{11} = & \sum_{a=0}^{\left[\frac{N}{2}\right]} \sum_{\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;N}} \left(\frac{\mathfrak{e}_{j_2} \dots \mathfrak{e}_{j_{2a}}}{\mathfrak{e}_{j_1} \dots \mathfrak{e}_{j_{2a-1}}} \right) \\ & \times \left\{ a_{N+1} A_{\boldsymbol{j}^{(a)}} + \frac{\mathfrak{e}_{N+1}}{\mathfrak{e}_N} b_{N+1} C_{\boldsymbol{j}^{(a)}} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=0}^{\left[\frac{N}{2}\right]} \sum_{\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;N}} \left(\frac{\mathfrak{e}_{j_2} \ldots \mathfrak{e}_{j_{2a}}}{\mathfrak{e}_{j_1} \ldots \mathfrak{e}_{j_{2a-1}}} \right) a_{N+1} A_{\boldsymbol{j}^{(a)}} \\
&\quad + \sum_{a=0}^{\left[\frac{N}{2}\right]} \sum_{\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;N-1}} \left(\frac{\mathfrak{e}_{j_2} \ldots \mathfrak{e}_{j_{2a}} \mathfrak{e}_{N+1}}{\mathfrak{e}_{j_1} \ldots \mathfrak{e}_{j_{2a-1}} \mathfrak{e}_N} \right) b_{N+1} C_{\boldsymbol{j}^{(a)}} \\
&\quad + \sum_{a=1}^{\left[\frac{N}{2}\right]} \sum_{\substack{\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;N} \\ j_{2a}=N}} \left(\frac{\mathfrak{e}_{j_2} \ldots \mathfrak{e}_{j_{2a-2}} \mathfrak{e}_N \mathfrak{e}_{N+1}}{\mathfrak{e}_{j_1} \ldots \mathfrak{e}_{j_{2a-1}} \mathfrak{e}_N} \right) b_{N+1} C_{\boldsymbol{j}^{(a)}}. \qquad \text{(B.4)}
\end{aligned}
$$

The result follows as the above sums can be seen organized in respect to the partition

$$
\begin{aligned}
&\bigcup_{a=1}^{[N+1/2]} \mathcal{B}_{a;N+1} = \left\{ \bigcup_{a=1}^{[N/2]} \mathcal{B}_{a;N} \right\} \\
&\bigcup \left\{ \bigcup_{a=1}^{[N/2]} \{1 \le j_1 < \cdots < j_{2a} \le N-1,\ j_{2a+1} = N,\ j_{2a+2} = N+1\} \right\} \\
&\bigcup \left\{ \bigcup_{a=1}^{[N/2]} \{1 \le j_1 < \cdots < j_{2a-1} \le N-1,\ j_{2a} = N+1\} \right\}. \qquad \text{(B.5)}
\end{aligned}
$$

□

Lemma B.2. *Let $\mathcal{F}_N(z_1, \ldots, z_N)$ be a holomorphic function on $\mathcal{D} = \mathcal{D}_{v,\delta_1} \times \cdots \times \mathcal{D}_{v,\delta_N}$, where $0 < \delta_N < \cdots < \delta_1$ and $v \in \mathbb{C}$. Let $\partial\mathcal{D} = \partial\mathcal{D}_{v,\delta_1} \times \cdots \times \partial\mathcal{D}_{v,\delta_N}$ be the skeleton of $\mathcal{D}$ and n_p a set of positive integers. Then, for λ lying outside of $\overline{\mathcal{D}}_{v,\delta_1}$, one has*

$$
\begin{aligned}
&\oint_{\partial\mathcal{D}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \frac{\mathcal{F}_N(z_1, \ldots, z_N)}{(\lambda - z_1) \prod_{k=1}^{N-1} (z_k - z_{k+1})} \cdot \prod_{p=1}^{N} \frac{1}{(z_p - v)^{n_p+1}} \\
&= \sum_{k_N=0}^{r_N} \cdots \sum_{k_1=0}^{r_1} \frac{1}{(\lambda - v)^{r_0}} \cdot \left\{ \frac{1}{k_1!} \frac{\partial^{k_1}}{\partial z_1^{k_1}} \cdots \frac{1}{k_N!} \frac{\partial^{k_N}}{\partial z_N^{k_N}} \mathcal{F}_N \right\}_{|z_p = v}. \qquad \text{(B.6)}
\end{aligned}
$$

where we agree upon

$$
r_p = \sum_{\ell=p}^{N} n_\ell + N - p - \sum_{\ell=p+1}^{N} k_\ell \quad \text{and} \quad n_0 = 0. \qquad \text{(B.7)}
$$

The proof is a straightforward induction. Note that the total highest possible order of derivatives of $\mathcal{F}_N$ that is produced by the above contour integral is $\sum_{\ell=1}^{N} n_\ell + N - 1$. It corresponds to no-derivation in respect to the variables $z_2, \dots, z_N$ and a derivative in respect to z_1 of order $\sum_{\ell=1}^{N} n_\ell + N - 1$. All other choices of the integers $\{k_a\}$ lead to a total order of the partial derivatives that is strictly smaller.

B.2 Proof of Proposition 6.1

We are now in position to prove Proposition 6.1.

Here, we will only discuss the case of the time-like regime. The proof in the case of the space-like regime goes very similarly, so that we omit it here.

We have already established that, for x-large enough, $\Pi(\lambda)$ is given in terms of a uniformly convergent Neumann series (6.8):

$$\Pi(\lambda) = I_2 + \sum_{N \geq 1} \frac{\Pi_N(\lambda)}{x^N} \quad \text{with}$$

$$\Pi_N(\lambda) = x^N \int_{\Sigma_\Pi^{(N)}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \frac{\Delta(z_N) \dots \Delta(z_1)}{(\lambda - z_1) \prod_{p=1}^{N-1} (z_p - z_{p+1})}. \tag{B.8}$$

Above each N-fold integral runs across the inslotted contour $\Sigma_\Pi^{(N)}$ as defined in figure 11 and the equality holds for λ uniformly away from the boundary Σ_Π. We remind that in this Neumann series the matrices $\Delta(z_k)$ are subordinate to the jump matrix $I_2 + \Delta(\lambda)$ for $\Pi(\lambda)$ solving the Π-type RHP associated with the jump contour $\Sigma_\Pi[z_k]$, cf. Definition 6.1.

To prove the claim of Proposition 6.1, we build on (B.8) so as to obtain a more precise form of the asymptotic expansion of $\Pi_N(\lambda)$.

Recall that each contour $\Sigma_\Pi[z_k]$ entering in the definition of the inlsotted contour $\Sigma_\Pi^{(N)}$ can be divided into its exterior part $\widetilde{\Gamma}[z_k]$ and three circles $\partial\mathcal{D}_{q,\delta_k} \cup \partial\mathcal{D}_{-q,\delta_k} \cup \partial\mathcal{D}_{\lambda_0,\delta_k}$. There $0 < \delta_N < \cdots < \delta_1$ and δ_1 is small enough, in particular it is such that $\delta_1 < |\lambda_0 \pm q|/2$. However, the very choice of the contour $\widetilde{\Gamma}$ implies that

$$\|\Delta\|_{L^\infty(\widetilde{\Gamma}[z_k])} + \|\Delta\|_{L^2(\widetilde{\Gamma}[z_k])} + \|\Delta\|_{L^1(\widetilde{\Gamma}[z_k])} = \mathrm{O}\left(x^{-\infty}\right). \tag{B.9}$$

Hence, from the point of view of the asymptotic expansion, one can drop all contributions to $\Pi_N(\lambda)$ stemming from those parts of the multiple integral in (B.8), where at least one variable is integrated along $\widetilde{\Gamma}$. Indeed, due to the estimates (B.9), such an integration would only produce $\mathrm{O}(x^{-\infty})$ terms.

The matrix $A_N(\lambda)$ appearing in (6.11) contains exactly these contributions, and hence $A_N(\lambda) = \mathrm{O}(x^{-\infty})$ uniformly away from Σ_Π.

It thus now remains to focus on the effect of the integration on the boundary of the three disks centered at $\pm q$ and λ_0. In other words,

$$\Pi_N(\lambda) = A_N(\lambda) + x^N \sum_{\boldsymbol{\epsilon}\in\mathcal{E}_N} \oint_{\partial\mathcal{D}_{\boldsymbol{\epsilon}}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \cdot \frac{\Delta(z_N)\dots\Delta(z_1)}{\prod_{k=1}^N (z_{k-1}-z_k)}, \quad \text{where} \quad z_0 = \lambda. \tag{B.10}$$

The above sum corresponds to summing up over all the possible choices of the integration contour $\partial\mathcal{D}_{v_{\epsilon_k},\delta_k}$ for each variables z_k. More precisely, one sums over all the N-dimensional vectors $\boldsymbol{\epsilon}$ belonging to

$$\mathcal{E}_N = \{\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1,\dots,\boldsymbol{\epsilon}_N) \ : \ \boldsymbol{\epsilon}_k \in \{\pm 1, 0\}\}. \tag{B.11}$$

We also agree upon the shorthand notation $v_+ = q$, $v_- = -q$ and $v_0 = \lambda_0$. Finally, the integration contour $\partial\mathcal{D}_{\boldsymbol{\epsilon}}$ in each summand corresponds to the Cartesian product of N-circles $\partial\mathcal{D}_{\boldsymbol{\epsilon}} = \partial\mathcal{D}_{v_{\epsilon_1},\delta_1}\times\cdots\times\partial\mathcal{D}_{v_{\epsilon_N},\delta_N}$ of decreasing radii $0<\delta_N<\cdots<\delta_1$, with δ_1 small enough.

We stress that there exists natural constraints on the possible choices of the ϵ_k. Indeed, if z_j and z_{j+1} both belong to a sufficiently small neighborhood of λ_0, then $\Delta(z_j)\Delta(z_{j+1}) = 0$. Hence, choices of N-dimensional vectors $\boldsymbol{\epsilon}$ having two neighboring coordinates (ϵ_j and ϵ_{j+1} for some j) equal to zero do not contribute to the sum in (B.10).

The asymptotic expansions of $\Delta(z)$ on each of the three disks all take the generic form:

$$\Delta(z) \simeq \sum_{n\geq 0} \frac{[\mathfrak{e}(z;\epsilon)]^{\frac{\sigma_3}{2}} \cdot \widetilde{\Delta}^{(n)}(z) \cdot [\mathfrak{e}(z;\epsilon)]^{-\frac{\sigma_3}{2}}}{x^{n+1}(z-v_\epsilon)^{n(2-|\epsilon|)+1}} \quad \text{uniformly in } z\in\mathcal{D}_{v_\epsilon,2\delta}\setminus\mathcal{D}_{v_\epsilon,\delta'} \quad \epsilon\in\{\pm1,0\}. \tag{B.12}$$

The radii are such that $\delta>\delta'>0$ and δ is taken sufficiently small, but are arbitrary otherwise. (B.12) is to be understood in the sense of an asymptotic expansion, i.e., up to a truncation to any given order in x. The detailed expression for the matrices $\widetilde{\Delta}^{(n)}(z)$ and $\mathfrak{e}(z;\epsilon)$ differ on each of the disks (i.e., for $\epsilon = \pm 1$ or 0). However, $\mathfrak{e}(z;\epsilon)$ are holomorphic on any sufficiently small neighborhood of $\pm q$ or λ_0. Also, the matrix $\widetilde{\Delta}^{(n)}(z)$ does not depend on x anymore. The function $\mathfrak{e}(z;\epsilon)$ contains a fractional power of x and also an oscillating term:

$$\mathfrak{e}(z;\epsilon) = \begin{cases} \mathrm{e}^{\mathrm{i}xu(q)} x^{-2\nu(z)} & \text{for } \epsilon = 1, \\ \mathrm{e}^{\mathrm{i}xu(-q)} x^{2\nu(z)} & \text{for } \epsilon = -1, \\ \mathrm{e}^{\mathrm{i}xu(\lambda_0)} x^{-\frac{1}{2}} & \text{for } \epsilon = 0. \end{cases} \tag{B.13}$$

We are now in position to establish the asymptotic expansion of the second term in (B.10).

Expanding each matrix $\Delta(z_n)$ into its asymptotic series (B.12), using that the latter is uniform on the compact domain of integration we obtain the asymptotic expansion of Π_N:

$$\Pi_N(\lambda) \simeq \sum_{r\geq 0} \frac{1}{x^r} \sum_{\boldsymbol{\epsilon}\in\mathcal{E}_N} \sum_{\boldsymbol{n}\in\mathcal{N}_N^{(r)}} \times \int_{\partial\mathcal{D}_{\boldsymbol{\epsilon}}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \cdot \frac{[\mathfrak{e}_N]^{\frac{\sigma_3}{2}} \widetilde{\Delta}^{(\boldsymbol{n}_N)}(z_N) [\mathfrak{e}_{N-1}/\mathfrak{e}_N]^{\frac{\sigma_3}{2}} \cdots [\mathfrak{e}_1/\mathfrak{e}_2]^{\frac{\sigma_3}{2}} \widetilde{\Delta}^{(\boldsymbol{n}_1)}(z_1) [\mathfrak{e}_1]^{-\frac{\sigma_3}{2}}}{(\lambda - z_1) \cdot \prod_{k=2}^{N} (z_{k-1} - z_k) \cdot \prod_{p=1}^{N} \left(z_p - v_{\boldsymbol{\epsilon}_p}\right)^{\boldsymbol{n}_p(2-|\boldsymbol{\epsilon}_p|)+1}}. \tag{B.14}$$

There appears a summation over N-dimensional integer valued vectors $\boldsymbol{n}$ belonging to

$$\mathcal{N}_N^{(r)} = \left\{ \boldsymbol{n} = (\boldsymbol{n_1}, \ldots, \boldsymbol{n}_N) \ : \ n_k \in \mathbb{N}, \ \Sigma_{p=1}^N \boldsymbol{n}_p = r \right\}. \tag{B.15}$$

Note that in order to lighten the notations slightly, we have set $\mathfrak{e}_k \equiv \mathfrak{e}(z_k;\boldsymbol{\epsilon}_k)$. Also, just as in (B.12), we did not make the remainder explicit.

Lemma B.1 ensures the existence of holomorphic functions $A^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}}(\{z_k\})$, $\ldots$, $D^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}}(\{z_k\})$ of $z_1, \ldots, z_N$ such that

$$[\mathfrak{e}_N]^{\frac{\sigma_3}{2}} \widetilde{\Delta}^{(\boldsymbol{n}_N)}(z_N) \left[\frac{\mathfrak{e}_{N-1}}{\mathfrak{e}_N}\right]^{\frac{\sigma_3}{2}} \cdots \left[\frac{\mathfrak{e}_1}{\mathfrak{e}_2}\right]^{\frac{\sigma_3}{2}} \widetilde{\Delta}^{(\boldsymbol{n}_1)}(z_1) [\mathfrak{e}_1]^{-\frac{\sigma_3}{2}} = \sum_{a=0}^{[\frac{N}{2}]} \sum_{\boldsymbol{j}^{(a)}\in\mathcal{B}_{a;N}} \begin{pmatrix} A^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}}(\{z_k\}) & \mathfrak{e}_N B^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}}(\{z_k\}) \\ \mathfrak{e}_N^{-1} C^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}}(\{z_k\}) & D^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}}(\{z_k\}) \end{pmatrix} \left(\frac{\mathfrak{e}_{j_2}\cdots\mathfrak{e}_{j_{2a}}}{\mathfrak{e}_{j_1}\cdots\mathfrak{e}_{j_{2a-1}}}\right)^{\sigma_3}. \tag{B.16}$$

Due to the form taken by the matrices $\Delta^{(n)}(z)$, not all configurations of the $2a$-uples $\boldsymbol{j}^{(a)}$ appear in (B.16). Indeed, when $z_k \in \partial\mathcal{D}_{\lambda_0,\delta_k}$ (i.e., $\boldsymbol{\epsilon}_k = 0$) the matrix $\widetilde{\Delta}^{(n)}(z_k)$ is proportional to σ^- (cf. (5.14)). It appears in (B.14) with a pre-factor $\mathfrak{e}_k^{-1}$. Therefore, for $z_k \in \partial\mathcal{D}_{\lambda_0,\delta_k}$ the only non-vanishing terms in the sum over $\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;n}$ are those corresponding to choices of $2a$-uples $\boldsymbol{j}^{(a)} = (j_1, \dots, j_{2a})$ such that $j_p = k$ for some p. In other words, each time an integration variable belongs to $\partial\mathcal{D}_{\lambda_0,\delta_k}$ for some k, the associated oscillating exponent $\mathfrak{e}^{-1}(\lambda_0;0)$ is always present. Moreover, all matrix entries in the expansion (B.16) that appear (after taking the matrix products) in front of the monomials $(\mathfrak{e}_{j_2} \dots \mathfrak{e}_{j_{2a}})^{\pm 1} / (\mathfrak{e}_{j_1} \dots \mathfrak{e}_{j_{2a-1}})^{\pm 1}$ vanish whenever a function $\mathfrak{e}_{j_p} \equiv \mathfrak{e}\left(z_{j_p}; \boldsymbol{\epsilon}_{j_p}\right)$ corresponding to $z_{j_p} \in \partial\mathcal{D}_{\lambda_0,\delta_{j_p}}$ would appear in the numerator. More precisely, if there exists a p such that $j_p = k$ then

- for $p \in 2\mathbb{N}+1$ (i.e., $\boldsymbol{\epsilon}_{j_{2\ell+1}} = 0$ for some ℓ), one has $B^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}} = D^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}} = 0$;
- for $p \in 2\mathbb{N}$ (i.e., $\boldsymbol{\epsilon}_{j_{2\ell}} = 0$ for some ℓ), one has $A^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}} = C^{(\boldsymbol{n})}_{\boldsymbol{j}^{(a)}} = 0$.

Putting together (B.14) and (B.16) leads to the below form of the asymptotic expansion for Π_N

$$\Pi_N(\lambda) \simeq \sum_{r \geq 0} \frac{1}{x^r} \sum_{\boldsymbol{n} \in \mathcal{N}_N^{(r)}} \sum_{\boldsymbol{j}^{(a)} \in \mathcal{B}_{a;N}} \sum_{\boldsymbol{\epsilon} \in \mathcal{E}_N} I_{\boldsymbol{\epsilon};\boldsymbol{n}} \left[M_{\boldsymbol{j}^{(a)}} \right]. \tag{B.17}$$

There $M_{\boldsymbol{j}^{(a)}}$ stands for the matrix appearing in the expansion (B.16) and $I_{\boldsymbol{\epsilon};\boldsymbol{n}}$ is a functional depending on the choices of the entries of the N-dimensional vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{n}$. It acts on holomorphic functions (or matrices in the sense of entrywise action) $\mathcal{D}_{\boldsymbol{\epsilon}} = \partial\mathcal{D}_{v_{\boldsymbol{\epsilon}_1},\delta_1} \times \dots \times \partial\mathcal{D}_{v_{\boldsymbol{\epsilon}_N},\delta_N}$ according to:

$$I_{\boldsymbol{\epsilon};\boldsymbol{n}}[\mathcal{F}_N] = \oint_{\partial\mathcal{D}_{\boldsymbol{\epsilon}}} \frac{d^N y}{(2\mathrm{i}\pi)^N} \cdot \frac{\mathcal{F}_N(y_1, \dots, y_N)}{(\lambda - y_1) \cdot \prod_{s=2}^{N} (y_{s-1} - y_s) \cdot \prod_{s=1}^{N} (y_s - v_{\boldsymbol{\epsilon}_s})^{(2-|\boldsymbol{\epsilon}_s|)\boldsymbol{n}_s + 1}}. \tag{B.18}$$

The functional $I_{\boldsymbol{n};\boldsymbol{\epsilon}}$ can be estimated by computing the residues at $v_{\boldsymbol{\epsilon}_s}$. This produces partial derivatives of $\mathcal{F}_N$ at the points $y_s = v_{\boldsymbol{\epsilon}_s}$. From now on, we focus on the analysis of the action of $I_{\boldsymbol{n};\boldsymbol{\epsilon}}$ on the 11 entry of the matrix $M_{\boldsymbol{j}^{(a)}}$. The case of all the other entries can be treated similarly.

Depending on the choice of the components of the N-dimensional vector $\boldsymbol{\epsilon}$ and hence of the evaluation points $v_{\boldsymbol{\epsilon}_p}$, after performing the integration induced by $I_{\boldsymbol{n};\boldsymbol{\epsilon}}$ (and having computed the eventual derivatives) the ratio $(\mathfrak{e}_{j_2}\dots\mathfrak{e}_{j_{2a}})\,/\,(\mathfrak{e}_{j_1}\dots\mathfrak{e}_{j_{2a-1}})$ present in the 11 entry of (B.17) reduces to:

- $\dfrac{\mathfrak{e}^m(q;+)}{\mathfrak{e}^m(-q;-)}$ for some $-a\le m\le a$;
- or $\dfrac{\mathfrak{e}^p(q;+)\,\mathfrak{e}^{b-p}(-q;-)}{\mathfrak{e}^b(\lambda_0;0)}\cdot\left(\dfrac{\mathfrak{e}(q;+)}{\mathfrak{e}(-q;-)}\right)^m$
for some $1\le b\le m$, $\quad 0\le p\le b$, $-(a-b)\le m\le a-b$;

Hence, we get that there exists two sets of constant $c^{(m)}_{\boldsymbol{j}_\ell}$, and $c^{(m,\,b,\,p)}_{\boldsymbol{j}_\ell}$,

$$\sum_{\boldsymbol{n}\in\mathcal{N}_N^{(r)}}\sum_{\boldsymbol{\epsilon}\in\mathcal{E}_N} I_{\boldsymbol{n};\boldsymbol{\epsilon}}\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right]=\sum_{m=-a}^{a}\left(\frac{\mathfrak{e}\,(q;+)}{\mathfrak{e}\,(-q;-)}\right)^m c^{(m)}_{\boldsymbol{j}^{(a)}}$$
$$+\sum_{b=1}^{\ell}\sum_{p=0}^{b}\sum_{m=b-a}^{a-b}\frac{\mathfrak{e}^p(q;+)\,\mathfrak{e}^{b-p}(-q;-)}{\mathfrak{e}^b(\lambda_0;0)}\cdot\left(\frac{\mathfrak{e}(q;+)}{\mathfrak{e}(-q;-)}\right)^m c^{(m,\,b,\,p)}_{\boldsymbol{j}^{(a)}}. \qquad \text{(B.19)}$$

Each derivative of the factor $\mathfrak{e}_k$ in respect to z_k, when z_k is in a neighborhood of $\pm q$, produces one power of $\log x$. This $\log x$ term appears due to a differentiation of the exponent $x^{-2\epsilon_k\nu(z_k)}$. It thus follows that the coefficients $c^{(m)}_{\boldsymbol{j}^{(a)}}$ and $c^{(m,\,b,\,p)}_{\boldsymbol{j}^{(a)}}$ are polynomials in $\log x$. In the following, we determine the degree of these polynomials. This will allow us to show that

$$\max\deg\left(c^{(m)}_{\boldsymbol{j}^{(a)}}\right)=r+N-2m-\delta_{m,0}\quad\text{and}$$
$$\max\deg\left(c^{(m,b,p)}_{\boldsymbol{j}^{(a)}}\right)=r+N-2\,(b+m) \qquad \text{(B.20)}$$

where the sup is taken over all possible choices of $\boldsymbol{n}$, $\boldsymbol{\epsilon}$, $\boldsymbol{j}^{(a)}$. Once that (B.20) is established we get the claim.

The degree of $c^{(m)}_{\boldsymbol{j}^{(a)}}$

As already argued, when $\epsilon_k=0$, there necessarily appears $\mathfrak{e}\,(z_k;0)$ in the denominator of $\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}$. As no function $\mathfrak{e}\,(z_\ell;0)$, $\ell\neq k$, can appear in the numerator, this implies that in such a situation $\mathfrak{e}\,(\lambda_0,0)$ appears with some strictly positive exponent after computing the integrals. Therefore, one ends-up with a term that does not corresponds to the coefficient $c^{(m)}_{\boldsymbol{j}^{(a)}}$. Hence, contribution to the coefficients $c^{(m)}_{\boldsymbol{j}^{(a)}}$ can only stem from these choice of N-dimensional vectors $\boldsymbol{\epsilon}$ whose entries are in $\{\pm1\}$. This means that when focusing on $c^{(m)}_{\boldsymbol{j}^{(a)}}$, all "admissible" choices of the N-dimensional vector $\boldsymbol{\epsilon}$ can be parameterized as

$$\boldsymbol{\epsilon} = (\underbrace{\epsilon_1, \dots, \epsilon_1}_{\ell_1}, \underbrace{\epsilon_2, \dots, \epsilon_2}_{\ell_2}, \dots, \underbrace{\epsilon_p, \dots, \epsilon_p}_{\ell_p}) \quad \text{with}$$

$$\epsilon_s = (-1)^{s-1}\,\epsilon_1 \; \epsilon_1 \in \{\pm 1\} \quad \text{for some } p \le N. \tag{B.21}$$

We now compute explicitly the action of the functional $I_{\boldsymbol{\epsilon};\boldsymbol{n}}$ corresponding to some $\boldsymbol{\epsilon}$ given by (B.21). For this purpose, it is convenient to relabel the integration variables y_i appearing in (B.18) in a form that is subordinate to such a representation of the vector $\boldsymbol{\epsilon}$. Namely,

$$(y_1, \dots, y_N) = \left\{z_{1,1}, \dots, z_{1,\ell_1},\, z_{2,1}, \dots, z_{s,t}, \dots, z_{p,\ell_p}\right\}, \quad \textit{i.e.},$$

$$z_{s,t} = y_{\overline{\ell}_s + t}, \quad \text{where} \quad \overline{\ell}_s = \sum_{r=1}^{s-1} \ell_r. \tag{B.22}$$

We relabel the entries of the vector $\boldsymbol{n}$ in a similar way, i.e., $n_{s,t} = \boldsymbol{n}_{\overline{\ell}_s + t}$. Then, the functional $I_{\boldsymbol{\epsilon};\boldsymbol{n}}$ reads

$$\begin{aligned} I_{\boldsymbol{\epsilon};\boldsymbol{n}} &\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right] \\ &= \oint_{\partial\mathcal{D}_{\boldsymbol{\epsilon}}} \frac{d^N z}{(2\mathrm{i}\pi)^N} \cdot \frac{\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}(\{z\})}{\prod\limits_{s=1}^{p}\left\{\left(z_{s-1,\ell_{s-1}} - z_{s,1}\right) \cdot \prod\limits_{t=1}^{\ell_s - 1}\left(z_{s,t} - z_{s,t+1}\right)\right\}} \\ &\quad \cdot \prod_{s=1}^{p}\prod_{t=1}^{\ell_s} \frac{1}{\left(z_{s,t} - v_{\boldsymbol{\epsilon}_s}\right)^{n_{s,t}+1}}. \end{aligned} \tag{B.23}$$

Here we agree upon $\ell_0 = 0$ and $z_{0,0} = \lambda$. The above integral is directly computed by an inductive application of Lemma B.2:

$$\begin{aligned} I_{\boldsymbol{\epsilon};\boldsymbol{n}} &\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right] \\ &= \sum_{k_{s,t}=0}^{r_{s,t}} \prod_{s=1}^{p}\prod_{t=1}^{\ell_s} \left\{\frac{1}{k_{s,t}!}\frac{\partial^{k_{s,t}}}{\partial z_{s,t}^{k_{s,t}}}\right\} \\ &\quad \cdot \left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}(\{z\}) \prod_{s=1}^{p}\left(z_{s-1,\ell_{s-1}} - v_{\epsilon_s}\right)^{-r_{s,0}}\right]_{z_{s,t} = v_{\epsilon_s}}. \end{aligned} \tag{B.24}$$

In (B.24) one sums over integers $k_{s,t}$ with $s=1,\ldots,p$ and $t=1,\ldots,\ell_s$ where each $k_{s,t}$ is summed from 0 to

$$r_{s,t}=\sum_{j=t}^{\ell_s} n_{s,j}+\ell_s-t \;-\; \sum_{j=t+1}^{\ell_p} k_{s,t} \quad \text{with} \quad n_{s,0}=0. \tag{B.25}$$

It follows that each block of variables $(z_{s,1},\ldots,z_{s,\ell_s})$ associated to the same ϵ_s, is subject to partial derivatives of total order $\sum_{t=1}^{\ell_s} k_{s,t}$. Hence, the maximal total order of all the derivatives that may act on this block of variables is $r_{s,1}^{\max}=\sum_{j=1}^{\ell_s} n_{s,j}+\ell_s-1$. The unique way of realizing this maximal order is through a single derivative of order $r_{s,1}^{\max}$ with respect to the variable $z_{s,1}$. We stress that in this case, all the other variables of the block are simply set equal to v_{ϵ_s}. Very similarly, the maximal total order of all the partial derivatives that may act on a sub-block of variables $(z_{s,t},\ldots,z_{s,\ell_s})$ associated to the same ϵ_s is $r_{s,t}^{\max}=\sum_{j=t}^{\ell_s} n_{s,j}+\ell_s-t$. The unique way of realizing this maximal order is by a derivative of order $r_{s,t}^{\max}$ with respect to the variable $z_{s,t}$. Then $z_{s,t+1},\ldots,z_{s,\ell_s}$ should be set equal to v_{ϵ_s}.

As we have already mentioned, the function $\mathfrak{e}_{j_k}\equiv\mathfrak{e}\left(y_{j_k};\epsilon_k\right)$ depends on x. Hence, its derivative in respect to y_{j_k} generates powers of $\log x$. Therefore, the highest degree in $\log x$ appearing in $I_{\boldsymbol{\epsilon};\boldsymbol{n}}\big[M_{\boldsymbol{j}^{(a)}}\big]$ will be generated by a derivative of the highest order possible in respect to the variables y_{j_k}, with $k=1,\ldots,2a$.

Hence, to be able to determine this maximal degree in $\log x$, we first have to order the indices j_k according to the block to which they belong. For this purpose, we set

$$\mathcal{A}_s=\left\{k \;:\; j_k\in[\![\,\overline{\ell}_s+1\,;\,\overline{\ell}_{s+1}\,]\!]\right\} \quad \text{and} \quad a_s=\min\left\{k \;:\; k\in\mathcal{A}_s\right\}, \tag{B.26}$$

see figure 15 for an example. Suppose that one deals with a block of variables $(z_{s,1},\ldots,z_{s,\ell_s})$ such that $\#\mathcal{A}_s\neq 0$. Then the highest possible power of $\ln x$ that an integration over the variables of this block can produce will be issued by the action of a derivative of the highest order possible on the variable $z_{s,j_{a_s}-\overline{\ell}_s}$. Thence, an integration over this block of variables generates a polynomial in $\ln x$ of degree $r^{\max}_{s,j_{a_s}-\overline{\ell}_s}$. Clearly, if $\#\mathcal{A}_s=0$, its associated set of variables and functions cannot generate, once upon being integrated, any power of $\log x$.

We now characterize the oscillating term appearing in $I_{\boldsymbol{\epsilon};\boldsymbol{n}}\left[M_{\boldsymbol{j}^{(a)}}\right]$. If a given block $(z_{s,1},\ldots,z_{s,\ell_s})$ corresponds to a set $\mathcal{A}_s$ having an even number

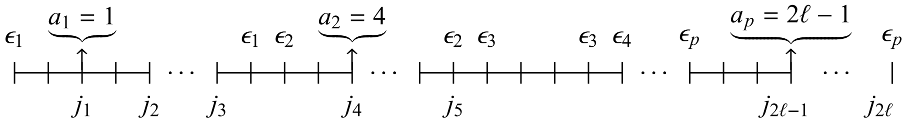

Figure 15: Definition of the sets $\mathcal{A}_s$ and of its minimal element a_s. One has $\mathcal{A}_1 = \{1, 2, 3\}$, $\mathcal{A}_2 = \{4, 5\}$, $\mathcal{A}_3 = \emptyset$, $\dots$, $\mathcal{A}_p = \{2\ell - 1, 2\ell\}$. The ϵ_k delimit the block of variables of length ℓ_k associated to ϵ_k.

of elements ($\#\mathcal{A}_s \in 2\mathbb{N}$), then after taking the derivatives and once upon evaluating at $z_{s,t} = v_{\epsilon_s}$, the associated ratio of the functions $\mathfrak{e}_j$ cancels out. Indeed, there are as many identical factors in the denominator that in the numerator. For instance, when a_s is even one has

$$\frac{\mathfrak{e}_{j_{a_s}} \cdots \mathfrak{e}_{j_{a_s + \#\mathcal{A}_s}}}{\mathfrak{e}_{j_{a_s+1}} \cdots \mathfrak{e}_{j_{a_s + \#\mathcal{A}_s - 1}}}\Bigg|_{z_{s,t} = v_{\epsilon_s}} = 1. \tag{B.27}$$

However, if a given block $(z_{s,1}, \dots, z_{s,\ell_s})$ corresponds to a set $\mathcal{A}_s$ with an odd number of elements ($\#\mathcal{A}_s \in 2\mathbb{N} + 1$), then after taking the derivatives, the associated ratio of the functions $\mathfrak{e}_j$ reduces to $[\mathfrak{e}\,(v_{\epsilon_s}; \epsilon_s)]^{(-1)^{a_s}}$. Indeed

$$\begin{aligned} a_s \in 2\mathbb{N} \Rightarrow \; & \frac{\mathfrak{e}_{j_{a_s}} \cdots \mathfrak{e}_{j_{a_s + \#\mathcal{A}_s - 1}}}{\mathfrak{e}_{j_{a_s+1}} \cdots \mathfrak{e}_{j_{a_s + \#\mathcal{A}_s - 2}}}\Bigg|_{z_{s,t} = v_{\epsilon_s}} = \mathfrak{e}\,(v_{\epsilon_s}; \epsilon_s) \quad \text{and} \\ a_s \in 2\mathbb{N} + 1 \Rightarrow \; & \frac{\mathfrak{e}_{j_{a_s+1}} \cdots \mathfrak{e}_{j_{a_s + \#\mathcal{A}_s - 2}}}{\mathfrak{e}_{j_{a_s}} \cdots \mathfrak{e}_{j_{a_s + \#\mathcal{A}_s - 1}}}\Bigg|_{z_{s,t} = v_{\epsilon_s}} = \frac{1}{\mathfrak{e}\,(v_{\epsilon_s}; \epsilon_s)}. \end{aligned} \tag{B.28}$$

Therefore, we obtain that

$$\begin{aligned} & I_{\boldsymbol{\epsilon};\boldsymbol{n}}\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right] = P^{(\boldsymbol{j}^{(a)})}_{\boldsymbol{\epsilon};\boldsymbol{n}}(\log x) \cdot \prod_{\substack{s=1 \\ \#\mathcal{A}_s \in 2\mathbb{N}+1}}^{p} [\mathfrak{e}\,(v_{\epsilon_s}; \epsilon_s)]^{(-1)^{a_s}} \\ & \text{where} \quad \deg\left(P^{(\boldsymbol{j}^{(a)})}_{\boldsymbol{\epsilon};\boldsymbol{n}}\right) = \sum_{\substack{s=1 \\ \#\mathcal{A}_s \neq 0}}^{p} \left\{ \sum_{k = j_{a_s} - \overline{\ell}_s}^{\ell_s} n_{s,k} + \ell_s - \left(j_{a_s} - \overline{\ell}_s\right) \right\}. \end{aligned} \tag{B.29}$$

Now, in order to obtain the coefficient $c^{(m)}_{\boldsymbol{j}^{(a)}}$ we should sum up (B.29) over $\boldsymbol{n} \in \mathcal{N}^{(r)}_N$ and also over all the possible configurations of vectors $\boldsymbol{\epsilon}$ parameterized as in (B.21) and such that we eventually generate the power $(\mathfrak{e}\,(q; +) / \mathfrak{e}\,(-q; -))^m$. Then, among such configurations, we should look for those that correspond to a polynomial $P_{\boldsymbol{\epsilon};\boldsymbol{n}}\,(\log x)$ of highest degree.

Given a fixed number of flips p in (B.21), one maximizes the degree in (B.29) by choosing the lengths ℓ_s is such a way that $\#\mathcal{A}_s \neq 0$ for any s and such that $j_{a_s} = \overline{\ell}_s + 1$. One can do so for all s, but $s = 1$. Indeed, in the latter case one necessarily has $j_{a_1} = j_1 \geq 1$. Therefore, for such a choice of lengths ℓ_s, once upon choosing $n_{1,t} = 0$ for $t = 0, \ldots, j_1 - 1$ one obtains that this maximal degree of is $r + N - p - (j_1 - 1)$. Note that, we have used $\sum_{s,t} n_{s,t} = r$ and $\sum_s \ell_s = N$.

There is also a condition on the number of flips that are necessary to generate the oscillatory factors $(\mathfrak{e}(q;+)/\mathfrak{e}(-q;-))^m$. Due to the form of the oscillatory factor in (B.29), we get that one sequence $(\epsilon_s, \ldots, \epsilon_s)$ generates at most one factor $[\mathfrak{e}(\epsilon_s q; \epsilon_s)]^\tau$, $\tau = \pm 1$. Hence, if $m \neq 0$ there are at least $2m$ flips necessary to generate the factors $(\mathfrak{e}(q;+)/\mathfrak{e}(-q;-))^m$. If $m = 0$, then one still has one sequence $(\epsilon_1, \ldots, \epsilon_1)$ of length $\ell_1 = N$. Therefore, one has $p \geq \max(2m, 1)$. Taking the lowest possible value, i.e., $p = \max(2m, 1)$, we get that

$$\max_{\boldsymbol{\epsilon}\,;\,\boldsymbol{n}} \deg \left(P^{(\boldsymbol{j}^{(a)})}_{\boldsymbol{\epsilon};\boldsymbol{n}} \right) = n + N - 2m - \delta_{m,0} + (j_1 - 1)\,. \tag{B.30}$$

In order to obtain estimates for $[\Pi_N]_{11}$ one should still sum up over all the possible configurations of $2a$-uples $\boldsymbol{j}^{(a)}$. Therefore the highest degree in $\log x$ of $\left[\Pi^{(m)}_{N,r}\right]_{11}$ is obtained by setting $j_1 = 1$. That is, we reproduce (B.20).

The degree of $c^{(m,\,b,\,p)}_{\boldsymbol{j}^{(a)}}$

It follows from the previous discussions that each time an integration over $\partial\mathcal{D}_{\lambda_0,\delta}$ occurs in (B.17)–(B.18), there appears, once upon integrating, a factor $\mathfrak{e}^{-1}(\lambda_0, 0)$. Hence, the oscillating factor in front of $c^{(m,\,b,\,p)}_{\boldsymbol{j}^{(a)}}$ is necessarily generated by these choices of N-dimensional vectors $\boldsymbol{\epsilon}$ where precisely b entries are equal to zero (i.e., there are exactly b integrations over $\partial\mathcal{D}_{\lambda_0,\delta}$).

Taking into account the fact that, as previously argued, two neighboring entries of the vector $\boldsymbol{\epsilon}$ cannot simultaneously vanish, we get that such vectors $\boldsymbol{\epsilon}$ can be parameterized as

$$\boldsymbol{\epsilon} = (\underbrace{\epsilon_1, \ldots, \epsilon_1}_{\ell_1}, \ldots, \underbrace{\epsilon_{\tau_1}, \ldots, \epsilon_{\tau_1}}_{\ell_{\tau_1}}, 0, \epsilon_{\tau_1+1}, \ldots, \epsilon_{\tau_b}, 0, \ldots, \underbrace{\epsilon_p, \ldots, \epsilon_p}_{\ell_p}),$$

$$\text{where} \quad \sum_{r=1}^{p} \ell_r = N - b. \tag{B.31}$$

We relabel the integration variables y_i appearing in (B.18) in a form subordinate to (B.31):

$$\begin{aligned}&(y_1,\ldots,y_N)=\left\{z_{1,1},\ldots z_{\tau_a,\ell_{\tau_a}},\omega_a,z_{\tau_a+1,1}\ldots,z_{p,\ell_p}\right\},\\ &i.e.,\quad z_{s,t}=y_{\overline{\ell}_s+t},\quad \omega_a=y_{\overline{\ell}_{\tau_a}+1}\end{aligned}\tag{B.32}$$

where we agree upon

$$\overline{\ell}_s=\sum_{r=1}^{s-1}\ell_r+\#\{\,k\;:\;\tau_k<s\,\}.\tag{B.33}$$

We also relabel the entries of the vector $\boldsymbol{n}$ in a similar way, i.e., $n_{s,t}=\boldsymbol{n}_{\overline{\ell}_s+t}$ and $n_a^{(0)}=\boldsymbol{n}_{\overline{\ell}_{\tau_a}+1}$. The action of the associated functional $\boldsymbol{I}_{\boldsymbol{\epsilon};\boldsymbol{n}}$ takes the form

$$\begin{aligned}\boldsymbol{I}_{\boldsymbol{\epsilon};\boldsymbol{n}}\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right]=&\oint_{\partial\mathcal{D}_{\boldsymbol{\epsilon}}}\frac{d^{N-b}z}{(2\mathrm{i}\pi)^{N-b}}\cdot\frac{d^b\omega}{(2\mathrm{i}\pi)^b}\prod_{\substack{s=1\\ s\neq\tau_a+1}}^{p}\frac{1}{\left(z_{s-1,\ell_{s-1}}-z_{s,1}\right)}\\ &\cdot\prod_{s=1}^{p}\prod_{t=1}^{\ell_s-1}\frac{1}{(z_{s,t}-z_{s,t+1})}\frac{\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}(\{z\};\{\omega\})}{\prod_{s=1}^{p}\prod_{t=1}^{\ell_s}(z_{s,t}-v_{\epsilon_s})^{n_{s,t}+1}}\\ &\cdot\prod_{a=1}^{b}\frac{1}{\left(z_{\tau_a,\ell_{\tau_a}}-\omega_a\right)(\omega_a-z_{\tau_a+1,1})}\cdot\prod_{a=1}^{b}\frac{1}{(\omega_a-\lambda_0)^{2n_a^{(0)}+1}}.\end{aligned}\tag{B.34}$$

The integrals over ω_a are readily computed. We set

$$\mathcal{G}_{N-b}(\{z\})=\prod_{a=1}^{b}\left\{\frac{1}{\left(2n_a^{(0)}\right)!}\frac{\partial^{2n_a^{(0)}}}{\partial\omega_a^{2n_a^{(0)}}}\right\}\cdot\left[\frac{\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}(\{z\};\{\omega\})}{\left(z_{\tau_a,\ell_{\tau_a}}-\omega_a\right)(\omega_a-z_{\tau_a+1,1})}\right]_{\omega_a=\lambda_0}.\tag{B.35}$$

Then, the analysis boils down to the case previously studied:

$$\begin{aligned}\boldsymbol{I}_{\boldsymbol{\epsilon};\boldsymbol{n}}\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right]=&\oint_{\partial\mathcal{D}_{\boldsymbol{\epsilon}}^{\mathrm{red}}}\frac{d^{N-b}z}{(2\mathrm{i}\pi)^{N-b}}\mathcal{G}_{N-b}(\{z\})\prod_{\substack{s=1\\ s\neq\tau_a+1}}^{p}\frac{1}{\left(z_{s-1,\ell_{s-1}}-z_{s,1}\right)}\\ &\times\prod_{s=1}^{p}\prod_{t=1}^{\ell_s-1}\frac{1}{(z_{s,t}-z_{s,t+1})}\cdot\prod_{s=1}^{p}\prod_{t=1}^{\ell_s}\frac{1}{(z_{s,t}-v_{\epsilon_s})^{n_{s,t}+1}}.\end{aligned}\tag{B.36}$$

The integrals runs over the contour $\partial\mathcal{D}_{\boldsymbol{\epsilon}}^{\mathrm{red}}$, which corresponds to that part of the initial contour $\partial\mathcal{D}_{\boldsymbol{\epsilon}}$ where the integrals over the variable ω_a have been suppressed. Therefore

$$I_{\boldsymbol{\epsilon};\boldsymbol{n}}\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right] = \sum_{k_{s,t}=0}^{r_{s,t}} \prod_{s=1}^{p}\prod_{t=1}^{\ell_s}\left\{\frac{1}{(k_{s,t})!}\frac{\partial^{k_{s,t}}}{\partial z_{s,t}^{k_{s,t}}}\right\}\cdot\left[\frac{\mathcal{G}_{N-m}\left(\{z\}\right)}{\prod\limits_{\substack{s=1\\ s\neq\tau_a+1}}^{p}\left(z_{s-1,\ell_{s-1}}-v_{\epsilon_s}\right)^{r_{s,0}}}\right]_{|z_{s,t}=v_{\epsilon_s}}. \tag{B.37}$$

The sum over the integers $k_{s,t}$ runs from 0 to

$$r_{s,t}=\sum_{j=t}^{\ell_s} n_{s,j}+\ell_s-t-\sum_{j=t+1}^{\ell_s}k_{s,t}.$$

Similarly to the previous analysis, we set

$$\mathcal{A}_s=\left\{k : j_k\in [\![\,\overline{\ell}_s+1\,;\,\overline{\ell}_s+\ell_s\,]\!]\right\} \quad \text{and} \quad a_s=\min\left\{k:k\in\mathcal{A}_k\right\}. \tag{B.38}$$

It is then easy to see by using similar arguments to those invoked for $c_{\boldsymbol{j}^{(a)}}^{(m)}$ that

$$I_{\boldsymbol{\epsilon};\boldsymbol{n}}\left[\left[M_{\boldsymbol{j}^{(a)}}\right]_{11}\right]=\frac{P_{\boldsymbol{\epsilon};\boldsymbol{n}}^{(\boldsymbol{j}^{(a)})}(\log x)}{\mathfrak{e}^{b}\left(\lambda_0;0\right)}\cdot\prod_{\substack{s=1\\ \#\mathcal{A}_s\in 2\mathbb{N}+1}}^{p}\left[\mathfrak{e}\left(v_{\epsilon_s};\epsilon_s\right)\right]^{(-1)^{a_s}}$$

$$\text{where}\quad \deg\left(P_{\boldsymbol{\epsilon};\boldsymbol{n}}^{(\boldsymbol{j}^{(a)})}\right)=\sum_{\substack{s=1\\ \#\mathcal{A}_s\neq 0}}^{p}\left\{\sum_{k=j_{a_s}-\overline{\ell}_s}^{\ell_s}n_{s,k}+\ell_s-\left(j_{a_s}-\overline{\ell}_s\right)\right\}. \tag{B.39}$$

In order to obtain the maximal degree in $\log x$ associated to the oscillating term

$$\frac{\mathfrak{e}^{p}(q,+)\,\mathfrak{e}^{b-p}(-q,-)}{\mathfrak{e}^{b}(\lambda_0;0)}\cdot\left(\frac{\mathfrak{e}(q,+)}{\mathfrak{e}(-q;-)}\right)^{m} \tag{B.40}$$

present in $[\Pi_N]_{11}$, we should maximize the degree of the previous polynomial in (B.39) under the constraint that the sequence ϵ_a in (B.31) ought to change its value at least $b+m$ times (this in order to produce the sought form of the oscillatory term with its associated power-law behavior) and that these changes are such that eventually (B.40) is generated.

We should also maximize this degree in respect to all the possible choices of $2a$-uples $\boldsymbol{j}^{(a)}$ of various lengths $2a$ and over the allowed vectors $\boldsymbol{n} \in \mathcal{N}_N^{(r)}$. In order to obtain this maximal degree, one should choose a minimal number of flips $(m+b)$, choose the lengths ℓ_k and the j_k in such a way that $j_{a_s} = \overline{\ell}_s + 1$. Finally, one should also take $n_a^{(0)} = 0$ for all a. This leads to the conclusion that the maximal degree in $\ln x$ is $r + N - 2\,(m+b)$. □

Appendix C. Fine bounds on Π_N

In this appendix we provide bounds for the matrices $\Pi_N^{(m,b,p)}$ entering in the decomposition for Π_N given in Proposition 6.1.

Proposition C.1. *Let Σ_Π be a contour appearing in the RHP for Π and U any open set such that $\mathrm{d}\,(U, \Sigma_\Pi) > 0$. Let Π_N be as defined by (6.10) and, agreeing upon $\boldsymbol{\eta} = 1$ in the space-like regime and $\boldsymbol{\eta} = -1$ in the time-like regime, let*

$$\mathfrak{e}\,(z;\epsilon) = \begin{cases} \mathrm{e}^{\mathrm{i}xu(q)} x^{-2\nu(z)} & \text{for} \quad \epsilon = 1, \\ \mathrm{e}^{\mathrm{i}xu(-q)} x^{2\nu(z)} & \text{for} \quad \epsilon = -1, \\ \mathrm{e}^{\mathrm{i}xu(\lambda_0)} x^{-\boldsymbol{\eta}\frac{1}{2}} & \text{for} \quad \epsilon = 0. \end{cases} \tag{C.1}$$

Then the matrix $\Pi_N(\lambda)$ admits the representation

$$\Pi_N\,(\lambda) = A_N\,(\lambda) + \sum_{b=0}^{[N/2]} \sum_{p=0}^{b} \sum_{m=b-[N/2]}^{[N/2]-b} \left(\frac{\mathfrak{e}\,(q;+)}{\mathfrak{e}\,(-q;-)} \right)^{m-\boldsymbol{\eta}p} \left(\frac{\mathfrak{e}\,(\lambda_0;0)}{\mathfrak{e}\,(-q;-)} \right)^{\boldsymbol{\eta}b} \Pi_N^{(m,b,p)}\,(\lambda)\,. \tag{C.2}$$

For x-large enough, the matrices $\Pi_N^{(m,b,p)}(\lambda)$ and $A_N(\lambda)$ depend smoothly on x and holomorphically on $\lambda \in U$. One has the decomposition

$$\Pi_N^{(m,b,p)}\,(\lambda) \;=\; \sum_{\epsilon \in \{\pm 1, 0\}} \left[\mathfrak{e}\,(v_\epsilon;\epsilon)\right]^{\frac{\sigma_3}{2}} \Pi_{N;\epsilon}^{(m,b,p)}\,(\lambda) \left[\mathfrak{e}\,(v_\epsilon;\epsilon)\right]^{-\frac{\sigma_3}{2}}$$
$$\text{where} \quad v_\pm = \pm q \quad \text{and } v_0 = \lambda_0. \tag{C.3}$$

The matrix $\Pi_{N;\epsilon}^{(m,b,p)}\,(\lambda)$ is such that it does not contain any oscillating factor in its entries. Moreover, for all $k \in \mathbb{N}$ there exists an N-independent constant

$C > 0$ such that

$$\|A_N\|_{L^\infty(U)} \le \frac{C^N}{x^k} \quad \text{and} \quad \left\|\Pi^{(m,b,p)}_{N;\epsilon}\right\|_{L^\infty(U)} \le C^N x^{\widetilde{w}N}$$
$$\text{with} \quad \widetilde{w} = 2 \max_{\epsilon=\pm}\left\{ \sup_{\partial\mathcal{D}_{\epsilon q,\delta}} |\Re(\nu - \nu(\epsilon q))| \right\}. \tag{C.4}$$

These estimates also hold for the first-order partial derivatives (in respect to x or λ).

Proof. Recall that the matrix Π_N can be represented in terms of Cauchy transforms (or their + boundary values) on Σ_Π:

$$\Pi_N(\lambda) = \mathcal{C}^{\Delta}_{\Sigma_\Pi} \circ \cdots \circ \mathcal{C}^{\Delta}_{\Sigma_\Pi}[I_2](\lambda) = \left\{\mathcal{C}^{\Delta}_{\Sigma_\Pi}\right\}^N [I_2](\lambda). \tag{C.5}$$

Above and in the following, $\mathcal{C}^{\Delta}_{\mathscr{C}}[M](\lambda)$ for $\lambda \notin \mathscr{C}$ corresponds to the case where in the integral representation (6.3) for this operator we substitute the + boundary value with $\lambda \notin \mathscr{C}$. This is clearly a well defined expression. We decompose the jump contour for Π according to $\Sigma_\Pi = \partial\mathcal{D} \cup \widetilde{\Sigma}_\Pi$ with $\partial\mathcal{D} = \partial\mathcal{D}_{q,\delta} \cup \partial\mathcal{D}_{-q,\delta} \cup \partial\mathcal{D}_{\lambda_0,\delta}$.

The exponentially small in x terms gathered in A_N can be written as

$$A_N(\lambda) = \sum_{k=0}^{N-1} \left\{\mathcal{C}^{\Delta}_{\partial\mathcal{D}}\right\}^{N-1-k} \circ \left\{\mathcal{C}^{\Delta}_{\widetilde{\Sigma}_\Pi}\right\} \circ \left\{\mathcal{C}^{\Delta}_{\Sigma_\Pi}\right\}^k [I_2](\lambda), \tag{C.6}$$

whereas

$$\Pi_N(\lambda) - A_N(\lambda) = \sum_{\boldsymbol{\epsilon} \in \mathcal{E}_N} \mathcal{C}^{\Delta}_{\partial\mathcal{D}_{v_{\epsilon_1},\delta}} \circ \cdots \circ \mathcal{C}^{\Delta}_{\partial\mathcal{D}_{v_{\epsilon_N},\delta}}. \tag{C.7}$$

There, the sum runs through $\boldsymbol{\epsilon} \in \mathcal{E}_N = \{\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_N) \ : \ \boldsymbol{\epsilon}_s \in \{\pm 1, 0\}\}$. One can readily convince oneself that for any matrix function M such that $\Delta M \in L^2\left(\widetilde{\Sigma}_\Pi\right)$ there exists a constant c' such that

$$\left\|\mathcal{C}^{\Delta}_{\widetilde{\Sigma}_\Pi}[M]\right\|_{L^2(\partial\mathcal{D})} \le c' \|\Delta M\|_{L^2(\widetilde{\Sigma}_\Pi)}. \tag{C.8}$$

Thence setting $c = \max\{c', c(\Sigma_\Pi), c(\partial\mathcal{D})\}$ (we recall that for a curve Γ, $c(\Gamma)$ stands for the norm of the + boundary value of the Cauchy operator on $L^2(\Gamma)$), one gets

$$\|A_N\|_{L^\infty(U)} \leq N \frac{[2c\mathcal{N}_{\Sigma_\Pi}(\Delta)]^{N-1}}{\pi^2 \mathrm{d}(U,\Sigma_\Pi)} \mathcal{N}_{\widetilde{\Sigma}_\Pi}(\Delta)$$
$$\text{with} \quad \mathcal{N}_{\mathscr{C}}(\Delta) = \|\Delta\|_{L^2(\mathscr{C})} + \|\Delta\|_{L^\infty(\mathscr{C})} .$$

Thus, the claim follows for A_N as, by construction, $\mathcal{N}_{\widetilde{\Sigma}_\Pi}(\Delta) = \mathrm{O}(x^{-\infty})$.

It remains to obtain estimates for the remaining, algebraically small in x, part. For this we set

$$\widehat{\Delta}(z) = [\mathfrak{e}(\varphi(z);\epsilon(z))]^{-\frac{\sigma_3}{2}} \Delta(z) [\mathfrak{e}(\varphi(z);\epsilon(z))]^{\frac{\sigma_3}{2}}$$
$$\text{with} \quad \begin{cases} \varphi(z) = q\mathbf{1}_{\partial\mathcal{D}_{q,\delta}} - q\mathbf{1}_{\partial\mathcal{D}_{-q,\delta}} + \lambda_0 \mathbf{1}_{\partial\mathcal{D}_{\lambda_0,\delta}} \\ \epsilon(z) = \mathbf{1}_{\partial\mathcal{D}_{q,\delta}} - \mathbf{1}_{\partial\mathcal{D}_{-q,\delta}} \end{cases} .$$

Then, by carrying out similar expansions to (B.14) and (B.16) it is easy to convince oneself that

$$I_2 \cdot \left[\Pi_{N;\epsilon}^{(m,b,p)}\right]_{k_1,k_{N+1}}(\lambda) = \sum_{\boldsymbol{\epsilon}\in\mathcal{E}_N}{}' \sum_{k_a=1}^{2}{}' \, \mathcal{C}_{\partial\mathcal{D}_{v_{\epsilon_1},\delta}}^{\widehat{\Delta}_{k_2k_1}} \circ \cdots \circ \mathcal{C}_{\partial\mathcal{D}_{v_{\epsilon_N},\delta}}^{\widehat{\Delta}_{k_{N+1},k_N}}[I_2] . \quad \text{(C.9)}$$

Above $\widehat{\Delta}_{ab}$ stands for the ab entry of $\widehat{\Delta}$. Also, there appears $\widehat{\Delta}$ instead of Δ as the oscillating factors have been already pulled-out, as in (C.2)–(C.3). Also the primes $'$ in front of the two sums are there to indicate that these are constrained. Namely, one should sum-up only over those choices of $\boldsymbol{\epsilon} \in \mathcal{E}_N$ and $k_a \in \{1,2\}$, $a = 2,\ldots,N$ which, upon the replacement $\widehat{\Delta} \mapsto \Delta$ would give rise to the oscillating factor associated with $\left[\Pi_{N;\epsilon}^{(m,b,p)}\right]_{k_1,k_{N+1}}$. By using the continuity of the + boundary value Cauchy operator on $L^2(\partial\mathcal{D})$, one shows that there exists a constant c such that for any $\epsilon,\tau \in \{\pm 1, 0\}$ and any $f \in L^2(\partial\mathcal{D})$:

$$\left\|\mathcal{C}_{\partial\mathcal{D}_{v_\epsilon,\delta}}^{I_2}[fI_2]\right\|_{L^2(\partial\mathcal{D}_{\tau,\delta})} \leq c\,\|f\|_{L^2(\partial\mathcal{D}_{v_\epsilon,\delta})} . \quad \text{(C.10)}$$

Then,

$$\begin{aligned}\left\|\Pi_{N;\epsilon}^{(m,b,p)}\right\|_{L^\infty(U)} &\leq \frac{c^{N-1}}{\mathrm{d}\left(U,\Sigma_\Pi\right)} \sum_{\epsilon\in\mathcal{E}_N}{}' \sum_{k_a=1}^{2}{}' \left\|\widehat{\Delta}_{k_2k_1}\right\|_{L^2\left(\partial\mathcal{D}_{v_{\epsilon_1},\delta}\right)} \\ &\quad\times \prod_{a=2}^{N-1} \left\|\widehat{\Delta}_{k_{a+1}k_a}\right\|_{L^\infty\left(\partial\mathcal{D}_{v_{\epsilon_a},\delta}\right)} \left\|\widehat{\Delta}_{k_{N+1}k_N}\right\|_{L^2\left(\partial\mathcal{D}_{v_{\epsilon_N},\delta}\right)} \\ &\leq \frac{(2c)^{N-1}}{\mathrm{d}\left(U,\Sigma_\Pi\right)} \mathcal{N}_{\partial\mathcal{D}}\left(\widehat{\Delta}\right). \end{aligned} \tag{C.11}$$

Since there exists $c'>0$ such that $||\, \widehat{\Delta}\, ||_{L^2(\partial\mathcal{D})} \leq c' x^{\widetilde{w}}$, the claim follows. Also, we stress that, by construction, $\Pi_{N;\epsilon}^{(m,b,p)}$ does not contain any oscillating terms in x in its asymptotic expansion when $x\to+\infty$. $\square$

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