Adv. Theor. Math. Phys. **15** (2011) 1373–1410

Division algebras and supersymmetry II

John C. Baez and John Huerta

Department of Mathematics, University of California, Riverside, CA 92521, USA
baez@math.ucr.edu

Abstract

Starting from the four normed division algebras — the real numbers, complex numbers, quaternions and octonions — a systematic procedure gives a 3-cocycle on the Poincaré Lie superalgebra in dimensions 3, 4, 6 and 10. A related procedure gives a 4-cocycle on the Poincaré Lie superalgebra in dimensions 4, 5, 7 and 11. In general, an $(n + 1)$-cocycle on a Lie superalgebra yields a "Lie n-superalgebra": that is, roughly speaking, an n-term chain complex equipped with a bracket satisfying the axioms of a Lie superalgebra up to chain homotopy. We thus obtain Lie 2-superalgebras extending the Poincaré superalgebra in dimensions 3, 4, 6 and 10, and Lie 3-superalgebras extending the Poincaré superalgebra in dimensions 4, 5, 7 and 11. As shown in Sati, Schreiber and Stasheff's work on higher gauge theory, Lie 2-superalgebra connections describe the parallel transport of strings, while Lie 3-superalgebra connections describe the parallel transport of 2-branes. Moreover, in the octonionic case, these connections concisely summarize the fields appearing in 10- and 11-dimensional supergravity.

e-print archive: http://lanl.arXiv.org/abs/1003.3436v2

1 Introduction

There is a deep connection between supersymmetry and the four normed division algebras: the real numbers, complex numbers, quaternions and octonions. This can be seen in super-Yang–Mills theory, in superstring theory, and in theories of supermembranes and supergravity. Most simply, the connection is visible from the fact that the normed division algebras have dimensions 1, 2, 4 and 8, while classical superstring theories and minimal super-Yang–Mills theories live in spacetimes of dimension *two* higher: 3, 4, 6 and 10. The simplest classical super-2-brane theories make sense in spacetimes of dimensions *three* higher: 4, 5, 7 and 11. Classical supergravity makes sense in all of these dimensions, but the octonionic cases are the most important: in 10 dimensions supergravity is a low-energy limit of superstring theory, while in 11 dimensions it is believed to be a low-energy limit of "M-theory", which incorporates the 2-brane.

These numerical relationships are far from coincidental. They arise because we can use the normed division algebras to construct the spacetimes in question, as well as their associated spinors. A certain spinor identity that holds in dimensions 3, 4, 6 and 10 is an easy consequence of this construction, as is a related identity that holds in dimensions 4, 5, 7 and 11. These identities are fundamental to the physical theories just listed.

In a bit more detail, suppose $\mathbb{K}$ is a normed division algebra of dimension n. There are just four examples:

- the real numbers $\mathbb{R}$ ($n = 1$),
- the complex numbers $\mathbb{C}$ ($n = 2$),
- the quaternions $\mathbb{H}$ ($n = 4$),
- the octonions $\mathbb{O}$ ($n = 8$).

Then we can identify vectors in $(n + 2)$-dimensional Minkowski spacetime with 2×2 Hermitian matrices having entries in $\mathbb{K}$. Similarly, we can identify spinors with elements of $\mathbb{K}^2$. Matrix multiplication then gives a way for vectors to act on spinors. There is also an operation that takes two spinors ψ and ϕ and forms a vector $\psi \cdot \phi$. Using elementary properties of normed division algebras, we can prove that

$$(\psi \cdot \psi)\psi = 0.$$

Following Schray [30], we call this identity the "3-ψ's rule". This identity is an example of a "Fierz identity" — roughly, an identity that allows one to reorder multilinear expressions made of spinors. This can be made more visible in the 3-ψ's rule if we polarize the above cubic form to extract a genuinely trilinear expression:

$$(\psi \cdot \phi)\chi + (\phi \cdot \chi)\psi + (\chi \cdot \psi)\phi = 0.$$

In fact, the 3-ψ's rule holds *only* when Minkowski spacetime has dimension 3, 4, 6 or 10. Moreover, it is crucial for super-Yang–Mills theory and superstring theory in these dimensions. In minimal super-Yang–Mills theory, we need the 3-ψ's rule to check that the Lagrangian is supersymmetric, thanks to an argument reviewed in our previous paper [6]. In superstring theory, we need it to check the supersymmetry of the Green–Schwarz Lagrangian [21, 22]. But the 3-ψ's rule also has a deeper significance, which we study here.

This deeper story involves not only the 3-ψ's rule but also the "4-Ψ's rule", a closely related Fierz identity required for super-2-brane theories in dimensions 4, 5, 7 and 11. To help the reader see the forest for the trees, we present a rough summary of this story in the form of a recipe:

1. Spinor identities that come from division algebras are *cocycle conditions*.
2. The corresponding cocycles allow us to extend the Poincaré Lie superalgebra to a higher structure, a *Lie n-superalgebra*.
3. Connections valued in these Lie n-superalgebras describe the *field content* of superstring and super-2-brane theories.

To begin our story in dimensions 3, 4, 6 and 10, let us first introduce some suggestive terminology: despite our notation, we shall call $\psi \cdot \phi$ the **bracket of spinors**. This is because this function is symmetric, and it defines a Lie superalgebra structure on the supervector space

$$\mathcal{T} = V \oplus S,$$

where the even subspace V is the vector representation of $\mathrm{Spin}(n+1, 1)$, while the odd subspace S is a certain spinor representation. This Lie superalgebra is called the **supertranslation algebra**.

There is a cohomology theory for Lie superalgebras, sometimes called Chevalley–Eilenberg cohomology. The cohomology of $\mathcal{T}$ will play a central role in what follows. Why? First, because the 3-ψ's rule is really a cocycle condition, for a 3-cocycle α on $\mathcal{T}$, which eats two spinors and a vector and produces a number as follows:

$$\alpha(\psi, \phi, A) = \langle \psi, A\phi \rangle.$$

Here, $\langle -,- \rangle$ is a pairing between spinors. Since this 3-cocycle is Lorentz-invariant, it extends to a cocycle on the Poincaré superalgebra

$$\mathfrak{siso}(n+1,1) \cong \mathfrak{so}(n+1,1) \ltimes \mathcal{T}.$$

In fact, we obtain a nonzero element of the third cohomology of the Poincaré superalgebra this way.

Just as 2-cocycles on a Lie superalgebra give ways of extending it to larger Lie superalgebras, 3-cocycles give extensions to *Lie 2-superalgebras*. To understand this, we need to know a bit about L_∞-algebras [26, 29]. An L_∞-algebra is a chain complex equipped with a structure like that of a Lie algebra, but where the laws hold only "up to d of something". A Lie n-algebra is an L_∞-algebra in which only the first n terms are nonzero. All these ideas also have "super" versions. In particular, we can use the 3-cocycle α to extend $\mathfrak{siso}(n+1,1)$ to a Lie 2-superalgebra of the following form:

$$\mathfrak{siso}(n+1,1) \xleftarrow{d} \mathbb{R}\ .$$

We call this the "superstring Lie 2-superalgebra", and denote it as $\mathfrak{superstring}(n+1,1)$.

The superstring Lie 2-superalgebra is an extension of $\mathfrak{siso}(n+1,1)$ by $\mathfrak{b}\mathbb{R}$, the Lie 2-algebra with $\mathbb{R}$ in degree 1 and everything else trivial. By "extension", we mean that there is a short exact sequence of Lie 2-superalgebras:

$$0 \to b\mathbb{R} \to \mathfrak{superstring}(n+1,1) \to \mathfrak{siso}(n+1,1) \to 0.$$

To see precisely what this means, let us expand it a bit. Lie 2-superalgebras are 2-term chain complexes, and writing these vertically, our short exact sequence looks like this:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, "d"'] & \mathbb{R} \arrow[r] \arrow[d, "d"'] & 0 \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathfrak{siso}(n+1,1) \arrow[r] & \mathfrak{siso}(n+1,1) \arrow[r] & 0
\end{tikzcd}$$

In the middle, we see $\mathfrak{superstring}(n+1,1)$. This Lie 2-superalgebra is built from two pieces: $\mathfrak{siso}(n+1,1)$ in degree 0 and $\mathbb{R}$ in degree 1. But since the cocycle α is nontrivial, these two pieces still interact in a nontrivial way. Namely, the Jacobi identity for three 0-chains holds only up to d of a 1-chain. So, besides its Lie bracket, the Lie 2-superalgebra $\mathfrak{superstring}(n+1,1)$ also involves a map that takes three 0-chains and gives a 1-chain. This map is just α.

What is the superstring Lie 2-algebra good for? The answer lies in a feature of string theory called the "Kalb–Ramond field", or "B field". The B field couples to strings just as the A field in electromagnetism couples to charged particles. The A field is described locally by a 1-form, so we can integrate it over a particle's worldline to get the interaction term in the Lagrangian for a charged particle. Similarly, the B field is described locally by a 2-form, which we can integrate over the worldsheet of a string.

Gauge theory has taught us that the A field has a beautiful geometric meaning: it is a connection on a U(1) bundle over spacetime. What is the corresponding meaning of the B field? It can be seen as a connection on a "U(1) gerbe": a gadget like a U(1) bundle, but suitable for describing strings instead of point particles. Locally, connections on U(1) gerbes can be identified with 2-forms. But globally, they cannot. The idea that the B field is a U(1) gerbe connection is implicit in work going back at least to the 1986 paper by Gawedzki [19]. More recently, Freed and Witten [18] showed that the subtle difference between 2-forms and connections on U(1) gerbes is actually crucial for understanding anomaly cancellation. In fact, these authors used the language of "Deligne cohomology" rather than gerbes. Later work made the role of gerbes explicit: see for example Carey *et al.* [9], and also Gawedzki and Reis [20].

More recently still, work on higher gauge theory has revealed that the B field can be viewed as part of a larger package. Just as gauge theory uses Lie groups, Lie algebras and connections on bundles to to describe the parallel transport of point particles, higher gauge theory generalizes all these concepts to describe parallel transport of extended objects such strings and membranes [7, 8]. In particular, Schreiber *et al.* [27] have developed a theory of "n-connections" suitable for describing parallel transport of objects with n-dimensonal worldvolumes. In their theory, the Lie algebra of the gauge roup is replaced by an Lie n-algebra — or in the supersymmetric context, a Lie n-superalgebra. Applying their ideas to $\mathfrak{superstring}(n+1,1)$, we get a 2-connection, which can be described locally using the following fields:

$\mathfrak{superstring}(n+1,1)$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 2-form
$\downarrow$	
$\mathfrak{siso}(n+1,1)$	$\mathfrak{siso}(n+1,1)$-valued 1-form

The $\mathfrak{siso}(n+1,1)$-valued 1-form consists of three fields which help define the background geometry on which a superstring propagates: the Levi–Civita connection A, the vielbein e, and the gravitino ψ. But the $\mathbb{R}$-valued 2-form is equally important in the description of this background geometry: it is the B field!

Next let us extend these ideas to Minkowski spacetimes one dimension higher: dimensions 4, 5, 7 and 11. In this case, a certain subspace of 4×4 matrices with entries in $\mathbb{K}$ will form the vector representation of $\mathrm{Spin}(n+2,1)$, while $\mathbb{K}^4$ will form a spinor representation. As before, there is a "bracket" operation that takes two spinors Ψ and Φ and gives a vector $\Psi \cdot \Phi$. As before, there is an action of vectors on spinors. This time the 3-ψ's rule no longer holds:

$$(\Psi \cdot \Psi)\Psi \neq 0.$$

However, we show that

$$\Psi \cdot ((\Psi \cdot \Psi)\Psi) = 0.$$

We call this the "4-Ψ's rule". This identity plays a basic role for the super-2-brane, and related theories of supergravity.

Once again, the bracket of spinors defines a Lie superalgebra structure on the supervector space

$$\mathcal{T} = \mathcal{V} \oplus \mathcal{S},$$

where now $\mathcal{V}$ is the vector representation of $\mathrm{Spin}(n+2,1)$, whereas $\mathcal{S}$ is a certain spinor representation of this group. Once again, the cohomology of $\mathcal{T}$ plays a key role. The 4-Ψ's rule is a cocycle condition — but this time for a 4-cocycle β which eats two spinors and two vectors and produces a number as follows:

$$\beta(\Psi, \Phi, \mathcal{A}, \mathcal{B}) = \langle \Psi, (\mathcal{A} \wedge \mathcal{B})\Phi \rangle.$$

Here, $\langle -, - \rangle$ denotes the inner product of two spinors, and the bivector $\mathcal{A} \wedge \mathcal{B}$ acts on Φ via the usual Clifford action. Since β is Lorentz-invariant, we shall see that it extends to a 4-cocycle on the Poincaré superalgebra $\mathfrak{siso}(n+2,1)$.

We can use β to extend the Poincaré superalgebra to a Lie 3-superalgebra of the following form:

$$\mathfrak{siso}(n+2,1) \xleftarrow{d} 0 \xleftarrow{d} \mathbb{R} \, .$$

We call this the "2-brane Lie 3-superalgebra", and denote it as 2-$\mathfrak{brane}(n+2,1)$. It is an extension of $\mathfrak{siso}(n+2,1)$ by $\mathfrak{b}^2\mathbb{R}$, the Lie 3-algebra with $\mathbb{R}$ in degree 2, and everything else trivial. In other words, there is a short exact sequence:

$$0 \to \mathfrak{b}^2\mathbb{R} \to 2\text{-}\mathfrak{brane}(n+2,1) \to \mathfrak{siso}(n+2,1) \to 0.$$

Again, to see what this means, let us expand it a bit. Lie 3-superalgebras are 3-term chain complexes. Writing out each of these vertically, our short exact sequence looks like this:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, "d"'] & \mathbb{R} \arrow[r] \arrow[d, "d"'] & 0 \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d, "d"'] & 0 \arrow[r] \arrow[d, "d"'] & 0 \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathfrak{siso}(n+2,1) \arrow[r] & \mathfrak{siso}(n+2,1) \arrow[r] & 0
\end{tikzcd}$$

In the middle, we see $2\text{-}\mathfrak{brane}(n+2,1)$.

The most interesting Lie 3-algebra of this type, $2\text{-}\mathfrak{brane}(10,1)$, plays an important role in 11-dimensional supergravity. This idea goes back to the work of Castellani, D'Auria and Fré [10, 12]. These authors derived the field content of 11-dimensional supergravity starting from a differential graded commutative algebra. Later Sati *et al.* [27] explained that these fields can be reinterpreted as a 3-connection valued in a Lie 3-algebra, which they called "$\mathfrak{sugra}(10,1)$". This is the Lie 3-algebra we are calling $2\text{-}\mathfrak{brane}(10,1)$. Our message here is that the all-important cocycle needed to construct this Lie 3-algebra arises naturally from the octonions, and has analogues for the other normed division algebras.

If we follow these authors and consider a 3-connection valued in $2\text{-}\mathfrak{brane}$ $(10,1)$, we find it can be described locally by these fields:

$2\text{-}\mathfrak{brane}(n+2,1)$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 3-form
$\downarrow$	
0	
$\downarrow$	
$\mathfrak{siso}(n+2,1)$	$\mathfrak{siso}(n+2,1)$-valued 1-form

Again, a $\mathfrak{siso}(n+2,1)$-valued 1-form contains familiar fields: the Levi–Civita connection, the vielbein and the gravitino. But now we also see a 3-form, called the C field. This is again something we might expect on physical grounds, at least in dimension 11. Although the case is less clear than in string theory, it seems that for the quantum theory of a 2-brane to be consistent, it must propagate in a background obeying the equations of 11-dimensional supergravity, in which the C field naturally shows up [34]. The work of Diaconescu *et al.* [14], as well as that of Aschieri and Jurco [2], is also relevant here.

Finally, we mention another use for the cocycles α and β. These cocycles are also used to build Wess–Zumino–Witten (WZW) terms for superstrings and 2-branes. For example, in the case of the string, one can extend the string's worldsheet to be the boundary of a three-dimensional manifold, and then integrate α over this manifold. This provides an additional term for the action of the superstring, a term that is required to give the action Siegel symmetry, balancing the number of bosonic and fermionic degrees of freedom. For the 2-brane, the WZW term is constructed in complete analogy — we just "add one" to all the dimensions in sight [1, 17].

Indeed, the network of relationships between supergravity, string and 2-brane theories, and cocycles constructed using normed division algebras is extremely tight. The Siegel symmetry of the string or 2-brane action constrains the background of the theory to be that of supergravity, at least in dimensions 10 and 11 [34], and without the WZW terms, there would be no Siegel symmetry. The WZW terms rely on the cocycles α and β. These cocycles also give rise to the Lie 2- and 3-superalgebras $\mathfrak{superstring}(9,1)$ and 2-$\mathfrak{brane}(10,1)$. And these, in turn, describe the field content of supergravity in these dimensions!

As further grist for this mill, WZW terms can also be viewed in the context of higher gauge theory. In string theory, the WZW term is the holonomy of a connection on a U(1) gerbe [20]. Presumably the WZW term in a 2-brane theory is the holonomy of a connection on a U(1) 2-gerbe [32]. This is a tantalizing clue that we are at the beginning of a larger but ultimately simpler story.

In what follows, we focus on the mathematics of constructing Lie n-superalgebras from normed division algebras, rather than the applications to physics that we have just described. We begin with a quick review of normed division algebras in Section 2. In Section 3, we recall how an n-dimensional normed division algebra can be used to describe vectors and spinors in $(n+2)$-dimensional spacetime, and how this description yields the 3-ψ's rule. All this material is treated in more detail in our previous paper [6]. In Section 4, we build on this work and use normed division algebras to describe vectors and spinors in $(n+3)$-dimensional spacetime. In Section 5, we use this description to prove the the 4-Ψ's rule. In Section 6, we describe the cohomology of Lie superalgebras, and show that the 3-ψ's rule and 4-Ψ's rule yield nontrivial 3-cocycles and 4-cocycles on supertranslation algebras. In Section 7, we describe Lie n-superalgebras, and prove that an $(n+1)$-cocycle on a Lie superalgebra gives a way to extend it to a Lie n-superalgebra. We thus obtain Lie 2-superalgebras and Lie 3-superalgebras extending supertranslation algebras. In Section 8, we conclude by constructing the superstring Lie 2-algebras and the 2-brane Lie 3-algebras.

2 Division algebras

We begin with a lightning review of normed division algebras. A **normed division algebra** $\mathbb{K}$ is a (finite-dimensional, possibly nonassociative) real algebra equipped with a multiplicative unit 1 and a norm $|\cdot|$ satisfying:

$$|ab| = |a||b|$$

for all $a, b \in \mathbb{K}$. Note this implies that $\mathbb{K}$ has no zero divisors. We will freely identify $\mathbb{R}1 \subseteq \mathbb{K}$ with $\mathbb{R}$. By a classic theorem of Hurwitz [23], there are only four normed division algebras: the real numbers, $\mathbb{R}$, the complex numbers, $\mathbb{C}$, the quaternions, $\mathbb{H}$, and the octonions, $\mathbb{O}$. These algebras have dimension 1, 2, 4 and 8.

In all four cases, the norm can be defined using conjugation. Every normed division algebra has a **conjugation** operator — a linear operator $*\colon \mathbb{K} \to \mathbb{K}$ satisfying

$$a^{**} = a, \quad (ab)^* = b^* a^*$$

for all $a, b \in \mathbb{K}$. Conjugation lets us decompose each element of $\mathbb{K}$ into real and imaginary parts, as follows:

$$\mathrm{Re}(a) = \frac{a + a^*}{2}, \quad \mathrm{Im}(a) = \frac{a - a^*}{2}.$$

Conjugating changes the sign of the imaginary part and leaves the real part fixed. We can write the norm as

$$|a| = \sqrt{aa^*} = \sqrt{a^* a}.$$

This norm can be polarized to give an inner product on $\mathbb{K}$:

$$(a, b) = \mathrm{Re}(ab^*) = \mathrm{Re}(a^* b).$$

The algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are associative. The octonions $\mathbb{O}$ are not. Instead, they are **alternative**: the subalgebra generated by any two octonions is associative. Note that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, being associative, are also trivially alternative. By a theorem of Artin [28], this is equivalent to the fact that the **associator**

$$[a, b, c] = (ab)c - a(bc)$$

is completely antisymmetric in its three arguments.

For any square matrix A with entries in $\mathbb{K}$, we define its **trace** $\mathrm{tr}(A)$ to be the sum of its diagonal entries. This trace lacks the usual cyclic property, because $\mathbb{K}$ is noncommutative, so in general $\mathrm{tr}(AB) \neq \mathrm{tr}(BA)$. Luckily, taking the real part restores this property:

Proposition 2.1. *Let A, B, and C be $k \times \ell$, $\ell \times m$ and $m \times k$ matrices with entries in $\mathbb{K}$. Then*

$$\mathrm{Re}\,\mathrm{tr}((AB)C) = \mathrm{Re}\,\mathrm{tr}(A(BC))$$

and this quantity is invariant under cyclic permutations of A, B and C. We call this quantity the **real trace** $\mathrm{Re}\,\mathrm{tr}(ABC)$.

Proof. This relies heavily on the alternativity of normed division algebras. See Proposition 4 of our previous paper [6]. □

3 Spacetime geometry in $n + 2$ dimensions

In this section, we recall the relation between a normed division algebra $\mathbb{K}$ of dimension n and Lorentzian geometry in $n + 2$ dimensions. Most of the material here is well-known [4, 11, 24, 25, 33], and we follow the treatment in Section 3 of our previous paper [6], much of which we learned from the papers of Schray and Manogue [30, 31]. The key facts are that one can describe vectors in $(n + 2)$-dimensional Minkowski spacetime as 2×2 Hermitian matrices with entries in $\mathbb{K}$, and spinors as elements of $\mathbb{K}^2$. In fact, there are two representations of $\mathrm{Spin}(n + 1, 1)$ on $\mathbb{K}^2$, which we call S_+ and S_-. The nature of these representations depends on $\mathbb{K}$:

- When $\mathbb{K} = \mathbb{R}$, $S_+ \cong S_-$ is the Majorana spinor representation of $\mathrm{Spin}(2, 1)$.
- When $\mathbb{K} = \mathbb{C}$, $S_+ \cong S_-$ is the Majorana spinor representation of $\mathrm{Spin}(3, 1)$.
- When $\mathbb{K} = \mathbb{H}$, S_+ and S_- are the Weyl spinor representations of $\mathrm{Spin}(5, 1)$.
- When $\mathbb{K} = \mathbb{O}$, S_+ and S_- are the Majorana–Weyl spinor representations of $\mathrm{Spin}(9, 1)$.

As usual, these spinor representations are also representations of the even part of the relevant Clifford algebras:

Even parts of Clifford algebras		
$\mathrm{Cliff}_{\mathrm{ev}}(2,1)$	$\cong$	$\mathbb{R}[2]$
$\mathrm{Cliff}_{\mathrm{ev}}(3,1)$	$\cong$	$\mathbb{C}[2]$
$\mathrm{Cliff}_{\mathrm{ev}}(5,1)$	$\cong$	$\mathbb{H}[2] \oplus \mathbb{H}[2]$
$\mathrm{Cliff}_{\mathrm{ev}}(9,1)$	$\cong$	$\mathbb{R}[16] \oplus \mathbb{R}[16]$

Here we see $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{H}^2$ and $\mathbb{O}^2$ showing up as irreducible representations of these algebras, albeit with $\mathbb{O}^2$ masquerading as $\mathbb{R}^{16}$. The first two algebras have a unique irreducible representation. The last two both have two irreducible representations, which correspond to left-handed and right-handed spinors.

Our discussion, so far, has emphasized the differences between the four cases. But the wonderful thing about normed division algebras is that they allow a unified approach that treats all four cases simultaneously! They also give simple formulas for the basic intertwining operators involving vectors, spinors and scalars.

To begin, let $\mathbb{K}[m]$ denote the space of $m \times m$ matrices with entries in $\mathbb{K}$. Given $A \in \mathbb{K}[m]$, define its **Hermitian adjoint** $A^\dagger$ to be its conjugate transpose:

$$A^\dagger = (A^*)^{\mathrm{T}}.$$

We say such a matrix is **Hermitian** if $A = A^\dagger$. Now take the 2×2 Hermitian matrices:

$$\mathfrak{h}_2(\mathbb{K}) = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} : t, x \in \mathbb{R},\ y \in \mathbb{K} \right\}.$$

This is an $(n+2)$-dimensional real vector space. Moreover, the usual formula for the determinant of a matrix gives the Minkowski norm on this vector space:

$$-\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = -t^2 + x^2 + |y|^2.$$

We insert a minus sign to obtain the signature $(n+1,1)$. Note this formula is unambiguous even if $\mathbb{K}$ is noncommutative or nonassociative.

It follows that the double cover of the Lorentz group, $\mathrm{Spin}(n+1,1)$, acts on $\mathfrak{h}_2(\mathbb{K})$ via determinant-preserving linear transformations. Since this is the "vector" representation, we will often call $\mathfrak{h}_2(\mathbb{K})$ simply V. The Minkowski metric

$$g\colon V \otimes V \to \mathbb{R}$$

is given by

$$g(A, A) = -\det(A).$$

There is also a nice formula for the inner product of two different vectors. This involves the **trace reversal** of $A \in \mathfrak{h}_2(\mathbb{K})$, defined by

$$\tilde{A} = A - (\mathrm{tr}A)1.$$

Note we indeed have $\mathrm{tr}(\tilde{A}) = -\mathrm{tr}(A)$.

Proposition 3.1. *For any vectors $A, B \in V = \mathfrak{h}_2(K)$, we have*

$$A\tilde{A} = \tilde{A}A = -\det(A)1$$

and

$$\frac{1}{2}\mathrm{Re}\ \mathrm{tr}(A\tilde{B}) = \frac{1}{2}\mathrm{Re}\ \mathrm{tr}(\tilde{A}B) = g(A, B).$$

Next we consider spinors. As real vector spaces, the spinor representations S_+ and S_- are both just $\mathbb{K}^2$. However, they differ as representations of $\mathrm{Spin}(n+1, 1)$. To construct these representations, we begin by defining ways for vectors to act on spinors:

$$\begin{array}{rccc} \gamma\colon & V \otimes S_+ & \to & S_- \\ & A \otimes \psi & \mapsto & A\psi. \end{array}$$

and

$$\begin{array}{rccc} \tilde{\gamma}\colon & V \otimes S_- & \to & S_+ \\ & A \otimes \psi & \mapsto & \tilde{A}\psi. \end{array}$$

We can also think of these as maps that send elements of V to linear operators:

$$\begin{array}{rccc} \gamma\colon & V & \to & \mathrm{Hom}(S_+, S_-), \\ \tilde{\gamma}\colon & V & \to & \mathrm{Hom}(S_-, S_+). \end{array}$$

Since vectors act on elements of S_+ to give elements of S_- and vice versa, they map the space $S_+ \oplus S_-$ to itself. This gives rise to an action of the Clifford algebra $\mathrm{Cliff}(V)$ on $S_+ \oplus S_-$:

Proposition 3.2. *The vectors $V = \mathfrak{h}_2(\mathbb{K})$ act on the spinors $S_+ \oplus S_- = \mathbb{K}^2 \oplus \mathbb{K}^2$ via the map*

$$\Gamma\colon V \to \mathrm{End}(S_+ \oplus S_-)$$

given by

$$\Gamma(A)(\psi, \phi) = (\widetilde{A}\phi, A\psi).$$

Furthermore, $\Gamma(A)$ satisfies the Clifford algebra relation:

$$\Gamma(A)^2 = g(A, A)1$$

and so extends to a homomorphism $\Gamma\colon \mathrm{Cliff}(V) \to \mathrm{End}(S_+ \oplus S_-)$, i.e., a representation of the Clifford algebra $\mathrm{Cliff}(V)$ on $S_+ \oplus S_-$.

As explained in our previous paper [6], the spaces S_+, S_- and V are representations of the spin group $\mathrm{Spin}(n+1, 1)$. Moreover:

Proposition 3.3. *The maps*

$$\begin{array}{rccc} \gamma\colon & V \otimes S_+ & \to & S_-, \\ & A \otimes \psi & \mapsto & A\psi \end{array}$$

and

$$\begin{array}{rccc} \tilde{\gamma}\colon & V \otimes S_- & \to & S_+, \\ & A \otimes \psi & \mapsto & \tilde{A}\psi \end{array}$$

are equivariant with respect to the action of $\mathrm{Spin}(n+1, 1)$.

Proposition 3.4. *The pairing*

$$\begin{array}{rccc} \langle -, - \rangle\colon & S_+ \otimes S_- & \to & \mathbb{R}, \\ & \psi \otimes \phi & \mapsto & \mathrm{Re}(\psi^\dagger \phi) \end{array}$$

is invariant under the action of $\mathrm{Spin}(n+1, 1)$.

With this pairing in hand, there is a manifestly equivariant way to turn a pair of spinors into a vector. Given $\psi, \phi \in S_+$, there is a unique vector $\psi \cdot \phi$ whose inner product with any vector A is given by

$$g(\psi \cdot \phi, A) = \langle \psi, \gamma(A)\phi \rangle.$$

Similarly, given $\psi, \phi \in S_-$, we define $\psi \cdot \phi \in V$ by demanding

$$g(\psi \cdot \phi, A) = \langle \tilde{\gamma}(A)\psi, \phi \rangle$$

for all $A \in V$. This gives us maps

$$S_\pm \otimes S_\pm \to V,$$

which are manifestly equivariant. In fact:

Proposition 3.5. *The maps* $\cdot : S_\pm \otimes S_\pm \to V$ *are given by:*

$$\begin{array}{cccc} \cdot : & S_+ \otimes S_+ & \to & V \\ & \psi \otimes \phi & \mapsto & \widetilde{\psi\phi^\dagger + \phi\psi^\dagger}, \end{array}$$

$$\begin{array}{cccc} \cdot : & S_- \otimes S_- & \to & V \\ & \psi \otimes \phi & \mapsto & \psi\phi^\dagger + \phi\psi^\dagger. \end{array}$$

These maps are equivariant with respect to the action of $\mathrm{Spin}(n+1,1)$*.*

Theorem 10 of our previous paper stated the fundamental identity, which allows supersymmetry in dimensions 3, 4, 6 and 10: the "3-ψ's rule". Our proof was based on an argument in the appendix of a paper by Dray *et al.* [16]:

Theorem 3.1. *Suppose* $\psi \in S_+$*. Then* $(\psi \cdot \psi)\psi = 0$*. Similarly, if* $\phi \in S_-$*, then* $(\widetilde{\phi \cdot \phi})\phi = 0$*.*

4 Spacetime geometry in $n + 3$ dimensions

In the last section, we recalled how to describe spinors and vectors in $(n+2)$-dimensional Minkowski spacetime using a division algebra $\mathbb{K}$ of dimension n. Here, we show how to boost this up one dimension, and give a division algebra description of vectors and spinors in $(n+3)$-dimensional Minkowski spacetime.

We shall see that vectors in $(n+3)$-dimensional Minkowski spacetime can be identified with 4×4 $\mathbb{K}$-valued matrices of this particular form:

$$\begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix},$$

where a is a real multiple of the 2×2 identity matrix and A is a 2×2 Hermitian matrix with entries in $\mathbb{K}$. Moreover, $\mathrm{Spin}(n+2,1)$ has a representation on $\mathbb{K}^4$, which we call $\mathcal{S}$. Depending on $\mathbb{K}$, this gives the following types of spinors:

- When $\mathbb{K} = \mathbb{R}$, $\mathcal{S}$ is the Majorana spinor representation of $\mathrm{Spin}(3,1)$.
- When $\mathbb{K} = \mathbb{C}$, $\mathcal{S}$ is the Dirac spinor representation of $\mathrm{Spin}(4,1)$.

- When $\mathbb{K} = \mathbb{H}$, $\mathcal{S}$ is the Dirac spinor representation of $\mathrm{Spin}(6,1)$.
- When $\mathbb{K} = \mathbb{O}$, $\mathcal{S}$ is the Majorana spinor representation of $\mathrm{Spin}(10,1)$.

Again, these spinor representations are also representations of the even part of the relevant Clifford algebra:

Even parts of Clifford algebras		
$\mathrm{Cliff}_{\mathrm{ev}}(3,1)$	$\cong$	$\mathbb{C}[2]$
$\mathrm{Cliff}_{\mathrm{ev}}(4,1)$	$\cong$	$\mathbb{H}[2]$
$\mathrm{Cliff}_{\mathrm{ev}}(6,1)$	$\cong$	$\mathbb{H}[4]$
$\mathrm{Cliff}_{\mathrm{ev}}(10,1)$	$\cong$	$\mathbb{R}[29]$

These algebras have irreducible representations on $\mathbb{R}^4 \cong \mathbb{C}^2$, $\mathbb{C}^4 \cong \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{O}^4 \cong \mathbb{R}^{32}$, respectively.

The details can be described in a uniform way for all four cases. We take as our space of "vectors" the following $(n+3)$-dimensional subspace of $\mathbb{K}[4]$:

$$\mathcal{V} = \left\{ \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix} : a \in \mathbb{R}, \quad A \in \mathfrak{h}_2(\mathbb{K}) \right\}.$$

In the last section, we defined vectors in $n+2$ dimensions to be $V = \mathfrak{h}_2(\mathbb{K})$. That space has an obvious embedding into $\mathcal{V}$, given by

$$\begin{aligned} V &\hookrightarrow \mathcal{V}, \\ A &\mapsto \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix}. \end{aligned}$$

The Minkowski metric

$$h \colon \mathcal{V} \otimes \mathcal{V} \to \mathbb{R}$$

is given by extending the Minkowski metric g on V:

$$h\left(\begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix}, \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix} \right) = g(A,A) + a^2.$$

From our formulae for g, we can derive formulae for h:

Proposition 4.1. *For any vectors $\mathcal{A}, \mathcal{B} \in \mathcal{V} \subseteq \mathbb{K}[4]$, we have*

$$\mathcal{A}^2 = h(\mathcal{A},\mathcal{A})1$$

and

$$\frac{1}{4}\mathrm{Re}\ \mathrm{tr}(\mathcal{A}\mathcal{B}) = h(\mathcal{A},\mathcal{B}).$$

Proof. For $\mathcal{A} = \left(\begin{smallmatrix} a & \tilde{A} \\ A & -a \end{smallmatrix}\right)$, it is easy to check:

$$\mathcal{A}^2 = \begin{pmatrix} a^2 + \tilde{A}A & 0 \\ 0 & A\tilde{A} + a^2 \end{pmatrix}.$$

By Proposition 3.1, we have $A\tilde{A} = \tilde{A}A = g(A, A)1$, and substituting this in establishes the first formula. The second formula follows from polarizing and taking the real trace of both sides. ☐

Define a space of "spinors" by $\mathcal{S} = S_+ \oplus S_- = \mathbb{K}^4$. To distinguish elements of $\mathcal{V}$ from elements of $\mathfrak{h}_2(\mathbb{K})$, we will denote them with calligraphic letters like $\mathcal{A}$, $\mathcal{B}$, ... Similarly, to distinguish elements of $\mathcal{S}$ from $S_\pm$, we will denote them with capital Greek letters like $\Psi, \Phi, \ldots$.

Elements of $\mathcal{V}$ act on $\mathcal{S}$ by left multiplication:

$$\begin{array}{rcl} \mathcal{V} \otimes \mathcal{S} & \to & \mathcal{S}, \\ \mathcal{A} \otimes \Psi & \mapsto & \mathcal{A}\Psi. \end{array}$$

We can dualize this to get a map:

$$\begin{array}{rrcl} \Gamma\colon & \mathcal{V} & \to & \mathrm{End}(\mathcal{S}), \\ & \mathcal{A} & \mapsto & L_{\mathcal{A}}. \end{array}$$

This induces the Clifford action of $\mathrm{Cliff}(\mathcal{V})$ on $\mathcal{S}$. Note that this Γ is the same as the map in Proposition 3.2 when we restrict to $V \subseteq \mathcal{V}$.

Proposition 4.2. *The vectors $\mathcal{V} \subseteq \mathbb{K}[4]$ act on the spinors $\mathcal{S} = \mathbb{K}^4$ via the map*

$$\Gamma\colon \mathcal{V} \to \mathrm{End}(\mathcal{S})$$

given by

$$\Gamma(\mathcal{A})\Psi = \mathcal{A}\Psi.$$

Furthermore, $\Gamma(\mathcal{A})$ satisfies the Clifford algebra relation:

$$\Gamma(\mathcal{A})^2 = h(\mathcal{A}, \mathcal{A})1$$

and so extends to a homomorphism $\Gamma\colon \mathrm{Cliff}(\mathcal{V}) \to \mathrm{End}(\mathcal{S})$*, i.e., a representation of the Clifford algebra* $\mathrm{Cliff}(\mathcal{V})$ *on $\mathcal{S}$.*

Proof. Here, we must be mindful of nonassociativity. For $\Psi = (\psi, \phi) \in \mathcal{S}$ and $\mathcal{A} = \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix} \in \mathcal{V}$, we have:

$$\Gamma(\mathcal{A})^2\Psi = \mathcal{A}(\mathcal{A}\Psi),$$

which works out to be:

$$\Gamma(\mathcal{A})^2\Psi = \begin{pmatrix} a^2\psi + \tilde{A}(A\psi) \\ A(\tilde{A}\phi) + a^2\phi \end{pmatrix}.$$

A quick calculation shows that the expressions $\tilde{A}(A\psi)$ and $A(\tilde{A}\phi)$ involve at most two nonreal elements of $\mathbb{K}$, so everything associates and we can write:

$$\Gamma(\mathcal{A})^2\Psi = \mathcal{A}^2\Psi.$$

By Proposition 4.1, we are done. □

This tells us how $\mathcal{S}$ is a module of $\mathrm{Cliff}(\mathcal{V})$, and thus a representation of $\mathrm{Spin}(\mathcal{V})$, the subgroup of $\mathrm{Cliff}(\mathcal{V})$ generated by products of pairs of unit vectors.

In the last section, we saw how to construct a $\mathrm{Spin}(V)$-invariant pairing

$$\langle -, - \rangle \colon S_+ \otimes S_- \to \mathbb{R}.$$

We can use this to build up to a $\mathrm{Spin}(\mathcal{V})$-invariant pairing on $\mathcal{S}$:

$$\langle (\psi, \phi), (\chi, \theta) \rangle = \langle \psi, \theta \rangle - \langle \chi, \phi \rangle.$$

To see this, let

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, because $\langle \psi, \phi \rangle = \mathrm{Re}(\psi^\dagger \phi)$, it is easy to check that:

$$\langle \psi, \theta \rangle - \langle \chi, \phi \rangle = \mathrm{Re}\left(\begin{pmatrix} \psi \\ \phi \end{pmatrix}^\dagger \Gamma^0 \begin{pmatrix} \chi \\ \theta \end{pmatrix} \right).$$

We can show this last expression is invariant by explicit calculation.

Proposition 4.3. *Define the nondegenerate skew-symmetric bilinear form*

$$\langle -,-\rangle \colon \mathcal{S}\otimes\mathcal{S}\to\mathbb{R}$$

by

$$\langle\Psi,\Phi\rangle = \mathrm{Re}(\Psi^\dagger\Gamma^0\Phi).$$

This form is invariant under $\mathrm{Spin}(\mathcal{V})$.

Proof. It is easy to see that, for any spinors $\Psi,\Phi\in\mathcal{S}$ and vectors $\mathcal{A}\in\mathcal{V}$, we have

$$\langle\mathcal{A}\Psi,\mathcal{A}\Phi\rangle = \mathrm{Re}((\Psi^\dagger\mathcal{A}^\dagger)\Gamma^0(\mathcal{A}\Phi)) = \mathrm{Re}(\Psi^\dagger(\mathcal{A}^\dagger\Gamma^0(\mathcal{A}\Phi))),$$

where in the last step we have used Proposition 2.1. Now, given that

$$\mathcal{A} = \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix}$$

a quick calculation shows:

$$\mathcal{A}^\dagger\Gamma^0 = -\Gamma^0\mathcal{A}.$$

So, this last expression becomes:

$$-\mathrm{Re}(\Psi^\dagger(\Gamma^0\mathcal{A}(\mathcal{A}\Phi))) = -\mathrm{Re}(\Psi^\dagger(\Gamma^0\Gamma(\mathcal{A})^2\Phi))) = -|\mathcal{A}|^2\mathrm{Re}(\Psi^\dagger\Gamma^0\Phi),$$

where in the last step we have used the Clifford relation. Summing up, we have shown:

$$\langle\mathcal{A}\Psi,\mathcal{A}\Phi\rangle = -|\mathcal{A}|^2\langle\Psi,\Phi\rangle.$$

In particular, when $\mathcal{A}$ is a unit vector, acting by $\mathcal{A}$ changes the sign at most. Thus, $\langle -,-\rangle$ is invariant under the group generated by products of pairs of unit vectors, which is $\mathrm{Spin}(\mathcal{V})$. It is easy to see that it is nondegenerate, and it is skew-symmetric because of Γ^0. □

With the form $\langle -,-\rangle$ in hand, there is a manifestly equivariant way to turn a pair of spinors into a vector. Given $\Psi,\Phi\in\mathcal{S}$, there is a unique vector $\Psi\cdot\Phi$ whose inner product with any vector $\mathcal{A}$ is given by

$$h(\Psi\cdot\Phi,\mathcal{A}) = \langle\Psi,\Gamma(\mathcal{A})\Phi\rangle.$$

It will be useful to have an explicit formula for this operation:

Proposition 4.4. *Given $\Psi = (\psi_1, \psi_2)$ and $\Phi = (\phi_1, \phi_2)$ in $\mathcal{S} = S_+ \oplus S_-$, we have:*

$$\Psi \cdot \Phi = \begin{pmatrix} \langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle & -\widetilde{\psi_1 \cdot \psi_2} + \widetilde{\phi_1 \cdot \phi_2} \\ -\psi_1 \cdot \psi_2 + \phi_1 \cdot \phi_2 & -\langle \psi_1, \phi_2 \rangle - \langle \phi_1, \psi_2 \rangle \end{pmatrix}.$$

Proof. Decompose $\mathcal{V}$ into orthogonal subspaces:

$$\mathcal{V} = \left\{ \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix} : A \in V \right\} \oplus \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

The first of these is just a copy of V, an $(n+2)$-dimensional Minkowski spacetime. The second is the single extra spatial dimension in our $(n+3)$-dimensional Minkowski spacetime, $\mathcal{V}$.

Now, use the definition of $\Psi \cdot \Phi$, but restricted to V. It is easy to see that, for any vector $A \in V$, we have:

$$h(\Psi \cdot \Phi, A) = -\langle \psi_1, \gamma(A)\phi_1 \rangle + \langle \tilde{\gamma}(A)\psi_2, \phi_2 \rangle.$$

Letting B be the component of $\Psi \cdot \Phi$ which lies in V, this becomes:

$$g(B, A) = -\langle \psi_1, \gamma(A)\phi_1 \rangle + \langle \tilde{\gamma}(A)\psi_2, \phi_2 \rangle.$$

Note that we have switched to the metric g on V, to which h restricts. By definition, this is the same as:

$$g(B, A) = g(-\psi_1 \cdot \phi_1 + \psi_2 \cdot \phi_2, A).$$

Since this holds for all A, we must have $B = -\psi_1 \cdot \phi_1 + \psi_2 \cdot \phi_2$.

It remains to find the component of $\Psi \cdot \Phi$ orthogonal to B. Since $\left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) : a \in \mathbb{R} \right\}$ is one-dimensional, this is merely a number. Specifically, it is the constant of proportionality in the expression:

$$h\left(\Psi \cdot \Phi, \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\right) = a(\langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle).$$

Thus, this component is $\langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle$. Putting everything together, we get

$$\Psi \cdot \Phi = \begin{pmatrix} \langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle & -\widetilde{\psi_1 \cdot \psi_2} + \widetilde{\phi_1 \cdot \phi_2} \\ -\psi_1 \cdot \psi_2 + \phi_1 \cdot \phi_2 & -\langle \psi_1, \phi_2 \rangle - \langle \phi_1, \psi_2 \rangle \end{pmatrix}.$$

□

5 The 4-Ψ's rule

Spinors in dimension 4, 5, 7 and 11 satisfy an identity, written in conventional notation as follows:

$$\overline{\Psi}\Gamma_{ab}\Psi\overline{\Psi}\Gamma^b\Psi = 0.$$

This identity shows up in two prominent places in the physics literature. First, it is required for the existence of 2-brane theories in these dimensions [1, 17]. This is because it allows the construction of a WZW term for these theories, which give these theories Siegel symmetry.

Yet it is known that 2-branes in 11 dimensions are intimately connected to supergravity. Indeed, the Siegel symmetry imposed by the WZW term constrains the 2-brane background to be that of 11-dimensional supergravity [34]. Hence, it should come as no surprise that this spinor identity also plays a crucial role in supergravity, most visibly in the work of D'Auria and Fré [12] and subsequent work by Sati *et al.* [27].

This identity is equivalent to the "4-Ψ's rule":

$$\Psi \cdot ((\Psi \cdot \Psi)\Psi) = 0.$$

To see this, note that we can turn a pair of spinors Ψ and Φ into a 2-form, $\Psi * \Phi$. This comes from the fact that we can embed bivectors inside the Clifford algebra $\mathrm{Cliff}(\mathcal{V})$ via the map

$$\mathcal{A} \wedge \mathcal{B} \mapsto \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \in \mathrm{Cliff}(\mathcal{V}).$$

These can then act on spinors using the Clifford action. Thus, define:

$$(\Psi * \Phi)(\mathcal{A}, \mathcal{B}) = \langle \Psi, (\mathcal{A} \wedge \mathcal{B})\Phi \rangle. \tag{5.1}$$

However, when $\Psi = \Phi$, we can simplify this using the Clifford relation:

$$\begin{aligned}(\Psi * \Psi)(\mathcal{A}, \mathcal{B}) &= \langle \Psi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Psi \rangle \\ &= \langle \Psi, 2\mathcal{A}\mathcal{B}\Psi \rangle - \langle \Psi, \Psi \rangle h(\mathcal{A}, \mathcal{B}) \\ &= 2\langle \Psi, \mathcal{A}\mathcal{B}\Psi \rangle,\end{aligned}$$

where we have used the skew-symmetry of the form. The index-ridden identity above merely says that inserting the vector $\Psi \cdot \Psi$ into one slot of the 2-form $\Psi * \Psi$ is zero, no matter what goes into the other slot:

$$(\Psi * \Psi)(\mathcal{A}, \Psi \cdot \Psi) = 2\langle \Psi, \mathcal{A}(\Psi \cdot \Psi)\Psi \rangle = 0$$

for all $\mathcal{A}$. By the definition of the $\cdot$ operation, this is the same as

$$2h(\Psi \cdot ((\Psi \cdot \Psi)\Psi), \mathcal{A}) = 0$$

for all $\mathcal{A}$. Thus, the index-ridden identity is equivalent to:

$$\Psi \cdot ((\Psi \cdot \Psi)\Psi) = 0$$

as required.

Now, let us prove this:

Theorem 5.1. *Suppose* $\Psi \in \mathcal{S}$. *Then* $\Psi \cdot ((\Psi \cdot \Psi)\Psi) = 0$.

Proof. Let $\Psi = (\psi, \phi)$. By Proposition 4.4,

$$\Psi \cdot \Psi = \begin{pmatrix} 2\langle \psi, \phi \rangle & -\widetilde{\psi \cdot \psi} + \widetilde{\phi \cdot \phi} \\ -\psi \cdot \psi + \phi \cdot \phi & -2\langle \psi, \phi \rangle \end{pmatrix}$$

and thus

$$(\Psi \cdot \Psi)\Psi = \begin{pmatrix} 2\langle \psi, \phi \rangle \psi - (\widetilde{\psi \cdot \psi})\phi + (\widetilde{\phi \cdot \phi})\phi \\ -(\psi \cdot \psi)\psi + (\phi \cdot \phi)\psi - 2\langle \psi, \phi \rangle \phi \end{pmatrix}.$$

Both $(\psi \cdot \psi)\psi = 0$ and $(\widetilde{\phi \cdot \phi})\phi = 0$ by the 3-ψ's rule, Theorem 3.1. So:

$$(\Psi \cdot \Psi)\Psi = \begin{pmatrix} 2\langle \psi, \phi \rangle \psi - (\widetilde{\psi \cdot \psi})\phi \\ (\phi \cdot \phi)\psi - 2\langle \psi, \phi \rangle \phi \end{pmatrix}.$$

The resulting matrix for $\Psi \cdot ((\Psi \cdot \Psi)\Psi)$ is large and unwieldy, so we shall avoid writing it out. Fortunately, all we really need is the $(1, 1)$ entry. Recall, this is the component of the vector $\Psi \cdot ((\Psi \cdot \Psi)\Psi)$ that is orthogonal to the subspace $V \subset \mathcal{V}$. Call this component a. A calculation shows:

$$\begin{aligned} a &= \langle \psi, (\phi \cdot \phi)\psi \rangle - \langle (\widetilde{\psi \cdot \psi})\phi, \phi \rangle \\ &= \mathrm{Re}\ \mathrm{tr}(\psi^\dagger (2\phi\phi^\dagger)\psi) - \mathrm{Re}\ \mathrm{tr}(\phi^\dagger (2\psi\psi^\dagger)\phi) \\ &= 0, \end{aligned}$$

where the two terms cancel by the cyclic property of the real trace, Proposition 2.1. Thus, this component of the vector $\Psi \cdot ((\Psi \cdot \Psi)\Psi)$ vanishes. However, since the map $\Psi \mapsto \Psi \cdot ((\Psi \cdot \Psi)\Psi)$ is equivariant with respect to the action of $\mathrm{Spin}(\mathcal{V})$, and $\mathcal{V}$ is an irreducible representation of this group, it follows that all components of this vector must vanish. □

6 Cohomology of Lie superalgebras

In this section, we explain more of the meaning of the 3-ψ's rule and 4-Ψ's rules: they are *cocycle conditions*. In any dimension, a symmetric bilinear intertwining operator that eats two spinors and spits out a vector gives rise to a "super-Minkowski spacetime". The infinitesimal translation symmetries of this object form a Lie superalgebra called the "supertranslation algebra" [15]. The cohomology of this Lie superalgebra is interesting and apparently rather subtle. We shall see that its 3rd cohomology is nontrivial in dimensions 3, 4, 6 and 10, thanks to the 3-ψ's rule. Similarly, its 4th cohomology is nontrivial in dimensions 4, 5, 7 and 11, thanks to the 4-Ψ's rule.

We begin by recalling the supertranslation algebra. Take V to be the space of vectors in Minkowski spacetime in any dimension, and take S to be any spinor representation in this dimension. Suppose that there is a symmetric equivariant bilinear map:

$$\cdot \colon S \otimes S \to V.$$

Form a super vector space $\mathcal{T}$ with

$$\mathcal{T}_0 = V, \qquad \mathcal{T}_1 = S.$$

We make $\mathcal{T}$ into a Lie superalgebra, the **supertranslation algebra**, by giving it a suitable bracket operation. This bracket will be zero except when we bracket a spinor with a spinor, in which case it is simply the operation

$$\cdot \colon S \otimes S \to V.$$

Since this is symmetric and spinors are odd, the bracket operation is super-skew-symmetric overall. Furthermore, the Jacobi identity holds trivially, thanks to the near triviality of the bracket. Thus $\mathcal{T}$ is indeed, a Lie superalgebra.

Despite the fact that $\mathcal{T}$ is nearly trivial, its cohomology is not. To see this, we must first recall how to generalize Chevalley–Eilenberg cohomology [3, 13] from Lie algebras to Lie superalgebras. Suppose $\mathfrak{g}$ is a Lie superalgebra and R is a representation of $\mathfrak{g}$. That is, R is a supervector space equipped with a Lie superalgebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(R)$. We now define the cohomology groups of $\mathfrak{g}$ with values in R.

First, of course, we need a cochain complex. We define the ***n*-cochains** $C^n(\mathfrak{g}, R)$ to be the vector space of super-skew-symmetric n-linear maps:

$$\Lambda^n \mathfrak{g} \to R.$$

In fact, the n-cochains $C^n(\mathfrak{g}, R)$ are a super vector space, in which parity-preserving elements are even, while parity-reversing elements are odd.

Next, we define the coboundary operator $d\colon C^n(\mathfrak{g}, R) \to C^{n+1}(\mathfrak{g}, R)$. Let ω be a homogeneous n-cochain and let $X_1, \ldots, X_{n+1}$ be homogeneous elements of $\mathfrak{g}$. Now define:

$$\begin{aligned} d\omega(X_1, \ldots, X_{n+1}) & \\ = \sum_{i=1}^{n+1} & (-1)^{i+1}(-1)^{|X_i||\omega|}\epsilon_1^{i-1}(i)\rho(X_i)\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}) \\ & + \sum_{i<j}(-1)^{i+j}(-1)^{|X_i||X_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j) \\ & \times \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{n+1}). \end{aligned}$$

Here, $\epsilon_i^j(k)$ is shorthand for the sign one obtains by moving X_k through $X_i, X_{i+1}, \ldots, X_j$. In other words,

$$\epsilon_i^j(k) = (-1)^{|X_k|(|X_i|+|X_{i+1}|+\cdots+|X_j|)}.$$

Following the usual argument for Lie algebras, one can check that

Proposition 6.1. *The Lie superalgebra coboundary operator d satisfies $d^2 = 0$.*

We thus say a R-valued n-cochain ω on $\mathfrak{g}$ is an ***n*-cocycle** or **closed** when $d\omega = 0$, and an ***n*-coboundary** or **exact** if there exists an $(n-1)$-cochain θ such that $\omega = d\theta$. Every n-coboundary is an n-cocycle, and we say an n-cocycle is **trivial** if it is a coboundary. We denote the super vector spaces of n-cocycles and n-coboundaries by $Z^n(\mathfrak{g}, V)$ and $B^n(\mathfrak{g}, V)$, respectively. The nth **Lie superalgebra cohomology of $\mathfrak{g}$ with coefficients in** R, denoted $H^n(\mathfrak{g}, R)$ is defined by

$$H^n(\mathfrak{g}, R) = Z^n(\mathfrak{g}, R)/B^n(\mathfrak{g}, R).$$

This super vector space is nonzero if and only if there is a nontrivial n-cocycle. In what follows, we shall be especially concerned with the even part of this super vector space, which is nonzero if and only if there is a nontrivial even n-cocycle. Our motivation for looking for even cocycles is simple: these parity-preserving maps can regarded as morphisms in the category of super vector spaces, which is crucial for the construction in Theorem 7.1 and everything following it.

Now consider Minkowski spacetimes of dimensions 3, 4, 6 and 10. Here Minkowski spacetime can be written as $V = \mathfrak{h}_2(\mathbb{K})$, and we can take our spinors to be $S_+ = \mathbb{K}^2$. Since from Section 3, we know there is a symmetric bilinear intertwiner $\cdot \colon S_+ \otimes S_+ \to V$, we obtain the supertranslation algebra $\mathcal{T} = V \oplus S_+$. We can decompose the space of n-cochains with coefficients in the trivial 1-dimensional representation of $\mathcal{T}$ into summands by counting how many of the arguments are vectors and how many are spinors:

$$C^n(\mathcal{T}, \mathbb{R}) \cong \bigoplus_{p+q=n} (\Lambda^p(V) \otimes \mathrm{Sym}^q(S_+))^*.$$

We call an element of $(\Lambda^p(V) \otimes \mathrm{Sym}^q(S_+))^*$ a $\boldsymbol{(p, q)}$**-form**. Since the bracket of two spinors is a vector, and all other brackets are zero, d of a $(p,\ q)$-form is a $(p-1,\ q+2)$-form.

Using the 3-ψ's rule we can show:

Theorem 6.1. *In dimensions 3, 4, 6 and 10, the supertranslation algebra $\mathcal{T}$ has a nontrivial even 3-cocycle taking values in the trivial representation $\mathbb{R}$, namely the unique $(1, 2)$-form with*

$$\alpha(\psi, \phi, A) = g(\psi \cdot \phi, A)$$

for spinors $\psi, \phi \in S_+$ and vectors $A \in V$.

Proof. First, note that α has the right symmetry to be a linear map on $\Lambda^3(V \oplus S_+)$. Second, note that α is a $(1, 2)$-form, eating one vector and two spinors. Thus $d\alpha$ is a $(0, 4)$-form.

Because spinors are odd, $d\alpha$ is a symmetric function of four spinors. By the definition of d, $d\alpha(\psi, \phi, \chi, \theta)$ is the totally symmetric part of $\alpha(\psi \cdot \phi, \chi, \theta) = \alpha(\chi, \theta, \psi \cdot \phi) = g(\chi \cdot \theta, \psi \cdot \phi)$. However, any symmetric 4-linear form can be obtained from polarizing a quartic form. In this, we polarize $g(\psi \cdot \psi, \psi \cdot \psi)$ to get $d\alpha$. Thus:

$$d\alpha(\psi, \psi, \psi, \psi) = g(\psi \cdot \psi, \psi \cdot \psi) = \langle \psi, (\psi \cdot \psi)\psi \rangle,$$

where we have used the definition of the dot operation to obtain the last expression, which vanishes due to the 3-ψ rule. Thus, α is closed.

It remains to show α is not exact. So suppose it is exact, and that

$$\alpha = d\omega.$$

By our remarks above we may assume ω is a $(2,0)$-form: that is, an antisymmetric bilinear function of two vectors. By the definition of d, this last equation says:

$$g(\psi \cdot \phi, A) = -\omega(\psi \cdot \phi, A).$$

However, since $S_+ \otimes S_+ \to V$ is onto, this implies

$$g = -\omega,$$

a contradiction, since g is symmetric while ω is antisymmetric. $\square$

Next consider Minkowski spacetimes of dimensions 4, 5, 7 and 11. In this case Minkowski spacetime can be written as a subspace $\mathcal{V}$ of the 4×4 matrices valued in $\mathbb{K}$, and we can take our spinors to be $\mathcal{S} = \mathbb{K}^4$. Since from Section 4 we know that there is a symmetric bilinear intertwiner $\cdot\colon \mathcal{S} \otimes \mathcal{S} \to \mathcal{V}$, we obtain a supertranslation algebra $\mathcal{T} = \mathcal{V} \oplus \mathcal{S}$. As before, we can uniquely decompose any n-cochain in $C^n(\mathcal{T}, \mathbb{R})$ into a sum of (p,q)-forms, where now a $\boldsymbol{(p,q)}$**-form** is an an element of $(\Lambda^p(\mathcal{V}) \otimes \mathrm{Sym}^q(\mathcal{S}))^*$. As before, d of a $\boldsymbol{(p,q)}$-form is a $(p-1, q+2)$-form. And using the 4-Ψ's rule, we can show:

Theorem 6.2. *In dimensions 4, 5, 7 and 11, the supertranslation algebra $\mathcal{T}$ has a nontrivial even 4-cocycle, namely the unique $(2,2)$-form with*

$$\beta(\Psi, \Phi, \mathcal{A}, \mathcal{B}) = \langle \Psi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Phi \rangle$$

for spinors $\Psi, \Phi \in \mathcal{S}$ and vectors $\mathcal{A}, \mathcal{B} \in \mathcal{V}$. Here the commutator $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is taken in the Clifford algebra of $\mathcal{V}$.

Proof. First, to see that β has the right symmetry to be a map on $\Lambda^4(\mathcal{V} \oplus \mathcal{S})$, we note that it is antisymmetric on vectors, and that because

$$\Gamma^0 \mathcal{A} = -\mathcal{A}^\dagger \Gamma^0,$$

we have:

$$\Gamma^0 \mathcal{A}\mathcal{B} = \mathcal{A}^\dagger \mathcal{B}^\dagger \Gamma^0.$$

Thus:

$$\langle \Psi, \mathcal{A}\mathcal{B}\Phi \rangle = \langle \mathcal{B}\mathcal{A}\Psi, \Phi \rangle = -\langle \Phi, \mathcal{B}\mathcal{A}\Psi \rangle,$$

so we have:

$$\langle \Psi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Phi \rangle = \langle \Phi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Psi \rangle.$$

Thus, β is symmetric on spinors.

Next note that $d\beta$ is a $(1, 4)$-form, symmetric on its four spinor inputs. It is thus proportional to the polarization of

$$\alpha(\Psi, \Psi, (\Psi \cdot \Psi), \mathcal{A}) = \Psi * \Psi(\Psi \cdot \Psi, \mathcal{A}).$$

We encountered this object in Section 5, where we showed that it is proportional to

$$h(\Psi \cdot [(\Psi \cdot \Psi)\Psi], \mathcal{A}).$$

Moreover, this last expression vanishes by the 4-Ψ's rule. So, β is closed.

Furthermore, β is not exact. To see this, consider the unit vector $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ orthogonal to $V \subseteq \mathcal{V}$. Taking the interior product of β with this vector, a quick calculation shows:

$$\beta(\Psi, \Phi, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathcal{A}) = 2\langle \psi_1, \gamma(A)\phi_1 \rangle + 2\langle \tilde{\gamma}(A)\psi_2, \phi_2 \rangle,$$

where we have decomposed $\Psi = (\psi_1, \psi_2)$ and $\Phi = (\phi_1, \phi_2)$ into their components in $\mathcal{S} = S_+ \oplus S_-$, and A is the component of $\mathcal{A}$ in V. Restricting to the subalgebra $V \oplus S_+ \subseteq \mathcal{V} \oplus \mathcal{S}$, we see this is just α, up to a factor.

So, it suffices to check that interior product with $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ preserves exactness. For then, if β were exact, it would contradict that fact that α is not. Indeed, let ω be an n-cochain on $\mathcal{T}$, and let $X_1, \ldots, X_n \in \mathcal{T}$. Then, by our formula for the coboundary operator, we have:

$$\begin{aligned} & d\omega(X, X_1, \ldots, X_n) \\ & \quad = \sum_{i<j} -(-1)^{i+j}(-1)^{|X_i||X_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j) \\ & \qquad \times \omega(X, [X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_n) \\ & \qquad + \sum_{i=1}^{n} (-1)^{1+i}\epsilon_1^{i-1}(i)\omega([X, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_n), \end{aligned}$$

where, taking care with signs, we have collected terms involving bracketing with X into the second summation. But X is a vector, so all brackets with it vanish, and the second summation is zero.

If we write $i_X\omega$ for the operation of taking the interior product of ω with X, we have just shown:

$$i_X d\omega = -d i_X \omega$$

for any ω. In particular, if $\omega = d\theta$ then $i_X\omega = d(-i_X\theta)$, and so interior product with X preserves exactness, as claimed. □

7 L_∞-superalgebras

In the last section, we saw that the 3-ψ's and 4-Ψ's rules are cocycle conditions for the cocycles α and β. This sheds some light on the meaning of these rules, but it prompts an obvious followup question: what are these cocycles good for?

There is a very general answer to this question: a cocycle on a Lie superalgebra lets us extend it to an "L_∞-superalgebra". As we touched on in the Introduction, this is a chain complex equipped with structure like that of a Lie superalgebra, but where all the laws hold only "up to chain homotopy". We give the precise definition below.

It is well known that that the 2nd cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in some representation R classifies "central extensions" of $\mathfrak{g}$ by R [3, 13]. These are short exact sequences of Lie algebras:

$$0 \to R \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

where $\tilde{g}$ is the arbitrary and R is treated as an abelian Lie algebra whose image lies in the center of $\tilde{g}$. The same sort of result is true for Lie superalgebras. But this is just a special case of an even more general fact.

Suppose $\mathfrak{g}$ is a Lie superalgebra with a representation on a supervector space R. Then we shall prove that an even R-valued $(n+2)$-cocycle ω on $\mathfrak{g}$ lets us construct an L_∞-superalgebra, say $\tilde{\mathfrak{g}}$, of the following form:

$$\mathfrak{g} \xleftarrow{d} 0 \xleftarrow{d} \cdots \xleftarrow{d} 0 \xleftarrow{d} R,$$

where only the 0th and and nth grades are nonzero. Moreover, $\tilde{\mathfrak{g}}$ is an **extension** of $\mathfrak{g}$: there is a short exact sequence of L_∞-superalgebras

$$0 \to \mathfrak{b}^n R \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Here $\mathfrak{b}^n R$ is the abelian L_∞-superalgebra with R as its nth grade and all the rest zero:

$$0 \xleftarrow{d} 0 \xleftarrow{d} \cdots \xleftarrow{d} 0 \xleftarrow{d} R.$$

Note that when $n = 0$ and our vector spaces are all purely even, we are back to the familiar construction of Lie algebra extensions from 2-cocycles.

Technically, we should be more general than this in defining extensions. Maps between L_∞-algebras admit homotopies among themselves, and this allows us to introduce a weakened notion of "short exact sequence": a fibration sequence in the $(\infty, 1)$-category of L_∞-algebras. In general, these fibration sequences give the right concept of extension for L_∞-algebras. However, for the very special extensions we consider here, ordinary short exact sequences are all we need.

It is useful to have a special name for L_∞-superalgebras whose nonzero terms are all of degree $< n$: we call them **Lie n-superalgebras**. In this language, the 3-cocycle α defined in Theorem 6.1 gives rise to a Lie 2-superalgebra

$$\mathcal{T} \xleftarrow{d} \mathbb{R}$$

extending the supertranslation algebra $\mathcal{T}$ in dimensions $3, 4, 6$ and 10. Similarly, the 4-cocycle β defined in Theorem 6.2 gives a Lie 3-superalgebra

$$\mathcal{T} \xleftarrow{d} 0 \xleftarrow{d} \mathbb{R}$$

extending the supertranslation algebra in dimensions $4, 5, 7$ and 11.

Of course, this raises yet another question: what are these Lie n-superalgebras good for? The answer lies in physics: following a suggestion of Urs Schreiber, we can apply his work with Sati *et al.* [27] and consider connections on "n-bundles". These are, roughly speaking, bundles where the fibers are smooth n-categories instead of smooth manifolds. If we do this for n-bundles — or more precisely, "super-n-bundles" — arising from the Lie n-superalgebras constructed here, we find that the connections are fields that show up naturally in supermembrane and supergravity theories.

But now let us turn to the business at hand. In what follows, we shall use **super chain complexes**, which are chain complexes in the category SuperVect of $\mathbb{Z}_2$-graded vector spaces:

$$V_0 \xleftarrow{d} V_1 \xleftarrow{d} V_2 \xleftarrow{d} \cdots .$$

Thus each V_p is $\mathbb{Z}_2$-graded and d preserves this grading.

There are thus two gradings in play: the $\mathbb{Z}$-grading by **degree**, and the $\mathbb{Z}_2$-grading on each vector space, which we call the **parity**. We shall require a sign convention to establish how these gradings interact. If we consider an object of odd parity and odd degree, is it in fact even overall? By convention, we assume that it is. That is, whenever we interchange something of parity p and degree q with something of parity p' and degree q', we introduce the sign $(-1)^{(p+q)(p'+q')}$. We shall call the sum $p+q$ of parity and degree the **overall grade**, or when it will not cause confusion, simply the grade. We denote the overall grade of X by $|X|$.

We require a compressed notation for signs. If $x_1, \ldots, x_n$ are graded, $\sigma \in S_n$ a permutation, we define the **Koszul sign** $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ by

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \cdots x_{\sigma(n)},$$

the sign we would introduce in the free graded-commutative algebra generated by $x_1, \ldots, x_n$. Thus, $\epsilon(\sigma)$ encodes all the sign changes that arise from permuting graded elements. Now define:

$$\chi(\sigma) = \chi(\sigma; x_1, \ldots, x_n) := \mathrm{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).$$

Thus, $\chi(\sigma)$ is the sign we would introduce in the free graded-anticommutative algebra generated by $x_1, \ldots, x_n$.

Yet we shall only be concerned with particular permutations. If n is a natural number and $1 \le j \le n-1$ we say that $\sigma \in S_n$ is an $\boldsymbol{(j, n-j)}$**-unshuffle** if

$$\sigma(1) \le \sigma(2) \le \cdots \le \sigma(j) \quad \text{and} \quad \sigma(j+1) \le \sigma(j+2) \le \cdots \le \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses. A *shuffle* of two ordered sets (such as a deck of cards) is a permutation of the ordered union preserving the order of each of the given subsets. An *unshuffle* reverses this process. We denote the collection of all $(j, n-j)$ unshuffles by $S_{(j,n-j)}$.

The following definition of an L_∞-algebra was formulated by Schlessinger and Stasheff in 1985 [29]:

Definition 7.1. An $\mathbf{L}_\infty$**-algebra** is a graded vector space V equipped with a system $\{l_k | 1 \le k < \infty\}$ of linear maps $l_k \colon V^{\otimes k} \to V$ with $\deg(l_k) = k-2$ which are totally antisymmetric in the sense that

$$l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \ldots, x_n) \tag{7.1}$$

for all $\sigma \in S_n$ and $x_1, \ldots, x_n \in V$, and, moreover, the following generalized form of the Jacobi identity holds for $0 \le n < \infty$:

$$\sum_{i+j=n+1} \sum_{\sigma \in S_{(i,n-i)}} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{7.2}$$

where the summation is taken over all $(i, n-i)$-unshuffles with $i \ge 1$.

The following result shows how to construct L_∞-superalgebras from Lie superalgebra cocycles. This is the "super" version of a result due to Baez and Crans [5]. In this result, we require our cocycle to be even so we can consider it as a morphism in the category of super vector spaces.

Theorem 7.1. *There is a one-to-one correspondence between L_∞-superalgebras consisting of only two nonzero terms V_0 and V_n, with $d = 0$, and quadruples $(\mathfrak{g}, V, \rho, l_{n+2})$ where $\mathfrak{g}$ is a Lie superalgebra, V is a super vector space, ρ is a representation of $\mathfrak{g}$ on V, and l_{n+2} is an even $(n+2)$-cocycle on $\mathfrak{g}$ with values in V.*

Proof. Given such an L_∞-superalgebra we set $\mathfrak{g} = V_0$. V_0 comes equipped with a bracket as part of the L_∞-structure, and since d is trivial, this bracket satisfies the Jacobi identity on the nose, making $\mathfrak{g}$ into a Lie superalgebra. We define $V = V_n$, and note that the bracket also gives a map $\rho\colon \mathfrak{g} \otimes V \to V$, defined by $\rho(x)f = [x, f]$ for $x \in \mathfrak{g}, f \in V$. We have

$$\begin{aligned}
\rho([x,y])f &= [[x,y],f] \\
&= (-1)^{|y||f|}[[x,f],y] + [x,[y,f]] \quad \text{by (3) of Definition 7.1} \\
&= (-1)^{|f||y|}[\rho(x)f, y] + [x, \rho(y)f] \\
&= -(-1)^{|x||y|}\rho(y)\rho(x)f + \rho(x)\rho(y)f \\
&= [\rho(x), \rho(y)]f
\end{aligned}$$

for all $x, y \in \mathfrak{g}$ and $f \in V$, so that ρ is indeed a representation. Finally, the L_∞ structure gives a map $l_{n+2}\colon \Lambda^{n+2}\mathfrak{g} \to V$ which is in fact an $(n+2)$-cocycle. To see this, note that

$$0 = \sum_{i+j=n+4} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n+3)}),$$

where we sum over $(i, (n+3)-i)$-unshuffles $\sigma \in S_{n+3}$. However, the only choices for i and j that lead to nonzero l_i and l_j are $i = n+2, j = 2$ and

$i = 2, j = n + 2$. Thus, the above becomes, with σ a $(n+2,1)$-unshuffle and τ a $(2, n+1)$-unshuffle:

$$\begin{aligned}
0 &= \sum_{\sigma} \chi(\sigma)(-1)^{n+2}[l_{n+2}(x_{\sigma(1)}, \ldots, x_{\sigma(n+2)}), x_{\sigma(n+3)}] \\
&\quad + \sum_{\tau} \chi(\tau) l_{n+2}([x_{\tau(1)}, x_{\tau(2)}], x_{\tau(3)}, \ldots, x_{\tau(n+3)}) \\
&= \sum_{i=1}^{n+3} (-1)^{n+3-i}(-1)^{n+2}\epsilon_{i+1}^{n+2}(i)[l_{n+2}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+3}), x_i] \\
&\quad + \sum_{1 \le i < j \le n+3} (-1)^{i+j+1}(-1)^{|x_i||x_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j) \\
&\quad \times l_{n+2}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+3}).
\end{aligned}$$

On the second line, we have explicitly specified the unshuffles and unwrapped the signs encoded by χ. Since l_{n+2} is a morphism in SuperVect, it preserves parity, and thus the element $l_{n+2}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+2})$ has parity $|x_1| + \cdots + |x_{i-1}| + |x_{i+1}| + \cdots + |x_{n+2}|$. So, we can reorder the bracket in the first term, at the cost of a sign:

$$\begin{aligned}
0 &= \sum_{i=1}^{n+3} -(-1)^{i+1}\epsilon_1^{i-1}(i)[x_i, l_{n+2}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+3})] \\
&\quad + \sum_{1 \le i < j \le n+3} -(-1)^{i+j}(-1)^{|x_i||x_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j) \\
&\quad \times l_{n+2}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+3}) \\
&= -dl_{n+2}.
\end{aligned}$$

Here, we have used the fact that $\epsilon_{i+1}^{n+2}(i)(-1)^{|x_i|(|x_1|+\cdots+|x_{i-1}|+|x_{i+1}|+\cdots+|x_{n+2}|)} = \epsilon_1^{i-1}(i)$. Thus, l_{n+2} is indeed a cocycle.

Conversely, given a Lie superalgebra $\mathfrak{g}$, a representation ρ of $\mathfrak{g}$ on a vector space V, and an even $(n+2)$-cocycle l_{n+2} on $\mathfrak{g}$ with values in V, we define our L_∞-superalgebra V by setting $V_0 = \mathfrak{g}$, $V_n = V$, $V_i = \{0\}$ for $i \neq 0, n$, and $d = 0$. It remains to define the system of linear maps l_k, which we do as follows: since $\mathfrak{g}$ is a Lie superalgebra, we have a bracket defined on V_0. We extend this bracket to define the map l_2, denoted by $[\cdot,\cdot]\colon V_i \otimes V_j \to V_{i+j}$ where $i, j = 0, n$, as follows:

$$[x, f] = \rho(x)f, \quad [f, y] = (-1)^{|y||f|}\rho(y)f, \quad [f, g] = 0$$

for $x, y \in V_0$ and $f, g \in V_n$. With this definition, the map $[\cdot,\cdot]$ satisfies condition (1) of Definition 7.1. We define $l_k = 0$ for $3 \leq k \leq n+1$ and $k > n+2$, and take l_{n+2} to be the given $(n+2)$ cocycle, which satisfies conditions (1) and (2) of Definition 7.1 by the cocycle condition. $\square$

As already mentioned, we call an L_∞-superalgebra whose nonzero terms are all of degree $< n$ a **Lie n-superalgebra**. Using the above theorem, we say a Lie n-superalgebra is **exact** if only V_0 and V_{n-1} are nonzero and the $(n+1)$-cocycle l_{n+1} is trivial. The point of this definition is that we may easily obtain an exact Lie n-superalgebra simply from a Lie superalgebra and a representation, simply by taking the $(n+1)$-cocycle to be zero. Nonexact Lie n-superalgebras are more interesting.

Corollary 7.1. *In dimensions 3, 4, 6 and 10, there exists a nonexact Lie 2-superalgebra corresponding to the cocycle α given in Theorem* 6.1*.*

Corollary 7.2. *In dimensions 4, 5, 7 and 11, there exists a nonexact Lie 3-superalgebra corresponding to the cocycle β given in Theorem* 6.2*.*

8 Superstring Lie 2-algebras, 2-brane Lie 3-algebras

We have now met all the stars of our story. First, we met the 3-cocycle α:

$$\begin{array}{rccc} \alpha\colon & \Lambda^3(\mathcal{T}) & \to & \mathbb{R} \\ & A \wedge \psi \wedge \phi & \mapsto & \langle \psi, A\phi \rangle \end{array}$$

and saw its cocycle condition is the 3-ψ's rule in Theorem 6.1. Next, we encountered the 4-cocycle β:

$$\begin{array}{rccc} \beta\colon & \Lambda^4(\mathcal{T}) & \to & \mathbb{R} \\ & \mathcal{A} \wedge \mathcal{B} \wedge \Psi \wedge \Phi & \mapsto & \langle \Psi, (\mathcal{A} \wedge \mathcal{B})\Phi \rangle \end{array}$$

and saw that its cocycle condition is the 4-Ψ's rule in Theorem 6.2. We explored the meaning of these cocycles in Section 7: they allow us to create nontrivial extensions of the supertranslations $\mathcal{T}$ to Lie 2- and Lie 3-superalgebras, which we exhibited in Corollaries 7.1 and 7.2. But that is not the end of our story. We can go one step further with α and β, because both of them are invariant under the action of the corresponding Lorentz algebra: $\mathfrak{so}(n+1,1)$ in the case of α, and $\mathfrak{so}(n+2,1)$ for β. This is manifestly true, because α and β are built from equivariant maps.

As we shall see, this invariance implies that α and β are cocycles, not merely on the supertranslations, but on the full Poincaré superalgebra

— $\mathfrak{siso}(n+1,1)$ in the case of α:

$$\mathfrak{siso}(n+1,1) = \mathfrak{so}(n+1,1) \ltimes \mathcal{T}$$

and $\mathfrak{siso}(n+2,1)$ in the case of β:

$$\mathfrak{siso}(n+2,1) = \mathfrak{so}(n+2,1) \ltimes \mathcal{T}$$

We can extend α and β to these larger algebras in a trivial way: define the unique extension which vanishes unless all of its arguments come from $\mathcal{T}$. Doing this, α and β remain cocycles, even though the Lie bracket (and thus d) has changed. Moreover, they remain nontrivial. All of this is contained in the following proposition:

Proposition 8.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie superalgebras such that $\mathfrak{g}$ acts on $\mathfrak{h}$, and let R be a representation of $\mathfrak{g} \ltimes \mathfrak{h}$. Given any R-valued n-cochain ω on $\mathfrak{h}$, we can uniquely extend it to an n-cochain $\tilde{\omega}$ on $\mathfrak{g} \ltimes \mathfrak{h}$ that takes the value of ω on $\mathfrak{h}$ and vanishes on $\mathfrak{g}$. When ω is even, we have:*

1. *$\tilde{\omega}$ is closed if and only if ω is closed and $\mathfrak{g}$-equivariant.*
2. *$\tilde{\omega}$ is exact if and only if $\omega = d\theta$, for θ a $\mathfrak{g}$-equivariant $(n-1)$-cochain on $\mathfrak{h}$.*

Proof. As a vector space, $\mathfrak{g} \ltimes \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{h}$, so that

$$\Lambda^n(\mathfrak{g} \ltimes \mathfrak{h}) \cong \bigoplus_{p+q=n} \Lambda^p\mathfrak{g} \otimes \Lambda^q\mathfrak{h},$$

as a vector space. Thanks to this decomposition, we can uniquely decompose n-cochains on $\mathfrak{g} \ltimes \mathfrak{h}$ by restricting to the summands. In keeping with our prior terminology, we call an n-cochain supported on $\Lambda^p\mathfrak{g} \otimes \Lambda^q\mathfrak{h}$ a (p,q)-form. Note that $\tilde{\omega}$ is just the n-cochain ω regarded as a $(0,n)$-form on $\mathfrak{g} \ltimes \mathfrak{h}$. We shall denote the space of (p,q)-forms by $C^{p,q}$.

We have two actions to distinguish: the action of $\mathfrak{g} \ltimes \mathfrak{h}$ on R, which we denote by ρ, and the action of $\mathfrak{g}$ on $\mathfrak{h}$, which we shall denote simply by the bracket, $[-,-]$. Inspecting the formula for the differential:

$$\begin{aligned} &d\tilde{\omega}(X_1, \ldots, X_{n+1}) \\ &\quad = \sum_{i=1}^{n+1} (-1)^{i+1}(-1)^{|X_i||\tilde{\omega}|}\epsilon_1^{i-1}(i)\rho(X_i)\tilde{\omega}(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}) \end{aligned}$$

$$+\sum_{i<j}(-1)^{i+j}(-1)^{|X_i||X_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j)$$
$$\times\,\tilde{\omega}([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{n+1}),$$

it is easy to see that

$$d\colon C^{p,q}\to C^{p,q+1}\oplus C^{p+1,q}.$$

In particular,

$$d\colon C^{0,n}\to C^{0,n+1}\oplus C^{1,n}.$$

Given an n-cochain ω on $\mathfrak{h}$, it is easy to see that the part of $d\tilde{\omega}$, which lies in $C^{0,n+1}$ is just $\widetilde{d\omega}$, the extension of the $(n+1)$-cochain $d\omega$ to $\mathfrak{g}\ltimes\mathfrak{h}$.

Let $e\omega$ denote the $(1,n)$-form part of $d\tilde{\omega}$. To express this explicitly, choose $Y_1\in\mathfrak{g}$ and $X_2,\ldots,X_{n+1}\in\mathfrak{h}$. By definition $e\omega(Y_1,X_2,\ldots,X_{n+1})=d\tilde{\omega}(Y_1,X_2,\ldots,X_{n+1})$, and inspecting the formula for the differential once more, we see this consists of only two nonzero terms:

$$\begin{aligned}
&e\omega(Y_1,X_2,\ldots,X_{n+1})\\
&\quad=(-1)^{|\tilde{\omega}||Y_1|}\rho(Y_1)\tilde{\omega}(X_2,\ldots,X_{n+1})\\
&\qquad+\sum_{i=2}^{n+1}(-1)^{i+1}\epsilon_2^{i-1}(i)\tilde{\omega}([Y_1,X_i],X_2,\ldots,\hat{X}_i,\ldots,X_{n+1})\\
&\quad=(-1)^{|\omega||Y_1|}\rho(Y_1)\omega(X_2\wedge\cdots\wedge X_{n+1})-\omega([Y_1,X_2\wedge\cdots\wedge X_{n+1}]).
\end{aligned}$$

In particular, note that for even ω, $e\omega=0$ if and only if ω is $\mathfrak{g}$-equivariant.

To summarize, for any n-cochain ω, we have that

$$d\tilde{\omega}=\widetilde{d\omega}+e\omega,$$

where the first d is defined on $\mathfrak{g}\ltimes\mathfrak{h}$, while the second is only defined on $\mathfrak{h}$. The proof of 1 is now immediate: for even ω, $d\tilde{\omega}=0$ if and only if $\widetilde{d\omega}=0$ and $e\omega=0$, which happens if and only $d\omega=0$ and ω is $\mathfrak{g}$-equivariant.

To prove 2, suppose ω is even. Assume $\tilde{\omega}=d\chi$, for some $(n-1)$-cochain χ on $\mathfrak{g}\ltimes\mathfrak{h}$. Because $d\chi$ is an even $(0,n)$-form, we may assume χ is an even $(0,n-1)$-form, as any other part of χ is closed and does not contribute to

$d\chi$. Thus χ is the extension of an even $(n-1)$-cochain θ on $\mathfrak{h}$. By our prior formula, we have:

$$\tilde{\omega} = d\tilde{\theta} = \widetilde{d\theta} + e\theta.$$

The left-hand side is a $(0,n)$-form, and thus the $(1,n-1)$-form part of the right-hand side, $e\theta$, vanishes. Thus θ is $\mathfrak{g}$-equivariant, and $\tilde{\omega} = \widetilde{d\theta}$, which implies $\omega = d\theta$. On the other hand, if $\omega = d\theta$ and θ is $\mathfrak{g}$-equivariant, then $e\theta = 0$ and thus $\tilde{\omega} = d\tilde{\theta}$. □

Thus we can extend α and β to nonexact cocycles on the Poincaré Lie superalgebra. Thanks to Theorem 7.1, we know that α lets us extend $\mathfrak{siso}(n+1,1)$ to a nonexact Lie 2-superalgebra:

Theorem 8.1. *In dimensions 3, 4, 6 and 10, there exists a nonexact Lie 2-superalgebra formed by extending the Poincaré superalgebra* $\mathfrak{siso}(n+1,1)$ *by the 3-cocycle* α*, which we call we the* **superstring Lie 2-superalgebra***,* $\boldsymbol{\mathfrak{superstring}(n+1,1)}$.

Likewise, in dimensions one higher, β lets us extend $\mathfrak{siso}(n+2,1)$ to a nonexact Lie 3-superalgebra. In the 11-dimensional case, this coincides with the Lie 3-superalgebra which Sati *et al.* call $\mathfrak{sugra}(10,1)$ [27], which is the Koszul dual of an algebra defined by D'Auria and Fré [12].

Theorem 8.2. *In dimensions 4, 5, 7 and 11, there exists a nonexact Lie 3-superalgebra formed by extending the Poincaré superalgebra* $\mathfrak{siso}(n+2,1)$ *by the 4-cocycle* β*, which we call the* **2-brane Lie 3-superalgebra***,* **2-**$\boldsymbol{\mathfrak{brane}(n+2,1)}$.

Acknowledgments

We thank Tevian Dray, Robert Helling, Corinne Manogue, Chris Rogers, Hisham Sati, James Stashef and Riccardo Nicoletti for useful conversations. We especially thank Urs Schreiber for many discussions of higher gauge theory and L_∞-superalgebras. This work was supported by the FQXi under grant no. RFP2-08-04.

References

[1] A. Achúcarro, J.M. Evans, P.K. Townsend, and D.L. Wiltshire, *Super p-branes*, Phys. Lett. B **198** (1987), 441–446.

[2] P. Aschieri and B. Jurco, *Gerbes, M5-brane anomalies and E_8 gauge theory.* `arXiv:hep-th/0409200`.

[3] J. Azcárraga and J. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications to physics*, Cambridge University Press, Cambridge, 1995, 230–236.

[4] J. Baez, *The octonions*, Bull. Amer. Math. Soc. **39** (2002), 145–205; `arXiv:math/0105155`.

[5] J. Baez and A. Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–528. Available at http://www.tac.mta.ca/tac/volumes/12/15/12-15abs.html; `arXiv:math.QA/0307263`.

[6] J. Baez and J. Huerta, *Division algebras and supersymmetry I*, in 'Proceedings of the NSF/CBMS Conference on Topology, C* Algebras, and String Theory', eds. R. Doran *et al.*, Proc. Symp. Pure Math. **81**, AMS, Providence, 2010, 65–80. Also available as `arXiv:0909.0551`.

[7] J. Baez and J. Huerta, *An invitation to higher gauge theory*, Gen. Relativ. Grav **43** (2011), 2335–2392. Also available as `arXiv:1003.4485`.

[8] J. Baez and U. Schreiber, *Higher gauge theory*, in 'Categories in Algebra, Geometry and Mathematical Physics', eds. A. Davydov *et al.* Contemp. Math. **431**, AMS, Providence, 2007, 7–30. Also available as `arXiv:math/0511710`.

[9] A. Carey, S. Johnson, and M. Murray, *Holonomy on D-branes*, J. Geom. Phys. **52**(2) (2004), 186–216. Also available as `arXiv:hep-th/0204199`.

[10] L. Castellani, R. D'Auria, and P. Fré, *Supergravity and superstrings: a geometric perspective*, World Scientific, Singapore, 1991.

[11] K.-W. Chung and A. Sudbery, *Octonions and the Lorentz and conformal groups of ten-dimensional space-time*, Phys. Lett. B **198** (1987), 161–164.

[12] R. D'Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, Nucl. Phys. B **201** (1982), 101–140.

[13] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124.

[14] E. Diaconescu, D. Freed, and G. Moore, *The M theory 3-form and* E_8 *gauge theory.* Available as `arXiv:hep-th/0312069`.

[15] P. Deligne *et al.* (eds.), *Quantum fields and strings: a course for mathematicians*, vol 1, Amer. Math. Soc., Providence, Rhode Island, 1999.

[16] T. Dray, J. Janesky, and C. Manogue, *Octonionic Hermitian matrices with non-real eigenvalues*, Advances in Applied Clifford Algebras **10** (2000), 193–216. Also available as `arXiv:math/0006069`.

[17] M.J. Duff, *Supermembranes: the first fifteen weeks*, Class. Quantum Grav. **5** (1988), 189–205. Also available at http://ccdb4fs.kek.jp/cgi-bin/img_index?8708425.

[18] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, Asian J. Math. **3** (1999), 819–852. Also available as `arXiv:hep-th/9907189`.

[19] K. Gawedzki, *Topological actions in two-dimensional quantum field theories*, in 'Nonperturbative quantum field theory', eds. G. t'Hooft, A. Jaffe, G. Mack, P.K. Mitter, and R. Stora, Plenum, New York, 1988, 101–141.

[20] K. Gawedzki and N. Reis, *WZW branes and gerbes*, Rev. Math. Phys. **14** (2002), 1281–1334. Also available as `arXiv:hep-th/0205233`.

[21] M. Green and J. Schwarz, *Covariant description of superstrings*, Phys. Lett. B **136** (1984), 367–370.

[22] M. Green, J. Schwarz, and E. Witten, *Superstring theory*, Vol 1, Cambridge University Press, Cambridge, 1987. Section 5.1.2: The supersymmetric string action, 253–255.

[23] A. Hurwitz, *Über die Composition der quadratischen Formen von beliebig vielen Variabeln*, Nachr. Ges. Wiss. Göttingen (1898), 309–316.

[24] T. Kugo and P. Townsend, *Supersymmetry and the division algebras*, Nucl. Phys. B **221** (1983), 357–380. Also available at http://ccdb4fs.kek.jp/cgi-bin/img_index?198301032.

[25] C. Manogue and A. Sudbery, *General solutions of the covariant superstring equations of motion*, Phys. Rev. D **40** (1989), 4073–4077.

[26] M. Markl, S. Schnider, and J. Stasheff, *Operads in algebra, topology and physics*, AMS, Providence, Rhode Island, 2002.

[27] H. Sati, U. Schreiber, and J. Stasheff, *L_∞-algebras and applications to string - and Chern–Simons n-transport.* Available as `arXiv:0801.3480`.

[28] R.D. Schafer, *Introduction to non-associative algebras*, Dover, New York, 1995.

[29] M. Schlessinger and J. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Alg. **38** (1985), 313–322.

[30] J. Schray, *The general classical solution of the superparticle*, Class. Quantum Grav. **13** (1996), 27–38. Also available as `arXiv:hep-th/9407045`.

[31] J. Schray and C. Manogue, *Octonionic representations of Clifford algebras and triality*, Found. Phys. **26** (1996), 17–70. Also available as `arXiv:hep-th/9407179`.

[32] D. Stevenson, *The geometry of bundle gerbes*, Ph.D. Thesis, University of Adelaide, 2000. Also available as `arXiv:math/0004117`.

[33] A. Sudbery, *Division algebras, (pseudo)orthogonal groups and spinors*, J. Phys. A **17** (1984), 939–955.

[34] Y. Tanii, Introduction to supergravities in diverse dimensions. Available as `arXiv:hep-th/9802138`.