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Model building with *F*-theory

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Abstract

Despite much recent progress in model building with D-branes, it has been problematic to find a completely convincing explanation of gauge coupling unification. We extend the class of models by considering F-theory compactifications, which may incorporate unification more naturally. We explain how to derive the charged chiral spectrum and Yukawa couplings in $N = 1$ compactifications of F-theory with G-flux. In a class of models which admit perturbative heterotic duals, we show that the F-theory and heterotic computations match.

Contents

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1 Introduction

String theory is an extension of quantum field theory which incorporates quantum gravity. In the process it reformulates many questions about field theory into questions about the geometry of extra dimensions. Recently, ten-dimensional string backgrounds were found that reproduce the MSSM at low energies [1, 2].

However finding realizations of the MSSM is merely an intermediate step, because we would like to answer questions that the MSSM does not explain. Indeed there are quite a number of hard issues to tackle. Many of these have to do with the intrinsic difficulties of a theory of quantum gravity. Thus some of these issues may probably be resolved by a better conceptual understanding of quantum gravity. However as in [3] we would like to take a more practical perspective with regards to the phenomenological requirements which have a direct impact on particle physics. We will assume that they can be understood in a framework where four-dimensional gravity is effectively turned off. That is, we do not yet want to be pushed into having to specify a complete model of physics at the Planck scale, while there are still many issues in particle physics that presumably can be explained without referring to a full UV completion. Interestingly, string theory allows us to think in such a framework and in the process provides an intuitive geometric picture through the brane world scenario.

One of the first coincidences that one would like to address is the issue of gauge coupling unification. The most natural scenario is still some type of Grand Unified Theory (GUT). In particular, one would like to have realizations of such models in type IIb string theory, where most of the recent progress in moduli stabilization, mediation of SUSY breaking and other issues has recently taken place. There have in fact been attempts to construct D-brane GUT models, but these suffer from a number of inherent difficulties, such as the lack of a spinor representation for $SO(10)$ or the perturbative vanishing of top quark Yukawa couplings for $SU(5)$ models.

These difficulties arise because past constructions have relied on mutually local 7-branes. There is however a natural way to evade these obstacles, which is by incorporating mutually non-local 7-branes. This enlarged class of models goes under the name of F-theory [4]. In fact, in certain limits F-theory is dual to the heterotic string, which "explains" why F-theory should be able to circumvent the no-go theorems.

It is then surprising that, despite the potentially promising phenomenology of the F-theory set-up, some important issues in F-theory compactifications have not been addressed. Foremost among these, it is not currently known how to derive the spectrum of quarks and leptons. It is the purpose of this paper to explain the origin of charged chiral matter and to provide tools for computing the spectrum and the couplings. Our approach is to deduce everything from the eight-dimensional Yang–Mills–Higgs theory living on the 7-branes, and our results are therefore quite general. As expected from type IIb string theory, we can get chiral matter spread in the bulk of a 7-brane or localized on the intersection of 7-branes by turning on suitable fluxes. Also as expected, the Yukawa couplings are computed simply from the overlap of the chiral zero modes on the 7-brane. These results should be helpful in putting many extra-dimensional phenomenological models, in which localization of wave functions was used to explain differences in couplings, on a firmer footing. In order to make sure that our results are correct, we test our formulae for F-theory compactifications which are dual to the heterotic string. We will see that the computations on both sides of the duality match. Along the way we clarify several issues in heterotic/ F-theory duality.

In this paper, we emphasize mostly conceptual issues. In Section 2, we discuss the main model building ingredients of F-theory. In particular, we explain how charged chiral matter arises and how we can compute the spectrum and the supersymmetric Yukawa couplings. We also discuss constructive techniques for F-theory vacua with Grand Unified gauge groups. Section 3 is mostly devoted to F-theory/heterotic duality. After reviewing the spectral cover approach to constructing heterotic vacua, we show that the heterotic computation of massless matter matches exactly with our F-theory prescription. We also discuss the matching of superpotentials under the duality. In Section 4, we briefly discuss some simple explicit examples of GUT models. Finally in Section 5, we discuss how to break the GUT group to the Standard Model gauge group. In the appendices, we collect some properties of spinors and the Dirac operator that we will use in the text.

Note for revision: The present version of this paper includes significant clarifications and improvements over the original version of arXiv:0802.2969.

Most of these were designed to bring the paper more in line with the subsequent papers [5, 6]. We are also grateful to T. Watari for suggested improvements to Appendix C. Shortly after the posting of arXiv:0802.2969, similar results were obtained in the work [7] by the Harvard group.

2 Model building with F-theory

The purpose of this section is to discuss how to engineer gauge groups and charged chiral matter from F-theory.

2.1 Gauge fields

Let us consider an F-theory compactification to four dimensions with $N = 1$ supersymmetry. This consists of a Calabi–Yau four-fold Y_4, which is elliptically fibered $\pi : Y_4 \to B_3$ with a section $\sigma : B_3 \to Y_4$. The base B_3, or more precisely the section, is the space-time visible to type IIB, and the complex structure of the T^2 fibre encodes the dilaton and axion at each point on B_3:

$$\tau = \mathrm{e}^{-\phi}\,\mathrm{i} + C_{(0)}. \tag{2.1}$$

It is convenient to represent the four-fold in Weierstrass form:

$$y^2 = x^3 + fx + g. \tag{2.2}$$

Requiring the four-fold to be Calabi–Yau implies that f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$ respectively. The complex structure of the fibre is given by

$$j(\tau) = \frac{4(24\,f)^3}{\Delta}, \quad \Delta = 4f^3 + 27g^2. \tag{2.3}$$

At the discriminant locus $\{\Delta = 0\} \subset B_3$ the T^2 degenerates by pinching one of its cycles. Let us label the one-cycles by $(p, q) = p\,\alpha + q\,\beta$, and suppose we pick a local coordinate z on B_3 such that the $(1, 0)$-cycle pinches as $z \to 0$. Then τ has a monodromy around this locus:

$$\tau \sim \frac{1}{2\pi \mathrm{i}} \log(z). \tag{2.4}$$

This is a shift in the axion. It means that the brane at $z = 0$ is a source for one unit of RR-flux, and so we identify it with an ordinary D7-brane, a brane on which a $(1, 0)$ string (i.e., a fundamental string) can end. For more general (p, q) we can do an $Sl(2, \mathbf{Z})$ transform, and we find that the brane is a locus where a (p, q)-string can end. This is called a (p, q) 7-brane.

In perturbative string theory, the worldvolume of an isolated 7-brane contains a $U(1)$ gauge field A_μ, from quantising an open string with both ends on the brane. Non-perturbatively this gauge field is encoded in the so-called G-flux [8]. To see this, let us compactify on an extra S^1 with radius R. This is dual to M-theory on Y_4, where the area of the elliptic fiber is now proportional to R^{-1}. In M-theory gauge fields arise from expanding the three-form C_3 along harmonic two-forms ω, and the four-form flux of this field is called the G-flux. So the same must be true in the F-theory limit. Therefore on the F-theory side we formally introduce a three-form field C_3 and expand it along harmonic two-forms to get the gauge fields.

However, we should only expand around a subset of the harmonic two-forms on Y_4, because some of the modes of the M-theory compactification do not survive in the F-theory limit. The easiest way to see this is by following various BPS states through the duality. If C_3 has both spatial indices on B_3 then it couples to an $M2$-brane wrapped on a cycle α_2 in B_3. This gets mapped to a $D3$-brane wrapping $\alpha_2 \times S^1_R$, which becomes a string in four-dimensions as $R \to \infty$, therefore couples to a pseudo-scalar (more precisely its dual two-form field) but not a four-dimensional vector. Similarly if C_3 has two spatial indices on the elliptic fiber, it couples to an $M2$-brane wrapping this fiber which gets mapped to a fundamental string with momentum along S^1_R. Therefore it couples to the KK gauge field associated to S^1_R, and in the limit $R \to \infty$ we just recover a component of the four-dimensional metric, not a four-dimensional vector field. Finally, membranes wrapping the remaining cycles of Y_4 get mapped to (p, q)-strings. If they map to open strings, the ends of such a string are electric charges on the worldvolume of 7-branes, therefore they couple to the gauge fields on the 7-branes. If they map to closed strings, then they couple to some linear combination of the NS and RR two-forms with one index on B_3 and thus we get a gauge field in four dimensions also. Thus the relevant harmonic forms constitute the lattice

$$\Lambda = \{\omega \in H^2(Y_4) | \omega \cdot \alpha = 0 \text{ when } \alpha \in H_2(B_3) \text{ or } \alpha = [T^2]\}. \tag{2.5}$$

This lattice is the coroot lattice of the four-dimensional gauge group originating from the 7-branes.

From the above argument, we see that the three-form field actually has a simple physical interpretation: it gives the non-perturbative description of the NS and RR twoforms of type IIb. Recall that these two-forms transform as a doublet under the $Sl(2, \mathbf{Z})$ duality group. Due to the branch cuts of the axio-dilaton on B_3, they are hard to describe on the IIb space-time directly. However when we add the elliptic fibration, we can get a simple global description as follows. We define

$$\mathsf{H} = H_{\mathrm{RR}} - \tau H_{\mathrm{NS}} \tag{2.6}$$

Then the three-form fluxes in type IIb can be encoded in F-theory as [9]

$$\mathsf{G} = \mathsf{H} \wedge dz + \bar{\mathsf{H}} \wedge d\bar{z}, \tag{2.7}$$

where dz and $d\bar{z}$ are the normalized harmonic $(1,0)$ and $(0,1)$ forms on an elliptic curve.

Furthermore in F-theory we must identify the abelian gauge field on the 7-brane as a bound state of the closed string fields of the IIb theory; it cannot be added by hand. To see this, we note that in F-theory a supersymmetric 7-brane is described *as* a cosmic string solution, which is then further lifted to an elliptically fibered Calabi–Yau metric in two dimensions higher [10]. Since the brane is described as a soliton of IIb supergravity, all the degrees of freedom we associate with it must be made from $g_{ij}, \tau, H_{\mathrm{RR}}, H_{\mathrm{NS}}$ and the IIb spinors. And indeed this works in the standard way. At the center of the cosmic string, where an $S^1 \subset T^2$ shrinks to zero size, the Calabi–Yau geometry is similar to a Taub-NUT space and supports a harmonic two-form ω of type $(1,1)$ which peaks when the S^1 shrinks to zero. Then the collective coordinates of the 7-brane may be interpreted from the worldvolume perspective by expanding in this harmonic form. Thus the $U(1)$ gauge field on the 7-brane is obtained by expanding C_3 as

$$\mathsf{C}_3 = \mathsf{A} \wedge \omega. \tag{2.8}$$

The G-flux $\mathsf{G}_4 = d\mathsf{C}_3$ then describes the magnetic flux along the 7-brane. Note that this explicitly identifies the gauge field on a 7-brane as a collective coordinate constructed from the RR and NS two-form fields of type IIb supergravity.

We may similarly recover the remaining fields of the $8d$ supersymmetric Yang–Mills multiplet as collective coordinates of the 7-brane. The adjoint field Φ describing deformations of the 7-brane comes from deformations of the discriminant locus, that is from complex structure deformations δg_{ij} of the four-fold. Using the holomorphic $(4,0)$ form, these can also be written as harmonic forms $\alpha^{3,1}$ of Hodge type $(3,1)$. Expanding

$$\alpha^{3,1} = \Phi^{2,0} \wedge \omega \tag{2.9}$$

we see that the adjoint field corresponds to a section of the canonical bundle of the surface which the 7-brane wraps. Note this means that the $U(1)_R$ symmetry of the eight-dimensional gauge theory is identified with the structure group of the canonical bundle, not with the structure group of the normal bundle, as is the case for many lower dimensional branes. Finally the spinors will be sections of the spinor bundle of the wrapped surface tensored with the spinor bundle associated to the canonical bundle (to account for their R-charges). That is, they are sections of the gauge bundle tensored with

$$\Omega^{0,p}(K_{S_2}^{1/2}) \otimes (K_{S_2}^{-1/2} \oplus K_{S_2}^{1/2}) = \Omega^{0,p}(S_2) \oplus \Omega^{2,p}(S_2), \tag{2.10}$$

for $p = 0, 1, 2$, and S_2 is the surface that the 7-branes wrap. Sections related by Serre duality correspond to CPT conjugates rather than independent fields. Clearly the unique generator of $h^{0,0}(S_2)$ (together with its CPT conjugate in $h^{2,2}(S_2)$) corresponds to the four-dimensional gaugino, and the generators of $h^{2,0}(S_2)$ and $h^{0,1}(S_2)$ yield adjoint valued four-dimensional chiral superfields from deformations of the 7-branes and continuous moduli of the line bundles on the 7-branes respectively.

In a cosmic string background the two-form ω is not normalizable. This implies that the number of massless $U(1)$ gauge fields is always less than the number of singular fibers, counted with appropriate multiplicity. In fact since a configuration of 7-branes is labelled asymtotically by its total dyonic (p,q) charge, we expect that at least two non-normalizable modes will get lifted.

The allowed G-fluxes are constrained by the equations of motion [9, 11]. There is a superpotential coupling

$$W = \frac{1}{2\pi} \int \Omega^{4,0} \wedge \mathsf{G}. \tag{2.11}$$

Varying with respect to the complex structure moduli, we see that the flux should be of type $(2,2) + (4,0) + (0,4)$. If Y_4 is compact, supersymmetry and the requirement of a Minkowski vacuum also imply that $W = 0$, leading to the vanishing of the $(4,0)$ and $(0,4)$ parts. We also have to satisfy the D-terms. When Y_4 is smooth and there are no parametrically light states coupling to C_3, these are given by

$$J \wedge \mathsf{G} = 0, \tag{2.12}$$

i.e., G is required to be primitive with respect to J, where J is the Kähler form on B_3. Equivalently we can require that the contraction $\imath_J \mathsf{G} = 0$. This is a packaging of the D-terms for many $U(1)$ gauge fields into one equation. Note though that the conditions we imposed for validity are strictly necessary. We will find that when Y_4 is singular, there are corrections to this equation. Indeed we will typically be interested in models with $4d$ non-abelian gauge fields, in which case Y_4 is not smooth. If Y_4 is compact, there is a tadpole cancellation condition

$$N_{D3} = \frac{\chi(Y_4)}{24} - \frac{1}{8\pi^2} \int_{Y_4} \mathsf{G} \wedge \mathsf{G}, \tag{2.13}$$

where N_{D3} is the number of $D3$ branes filling $\mathbf{R}^4$, not including possible instantons which are already described by G. Finally, the G-flux must be properly quantized [12]

$$\left[\frac{\mathsf{G}}{2\pi}\right] - \frac{p_1(Y_4)}{4} \in H^4(Y_4, \mathbf{Z}). \tag{2.14}$$

Because $\mathsf{G}/2\pi$ is generally half-integer quantized (2.14), and the flux on the 7-branes is deduced by expanding $\mathsf{G} = \mathsf{F} \wedge \omega$, we anticipate that $\mathsf{F}/2\pi$ is also half-integer quantized, and so does not generally correspond to a good line bundle on the 7-brane. Indeed it has been argued in [13] that the induced flux is half-integer quantized precisely when the tangent bundle to the brane does not admit a spin structure. See also [14] where the shift in the quantization law of the 'spectral line bundle' (an object that we will discuss later) is related to the shift in the quantization law of the G-flux by $c_2/2 = p_1/4$ on the Calabi–Yau four-fold. Thus it is useful to split up the induced gauge field on the 7-brane into two pieces:

$$\mathsf{A} = \mathsf{A}_E - \frac{1}{2}\mathsf{A}_{K_S}, \tag{2.15}$$

where A_{K_S} is the connection on the canonical bundle of the cycle S that the 7-brane is wrapped on, and A_E is the connection for a well-defined bundle E on S. The shift in the quantization law of the gauge field on a brane for zero B-field is known as the Freed–Witten anomaly [15].

Similarly we may consider configurations with multiple branes. The $U(1)$ gauge field associated to each brane can be decomposed into a well-defined

Table 1: Abelian vector multiplets in F-theory compactifications.

Number of $U(1)$'s	Origin
$h^{1,1}(Y_4) - h^{1,1}(B_3) - 1$	7-branes and two-forms
$h^{2,1}(B_3)$	four-form RR-potential

Table 2: Moduli of F-theory. The axio-dilaton is usually stabilized and so not included here.

Number of moduli	Origin
$h^{2,1}(Y_4) - h^{2,1}(B_3) - 1$	Wilson lines on 7-branes and two-form periods
$h^{1,1}(B_3)$	Kähler moduli of B_3
$h^{3,1}(Y_4)$	complex structure of B_3 and 7-brane deformations

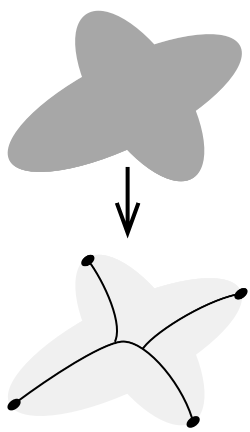


Figure 1: Multi-pronged strings in B_3 lift to curves in Y_4, allowing for matter and gauge groups which cannot be obtained from ordinary open strings.

piece and a correction given by half the connection of the canonical bundle of the four-cycle that the brane is wrapped on. The Cartan generators are linear combinations of these $U(1)$'s, so as long as all the branes are wrapped on the same cycle the shifts cancel out when we compare G-fluxes with line bundles, and may be ignored. However in an intersecting brane configuration the branes are wrapped on different four-cycles, and the shifts do not cancel.

Besides the $U(1)$ gauge fields from the 7-branes, we get additional $U(1)$ factors from expanding the RR four-form along harmonic three-forms. This is summarized in table 1. In addition, we will get neutral chiral fields from the moduli of the compactification. This is summarized in table 2 (see e.g., [14, 16, 17]).

Next we would like to discuss how non-abelian gauge symmetries are encoded in F-theory. We expect to find non-abelian gauge bosons from open strings stretching between 7-branes. It is well known that a perturbative open string has two ends and so cannot give rise to a spinor representation or an adjoint of an exceptional group. This gets evaded in F-theory because the branes are generically not mutually local, so the dilaton cannot be taken small and there is no perturbative description. Then the missing open string states which are needed to get a spinor or an exceptional adjoint can be realized as BPS junctions, i.e., open strings with multiple ends [18–20]. They correspond to minimal area two-cycles C in Y_4 which are projected to a multi-pronged string in B_3, see figure 1. When 7-branes approach each other, some of these minimal area cycles shrink to zero size and create an enhanced singularity. Given a set of generators $\vec{\omega}$ of the lattice Λ, the charges of these BPS states associated with vanishing curves under the Cartan generators in Λ are given by

$$\vec{w} = \int_C \vec{\omega}. \tag{2.16}$$

As the notation suggests, these vanishing curves will be in one-to-one correspondence with weights of some non-abelian Lie algebra (and also, as we will see in the next section, with weights of matter representations).

The dictionary between singularities of the elliptic fibration and enhanced gauge symmetries has been worked out in some detail. The basic starting point is the Kodaira classification of singular fibers which we have reproduced in table 3. To first approximation, we would associate an ADE gauge group to an ADE singularity. However, if the dimension of the base is larger than one then there can be monodromies which act as automorphisms on the algebra and reduce the group to a non-simply laced version. We will not review this in detail (see [21]) but we will quote some results on the form of the singularities in a moment.

Later we will be interested in comparison with the heterotic string. Such a comparison can be made using heterotic/F-theory duality in eight dimensions, which states that the heterotic string on T^2 is equivalent to F-theory on an elliptically fibered $K3$ surface with a choice of section. By fibering this duality over a complex surface B_2, we get a four-dimensional duality between the heterotic string on a Calabi–Yau three-fold $Z = (T^2 \to B_2)$ and F-theory on a Calabi–Yau four-fold $Y_4 = (K3 \to B_2)$ where $K3$ itself is elliptically fibered. One may match the analytic data on both sides of the duality in a certain limit on the boundary of moduli space, where the $K3$ surface undergoes a stable degeneration to two dP_9-surfaces glued along a common

Table 3: Kodaira classification

ord(f)	ord(g)	ord(Δ)	Fiber type	Singularity type
≥0	≥0	0	smooth	−
0	0	n	I_n	A_{n-1}
≥1	1	2	II	−
1	≥2	3	III	A_1
≥2	2	4	IV	A_2
2	≥3	$n+6$	I_n^*	D_{n+4}
≥2	3	$n+6$	I_n^*	D_{n+4}
≥3	4	8	IV^*	E_6
3	≥5	9	III^*	E_7
≥4	5	10	II^*	E_8

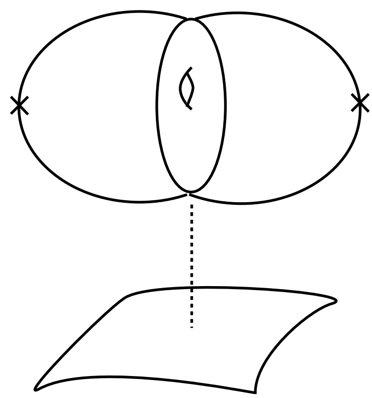


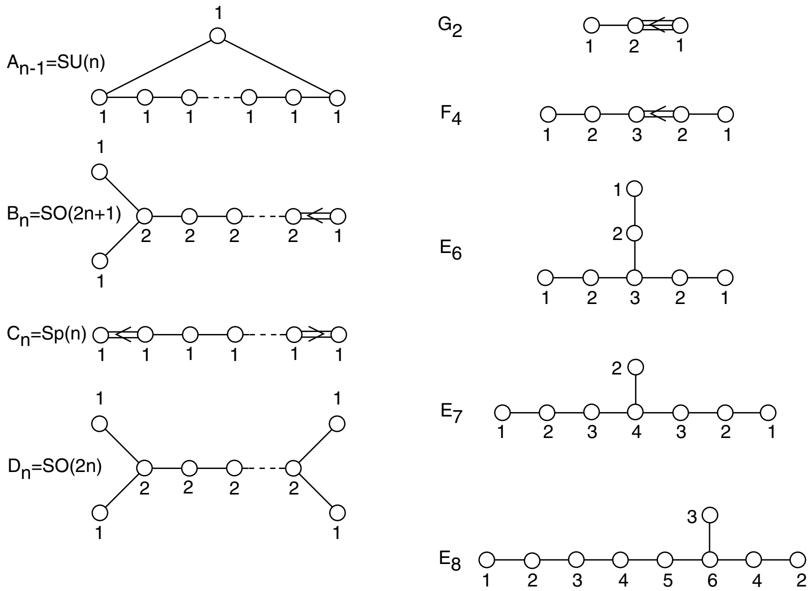
Figure 2: One can degenerate the $K3$ surface into two DP_9 surfaces glued along an elliptic curve, with non-abelian gauge symmetries localized at the crosses. In this limit one may compare with the $E_8 \times E_8$ heterotic string.

elliptic curve E [14,22–27].[1] On the heterotic side this corresponds to compactifying on an elliptic curve with the same complex structure as E and taking the limit where the volume of the T^2 goes to infinity. More details of this duality will be discussed in Section 3 after we review the construction of bundles on the heterotic side.

In the stable degeneration limit, we may choose good coordinates on the moduli space by unfolding a dP_9 surface with an E_8 singularity, keeping fixed

[1]The duality map is expected to receive various corrections away from this limit. Indeed, on the heterotic side T-dualities mix the bundle and geometric data for finite size T^2, so one cannot unambiguously reconstruct a geometry.

Table 4: Dynkin diagrams and Dynkin indices.



a canonical divisor E. We consider a degree six equation in $\mathbf{WP}^3_{(1,1,2,3)}$:

$$0 = y^2 + x^3 + \alpha_1 xyv + \alpha_2 x^2v^2 + \alpha_3 yv^3 + \alpha_4 xv^4 + \alpha_6 v^6 + p_i(v, x, y)u^i. \tag{2.17}$$

This is actually a dP_8 surface; one may obtain a dP_9 by blowing up the point $u = v = 0$. Intersection with the hyperplane $u = 0$ yields the elliptic curve E that we will keep fixed. The p_i, $i > 0$, are polynomials of degree $6 - i$ that describe the unfolding of the E_8 singularity, which lives at $v = x = y = 0$. As discovered in [21, 28], and further elucidated in [29–31], the coefficients in the p_i depend on the choice of a group H which will play a role similar to the holonomy group in the heterotic string.[2] Namely up to a change of variable they are parameterized by Looijenga's weighted projective space

$$\mathcal{M}_H = \mathbf{WP}^r_{s_0,\ldots,s_r}. \tag{2.18}$$

[2]The description using a weighted projective bundle in the following discussion does not quite apply to E_8 [23]. However, for all other cases except E_8 there is indeed a description by a weighted projective bundle.

The s_i are the Dynkin indices and are listed in table 4 (the non-simply laced cases will be relevant for compactifications to less than eight dimensions). This is of course also precisely the moduli space of flat H-bundles on T^2, which is how it will show up on the heterotic side. For instance for $H = SU(n)$, one has all $p_i = 0$ except

$$p_1 = v^{5-n}(a_0\, v^n + a_2\, xv^{n-2} + a_3\, yv^{n-3} + \cdots + a_n\, x^{n/2}), \tag{2.19}$$

(the last term being given by $yx^{(n-3)/2}$ when n is odd). Further dividing by the symmetry $u \to \lambda^{-1}u$, the coefficients a_j take values in

$$\mathcal{M}_{SU(n)} = \mathbf{WP}^n_{1,\dots,1}. \tag{2.20}$$

This set of deformations preserves a singularity corresponding to an enhanced gauge group G, which is the commutant[3] of H in E_8. Again consider the case $H = SU(n)$. If we turn off all the a_i for $i > 0$, then the geometry is of the form

$$y^2 = x^3 + xv^4 + v^6 + uv^5. \tag{2.21}$$

Near $x = y = v = 0$, we may drop the xv^4 and v^6 terms, and we get to leading order

$$y^2 = x^3 + v^5, \tag{2.22}$$

which is an E_8 singularity. On the other hand, suppose that we also turn on a_5, so that the geometry is of the form

$$y^2 = x^3 + xv^4 + v^6 + uv^5 + uxy. \tag{2.23}$$

After redefining $y \to y + \frac{1}{2}xu, x \to 2x$ and dropping subleading terms near $v = x = y = 0$, we get

$$y^2 = x^2 + v^5, \tag{2.24}$$

which is an $SU(5)$ singularity. Similarly in the intermediate cases we can get $SO(10), E_6, E_7$ singularities, which are the commutators of $SU(4), SU(3)$ and $SU(2)$ respectively.

[3]To be more precise, in eight dimensions we always have the $18 + 2\ U(1)$'s from expansion of C_3 along harmonic forms. On the heterotic side this arises because the holonomy group H on T^2 reduces to an abelian group, and so the commutator of H in E_8 contains extra $U(1)$'s. These extra $U(1)$'s are massive for generic compactifications below eight dimensions.

We can further fiber this degeneration over B_2, arriving at a stable degeneration of Y_4 into two dP_9 fibrations W_1, W_2 over B_2, glued along an elliptically fibered Calabi–Yau three-fold Z. We can write this as $Y_4 = W_1 \cup_Z W_2$, and Z will eventually be identified with the heterotic dual in the limit of large volume of the elliptic fiber. Then, $\{u, v, x, y\}$ can be taken as sections of $\{K_{B_2}^{-6}, \mathcal{N}, K_{B_2}^{-2} \otimes \mathcal{N}^2, K_{B_2}^{-3} \otimes \mathcal{N}^3\}$ respectively, where $\mathcal{N}$ is some sufficiently ample line bundle on B_2. The coefficients in equation (2.17) now become sections of line bundles over B_2 as well. However requiring an enhanced gauge group G over σ_{B_2} implies certain restrictions on these sections. Roughly speaking, just as requiring a singularity of type G on a dP_9 is equivalent to expressing the coefficients of (2.17) in terms of a reduced number of coefficients a_j, which take values $\mathcal{M}_H$, so is requiring a singularity of type G in W_1 along σ_{B_2} equivalent to expressing the coefficients of (2.17) in terms of a reduced number of sections a_j, such that $a_j(p)$ take values in $\mathcal{M}_H$ for any point p on the base B_2. Now G is allowed to be non-simply laced, and H is still the commutator of G in E_8. This is not an automatic consequence due to the issue of monodromies mentioned previously, however it turns out to be true anyways. The a_j are sections of the line bundles $\mathcal{N}^{s_j} \otimes K_{B_2}^{d_j}$. The d_j turn out to be precisely the degrees of the independent Casimirs of H (d_0 is taken to be zero), so the a_j should be thought of as the Casimirs of the adjoint field of the eight-dimensional gauge theory on the 7-branes.

So the upshot is that a dP_9 fibration W_1 with a fixed hyperplane section Z and a singularity of type G along the zero section is equivalent to a choice of the a_j, that is a choice of section $s : B_2 \to \mathcal{W}_H$ of the weighted projective bundle

$$\mathcal{W}_H = \mathbf{WP}\left(\mathcal{O} \oplus \bigoplus_{j>0} K_{B_2}^{d_j}\right), \tag{2.25}$$

where the weights are given by $a_j \to \lambda^{s_j} a_j$. The fiber of $\mathcal{W}_H \to B_2$ is given by $\mathcal{M}_H$. However the geometry W_1 specifies only part of the data of an F-theory compactification, because we are also allowed to turn on Wilson lines and fluxes along the 7-branes. That is, we can turn on periods of C_3 (which are typically trivial in a four-fold compactification however) and G-fluxes. This is called the 'twisting data' for the fibration [23] or 'Deligne cohomology'. It was first analyzed in the heterotic context in [24] and used in [14]. We will later return to the issue of which G-fluxes one is allowed to switch on for these geometries, after discussing how matter is engineered.

2.2 Charged chiral matter from intersecting branes

There are basically two ways to get charged chiral matter from 7-branes. In this section, we discuss intersecting 7-branes. Some properties of spinors and Dirac operators in complex geometry that will be heavily used in the following are collected in the appendices.

Given two 7-branes, with gauge bundles located on them, there will be massless matter from open string modes living on the intersection. The idea is very simple: when branes intersect at a small angle, we can think of them as a non-trivial field configuration in a gauge theory with an extended gauge group. The field content of a 7-brane is that of eight-dimensional maximally SUSY gauge theory. The fields consist of an eight-dimensional vector field, an adjoint valued complex scalar which is also sometimes called a Higgs field, and a Weyl spinor with R-charge $-1/2$. Let us first suppose that the eight-dimensional gauge theory has gauge group G. We turn on a constant adjoint VEV for the Higgs field, breaking G to a subgroup $H_1 \times H_2$. Let's suppose that the adjoint representation of G decomposes under $H_1 \times H_2$ as

$$R_{\text{adj}}(G) = \sum_a R_a(H_1) \otimes R'_a(H_2). \tag{2.26}$$

Then the fermions splits into the massless gauginos of H_1 and H_2 and massive fermions in the remaining representations appearing in (2.26). Geometrically this corresponds to separating the branes into two parallel stacks.

In order to describe intersecting branes, we need a slightly different configuration for the Higgs field. To set the notation, let us suppose we have a stack of branes wrapped on a complex surface with local coordinates z_1 and z_2. The holonomy on this surface will be denoted by $U(2)_S$. Further let us denote the local coordinate on the canonical bundle of S by z_3. The holonomy of the canonical bundle is contained in $U(1)_R$, and to preserve $N = 1$ supersymmetry this may be identified with $\det(U(2)_S)^{-2}$. Using separation of variables, the gaugino of the $8d$ gauge theory can be decomposed under $SO(3,1) \times U(2)_S \times U(1)_R$ as

$$\chi^{\pm}\psi_1^{\pm}\psi_2^{\pm}\psi_3^{\pm} \otimes \text{Ad}(\mathcal{G}). \tag{2.27}$$

Here we have introduced the following notations. $\mathcal{G}$ is the principle bundle over S with gauge group G, and ψ_i is a positive or negative chirality spinor for the z_i-plane. Recall from (2.10) that we can identify these spinors with forms, with $\psi_i^+ \sim 1$, $\psi_i^- \sim d\bar{z}_i$, and $d\bar{z}_3 \sim dz_1 \wedge dz_2$ using contraction with the holomorphic $(3,0)$-form, which is preserved by the holonomy. The spinors $\chi^{\pm}$ are $4d$ chiral/anti-chiral spinors. Furthermore the above spinor representation is reducible: only the states with an even number of pluses belong to the $8d$ gaugino.

In the non-degenerate case, we can describe a brane intersection as a configuration breaking a group G to a subgroup $H \times U(1)$. Therefore let us turn on an adjoint VEV which depends on the complex coordinate z_2 on the 7-brane, such that the gauge symmetry is restored as $z_2 \to 0$:

$$\Phi(z_1, z_2) = z_2\, T_{U(1)}\, d\bar{z}_3 \wedge, \tag{2.28}$$

where $T_{U(1)}$ is the Cartan generator breaking G to $H \times U(1)$. In other words, we consider a vortex configuration for the Higgs field Φ. It is well known that there are solutions of the Dirac equation localized on such a vortex, as we will now describe more explicitly. (In the language of Higgs bundles, we will construct a generator of the degree one hypercohomology of the Higgs bundle, localized at $z_2 = 0$).

To find the massless fermions, we use separation of variables to split the Dirac operator in a trivial six-dimensional part and a two-dimensional part, and then solve the Dirac equation on the z_2-plane with the z_2-dependent interaction term:

$$\begin{aligned} \bar{\partial}_{\bar{z}_2} \psi_2^+ \psi_3^- T^a + z_2 [T_{U(1)}, T^a] d\bar{z}_3 \wedge \psi_2^- \psi_3^+ &= 0, \\ \bar{\partial}_{\bar{z}_2} \psi_2^+ \psi_3^+ T^a - \bar{z}_2 [T_{U(1)}, T^a] \imath_{\bar{\partial}_{\bar{z}_3}} \psi_2^- \psi_3^- &= 0, \end{aligned} \tag{2.29}$$

as well as two more equations related by conjugation. By taking suitable complex linear combinations, we can always take T^a to be an eigenvector of $T_{U(1)}$, and we will do so in the remainder. The spinors $\psi_1^\pm, \chi^\pm$ are inert, and we only tensor with them in the end to get complete wave functions. Then we get two types of solutions. When the generator T^a commutes with $T_{U(1)}$, we get solutions which are holomorphic in z_2. They transform in the adjoint of H or $U(1)$ and their normalizability depends on more global information of the cycle on which the 7-branes are wrapped. They will be further discussed in the next subsection and we do not consider them here. Besides that, we may also take T^a to be a generator of the coset $G/(H \times U(1))$, with

$$[T_{U(1)}, T^a] = q_a\, T^a. \tag{2.30}$$

The precise normalization of the charge q_a is not so important here, only the sign is important. Then the second equation in (2.29) does not have normalizable solutions. The first equation in (2.29) on the other hand has solutions of the form

$$\begin{aligned}&\exp(-q_a z_2\bar{z}_2)(d\bar{z}_3 - d\bar{z}_2)T^a \quad \exp(+q_{\bar{a}} z_2\bar{z}_2)(d\bar{z}_3 + d\bar{z}_2)\overline{T^a}\\&\exp(+q_a z_2\bar{z}_2)(d\bar{z}_3 + d\bar{z}_2)T^a \quad \exp(-q_{\bar{a}} z_2\bar{z}_2)(d\bar{z}_3 - d\bar{z}_2)\overline{T^a}.\end{aligned} \tag{2.31}$$

When $q_a = -q_{\bar{a}}$ is positive, the solutions in the first line are normalizable but the solutions in the second line are not, hence should be thrown away. Conversely when q_a is negative, we throw away the solutions in the first line. In either case, we get zero modes with the shape of a gaussian localized at the core of the vortex, i.e., at the 7-brane intersection. Tensoring the normalizable solutions in (2.31) with constant modes of $\chi^\pm$ and $\psi_1^\pm$, subject to the constraint that we choose an even number of pluses in the superscripts, we therefore find fermion zero modes of the $8d$ gaugino charged in the off-diagonal representations appearing in (2.26), and localized at $z_2 = 0$, i.e., localized on the 7-brane intersection. These zero modes precisely fill out the fermionic content of a six-dimensional hypermultiplet: multiplying by $\chi^-\psi_1^+$, we get a chiral fermion transforming as T^a and another transforming as $\overline{T^a}$. Multiplying by $\chi^+\psi_1^-$, we get their CPT conjugates.

Therefore to find the massless open string spectrum living on the intersection of 7-branes, we simply have to know how the singularity of the elliptic fibration gets enhanced over the intersection locus of 7-branes, and decompose the corresponding adjoint representation. This procedure gives a generalization of the usual rules for intersecting 7-branes to the more general F-theory set-up [32].

Let us consider two examples, which are relevant for model building purposes. Suppose that we have an I_5 singularity corresponding to an $SU(5)$ gauge group, and we want to engineer matter by intersecting it with a matter brane. The minimal version, which does not introduce any extra gauge groups, is to add a locus of I_1 singularities. When the I_1 singularity intersects the I_5 singularity, it can get enhanced either to an I_6 singularity corresponding to an $SU(6)$ gauge group, or an I_1^* singularity corresponding to an $SO(10)$ gauge group. The adjoint representation of $SU(6)$ decomposes as

$$\mathbf{35} = \mathbf{24}_0 + \mathbf{5}_{-1} + \overline{\mathbf{5}}_1 + \mathbf{1}_0. \tag{2.32}$$

Thus we get a six-dimensional hypermultiplet in the fundamental of $SU(5)$ on the intersection locus with enhanced I_6. For the I_1^* enhancement, we use the decomposition

$$\mathbf{45} = \mathbf{24}_0 + \mathbf{10}_2 + \overline{\mathbf{10}}_{-2} + \mathbf{1}_0. \tag{2.33}$$

Therefore on this intersection locus we get a six-dimensional hypermultiplet in the **10** of $SU(5)$.

For the second example, consider an $SO(10)$ singularity enhanced to an E_6 singularity. Using the decomposition

$$\mathbf{78} = \mathbf{45}_0 + \mathbf{1}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3, \tag{2.34}$$

we see that we get a hypermultiplet in the **16** on the intersection.

In general, the fermions localized along the intersection of 7-branes further couple to the gauge bundles on the 7-branes. Again we can use separation of variables to analyze this question. Let us turn on a flux proportional to $T_{U(1)}$ (additional gauge fields could be turned on, but this will suffice for our purposes). This will give us a line bundle E supported along $z_3 = 0$ and a bundle F supported along $z_3 = \langle \Phi(z_2) \rangle$. Using (2.15), the actual gauge fields are then associated to fake bundles $\tilde{E} = E \otimes K_1^{-1/2}$ and $\tilde{F} = F \otimes K_2^{-1/2}$, where K_1 and K_2 are the canonical bundles of $z_3 = 0$ and $z_3 = \Phi(z_2)$ respectively.

Our solution is sharply localized in z_2 and z_3, so as long as the gauge field is smooth equations (2.29) only receive a small perturbation. The equations for $\psi_1^{\pm} T^a$ are modified to

$$\bar{\partial}_{\bar{z}_1} \psi_1^+ T^a + A_{\bar{z}_1} [T_{U(1)}, T^a] d\bar{z}_1 \wedge \psi_1^+ = 0 \tag{2.35}$$

and its conjugate. Now we may take $A_{\bar{z}_1}$ to be approximately independent of z_2, because the Higgs field still creates a potential localizing the solution near $z_2 = 0$. Further although we can locally pick $\psi_1^+ \sim 1$ and $\psi_1^- \sim d\bar{z}_1$, globally along the surface $\Sigma(z_1)$ parameterized by z_1 we should consider them to live in $K_\Sigma^{1/2}$ and $K_\Sigma^{-1/2}$. We see that ψ_1^+ and ψ_1^- behave as sections of

$$\psi_1^+ \in S_\Sigma^+ \otimes \tilde{E}^* \otimes \tilde{F}, \quad \psi_1^- \in S_\Sigma^- \otimes \tilde{E} \otimes \tilde{F}^*. \tag{2.36}$$

Then by the usual argument, the solutions to (2.35) and its conjugate are precisely counted by the Dolbeault cohomology groups

$$H^i(\Sigma, \mathcal{L}), \tag{2.37}$$

where

$$\mathcal{L} = R_a(\tilde{E}) \otimes R'_a(\tilde{F})|_\Sigma \otimes K_\Sigma^{1/2}. \tag{2.38}$$

Although $\mathcal{L}$ appears to contain various ill-defined bundles, one can always combine them into something sensible. In the case of mutually local branes, where R' and R correspond to the fundamental and anti-fundamental representation respectively, we can write

$$\begin{aligned}\mathcal{L} &= E^* \otimes F \otimes K_1^{1/2} \otimes K_2^{-1/2} \otimes K_\Sigma^{1/2}|_\Sigma \\ &= E^* \otimes F \otimes K_1|_\Sigma.\end{aligned} \tag{2.39}$$

Each solution of (2.35) must be dressed up with a solution of (2.29) in order to construct the full wave function, which is a zero mode of the $8d$ gaugino (2.27). Due to the constraints of normalizability and the number of plus signs in (2.27) being even, there is in fact a unique way to dress up each solution of (2.35) to a zero mode of the $8d$ gaugino, so the solutions are simply counted by (2.37). This concludes our derivation of the spectrum of fermion zero modes on 7-brane intersections.

Let us now examine the wave functions we have found in some more detail. An important point is that the degree i correlates with the four-dimensional chirality. To see this, note that a generator of the Dolbeault cohomology group $H^0(\Sigma, \mathcal{L})$ (i.e., a zero mode of the form $\psi_1^+ T^a$) yields a zero mode of the eight-dimensional Dirac operator of the form

$$\chi^- \psi_1^+ (\psi_2^+ \psi_3^- + \psi_2^- \psi_3^+) T^a \sim \chi^- \Phi^{2,0} T^a + \chi^- A^{0,1} T^a, \tag{2.40}$$

whereas a generator of $H^1(\Sigma, \mathcal{L})$ (i.e., a zero mode $\psi_1^- T^a$) yields a zero mode of the form

$$\chi^+ \psi_1^- (\psi_2^+ \psi_3^- + \psi_2^- \psi_3^+) T^a \sim \chi^+ \overline{A^{0,1}}\, T^a + \chi^+ \overline{\Phi^{2,0}}\, T^a. \tag{2.41}$$

Equivalently, it yields a chiral fermion transforming in the conjugate representation $\overline{T^a}$. At any rate, from the constraints of normalizability and the total number of plus signs being even, we see that the degree of the cohomology group and the $4d$ chirality are correlated, so that $i = 0$ corresponds to a chiral fermion and $i = 1$ corresponds to an anti-chiral fermion.[4] Further note that Serre duality maps a zero mode with $i = 0$ to a zero mode with opposite charges and $i = 1$, i.e., the opposite chirality for the four-dimensional chiral fermion. Because we started with a single Weyl spinor in eight dimensions, this means that generators related by Serre duality do not correspond to independent four-dimensional fields, but to fields related by CPT conjugation.

[4]Note incidentally that if there had been normalizable solutions near the intersection of the form $\psi_2^+\psi_3^+$ or $\psi_2^-\psi_3^-$, we would find zero modes of the form $\chi^+\psi_1^+\psi_2^+\psi_3^+T^a$ or $\chi^-\psi_1^-\psi_2^-\psi_3^-T^a$ on the intersection. These correspond to symmetries rather than deformations, and in the present context would be interpreted as ghosts [33]. Fortunately, we see that we cannot get them for intersecting branes.

Now that we understand the chirality of our zero modes, we can check when we can get a non-zero net chirality. For this purpose we can use the index theorem:

$$\begin{aligned} h^0(\Sigma, \mathcal{L}) - h^1(\Sigma, \mathcal{L}) &= \int_\Sigma \mathbf{ch}(\mathcal{L}) \wedge \mathbf{Todd}(T\Sigma) \\ &= \int_\Sigma c_1(\tilde{F}) - c_1(\tilde{E}). \end{aligned} \tag{2.42}$$

Thus the introduction of fluxes on the 7-branes is precisely what is needed to make the hypermultiplets on the intersection chiral. Further the net chirality is precisely given by the amount of flux through the intersection. In the type IIb limit, this agrees precisely with the known answer, which can be deduced e.g., from anomaly inflow arguments [34, 35] or from quantization of open strings stretching between D-branes [36]. In fact our notation for the spinors was chosen to emphasize the similarity with open string quantization.

Finally, we would like to explain how this is related to integrals of the G-flux. To compute the chiral spectrum we do not actually need E and F separately. All we actually need is the combination $R(\tilde{E}) \otimes R'(\tilde{F})|_\Sigma$. To be concrete let's discuss the case of $SU(5)$ gauge symmetry with matter in the 10 and $\bar{5}$. Consider first the local geometry for an intersecting I_5 and I_1 locus which gets enhanced to I_6. As we discussed, there is a vanishing (anti-)holomorphic curve on top of Σ for each weight of the matter representation associated to it. Let us assume that we have not turned on any holonomy for the $SU(5)$ gauge field so that the group is unbroken (if not, the procedure we will explain can be generalized by using all the vanishing curves instead of just one). Then the G-flux close to the intersection is of the form

$$\mathsf{G} \sim \mathsf{F}_1 \wedge \omega_1 + \mathsf{F}_2 \wedge \omega_2, \tag{2.43}$$

where $\omega_{1,2}$ are the two non-normalizable harmonic two-forms associated to the overall $U(1)$'s for the I_5 locus and I_1 locus respectively. The $U(1)$'s may not appear in the low energy theory, but a linear combination may correspond to a massive $U(1)$ and still appear in the G-flux as we will see in a later subsection. Since the $U(1)$ charges of the BPS states associated to the vanishing curves are given by ± 1, we have

$$\int_C \omega_1 = +1, \quad \int_C \omega_2 = -1, \tag{2.44}$$

and we can integrate the G-flux over a vanishing curve to obtain

$$\int_C \mathsf{G} = \mathsf{F}_1 - \mathsf{F}_2, \tag{2.45}$$

which we interpret as the curvature of $\tilde{E}^* \otimes \tilde{F}$. By further integrating over Σ and using (2.42), we get the net number of generations in the **5**.

The other case is when the singularity is enhanced to I_1^* along the intersection of an I_5 and I_1 locus. This is not a transversal intersection in B_3, however our arguments do not depend on the form of this intersection[5] , but rather on the intersection in the spectral cover picture. Again let us assume unbroken $SU(5)$ symmetry. Then we can pick one of the extra vanishing curves C over the intersection, and we obtain

$$\int_C \mathsf{G} = 2\mathsf{F}_1 - 2\mathsf{F}_2. \tag{2.46}$$

The reason for the factor of two is because in the decomposition of the I_1^* singularity into individual (p, q) 7-branes [20], the BPS junction representing C has two ends on the I_5 locus and one end on each of the two extra I_1-singularities, i.e., it has charge two under each of the two $U(1)$'s. Further integrating over $\Sigma_{\mathbf{10}}$ and using (2.42), we get the net number of generations in the **10**.

2.3 Charged chiral matter from coincident branes

There is a second way to get charged chiral matter, by considering coincident 7-branes rather than intersecting 7-branes. The reasoning is similar. We take a 7-brane with a non-abelian gauge symmetry wrapping a four-cycle B_2. So far we assumed that only fluxes on the matter brane were turned on, so as not to break any additional gauge symmetry on the gauge brane. However, we can also turn on generally non-abelian holonomy on the worldvolume of the gauge brane. This corresponds on the heterotic side to turning on a bundle on the trivial part of the spectral cover. For example, suppose we have an E_6 gauge symmetry along B_2 and we turn on a $U(1)$ bundle E so that the commutant in E_6 is given by $SO(10) \times U(1)$. The $U(1)$ gauge field will become massive by eating a closed string axion.[6] We decompose the

[5] The local form of many collisions has been worked out in [37].

[6] Schematically this lifting arises as follows. From the Chern–Simons couplings $\int C_{(4)} \wedge F \wedge F = -\int dC_{(4)} \wedge \omega_3(A)$ on the 7-brane we deduce the existence of a term $\int (*_8 dC_{(4)} - \omega_3(A))^2$. Then if we turn on a line bundle on B_2 with first Chern class $c_1(E)$ and denote

adjoint representation of E_6 under $SO(10) \times U(1)$ as

$$R_{\text{adj}}(E_6) = \mathbf{78} = \sum_a R'_a \otimes R_a = \mathbf{45}_0 + \mathbf{1}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3. \tag{2.47}$$

Then chiral matter transforming in the R'_a representation of $SO(10)$ is given by zero modes of the eight-dimensional Dirac equation. Using the spinor bundles in (2.10), we get a four-dimensional fermion for every generator of the cohomology groups

$$H^i(B_2, R_a(E) \otimes K_{B_2}) \oplus H^i(B_2, R_a(E)), \tag{2.48}$$

for $i = 0, 1, 2$.[7] As usual, generators related by Serre duality are CPT conjugates, rather than independent fields. In the above example, we would have

$$\begin{aligned} N_\chi(\mathbf{16}) &= h^0(B_2, L^{-3} \otimes K_{B_2}) + h^1(B_2, L^{-3}), \\ N_\chi(\overline{\mathbf{16}}) &= h^0(B_2, L^3 \otimes K_{B_2}) + h^1(B_2, L^3), \end{aligned} \tag{2.49}$$

where L is the line bundle corresponding to the $U(1)$ gauge field we turned on. These chiral fields clearly correspond to 7-brane deformations and gauge field deformations respectively, and their Serre duals are the corresponding anti-chiral fields. Generators of $H^0(B_2, L^3)$ or $H^0(B_2, L^{-3})$ do not correspond to deformations at all, but to symmetries. If these cohomology groups are non-zero, the compactification has ghosts and is inconsistent.

In addition to this spectrum, we must find the massless matter representations of E_6 originating from the intersection with other 7-branes using the procedure we explained before, and add them to the spectrum. An amusing feature is that this may effectively increase the number of generations obtained from the intersection with the matter brane. For instance, if we broke E_6 to a group G using an $SU(3)$ bundle, then from Higgsing it is clear that the number of generations of G obtained from the intersection is three times the number of generations of E_6. A very similar mechanism was used

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the dual four-form as α_4, the $U(1)$ gauge field will have a four-dimensional coupling of the form $\int (A_\mu - \partial_\mu a)^2$, where a is the RR axion obtained by expanding $C_{(4)}$ along α_4. This is completely analogous to a similar mechanism in the heterotic string.

[7] More generally, if we also have a non-zero VEV for Φ, we should solve equations of type $\bar{\partial}_A\, \psi_1^+ \psi_2^+ \psi_3^- + [\Phi, a_1 \psi_1^+ \psi_2^- \psi_3^+ + a_2 \psi_1^- \psi_2^+ \psi_3^+] = 0$. That is, we have a spectral sequence with $E_2^{p,q} = H^p(B_2, R_a(E) \otimes K_{B_2}^q)$, horizontal differential $E_2^{p,q} \to E_2^{p+1,q}$ given by $\bar{\partial} + A^{0,1}$, and vertical differential $E_2^{p,0} \to E_2^{p,1}$ given by $\Phi^{2,0}$. But when the d_2 differential $E_2^{0,1} \to E_2^{2,0}$ of this spectral sequence is zero, then we have $E_2 = E_\infty$ and we still get (2.48).

in [2] to obtain the three generation MSSM from a one generation model with an extended gauge group.

We can also ask about matter in real representations. We have already seen that four-dimensional adjoint-valued chiral fields come from $h^{0,1}(B_2)$ and $h^{2,0}(B_2)$. If we turn on a non-abelian bundle M on σ_{B_2}, we can ask for the number of bundle moduli. This is given by the number of zero modes of the Dolbeault operator acting on

$$\mathrm{Ad}(M) \otimes \Omega^{0,1}(K_{B_2}^{1/2}) \otimes K_{B_2}^{-1/2}, \tag{2.50}$$

with the last factor accounting for the R-charge. Thus it is given by the number of generators of $H^1(B_2, \mathrm{Ad}(M))$, in agreement with the heterotic result [25].

Finally, we have to give a prescription for relating line bundles on B_2 and G-flux. This is easy for coincident branes, the G-flux is simply of the form $\mathrm{Tr}(F/2\pi) \wedge \omega$ where $\omega \in \Lambda$.

2.4 Yukawa couplings

The form of the SUSY Yukawa couplings can be deduced from the reduction of the interaction terms in the ten-dimensional Yang–Mills action (B.1). Schematically they are given by

$$\int d^2\theta \, d^4x \, \mathrm{Tr}(\Phi_1 \Phi_2 \Phi_3) \int_S \mathrm{Tr}(\varphi_1 \xi_2 \xi_3), \tag{2.51}$$

where φ_i, ξ_i denote bosonic and fermionic zero modes on the 7-branes. Let us discuss the various special cases.

For coincident branes, the chiral fields came from generators of the form $A^{0,1}$ or $\Phi^{2,0}$. We can compose two generators of type $(0,1)$ and one of type $(2,0)$ to get a number:

$$\int d_{abc} \, A^a \wedge A^b \wedge \Phi^c. \tag{2.52}$$

This is just the cubic piece of the holomorphic Chern–Simons action for 7-branes. In fact, if we use the mode expansions for C_3 (2.8) and the complex structure moduli (2.9), it can also be interpreted as a non-abelian generalization of the flux superpotential (2.11). A similar coupling for matter in real representations was already discussed in [25]. We see that it holds more generally provided the three-fold tensor product of the group indices contains a singlet. (In the language of Higgs bundles, this corresponds to the Yoneda pairing on the hypercohomology of the Higgs bundle).

Next let us consider intersecting 7-branes. As we discussed around (2.29), chiral fermions living on the intersection Σ must be interpreted as zero modes of the $8d$ gaugino by dressing them with the normalizable wavefunctions for $\psi_2^+\psi_3^-$ and $\psi_2^-\psi_3^+$. Furthermore as we also discussed earlier, because of supersymmetry each such fermionic zero mode of the $8d$ gaugino is paired with a bosonic zero mode of the $8d$ fields $A^{0,1}$ and $\Phi^{2,0}$ with the same internal wave function. Concretely we have the following dictionary:

$$\begin{array}{llll} \chi^-\psi_1^+\psi_2^+\psi_3^- & \to \chi^-\Phi^{2,0} & \chi^-\psi_1^+\psi_2^-\psi_3^+ & \to \chi^- A^{0,1} \\ \chi^+\psi_1^-\psi_2^+\psi_3^- & \to \chi^+\overline{A^{0,1}} & \chi^+\psi_1^-\psi_2^-\psi_3^+ & \to \chi^+\overline{\Phi^{2,0}}. \end{array} \tag{2.53}$$

Then clearly the Yukawa couplings are given by the same cubic interaction inherited from the $8d$ gauge theory (2.52). The same formula also applies for the overlap of zero modes which are localized around the intersection with zero modes which are spread over all of the 7-branes. Further, it is not hard to see that the Yukawa couplings only depend on the Dolbeault cohomology classes of the zero modes and not on the explicit representatives, so the Yukawa coupling corresponds to a natural Yoneda product on the Dolbeault cohomology. By choosing manifestly holomorphic representatives of the Dolbeault cohomology classes of the zero modes, we can relate the integral (2.52) to a purely holomorphic computation at the intersection of the supports of the three zero modes, without ever doing any integrals. This is of course very analogous to computing a tree level three-point function in the heterotic string, which also corresponds to a Yoneda product on Dolbeault cohomology, and we will see later that it is in fact dual to it.

This picture implies some intriguing results. Suppose for instance we have an E_6 gauge group locally obtained from E_8 by turning on two abelian adjoint fields. The locally we have two additional unbroken $U(1)$'s, call them $U(1)_a \times U(1)_b$, which may be broken in another patch. From the decomposition of the adjoint of E_8 we get three copies of the **27** with charges:

$$\mathbf{27}_{(1,0)} + \mathbf{27}_{(0,1)} + \mathbf{27}_{(-1,-1)}. \tag{2.54}$$

Let us say that the adjoint field Φ_a vanishes for $z_1 = 0$, and Φ_b vanishes for $z_2 = 0$, and we let ψ_i be the spinor for the z_i-plane as before. Then we could have chiral fields $A_{\mathbf{27}_{(1,0)}} \sim \psi_1^-\psi_2^+\psi_3^+ \otimes \mathbf{27}_{(1,0)}$ propagating on $z_1 = 0$, $A_{\mathbf{27}_{(0,1)}} \sim \psi_1^+\psi_2^-\psi_3^+ \otimes \mathbf{27}_{(0,1)}$ propagating on $z_2 = 0$, and $\Phi_{\mathbf{27}_{(-1,-1)}} \sim \psi_1^+ \psi_2^+\psi_3^- \otimes \mathbf{27}_{(-1,-1)}$ propagating on $z_1 = z_2$. Then from the intersection $z_1 =$

$z_2 = 0$ we get a coupling of the form

$$\int \mathrm{Tr}(A_{\mathbf{27}_{(1,0)}} \wedge A_{\mathbf{27}_{(0,1)}} \wedge \Phi_{\mathbf{27}_{(-1,-1)}}). \tag{2.55}$$

Note that the indices of the forms and the gauge indices are precisely right to allow for a non-zero contribution. More precisely, the full wave function of a zero mode (2.40) is a linear combination of $A^{0,1}\,T^a$ and $\Phi^{2,0}\,T^a$, so we get a sum of terms of the above form with the gauge indices permuted. Hence we see that we could engineer certain interactions by playing with the matter curves, their intersections and the localization of the chiral fields in the extra dimensions. Similar ideas have played important roles in the phenomenology literature on extra-dimensional models (see e.g., [38] and references thereto).

As another example, suppose we have an $SO(10)$ gauge group on B_2, and chiral matter on $\Sigma_{\mathbf{16}}$ and $\Sigma_{\mathbf{10}}$. The Yukawa coupling for $\mathbf{16} \times \mathbf{16} \times \mathbf{10}$ clearly gets localized on $\Sigma_{\mathbf{16}} \cap \Sigma_{\mathbf{10}}$. Suppose that locally near such an intersection we can think of the $SO(10)$ as arising from an E_7 gauge group which is broken by two abelian adjoint fields Φ_a and Φ_b. Then from decomposing the adjoint of E_7 we get the following representations:

$$\mathbf{16}_{(-2,1)} + \mathbf{16}_{(0,-3)} + \mathbf{10}_{(2,2)} + \mathbf{1}_{(-2,4)}. \tag{2.56}$$

Again much like above we get a contribution to the Yukawa coupling of the form

$$\int \mathrm{Tr}(\Phi_{\mathbf{16}_{(-2,1)}} \wedge A_{\mathbf{16}_{(0,3)}} \wedge A_{\mathbf{10}_{(2,2)}}) \tag{2.57}$$

from intersection points of $\Sigma_{\mathbf{16}} \cap \Sigma_{\mathbf{10}}$. Again we may envisage geometric configurations that explain hierarchies in the Yukawa couplings.

In the above examples we described the Yukawa couplings on intersections of matter curves by configurations of two abelian adjoint fields Φ_a and Φ_b. This cannot always be done, it is more generic actually that the Yukawa couplings are described by a non-abelian adjoint field. Such more general configurations are outside the scope of this paper, however in principle the resulting Yukawa couplings still descend from the master formula (2.52).

There are several other phenomenological scenarios that depend on localization in the extra dimensions, and that can in principle be implemented in F-theory. Localization can be helpful in suppressing dangerous higher dimensions operators such as $\int d^2\theta\, QQQL$ [38]. It also provides scenarios for mediation of supersymmetry breaking, such as gaugino mediation [39,40]. For a review of some of the possibilities of extra-dimensional models, see [41].

Let us elaborate a bit on the remark above, that the Yukawa coupling can be localized exactly at the intersection of matter curves. This is a crucial difference between some of the old phenomenological scenarios and the F-theory models. We have seen that when the $8d$ gauge theory approximation is valid, localized wave functions approximately possess a gaussian tail, see e.g., equation (2.31). In the phenomenology literature cited above, if matter fields are localized at different positions, one imagines getting small but non-vanishing Yukawa couplings from the overlap of the tails, and this is an important ingredient for explaining hierarchies in the couplings (see e.g., [38]).

However as we already pointed out above, due to supersymmetry we do not have to estimate the overlap of the wave functions, we can calculate it exactly as a Yoneda pairing on Dolbeault cohomology. More precisely, we can calculate superpotential couplings exactly up to wave-function renormalization. The reason is familiar from type IIb and heterotic models: the Yukawa couplings as well as other F-term data depend only on the Dolbeault cohomology classes of the wave functions, but not on the explicit representatives of the Dolbeault cohomology classes. To compute the Kähler potential we need the physical wave function, i.e., the harmonic representative. But for the purpose of computing superpotential couplings we may instead choose a different representative. In order to do the computation exactly, it is more natural to choose a holomorphic representative, or we could use a completely different model for the cohomology, like Cech cohomology.[8]

Even better, if we are on an algebraic variety, as is the case for all the examples we consider, then the computation of the superpotential is equivalent to a purely algebraic computation. In the holomorphic world, the computation is reduced to a purely local calculation at the intersection. (In terms of the spectral cover, we can easily see this localization using the formulation in terms of Ext groups).

In any case, no overlap integrals need to ever be carried out to compute superpotential terms. The gaussian tails of the harmonic representatives have explicit anti-holomorphic dependence, and therefore cannot play

[8]It is helpful to consider de Rham cohomology by analogy. A de Rham cohomology class is uniquely specified by its periods on cycles (i.e., by the Poincaré duality pairing), so instead of taking the harmonic representative we may take a delta-function current localized on a Poincaré dual cycle. This representative is clearly independent of the metric and makes the localization of the triple product (i.e., the Yukawa couplings) manifest.

a role in the computation of superpotential couplings. Indeed, the anti-holomorphic dependence in the tails can simply be removed by a complexified gauge transformation. If the algebraic supports of the wave-functions do not intersect, then the overlap integral is simply zero. Thus the localization of wave functions is a much sharper effect in the presence of supersymmetry than in the old phenomenological models with extra dimensions.

It follows that just as in type IIb or the heterotic string, properties of the Yukawa matrices that are independent of wave-function renormalization, such as the rank of the matrix and texture zeroes, can be determined exactly by a purely holomorphic (and often even algebraic) computation. Furthermore, by the standard non-renormalization argument, there are no perturbative corrections to the superpotential. To see this, we define the complexified Kähler moduli by

$$T_a = \frac{m_{10}^4}{2}\int_{\mathrm{D}_a} J\wedge J - \mathrm{i}\int_{D_a} C_4. \tag{2.58}$$

The volume of the four-cycle is measured in ten-dimensional Planck units. The superpotential depends only holomorphically on the Kähler moduli. Then because of the shift symmetry in the imaginary part, T_{a} can only appear in an exponential, so the superpotential is only corrected non-perturbatively in the Kähler moduli. Such non-perturbative corrections can be interpreted as $D3$-instantons. We will give some additional discussion on such corrections in Section 3.4.

2.5 D-terms

Up to now we have discussed purely holomorphic properties of F-theory. We also have to say something about the D-term constraints. In smooth compactifications, one requires that the G-flux must be J-primitive:

$$J\wedge \mathsf{G} = 0. \tag{2.59}$$

Decomposing $\mathsf{G}\sim \mathsf{F}\wedge\omega$, this condition is reminiscent of

$$i^*J\wedge \mathsf{F} = 0 \tag{2.60}$$

i.e., the standard D-term equation on a smooth abelian 7-brane. In singular compactifications, or in the limit that the fiber is small compared to the base, the non-abelian degrees of freedom are light and we should use the non-abelian Hitchin's equations. Further we should make sure that all the fermion zero modes that parametrize symmetries correspond to gauginos. If not then the compactification has ghosts and is inconsistent [33].

Let us specialize to the $K3$ fibrations over B_2 which are dual to the heterotic string. Then the available Kähler forms are

$$J_{B_3} = t_1 \pi^* J_{B_2} + t_2 J_0, \tag{2.61}$$

where J_0 is the Poincaré dual of the zero section σ_{B_2}. Recall that F-theory is essentially ten-dimensional supergravity coupled to eight-dimensional Yang–Mills theory at certain singularities. Therefore it does not care about g_s and ℓ_s separately, but only about the combination given by the ten-dimensional Planck length $l_{10}^4 = g_s \ell_s^4$ and the Planck mass $m_{10} = 1/l_{10}$. In particular, in F-theory volumes must be measured in ten-dimensional Planck units, because these units are invariant under the S-duality transformations that we must apply across branch cuts of the axio-dilaton on the IIb space-time. For F-theory to be valid, both the volume of B_2 and the $\mathbf{P}^1$-base of the $K3$ should be large in the ten-dimensional Einstein frame.

On the other hand, the heterotic coupling is identified with the volume of the $\mathbf{P}^1$. To see this, a $D3$-brane wrapped on the base of the elliptically fibered $K3$ gets mapped to the fundamental string of the heterotic theory compactified on T^2. Its tension is

$$T \sim l_8^{-2} (V_{\mathbf{P}^1})^{2/3}, \quad T \sim l_8^{-2} \lambda_8^{2/3} \tag{2.62}$$

on the F-theory side and on the heterotic side respectively, where l_8 is the effective eight-dimensional Planck length, $V_{\mathbf{P}^1}$ is measured in ten-dimensional Planck units, and λ_8 is the eight-dimensional heterotic string coupling. Thus we find that $\lambda_8 = V_{\mathbf{P}^1}$ up to numerical factors. As expected, F-theory and the heterotic string have non-overlapping regimes of validity. In particular it is possible that heterotic constructions that were previously discarded correspond to valid F-theory compactifications.

Further, using the conventions of [36], in general F-theory compactifications we have

$$\alpha_{\rm GUT} = \frac{g_{\rm GUT}^2}{4\pi} = \frac{g_8^2}{4\pi V_{B_2}} = \frac{l_{10}^4}{V_{B_2}}, \tag{2.63}$$

$$G_N = \frac{1}{8\pi M_{Pl,4}^2} = \frac{2\kappa^2}{16\pi V_{B_3}} = \frac{\alpha_{\rm GUT}^2 V_{B_2}^2}{8(2\pi)^2 V_{B_3}}, \tag{2.64}$$

where we defined $l_{10}^8 = g_s^2 \ell_s^8$ to be the ten-dimensional Planck length. We also expect that

$$M_{\mathrm{GUT}} \sim V_{B_2}^{-1/4} \tag{2.65}$$

modulo threshold corrections. Note that in contrast to the heterotic string, it is now in principle possible to take $M_{\mathrm{GUT}}/M_{Pl,4}$ to zero while keeping α_{GUT} finite, since this only requires the ratio $V_{B_2}^{3/2}/V_{B_3}$ to go to zero. This is one of our main motivations for local models as we emphasized in the introduction.

Now let's consider the available G-fluxes for dP_9 fibrations over B_2. We could turn on fluxes for the Cartan generators of the non-abelian gauge group localized on σ_{B_2}. As we discussed in the context of coincident branes, this would partially break the gauge symmetry. One may consider this as a mechanism for breaking the GUT group to the Standard Model gauge group. However for testing our formula for chiral matter in F-theory by comparing with the heterotic string, we will also be interested in compactifications where such fluxes are not turned on. Generically, the remainder of the discriminant locus

$$\Delta' = \Delta - n[\sigma_{B_2}] \tag{2.66}$$

is an I_1 locus and does not generate a massless four-dimensional $U(1)$ vector multiplet, due to non-normalizability of the associated local harmonic two-form[9] , so it may seem at first sight that there are no other fluxes we could turn on. Said differently, a supersymmetric G-flux must be of type $(2,2)$ and rational, and therefore it is Poincaré dual to (a rational linear combination of) algebraic cycles in Y_4. At first sight, it seems that generically the only available algebraic cycles must be either (a) divisors in $\sigma(B_3)$ or (b) curves in $\sigma(B_3)$ with the elliptic fibration on top. Both of these two types of fluxes are turned off in the limit from M-theory to F-theory, hence do not exist in F-theory, and therefore generically it would seem that there are no supersymmetric fluxes available.

However the heterotic/F-theory duality map which we will discuss in Section 3 allows us to map the question of G-fluxes to an equivalent but more transparent problem. Namely, we get an isomorphism between G-fluxes and certain fluxes arising in the heterotic construction, with the explicit mapping being the push–pull formula known as the 'projected cylinder map' given in

[9] For duality with the heterotic string, we also assume that real codimension two singularities of the elliptic fibration are not localized on B_2, as this would correspond to a non-perturbative gauge symmetry on the heterotic side.

Appendix C. On the heterotic side it will be clear that even for generic values of the moduli, there is always an additional rank one lattice of holomorphic quantized fluxes, generated by a flux we will call G_γ. Moreover, for non-generic values of the moduli there will be additional G-fluxes besides this generic rank one lattice. (They were called the Noether–Lefschetz fluxes in [5]). The number of allowed linearly independent fluxes is given by the rank of the homology lattice $H^2(C, \mathbf{Z})$ of the spectral surface minus the rank of the homology lattice of the base surface. By tuning the complex structure moduli, these fluxes may also become of type $(1, 1)$.

A rough counting of the rank in a typical model, even requiring three generations, is obtained as follows. We use Riemann–Hurwitz to relate the Euler character of the spectral cover C to the Euler character of the base B_2. By Lefschetz, the odd cohomologies of C tend to vanish, and it is evident that the main contribution to $H^2(C, \mathbf{Z})$ comes from the Euler character of the branch locus of the spectral cover. Imposing constraints of three generations and vanishing c_1 of the spectral line bundle only changes the number by order one. Taking some simple degree five spectral covers over $B_2 = \mathbf{CP}^2$, and using the degree/genus formula to relate the degree of the discriminant to the Euler character, we get numbers of order 10^3 to 10^5. So there seems to be quite some flexibility in local models.

Following [42, 43], the number of supersymmetric solutions is then estimated to be given roughly by the volume of a ball in Euclidean space with dimension r given by the rank of this flux lattice, and a radius $\sqrt{2}\Lambda^{1/2}$ that is set by global tadpole cancellation. It is estimated (and to some extent verified in toy models) that for a generic such flux, there is an $\mathcal{O}(1)$ number of stable solutions. Thus the number of solutions is roughly given by

$$\#\,\text{solutions} \simeq \frac{1}{(r/2)!}\,(2\pi\Lambda)^{\mathrm{r}/2} \simeq \left(\frac{4\pi \mathrm{e}\Lambda}{r}\right)^{r/2}. \tag{2.67}$$

The tadpole cut-off Λ is of order $\Lambda \sim \chi(Y_4)/24 \gtrsim r/24$. It can be larger because in r we only included the contribution to rank of the lattice due to the visible sector. Even with conservative values, this can easily lead to the order of 10^{1000} possible solutions in our local F-theory models. In heterotic language, this is a rough estimate for the number of solutions to the hermitian Yang–Mills equations on a rank five bundle on an elliptically fibered Calabi–Yau with three generations. Clearly, the 'universal' flux lattice generated by G_γ is a rather special sublattice.

At any rate, let us here focus on the special flux G_γ. By construction it is guaranteed to be of Hodge type $(2, 2)$ and rational for general values of the moduli. However it is not a priori clear that G_γ is also J-primitive for some Kähler class J inside the Kähler cone. In Appendix C, we analyze conditions under which G_γ is J-primitive. From the analysis it follows that under certain assumptions, G_γ is actually J-primitive for all available Kähler classes J:

$$\pi^* J_{B_2} \wedge G_\gamma = 0, \quad J_0 \wedge G_\gamma = 0. \tag{2.68}$$

The explanation for this is essentially that for generic moduli, the light gauge fields sit in an unbroken non-abelian group and so there are no Fayet–Iliopoulos parameters. Moreover, we will see in Section 3 that the computation of the spectrum calculated on the F-theory side using the gauge theory approach agrees exactly with the heterotic calculation. In the case of $SU(5)_{\mathrm{GUT}}$ models, the net number of generations in the presence of the flux $\lambda\, G_\gamma$ is given by

$$N_{gen} = \lambda \int_{\Sigma_{\mathbf{10}}} \gamma = -\lambda(6c_1 - t)\cdot(c_1 - t), \tag{2.69}$$

where $\lambda \in \mathbf{Z} + \frac{1}{2}$ due to quantization constraints. We will use these fluxes to construct some simple F-theoretic GUT models in Section 4.

2.6 Soliton quantization

Our derivations have mostly relied on the field theory approach. The other idea we could have tried to use is to resolve the singularities. Physically speaking, this corresponds to compactifying to three dimensions. The $4d$ vector multiplet then gains a pseudo-scalar corresponding to a Wilson line around the extra S^1, and its expectation value parametrizes a Coulomb branch. Going out on this $3d$ Coulomb branch corresponds geometrically to the resolution mentioned above. When the resolved cycles are large, the BPS $M2$-branes wrapped on them are heavy and we can quantize them as solitons, as in [44]. We can use this for example to rederive the localized matter at loci of enhanced gauge symmetry: assuming the background C_3 is trivial, one tends to get one resolved cycle for every pair of generators of $G/H \times U(1)$, and quantizing a wrapped $M2$-brane on each such cycle yields a half-hypermultiplet. In fact, we could even use this to rederive our formulae for charged chiral matter. The basic idea is that in the limit that the $M2$-branes are heavy, the dynamics is reduced to that of charged particles moving in a magnetic field on the moduli space of the $M2$-brane, here given by the matter curve. Quantizing such $M2$-branes leads to the Dolbeault cohomology groups on the matter curve discussed previously. This approach is to some extent implicit in our discussion at the end of Section 2.2.

The field theory (or Higgs bundle) approach and the soliton approach have different ranges of validity. The soliton approach requires us to go to M-theory and resolve the singularities, where we get a quantum mechanics problem of massive BPS particles. However when the resolved cycles are small, the Compton wave-length of the BPS particles is large and we cannot use this approach. The field theory approach instead smooths the singularities not by resolving, but by incorporating the non-abelian degrees of freedom in the effective action. Here we are writing equations on a different mathematical object (a Higgs bundle), rather than on the F-theory Calabi–Yau itself. But we can recover the singular Calabi–Yau using spectral cover techniques, in particular the cylinder map. This approach is justified when the Higgs field is slowly varying. In brane language, this is the limit where the angles between intersecting branes are small.

As long as Kähler and complex structure moduli are decoupled, both approaches should give the same answer at the level of F-terms. The ability to compare the different approaches gives very useful cross-checks on the results. But for certain questions this decoupling may fail. One example is the issue of poly-stability. Indeed, wall-crossing in the Hitchin system is well studied and it is known that there tend to be multiple chambers. In contrast, such chambers have not been observed in the M-theory approach. The reason is that the flux is required to satisfy $J \wedge G = 0$ in the smooth M-theory regime, which is a closed condition. Thus if we had a sequence of Kähler moduli J_i accumulating to J_∞, such that $G \wedge J_i = 0$ for each i, then we also have $G \wedge J_\infty = 0$, so there cannot be a wall in the regime of validity of these equations. In contrast, the D-terms for an irreducible Higgs bundle are equivalent to stability of the Higgs bundle. This is an open condition, which allows for walls. For a more detailed analysis of what happens near a wall, see [6]. This is not the only issue. A more serious problem is that the blow-up modes may get lifted when $M2$-branes wrapped on the corresponding vanishing cycles are condensed. This cannot be decided based on the geometry alone, as it depends on the three-form configuration [6]. In this case, we cannot compare with a smooth supergravity solution.

Although it is useful to keep both pictures in mind, for our purposes here the Higgs bundle approach has a number of advantages. It allows for a much simpler analysis of the coupling to the background three-form field C_3. It allows us to derive interactions, which we don't know how to do properly in the soliton approach. And the regime of validity of the Higgs bundle approach can be attained in F-theory, whereas the soliton approach always requires us to extrapolate and go to a smooth M-theory compactification. This may not be possible and when it is, information about the D-terms derived from the soliton approach is not protected under the extrapolation.

2.7 Summary of local F-theory constructions

To summarize, we can construct a class of local F-theory compactifications for intersecting 7-branes with only the following three ingredients:

1. We take the four-fold to be a dP_9 fibration over a base B_2. For duals of heterotic spectral cover constructions B_2 can be an Enriques surface, a Del Pezzo surface, a Hirzebruch surface or blow-up thereof.
2. The dP_9 fibration is specified by a section $s : B_2 \to \mathcal{W}$ of a weighted projective bundle $\mathcal{W} \to B_2$. This determines the $\mathbf{P}^1$ fibration $B_3 \to B_2$ and the discriminant locus Δ, and hence the positions and intersections of the 7-branes.
3. In addition we can turn on a G-flux, where $[\mathsf{G}/2\pi]$ is of type $(2,2)$, (half-)integral, primitive. This specifies the magnetic fluxes on the 7-branes. For generic values of the moduli there is at least a rank one lattice of fluxes which do not further break the gauge symmetry, and for special values of the moduli there may be additional supersymmetric fluxes.

3 Duality between F-theory and the heterotic string

3.1 Spectral cover construction for heterotic bundles

3.1.1 Fourier–Mukai transform

To specify an $N = 1$ heterotic compactification in the supergravity approximation we need a Calabi–Yau three-fold Z, and two bundles V_1, V_2 on Z with structure group in $E_8 \times E_8$ and satisfying the Hermitian Yang–Mills equations[10] :

$$F^{2,0} = F^{0,2} = 0, \quad F_{i\bar{j}}\, g^{i\bar{j}} = 0. \tag{3.1}$$

We must further satisfy

$$dH = \frac{\alpha'}{4}\,\mathrm{tr}(R \wedge R) - \frac{\alpha'}{4}\,\mathrm{tr}_{E_8\times E_8}(F \wedge F). \tag{3.2}$$

[10]The first equation is the F-term associated with the four-dimensional superpotential $W = \int_{CY} \Omega^{3,0} \wedge \omega^3_{CS}(A)$, and the second can be interpreted as a four-dimensional D-term $F \wedge J \wedge J = 0$ [45].

The topological obstruction to solving this equation is

$$c_2(Z) = c_2(V_1) + c_2(V_2). \tag{3.3}$$

However even if this topological condition is satisfied, clearly we generally must turn on non-zero H. One may argue that a solution may be constructed order by order in the α' expansion starting with a Calabi–Yau metric and a solution of the Hermitian Yang–Mills equations (3.1) [45]. For some special cases the existence of exact solutions may be inferred from dualities or even proved mathematically [46]. In addition one sometimes adds some five-branes wrapped on effective curves in Z, even though this does not lead to a smooth supergravity background. Such 5-branes correspond to zero size instantons and give further singular contributions to (3.2) and (3.3).

In general constructing bundles satisfying (3.1) is not an easy matter. However if the three-fold admits an elliptic fibration $\pi : Z \to B_2$ with a section $\sigma_{B_2} : B_2 \to Z$, then an interesting class of bundles can be constructed using spectral covers. The idea is very simple: suppose we have a stable $SU(n)$-bundle V over Z. First we restrict V to the elliptic fibers and learn how to describe bundles on each T^2, and then we fiber this data over the base.

Restricting (3.1) to a T^2-fiber suggest that the bundle should be flat along fibers. This is actually not necessarily true and even fails along at least a codimension one locus in the base in all interesting examples, but let us assume we are in this situation to build some intuition. Flat bundles on T^2 are classified by a map $\pi_1(T^2) \to SU(n)$, that is by the Wilson lines around the T^2. The fundamental group of T^2 is abelian, so these Wilson loops commute, and by a gauge transformation the Wilson loops can be taken to lie in the Cartan of $SU(n)$. Therefore, the restriction of V to the generic elliptic fibre splits as a sum of n line bundles of degree zero. Each line bundle is characterized by a point on the dual T^2 (which parameterizes the holonomies), up to residual symmetries which form the Weyl group, therefore the moduli space is

$$\mathcal{M}_{SU(n)} = [\Lambda^c_{SU(n)} \otimes T^2]/W = \mathbf{WP}^n_{1,1,\ldots,1}, \tag{3.4}$$

where $\Lambda^c_{SU(n)}$ is the coroot lattice of $SU(n)$. The restriction that the bundle be $SU(n)$ rather than $U(n)$ means that the n points on the dual torus are required to sum to zero under the group law. Also, we may canonically identify the torus with its dual. Similar results hold for bundles with other structure groups.

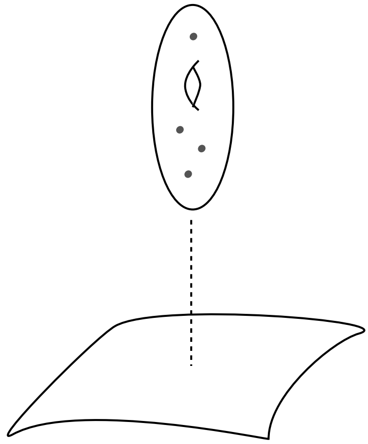

Figure 3: Part of the heterotic compactification data consists of an elliptically fibered Calabi–Yau, together with a set of points on each elliptic fibre describing the Wilson lines of the ten-dimensional gauge group.

Fibering this data over the base, we see that an $SU(n)$ bundle can be described by a set of n points on the elliptic fibre summing to zero, varying holomorphically over the base B_2, and thus sweeping out a holomorphic surface C which is an n-fold cover of B_2. This is called the spectral cover. Intuitively this is familiar to string theorists from T-duality of D-branes, which in this case maps an $SU(n)$ "9-brane" to a "7-brane" by T-dualizing along the elliptic fibre. Even though there are no physical branes in the game, it is useful to keep this picture in mind. Moreover, as also familiar from T-duality, to each of the n points on the dual T^2 we can associate a line in the n-dimensional vector space $H^0(V \otimes \mathcal{O}(\sigma_{B_2})|_{T^2})$. These lines fit together in a non-trivial holomorphic line bundle on C.

In order to describe this more explicitly, we may proceed as follows [23]. We represent the three-fold Z as a Weierstrass equation:

$$y^2 = x^3 + f\, xv^4 + gv^6. \qquad (3.5)$$

Here $\{v, x, y\}$ are taken as sections of $\{\mathcal{O}, K_{B_2}^{-2}, K_{B_2}^{-3}\}$ respectively, and $\{f, g\}$ are sections of $\{K_{B_2}^{-4}, K_{B_2}^{-6}\}$ respectively. Then the data of n points on each elliptic fiber summing to zero can be encoded by writing an equation on the fiber which has exactly these points as its solutions, and a pole of order n at $v = 0$ which we identify with the intersection of the elliptic curve with the section σ_{B_2}. Such an equation is a generic nth order polynomial in x and y:

$$a_0\, v^n + a_2\, xv^{n-2} + a_3\, yv^{n-3} + \cdots + a_n\, x^{n/2} = 0 \qquad (3.6)$$

(if n is odd, the last term is $yx^{(n-3)/2}$). In order for this equation to make sense globally on B_2, it follows that the a_i must be sections of $\mathcal{N} \otimes K^i_{B_2}$ where $\mathcal{N}$ is a line bundle on B_2. Since the a_i's are defined only up to multiplication on each fiber, they determine a section of the weighted projective bundle over B_2

$$\mathcal{W}_{SU(n)} = \mathbf{P}(\mathcal{O} \oplus K^2_{B_2} \oplus \cdots \oplus K^n_{B_2}) \tag{3.7}$$

with fiber $\mathcal{M}_{SU(n)}$. An analogous construction also works for more general bundles. Thus the spectral cover C is equivalent to a section $s : B_2 \to \mathcal{W}_{SU(n)}$. This description will provide an easy comparison of the analytic data under F-theory/heterotic duality.

The relation between the spectral cover and the bundle V on Z can be put in a precise algebraic–geometric form which is known as the Fourier–Mukai transform. The homology class of the spectral cover C can be expressed as

$$[C] = n[B_2] + \pi^*[\eta] \quad \in \quad H^{1,1}(Z, \mathbf{C}) \cap H^2(Z, \mathbf{Z}) = \mathrm{Pic}(Z), \tag{3.8}$$

where $[\eta]$ is a class in $H_2(B_2, \mathbf{Z})$, and we used Poincaré duality to identify the dual cohomology class. Comparing with the description of C using projective bundles over B_2, the homology class of the zero set of a section agrees with (3.8) provided $c_1(\mathcal{N}) = [\eta]$. Further, we need a line bundle L on C. In order for the bundle V to have holonomy $SU(n)$ rather than $U(n)$, the line bundle L is required to satisfy

$$c_1(V) = \pi_* c_1(L) + \frac{1}{2}(c_1(C) - \pi^* c_1(B_2)) \equiv 0. \tag{3.9}$$

Therefore, $c_1(L)$ is of the form

$$c_1(L) = -\frac{1}{2}(c_1(C) - p_C^* c_1(B_2)) + \lambda\,\gamma, \quad \pi_*\gamma = 0, \quad \gamma \in \mathrm{Pic}(C), \tag{3.10}$$

where p_C is the natural projection $C \to B_2$, and λ is a (half-)integer. Generically the Picard group of C is two dimensional: one generator for the pullback of the Kähler class of Z, and the other generator given by $\Sigma = C \cap \sigma_{B_2}$ which must be effective. Therefore when the complex structure moduli take generic values, the only "traceless" classes satisfying $\pi_*\gamma = 0$ that exist in general must be multiples of the natural generator:

$$\gamma = n[\sigma_{B_2} \cdot C] - p_C^*[\eta - n\, c_1(B_2)]. \tag{3.11}$$

With this choice of γ we can write $c_1(L)$ as

$$c_1(L) = n\left(\lambda + \frac{1}{2}\right)[\sigma_{B_2} \cdot C] + \left(\frac{1}{2} - \lambda\right) p_C^*\eta + \left(n\lambda + \frac{1}{2}\right) p_C^* c_1(B_2). \tag{3.12}$$

For n odd (such as $n = 5$), this is guaranteed to be an integral class when $\lambda - \frac{1}{2}$ is an integer.

For completeness let us briefly indicate how the bundle V may be reconstructed from this data in the case of $SU(n)$ holonomy. We first introduce the space $\hat{Z} = Z \times_{B_2} Z$. There are three natural divisors given by $\sigma_1 = \sigma \times_{B_2} Z$, $\sigma_2 = Z \times_{B_2} \sigma$, and the diagonal divisor Δ (not to be confused with the discriminant locus). We further define $\hat{C} = C \times_{B_2} Z$ and the Poincaré line bundle $\mathcal{P}$ on $\hat{C}$ as

$$\mathcal{P} = \mathcal{O}(\Delta - \sigma_1 - \sigma_2) \otimes p_{B_2}^* K_{B_2}|_{\hat{C}}. \tag{3.13}$$

Then the bundle V may be reconstructed by the Fourier–Mukai transform

$$V = p_{Z*}(p_{\hat{C}}^* L \otimes \mathcal{P}), \tag{3.14}$$

where $p_Z, p_{\hat{C}}$ denote the natural projections. With this expression for V one may compute the Chern classes of V [23, 47]. The result for $c_1(V)$ was quoted in (3.25), and one finds $\pi_* c_2(V) = \eta$. For the third Chern class one finds

$$c_3(V) = 2\lambda\, \eta \cdot (\eta - nc_1(B_2)). \tag{3.15}$$

The third Chern class is an important characteristic of the model as we will review in a moment.

So far we have discussed solving the F-terms on Z, that is we have discussed the construction of holomorphic bundles V whose curvature satisfies $F^{2,0} = F^{0,2} = 0$ and which admit a connection which satisfies $F_{i\bar{j}}\, g^{i\bar{j}} = 0$ when restricted to elliptic fibers. We must further show that it is possible to solve the D-terms, $F_{i\bar{j}}\, g^{i\bar{j}} = 0$ on all Z. As is well known, in an algebro-geometric setting one may argue that there exists a unique solution provided the bundle V is (poly-)stable. Since Fourier–Mukai is an equivalence of categories, the bundle V is stable with respect to an appropriate Kähler class when L has rank 1 and C is irreducible. According to the authors in [48,49], stability holds for

$$J = t_1\, \pi^* J_{B_2} + t_2\, J_0, \tag{3.16}$$

where J_0 is the Poincaré dual of the section, and $t_1 \gg t_2$. That is, the base should be large compared to the T^2 fiber. Note that both the fiber and base need to be large compared to the string scale in order to keep α'-corrections small.

Given a Calabi–Yau Z and a bundle V satisfying the Hermitian Yang–Mills equations, we may deduce the low energy spectrum as follows. We start with the ten-dimensional gaugino which transforms in the adjoint of E_8, and we will concentrate on one E_8 factor only. Then the four-dimensional fermions are zero modes of the Dirac operator on Z in the background with $SU(n)$ holonomy. Since Z is a complex manifold, the zero modes of the Dirac operator are zero modes of the Dolbeault operator coupled to the bundle V. Let us denote the commutator of $H = SU(n)$ in E_8 as G, and decompose the adjoint representation of E_8 as

$$\mathbf{248} = \sum_a R_a(H) \otimes R'_a(G). \tag{3.17}$$

Then the zero modes of the Dolbeault operator are given by the generators of the cohomology groups

$$H^p(Z, R_a(V)) \otimes R'_a(G). \tag{3.18}$$

Assuming V stable, zero modes of grade $p = 0, 3$ occur only when R_a is the trivial representation. These are paired with four-dimensional gauginos in the adjoint of G. Zero modes with $p = 1$ get paired with a left-handed four-dimensional chiral fermion in the representation $R'_a(G)$, and zero modes with $p = 2$ get paired with a right-handed chiral fermion. Since supersymmetry was preserved, we get a four-dimensional $N = 1$ SUSY gauge theory with a gauge group G and matter in various representations $R'_a(G)$. The net number of generations is given by

$$N_{\text{gen}} = H^1(Z, V) - H^2(Z, V) = -\frac{1}{2} c_3(V) \tag{3.19}$$

assuming $H^p(Z, V) = 0$ for $p = 0, 3$, which holds for stable bundles. In addition, the reduction of the gravity multiplet on Z gives various other fields neutral under G.

As would be expected from the brane-like interpretation for elliptically fibered Calabi-Yaus Z, chiral matter is localized on the intersection of the "7-branes." This can easily be seen from the Leray spectral sequence:

$$H^1(Z,V) \sim H^0(B_2,R^1), \tag{3.20}$$

where for each point p on B_2

$$R^1_p = H^1(T^2_p, V|_{T^2_p}). \tag{3.21}$$

Now recall that $V|_{T^2}$ splits as a sum of degree zero line bundles $\sum_i L_i$, and $H^p(T^2, L_i)$ vanish unless L_i is the trivial line bundle. So the only contributions come from the locus where one of the L_i becomes a trivial line bundle, so that its Wilson lines around the cycles of the T^2 vanish. This is precisely the locus $\Sigma = C \cap \sigma_{B_2}$ where the spectral cover intersects the section, and it is sometimes called the "matter curve". More precisely one can show that [47, 50]

$$H^1(Z,V) = \mathrm{Ext}^1(i_*\mathcal{O}_{B_2}, j_*L) = H^0(\Sigma, L \otimes N_{B_2}|_\Sigma). \tag{3.22}$$

Here N_{B_2} is the normal bundle of B_2 in Z, and N_C is the normal bundle to C. Since Z is Calabi–Yau, we have $N_{B_2} = K_{B_2}, N_C = K_C$. For later comparison with F-theory, it is useful to decompose the line bundle L by separating out the traceless piece:

$$L|_\Sigma = L^\lambda_\gamma \otimes N_{B_2}^{-1/2} \otimes N_C^{1/2}|_\Sigma, \quad c_1(L_\gamma) = \gamma, \tag{3.23}$$

so that we can express the number of chiral fields as

$$h^0(\Sigma, L^\lambda_\gamma|_\Sigma \otimes K_\Sigma^{1/2}). \tag{3.24}$$

3.1.2 Summary of the heterotic construction

Suppose we are given a Calabi–Yau three-fold Z with an elliptic fibration $\pi : Z \to B_2$, and a section $\sigma_{B_2} : B_2 \to Z$. Then an interesting class of $SU(n)$ bundles (and in fact also bundles with more general structure groups) can be constructed with only the following ingredients:

1. An elliptically fibred threefold $\pi : Z \to B_2$, and a section $\sigma_{B_2} : B_2 \to Z$.
2. An n-fold covering $p_{B_2} : C \to B_2$ with the homology class $[C] = n[\sigma_{B_2}] + [\pi^*\eta] \in H_4(Z, \mathbf{Z})$. Equivalently, we may specify a section of a weighted projective bundle $s : B_2 \to \mathcal{W}_{SU(n)}$. This involves specifying a line

bundle $\mathcal{N}$ on B_2 with $c_1(\mathcal{N}) = \eta$. The spectral cover describes the Wilson lines of the bundle V along the T^2 fibres.

3. A line bundle L over C. Generically the only allowed line bundles on C have a first Chern class of the form

$$c_1(L) = -\frac{1}{2}(c_1(C) - p^* c_1(B_2)) + \lambda\,\gamma \tag{3.25}$$

with γ defined in (3.11). Thus for generic complex structure moduli choosing the line bundle L amounts to specifying λ, which must be chosen so that $c_1(L)$ is integer quantized. In addition, one may turn on bundles on σ_{B_2} (the reducible part of the spectral cover), which will further break the observed four-dimensional gauge symmetry.

3.2 Duality map in the stable degeneration limit

3.2.1 Matching the holomorphic data

The heterotic string compactified over T^2 is characterized by a vector in an even self-dual lattice of signature $(18, 2)$. However we are only interested in a subset of this data, namely a bundle with holonomy in a subgroup H of E_8. These data may be isolated from the other geometric data in the limit of large T^2. Recall that the moduli space of stable H-bundles on T^2 is given by the Looijenga weighted projective space

$$\mathcal{M}_H = \mathbf{WP}^r_{s_0,\ldots,s_r}, \tag{3.26}$$

where s_i are the Dynkin indices of the affine Dynkin diagram of H, and r is the rank of H. We can further fiber this data over a base B_2, yielding a weighted projective bundle called $\mathcal{W}_H$. An H bundle over Z (which is semi-stable on fibers) determines a holomorphic section $s : B_2 \to \mathcal{W}_H$, or equivalently a spectral cover C which is identified with the zero locus of the section. To reconstruct the bundle on Z, we also need the twisting data. This is given by a line bundle on C. The line bundle can be represented through its first Chern class. To make the correspondence with F-theory clearer, the fiber of the covering $C \to B_2$ is a discrete set of points which we denote by f. We can use the Leray spectral sequence to identify

$$H^2(C, \mathbf{Z}) \sim H^2(B_2, H^0(f)). \tag{3.27}$$

This means that the flux can be represented as

$$F = F_I \wedge \omega_0^I, \tag{3.28}$$

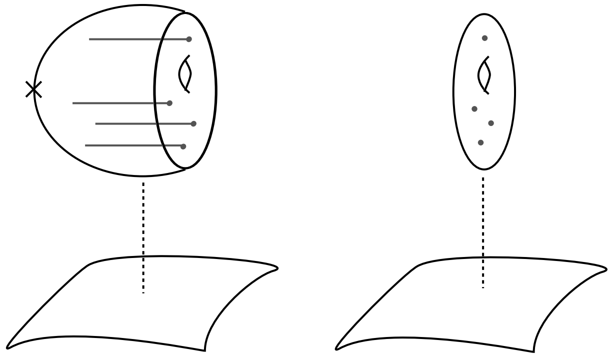

Figure 4: To every dP_9-surface we may associate an elliptic curve with a set of points on it by intersecting a fixed elliptic fiber of the dP_9 with the set of -1-curves. Conversely by taking an elliptic curve with a set of points and thickening the points to $\mathbf{P}^1$'s, we obtain a dP_9 surface.

where F_I is a flux on B_2, and ω_0^I is a set of generators of $H^0(f)$ which vary over B_2. It will be convenient to take ω_0^0 to be the diagonal generator which is the pull-back of a zero-form on B_2, and let the remaining generators satisfy $\pi_*\omega_0^I = 0$. In particular, with

$$c_1(L) = -\frac{1}{2}(c_1(C) - p^*c_1(B_2)) + \lambda\gamma \tag{3.29}$$

then the first two terms are proportional to ω_0^0, and γ is built of the ω_0^I with $I \neq 0$.

On the F-theory side we recovered the same ingredients, but with a different interpretation. In the stable degeneration limit, the $K3$ fibration degenerates into two dP_9 fibrations W_1, W_2 over B_2, glued along an elliptically fibered Calabi–Yau three-fold Z which is identified with the heterotic three-fold. Concentrating on W_1, we consider the unfolding of a dP_9 surface with an E_8 singularity, keeping a canonical divisor fixed. This can be expressed by the degree six equation in $\mathbf{WP}_{1,1,2,3}$:

$$0 = p_i(v, x, y)\, u^i, \tag{3.30}$$

where p_i is of degree $6 - i$ and $p_0 = 0$ describes the distinguished T^2-fiber. As we discussed in Section 2, requiring a section of singularities corresponding to an enhanced gauge group G implies certain restrictions on the $p_i, i > 0$. The coefficients in the p_i are also determined by a choice of section $s : B_2 \to \mathcal{W}_H$, up to a change of variables. In fact if u appears only linearly, we can integrate out the variable u without loosing any information about the complex structure moduli [30,31]. That is, the same information is contained in the pair of equations

$$p_0(v, x, y) = 0, \quad p_1(v, x, y) = 0. \tag{3.31}$$

This yields a collection of points on the T^2 at $u = 0$, which we interpret as the spectral cover. Conversely the dP_9 surface may be obtained as follows: we take the elliptic curve $p_0 = 0$ with a collection of points in it determined by the heterotic bundle and encoded as an equation $p_1 = 0$. Then we thicken each of these points to lines by adding the variable u, with each line intersecting the T^2 at $u = 0$ in a point[11] . This yields $p_0 + up_1 = 0$. Thus we have a completely explicit dictionary.

The twisting data is interpreted as turning on a C_3 field with non-zero G-flux. We have seen very explicitly above that there is a canonical map which associates to each point in the fiber f of the spectral cover $C \to B_2$ a $\mathbf{P}^1 \subset DP_9$, and dually with each zero-form in $H^0(f)$ a two-form in $H^{1,1}(DP_9)$. Thus we have a natural map

$$\begin{array}{ccc} H^{i,j}(C) & \longrightarrow & H^{i+1,j+1}(Y_4) \\ \updownarrow & & \uparrow \\ H^{i,j}(B_2, H^0(f)) & \longrightarrow & H^{i,j}(B_2, H^{1,1}(dP_9)). \end{array} \tag{3.32}$$

The map is actually somewhat ambiguous for ω^0, because dP_9 has two two-forms (dual to the base and the fiber) that it could get mapped to. But as we discussed in Section 2, the corresponding G-fluxes do not exist in F-theory anyways even off-shell, so requiring that we get a sensible flux eliminates the ambiguity. In particular the "traceless" piece of the magnetic flux on the spectral cover gets mapped unambiguously to a non-zero G-flux on the dP_9 fibration. For more details of the mapping between the spectral line bundle and the G-flux, see [14] and Appendix C. In a similar vein, the continuous moduli of the spectral line bundle, which live in $h^{0,1}(C)$, and deformations of the spectral cover, which live in $h^{2,0}(C)$, get mapped in F-theory to continuous moduli of the 7-brane gauge fields and deformations of the 7-branes, which live in $h^{1,2}(Y_4)$ and $h^{3,1}(Y_4)$ respectively as was summarized in table 2.

3.2.2 Matching the spectrum and Yukawa couplings

Now we would like to argue that the computation of the spectrum agrees with heterotic computations for F-theory duals of spectral cover

[11] This construction generalizes for spectral covers for groups other than $SU(n)$, and is called the cylinder map [14].

constructions, in the stable degeneration limit. (The arguments in this section can be understood more concisely as saying that the spectrum and superpotential computed from the Higgs bundle are the same as the spectrum and superpotential computed from the spectral cover).

In F-theory we have a dP_9 fibration over a base B_2, with a certain section of singularities leading to a four-dimensional gauge group G, but of generic type I_1 elsewhere. Suppose we want to compute the number of chiral fields in the representation $R(G)$. As we have discussed in Section 2, these are localized along a curve Σ where the singularity gets enhanced. This means that the 7-branes wrapping B_2 (which we called the gauge branes) intersect another 7-brane (which we called the matter brane) over a curve $\Sigma \subset B_2$. On the heterotic side we must get the corresponding gauge symmetry enhancement over the same curve $\Sigma \subset B_2$. Thus it coincides with one of the matter curves on the heterotic side, the locus where one of the spectral covers C (analogous to our matter brane) intersects the section σ_{B_2} (analogous to the gauge 7-branes).

Now we need the magnetic fluxes on the 7-branes, restricted to Σ. We consider first the matter curves where the **10** of $SU(5)$, the **16** of $SO(10)$, the **27** of E_6 and the **56** of E_7 are localized. On the heterotic side this corresponds to the intersection of σ_{B_2} with the spectral cover for the fundamental representation of the $SU(n)$ holonomy group, where $n = 5, 4, 3, 2$ respectively. The F-theory fluxes were described on the heterotic side by a line bundle L on the spectral cover, with first Chern class

$$c_1(L) = -\frac{1}{2}(c_1(C) - p^* c_1(B_2)) + \lambda\gamma. \tag{3.33}$$

According to the discussion in the previous subsection, using the identification $H^{1,1}(C) \sim H^{1,1}(B_2, H^0(f))$, the flux γ gets mapped to

$$\gamma = F_I \wedge \omega_0^I \to G_\gamma = F_I \wedge \omega_2^I \in H^{2,2}(Y_4), \tag{3.34}$$

where F_I is a flux on B_2, and the index I labels the generators of $H^0(f)$. Further, the remaining piece of $c_1(L)$ gets mapped to zero. Thus it is evident that the magnetic flux for $R_a(\tilde{E}) \otimes R'_a(\tilde{F})|_\Sigma$ extracted from the G-flux, using the rules described in Section 2, is exactly given by $\lambda\gamma|_\Sigma = -\lambda\eta \cdot \Sigma$. As for the heterotic string, we denote the line bundle on Σ whose first Chern class is γ as L_γ. Now plugging into our formula for the number of zero modes (2.37), we obtain

$$h^i(\Sigma, L_\gamma^\lambda \otimes K_\Sigma^{1/2}|_\Sigma). \tag{3.35}$$

This is exactly the same as the answer we obtained on the heterotic side (3.24).

In the $SU(5)$ case it is also interesting to consider the spectral cover $C_{\mathbf{10}}$ for the anti-symmetric representation of $SU(5)$. The intersection $\Sigma' = C_{\mathbf{10}} \cap \sigma_{B_2}$ is the locus in B_2 where the gauge symmetry gets enhanced from $SU(5)$ to $SU(6)$, so this corresponds on the F-theory side to the locus where I_5 and I_1 collide transversally to create an I_6 singularity.

The heterotic prediction for the amount of chiral matter in the $\mathbf{5}$ or $\overline{\mathbf{5}}$ of $SU(5)$ is

$$H^p(Z, \Lambda^2 V) = H^{p-1}(\Sigma', M \otimes K_{B_2}|_{\Sigma'}), \tag{3.36}$$

where M is a rank one sheaf on $C_{\mathbf{10}}$ obtained by Fourier–Mukai transform from $\Lambda^2 V$. The spectral cover $C_{\mathbf{10}}$ is singular along a codimension one locus and M may fail to be a line bundle there. This singular locus intersects Σ' in a finite number of points so $M|_{\Sigma'}$ may also fail to be a line bundle. Nevertheless because the holonomy group is $SU(5)$ rather than $U(5)$, the anti-symmetric sits in $SU(10)$ rather than $U(10)$, and we may again decompose

$$c_1(M) = -\frac{1}{2}(c_1(C_{\mathbf{10}}) - p^*_{C_{\mathbf{10}}} c_1(B_2)) + \lambda' \kappa, \tag{3.37}$$

where κ is a class in $H^{1,1}(C_{\mathbf{10}})$ with $p_{C_{\mathbf{10}}*}\kappa = 0$, and λ' is a (half-)integer. Since M is not a line bundle, its first Chern class is somewhat ambiguous, but with the appropriate definition this formula should be satisfied. The G-flux constructed from $\lambda' \kappa$ should be the same as the G-flux constructed from the class $\lambda \gamma$ on the spectral cover associated to the fundamental representation. Thus the difference between the 7-brane fluxes on the F-theory side should be given by $\lambda' \kappa|_{\Sigma'}$. Following our previous arguments then, the cohomology groups on both sides of the duality simplify to

$$H^{p-1}(\Sigma', L_\kappa^{\lambda'} \otimes K_{\Sigma'}^{1/2}), \tag{3.38}$$

for $p = 1, 2$, where L_κ satisfies $c_1(L_\kappa) = \kappa|_{\Sigma'}$.

We can also check that the chiral spectrum from coincident 7-branes agrees with the chiral spectrum computed on the heterotic side. The Freed–Witten shift can be ignored in this case because the branes are wrapped on the same four-cycle. On the heterotic side we have a reducible spectral cover consisting of multiple copies of σ_{B_2}, together with the bundle $R_a(E)$

on it. The sheaf $\sigma_{B_2 *} R_a(E)$ is the Fourier–Mukai transform[12] of the bundle $V = \pi^* R_a(E)$ on Z, so the heterotic answer in this case is

$$\begin{aligned} H^p(Z, \pi^* R_a(E)) &= \mathrm{Ext}^p(\mathcal{O}_Z, \pi^* R_a(E)) \\ &= \mathrm{Ext}^p(\sigma_{B_2 *}\mathcal{O}_{B_2}, \sigma_{B_2 *} R_a(E)) \\ &\sim H^p(B_2, R_a(E)) \oplus H^{p-1}(B_2, R_a(E) \otimes K_{B_2}). \end{aligned} \tag{3.39}$$

Here we used the fact that the Fourier–Mukai transform preserves the Ext groups. Again this agrees with what we obtained in F-theory.

Finally, me may check that the Yukawa couplings computed on both sides must agree. After Fourier–Mukai transform, the Yukawa couplings on the heterotic side take the same form as (2.52):

$$\int_{B_2} d_{abc}\, A^a \wedge A^b \wedge \Phi^c. \tag{3.40}$$

Here Φ takes values in K_{B_2} on both sides of the duality. Further, the procedure we have given for computing the wave functions of $A^{0,1}$ and $\Phi^{2,0}$ on B_2 only used B_2 itself, the data of where on B_2 gauge symmetry gets enhanced (i.e., the matter curves), and the fluxes on the matter curves. Thus we manifestly end up with the same wave functions on B_2, and the Yukawa couplings must agree as well.

3.3 Duality of the classical superpotential

We would like to briefly discuss the behaviour of the flux superpotential under F-theory/heterotic duality. Recall that on the F-theory side we had

$$W_{\text{flux}} = \frac{1}{2\pi} \int \Omega^{4,0} \wedge \mathsf{G} \tag{3.41}$$

and further, on large and smooth four-folds we had a set of D-terms

$$J \wedge \mathsf{G} = 0. \tag{3.42}$$

Generically this stabilizes all but one of the moduli. The solutions of the F-terms equations are given by integer quantized $(2, 2)$-classes. These are somewhat rare, and one typically has to tune all the complex structure moduli to find them (i.e., solve a Noether–Lefschetz problem). All but one of the Kähler moduli can in principle be stabilized with the D-term potential. Note however that this does not provide a potential for the volume modulus,

[12] This differs slightly from some of the literature because we included a factor of K_{B_2} in our Poincaré sheaf $\mathcal{P}$ (3.13).

because if $J \wedge \mathsf{G} = 0$, then $xJ \wedge \mathsf{G} = 0$ for all x. However these equations will receive corrections, which we may try to use to stabilize also this last modulus.

Now consider F-theory on $K3$. First note that it is not possible to turn on any internal fluxes, since $K3$ has only even dimensional harmonic forms and flux proportional the volume form of $K3$ is forbidden. Moreover a G-flux that lives purely in eight dimensions does not exist in F-theory. So all G-fluxes can be interpreted as gauge field fluxes for the $18+2$ gauge fields in eight dimensions[13] . Similarly, the $\Omega^{(4,0)}$ form must be decomposed into the internal $(2,0)$ form of the $K3$ and a $(2,0)$ form in eight dimensions. The flux superpotential reduces to the natural pairing of this $(2,0)$ form and the $18+2$ abelian fluxes. This data can be further fibered over a base B_2.

Analogously, in the heterotic string in ten dimensions we have the superpotential:

$$W = \int \Omega^{3,0} \wedge (H + i\, dJ) + \int \Omega^{3,0} \wedge \omega_3(A), \tag{3.43}$$

where $dJ \neq 0$ allows for the possibility of torsion [51, 52], and $\omega_3(A)$ is the holomorphic Chern-Simons form

$$\omega_3(A) = \epsilon^{\bar{i}\bar{j}\bar{k}} \left(A_{\bar{i}} \partial_{\bar{j}} A_{\bar{k}} + \frac{2}{3} A_{\bar{i}} A_{\bar{j}} A_{\bar{k}} \right). \tag{3.44}$$

It simply reduces to $\epsilon^{\bar{i}\bar{j}\bar{k}} A_{\bar{i}} F_{\bar{j}\bar{k}}$ for abelian gauge fields. (There is also a gravitational Chern-Simons term, which we ignore here). After compactification on T^2, we can turn on certain fluxes corresponding to $\partial_\lambda g_{\mu\nu}$ or $\partial_\lambda B_{\mu\nu}$ with two indices on the T^2, or $F_{\mu\nu}$ with one index on the T^2. However this corresponds to varying the Narain moduli over the eight-dimensional space-time and the same data exists on the F-theory side also as we have discussed, but is not interpreted as G-flux. Flux of type $F_{\mu\nu}$ with two indices on the T^2 might be allowed a priori, but is of type $(1,1)$ and the superpotential does not depend on it. The remaining fluxes can be interpreted as fluxes for the $18+2$ gauge fields in eight dimensions coming from the Cartan of $E_8 \times E_8$ and from modes of the metric and B-field on T^2. The $(3,0)$ form reduces to a $(2,0)$ form in eight dimensions, and from reduction of the ten-dimensional superpotential we get a pairing between this $(2,0)$ form and the $18+2$ fluxes. The data can be further fibered over B_2. Thus the flux superpotentials match qualitatively under the duality. A more precise analysis can be done by employing Hodge theoretic techniques in the stable degeneration limit.

[13]This includes the possibility of fluxes for the NS and RR three-forms.

This brings up the following puzzle: in F-theory, we can stabilize all the complex structure moduli at tree level. On the heterotic side, stabilizing the complex structure and vector bundle moduli has been problematic, and one usually invokes worldsheet instanton effects. Clearly, if the F-theory arguments are correct, we must be able to stabilize these moduli at tree level also, because as we just saw the superpotentials on both sides are isomorphic. The fact that H or $dJ \neq 0$ stabilizes some moduli has already received some attention. Here we would like to focus on the holomorphic Chern–Simons superpotential.

The moduli of the heterotic bundle V translate into continuous moduli of the line bundle on the spectral cover, which are absent in generic models, and deformations of the spectral cover. In constructions in the heterotic literature, the spectral cover moduli are flat directions for the holomorphic Chern–Simons superpotential, and they are not stabilized perturbatively. Instead one invokes worldsheet instanton effects, as their one-loop pre-factor depends on vector bundle moduli (as well as complex structure moduli), but they have the disadvantage of being exponentially suppressed at large volume and hard to compute. Clearly we would prefer to stabilize these moduli perturbatively, and duality with F-theory predicts that this must be possible. So why exactly are the spectral moduli unobstructed? Let us examine this issue in F-theory/7-brane language.

Varying the flux superpotential with respect to vector bundle moduli, we find that

$$0 = DW = \frac{1}{2\pi} \int_{S_2} \Phi^{2,0} \wedge \mathsf{F} + \cdots , \tag{3.45}$$

where we used $\delta\Omega \sim \Phi \wedge \omega$ and $\mathsf{G} \sim \mathsf{F} \wedge \omega$. Generally, F has both $(0,2)$ components and $(1,1)$ components. However, the critical critical locus of the superpotential is the locus in moduli space where F is purely of type $(1,1)$. Therefore, moduli of the Higgs field (which correspond to vector bundle moduli in the heterotic language) in fact *can* be stabilized if we choose fluxes that have a $(0,2)$-component for generic values of the moduli. The number of equations is the same as the number of variables, so the generic solution is a completely rigid bundle. Actually, if we change the complex structure of the Calabi–Yau then we generically destroy this solution, so we should really think of it as stabilizing combinations of complex structure and vector bundle moduli. (The same is true for the one-loop pre-factor of an instanton contribution to the superpotential. The pre-factor depends on both complex structure and vector bundle moduli, and therefore generates a potential for combinations of them).

Indeed, upon reflection it is obvious that the spectral line bundles that are normally used in heterotic constructions are very special: they are actually obtained by restricting line bundles on the Calabi–Yau three-fold to the spectral cover. They are therefore of type $(1,1)$ for any values of the spectral cover moduli, and so do not induce a potential for these moduli. This explains why it is frequently stated in the heterotic literature that one needs worldsheet instantons to stabilize such moduli.

However it is now clear that by choosing more general spectral line bundles which are not generically holomorphic, we can induce a potential for these moduli at tree level. Moreover, we see that we get the same type of exponential growth in the number of solutions as on the F-theory side, simply by compactifying the $10d$ E_8 Yang–Mills theory. This should not be surprising, because the second betti number $b_2(C)$ for the two E_8 spectral covers tends to provide the main contribution to $b_4(Y_4)$ under heterotic/F-theory duality, but it does not appear to have been previously appreciated. It implies that the F-theory landscape has so far been missed on the heterotic side, and moreover the exponential growth in the number of solutions can be seen directly in the visible sector, predicting an exponentially large number of solutions with the spectrum of the MSSM. We gave a rough estimate for the number of solutions in Section 2.5. For more discussion of these new heterotic vacua, including some toy constructions, see [5].

The heterotic string also has a set of D-terms $F \wedge J \wedge J = 0$ in ten dimensions. In eight dimensions we get a term $J \wedge F = 0$ where F are E_8 fluxes. Compatibility with T-duality suggests there should be such a term for all $18 + 2$ fluxes.

3.4 Non-perturbative corrections to the superpotential

As we reviewed earlier, the classical superpotential in F-theory does not receive corrections to any order in large volume perturbation theory, however it may receive non-perturbative corrections due to D-instantons. Let us discuss the possibilities and their heterotic analogues. Our discussion is similar to [53].

The easiest way to get the correspondence is to follow BPS states across a chain of dualities in seven dimensions:

$$F\text{-theory}/K3 \times S^1_R = M\text{-theory}/K3 = \text{Heterotic}/T^3,$$

which we will further compactify to four dimensions by fibering over a base B_2 and taking $R \to \infty$. Let us first consider the equivalence on the left. We could get non-trivial instanton effects from $M2$-branes wrapping a three-cycle which includes one of the circles of the T^2. These would correspond to instantons made of (p, q) strings on the F-theory side. Such three-cycles are rare however. If the three-cycle lives in B_3 completely then the instanton will have infinite action as $R \to \infty$. Therefore we can concentrate on the $M5$-branes. An $M5$-brane wrapped on $K3$ gets mapped to a $D3$-brane wrapping the $\mathbf{P}^1$ base of the $K3$, and we can wrap it on an additional curve $\alpha_2 \subset B_2$ to get an instanton. The other option is to wrap the $M5$-brane on the T^2 fiber of the $K3$ and some additional four-cycle α_4 which does not contain the $\mathbf{P}^1$-base of the $K3$. This gets mapped to a $D3$-brane instanton which wraps the four-cycle α_4. If this coincides with the location of gauge 7-branes and if the bundles on the four-cycle also agree, such instantons may be interpreted as gauge theory instantons.

Now we consider the equivalence on the right. The $M2$-brane instantons, if they exist, get mapped to instanton versions of Dabholkar–Harvey states, whose worldline wraps a (possibly trivial) one-cycle in B_2. This includes ordinary worldsheet instanton effects obtained from wrapping a string worldsheet on a geometric curve. An $M5$-brane wrapped on $K3$ gets mapped to the heterotic fundamental string. An $M5$-brane wrapping any other cycle of the $K3$ gets mapped to an $NS5$-brane wrapping some cycle in T^3. Therefore, the $D3$-instantons wrapping $\alpha_2 \times \mathbf{P}^1$ get mapped to worldsheet instantons wrapping α_2 in the heterotic string, and the $D3$-instanton wrapping α_4 get mapped to space-time instanton effects in the heterotic string.

On the heterotic side, we could also consider worldsheet instantons wrapping the T^2 fiber. The volume of this T^2 is a Kähler modulus which gets mapped to a "transcendental" complex structure parameter on the F-theory side, and the limit of infinite volume for T^2 corresponds to the stable degeneration limit on the F-theory side where the $K3$ degenerates into two dP_9 surfaces (and hence becomes algebraic). The associated worldsheet instanton corrections on the heterotic side correspond to classical contributions to the superpotential on the F-theory side which vanish in the stable degeneration limit.

The rules for D-instanton contributions to the superpotential were originally given in [54–56]. The prescription is simply to calculate the partition function of the instanton obtained by integrating over all the collective coordinates. In type IIa or type IIb backgrounds, where the instanton typically intersects wrapped branes, this includes localized degrees of freedom on the intersection of the instanton with the background branes. The collective coordinates, including the localized ones, can be described using the concept of "Ganor strings" [57]. This calculus has recently been clarified in [58–60].

For finite string coupling, the presence of branch-cuts of the axio-dilaton seems to pose problems for the D-instanton approach. The gauge field on the $D3$-instanton is not invariant under $Sl(2,\mathbf{Z})$ monodromies, so the sum over electric fluxes does not make sense. Further, there are no weakly coupled Ganor strings that we could quantize, so the partition function of $D3$-$D7$ strings ceases to make sense for finite string coupling as well.

These issues have been resolved recently in [61, 62]. Let us briefly summarize some of the results. The instanton contribution consists of an exponential part and a pre-factor:

$$\Delta W = f(m)\, \mathrm{e}^{-T}, \tag{3.46}$$

where T denotes Kähler moduli and m denotes complex structure moduli. The crucial point is that the pre-factor does not depend on Kähler moduli, due to shift symmetries. Therefore by varying Kähler moduli we can extrapolate to a regime where we get a welldefined computation without changing the pre-factor. There are currently two such limits known. Namely we could try to extrapolate the calculation either to $11d$ supergravity, or to a heterotic computation (even in certain cases when there is no heterotic dual). One may also contemplate extrapolating to perturbative IIb, but by contrast to the above limits, the IIb limit involves changing the complex structure. This changes the pre-factor, so the perturbative IIb results will be modified.

In the $11d$ supergravity approach we get an $M5$-instanton, and assuming the instanton is smooth then by the general prescription of [54–56] the instanton contribution to the superpotential is given by the partition function. The $M5$ worldvolume theory contains scalars ϕ, fermions ψ and a chiral two-form B^+, so the partition function is of the form

$$Z_{M5} = Z^M_\phi\, Z^M_\psi\, Z^M_{B+} \tag{3.47}$$

and the contribution to the superpotential is obtained by factoring out four universal bosonic and two universal fermionic zero modes. (Here we ignored the fact that Z^M_ψ and Z^M_{B+} are partition vectors, so it would be more accurate to write $Z_{M5} = Z^M_\phi\, \langle Z^M_\psi, Z^M_{B+}\rangle$ for a suitable inner product). By contrast, the $D3$ partition function is of the form

$$Z_{D3} = Z_\phi\, Z_\psi\, Z_F\, Z_{\lambda_{37}}, \tag{3.48}$$

where ϕ, ψ, F denote the $D3$ worldvolume fields, and λ_{37} denotes fermions obtained from quantizing the $D3$-$D7$ Ganor strings. Although this seems to look different, the expressions can nevertheless be matched. Roughly speaking, one finds

$$Z_\phi^M \to Z_\phi^{(5)}, \quad Z_\psi^M \to Z_\psi, \quad Z_{B^+}^M \to Z_\phi^{(1)}\, Z_F\, Z_{\lambda_{37}}. \tag{3.49}$$

Here we split the six scalars on the $D3$ as $1+5$, and correspondingly Z_ϕ as $Z_\phi^{(1)} Z_\phi^{(5)}$. In other words, it was found that in the IIb limit, the chiral two-form incorporates the $D3$ gauge field F, its magnetic dual $\tilde{F}$ as well as the localized fermionic degrees of freedom λ_{37} due to $D3$-$D7$-strings seen in the IIb weak coupling limit in a single, globally well-defined entity. Furthermore, the relation between $Z_{\lambda_{37}}$ and $Z_{B^+}^M$ is given by a cylinder mapping. Thus by replacing $Z_{F,\tilde{F}}\, Z_{\lambda_{37}}$ with Z_{B^+} for finite string coupling, it is possible to address both of the afore-mentioned problems and make sense of $D3$-instantons in a global way.

The partition function Z_ϕ^M consists of a classical piece $\exp(-\mathrm{vol}(M5))$ and a one-loop determinant, given by the determinant of the Laplacian acting on the scalars. Similarly, Z_ψ^M is given by the determinant of $\not{D} + \not{G}$, the Dirac operator extended by a term involving the G-flux. When the G-flux vanishes, this is related to Ray–Singer torsion. The partition function Z_{B^+} is more subtle. It was first treated in [63], where it was essentially identified with the theta function $\Theta(\tau, z)$ on the intermediate Jacobian of the $M5$-brane (pulled back to the moduli space of the compactification), and further aspects relevant for phenomenology were discussed in [62]. In particular, in perturbative IIb, and also in the heterotic string, there is an interesting relation between anomalous $U(1)$ gauge symmetries, selection rules for the superpotential, and charged fermionic zero modes. It was found to be beautifully transported to the presence or absence of G-flux induced tadpoles for Z_{B^+} in M-theory. (Note there are no charged fermionic zero modes in this picture, as B^+ is the only field that transforms under gauge transformations). A downside of the $M5$ approach however is that in practice the partition function is still hard to calculate explicitly, as one needs to know the complex structure τ of the intermediate Jacobian and the periods z of the F-theory three-form field as a function of the moduli, or calculate the zero locus of Z_{B^+} on the moduli space (i.e., the pull-back of the theta divisor).

Alternatively, we could use heterotic/F-theory duality [62]. Then the $D3$-instanton gets mapped to a worldsheet instanton, whose partition function is of the form

$$Z_{\mathrm{WS}} = Z_\phi\, Z_\psi\, Z_\lambda, \tag{3.50}$$

where ϕ, ψ denote world-volume scalars and right-moving fermions, and λ denotes the left-moving fermions. Again, this expression can be matched with Z_{M5}. The heterotic partition function Z_ϕ should be factorized into a contribution $Z_\phi^{(3)}$ from three of the scalars and a contribution $Z_\phi^{(5)}$ from the remaining scalars. Then we have

$$Z_\phi^M \to Z_\phi^{(5)}, \quad Z_\psi^M \to Z_\psi, \quad Z_{B+}^M \to Z_\phi^{(3)}\, Z_\lambda. \tag{3.51}$$

In the heterotic picture, the most interesting aspects are reproduced by the partition function Z_λ of the left-movers. This approach allows one to make contact with explicit computations of Z_λ as a function of some of the moduli [64], and agreement with the $M5$ picture is a consequence $2d$ bosonization and a cylinder mapping, this time applied to the instanton worldvolume. It also resolve some tensions with vanishing results from perturbative type IIb. The resolution is that these vanishing results are caused by extra light anomalous $U(1)$ gauge bosons that appear in the IIb limit but are absent (or rather very massive) in F-theory [62]. This is quite similar to the way that certain Yukawa couplings are forced to vanish in IIb, but are non-vanishing in F-theory, and further confirms that perturbative IIb results should be interpreted with care.

We may try to use non-perturbative effects to stabilize the last modulus (denoted by S) that we could not stabilize in Section 3.3. This last modulus is by far the most interesting one, because its VEV also serves as our expansion parameter, and it remains somewhat controversial whether the associated conceptual problems have really been solved.

Including the leading non-perturbative correction, the superpotential is of the form

$$W = W_0 + \mathrm{e}^{-S}. \tag{3.52}$$

As is by now well known on the F-theory side, this yields an AdS vacuum with all moduli stabilized, with $\langle S\rangle$ large provided W_0 is extremely small [65]. On the heterotic side, this is exactly the old gaugino condensation story [66], but here is is often stated that W_0 is not small enough to trust the solution. This is mainly a limitation of the explicit models considered. From the viewpoint of heterotic/F-theory duality, we can make W_0 (or really $e^{\mathcal{K}/2}|W_0|$) equally small provided we include the full landscape of constructions on the heterotic side. F-terms and D-term supersymmetry breaking and uplifting to de Sitter space can also be studied on both sides the duality. Whether these effects are really small enough would need to be investigated in more detail, and this is still controversial even on the F-theory side.

4 Examples with GUT groups

In this section, we consider some explicit three generation $SU(5)$ and $SO(10)$ GUT models in F-theory. One may easily come up with some models by lifting from the heterotic literature and translating into F-theory language. These examples are not yet realistic for a number of reasons. The primary reason is that we still need to specify a mechanism to break the GUT group to the standard model gauge group. In addition, generically these models will have extra non-chiral matter, R-parity violating couplings, and so on. Nevertheless we think these examples are useful to illustrate the ideas, and leave a more detailed analysis of the phenomenology for the future.

4.1 Examples with $SU(5)$ gauge group

We take a dP_9 fibration over a base B_2, and denote by N_{B_2} the normal bundle for B_2 in B_3, with Chern class $c_1(N_{B_2}) = -t$. The slightly odd notation is essentially for "historical" reasons, as it matches with the notation of [23]. We use s to denote a coordinate on the normal bundle. In Section 2, the singularity was located at $v = 0$ and v/u can be taken the coordinate on the normal bundle. Comparing the line bundles, we see that $N_{B_2} = K^6_{B_2} \otimes \mathcal{N}|_{\sigma B_2}$ and hence the relation between t and η is given by

$$t = 6c_1(B_2) - \eta. \tag{4.1}$$

In order to write down a model, we need to do two things: first, we need to specify a suitable Weierstrass equation, and second we need to specify a flux. The Weierstrass equation for dP_9 is of the form

$$y^2 = x^3 + f\,x + g, \tag{4.2}$$

where f and g are sections of $K^{-4}_{B_3}$ and $K^{-6}_{B_3}$ respectively. Near $\sigma(B_2)$ we have $K_{B_3} \sim K_{B_2} \otimes N^{-1}_{B_2}$ and we can expand the Weierstrass equation

$$y^2 = x^3 + x\sum_{i=0}^{4} f_{4c_1+(i-4)t}\, s^i + \sum_{j=0}^{6} g_{6c_1+(j-6)t}\, s^j. \tag{4.3}$$

The f_{4c_1-nt} are sections of a line bundle over B_2 with Chern class $4\,c_1(B_2) - n\,t$, and the g_{6c_1-nt} are sections of line bundles with Chern class $6\,c_1(B_2) -$

$n\,t$. In order to specify an $SU(5)$ singularity along B_2, we need a section of the projective space bundle $\mathcal{W}_{SU(5)} \to B_2$ with fibers

$$\mathcal{M}_{SU(5)} = \mathbf{WP}^4_{(1,1,1,1,1)}. \tag{4.4}$$

That is, we need to specify five sections of line bundles with appropriate Chern classes as discussed in Section 2. In [21], these sections are denoted as

$$h_{c_1-t}, \quad H_{2c_1-t}, \quad q_{3c_1-t}, \quad f_{4c_1-t}, \quad g_{6c_1-t}. \tag{4.5}$$

In Section 2, we instead denoted them as

$$a_5, \quad a_4, \quad a_3, \quad a_2, \quad a_0. \tag{4.6}$$

The f's and g's are expressed in terms of these five sections as [21]

$$g_{6c_1-6t} \sim h^6_{c_1-t}, \quad f_{4c_1-4t} \sim h^4_{c_1-t}, \quad \dots \tag{4.7}$$

Near σ_{B_2}, i.e., to leading order in s, the discriminant locus can be expressed as [21]

$$\Delta \sim s^5\, a_5^4\, P_{8c_1-3t} + \mathcal{O}(s^6), \quad f \sim a_5^4, \quad g \sim a_5^6, \tag{4.8}$$

where P_{8c_1-3t} is a section of a line bundle with $c_1 = 8c_1(B_2) - 3t$. Explicitly, we have [23]

$$P_{8c_1-3t} = a_0 a_5^2 - a_2 a_3 a_5 + a_3^2 a_4. \tag{4.9}$$

Note that this section is quadratic in a_3 and a_5, so it is certainly not a generic representative of $8c_1 - 3t$.

From the discriminant, we can easily read off the equations of the matter curves. Using the Kodaira classification, the zero locus of a_5 corresponds to an enhancement from $SU(5) \to SO(10)$, so anti-symmetric matter is localized here. The curve $\{a_5 = 0\}$ is denoted by $\Sigma_{\mathbf{10}}$. The zero locus of P_{8c_1-3t} corresponds to the enhancement $SU(5) \to SU(6)$, so this is where fundamental matter is localized. The curve $\{P_{8c_1-3t} = 0\}$ is denoted by $\Sigma_{\mathbf{5}}$.

As an example [67], let us take B_2 to be a dP_8 surface, and $\eta = 6c_1(B_2)$. Then there exist holomorphic sections (4.5) with the required Chern classes, so the spectral cover exists, and $[\Sigma] = \eta - 5c_1(B_2) = c_1(B_2)$ is effective, in fact it is just the canonical class (which is an elliptic curve). From equation (2.69), the net number of generations is given by

$$N_{\rm gen} = -\lambda\eta \cdot (\eta - 5c_1(B_2)) = -6\lambda \tag{4.10}$$

so taking $\lambda = -\frac{1}{2}$ we get three generations.

As a second example, let us take B_2 to be an Enriques surface. We refer to [68] for facts about Enriques surfaces. A generic Enriques surface always contains two effective divisors D_1, D_2 with intersection numbers

$$D_1^2 = D_2^2 = 0, \qquad D_1 \cdot D_2 = 1, \qquad c_1 \cdot D_1 = c_1 \cdot D_2 = 0. \tag{4.11}$$

To construct an $SU(5)$ GUT model, we may take eg. $\eta = D_1 + 3D_2 + 5c_1 \sim D_1 + 3D_2 + c_1$. Then there exist sufficient holomorphic sections (4.5) with the required Chern classes. If the spectral cover is smooth, then we may apply formula (2.69):

$$N_{\rm gen} = -\lambda\eta \cdot (\eta - 5c_1(B_2)) = -6\lambda. \tag{4.12}$$

For $\lambda = -1/2$ we get exactly three chiral generations.

After the first version of this paper appeared, where this model was included in a footnote, we have performed a more detailed analysis which shows that the spectral cover in this homology class may generically be taken to be smooth. The argument is given below.

There are several ways to represent Enriques surfaces. We will represent an Enriques surface as an elliptically fibered surface with a rational base. Let us discuss some of its properties. First we recall the notion of a multiple fiber. Given an elliptic fibration $\pi : S \to \mathbf{P}^1$ whose generic fibers are smooth, a multiple fiber $\pi^{-1}(p)$ is a special fiber such that a multiple $m \cdot \pi^{-1}(p)$ is linearly equivalent to the generic fiber, with $m \geq 2$. An Enriques surface admits an elliptic fibration over $\mathbf{P}^1$, and such a fibration has exactly two multiple fibers $2F$ and $2F'$, with

$$K_S = \mathcal{O}(F - F'). \tag{4.13}$$

The divisors F and F' are often called a half-pencil.

Furthermore, one may show that the elliptic fibration admits a two-section, that is an irreducible curve G with $G \cdot f = 2$ for every fiber f. Now there are two possibilities for G. Either $G^2 = -2$, which is non-generic because the existence of -2-curves requires tuning the complex structure moduli (and moreover means S is nodal, i.e., singular); or $G^2 = 0$. In the latter case, the linear system $|2G|$ has dimension one and the generic element is an elliptic curve, hence defines another elliptic fibration for which G is one of the half-pencils. Moreover $G \cdot F = 1$ where F was a half-pencil of the first elliptic fibration. Therefore a generic Enriques has two effective divisors D_1 and D_2, with $D_1^2 = D_2^2 = 0$, $D_1 \cdot D_2 = 1$, where D_1 and D_2 can be explicitly thought of as half-pencils of two distinct elliptic fibrations π_1 and π_2. We will use the explicit description of these divisors given above for constructing models.

Above we gave an explicit description of a certain sublattice of the Picard lattice of a generic Enriques surface. Let D_1, D_1' be the two half-pencils corresponding to a pencil E_1. Then

$$D_1' - D_1 = c_1, \quad E_1 = 2\,D_1. \tag{4.14}$$

Similarly let D_2, D_2' correspond to a second pencil E_2, with

$$D_1 \cdot D_2 = 1. \tag{4.15}$$

In the following, it will be useful to have explicit names for the intersection points. Hence we will define the following four distinct points on S:

$$D_1 \cdot D_2' = \{p\}, \quad D_1' \cdot D_2 = \{p'\}, \quad D_1 \cdot D_2 = \{q\}, \quad D_1' \cdot D_2' = \{q'\}. \tag{4.16}$$

Our explicit model was given by

$$\eta = D_1 + 3D_2, \quad \lambda = 1/2. \tag{4.17}$$

This is closely related to the model mentioned in the first version of this paper, which had $\eta = D_1 + D_2, \lambda = 3/2$. Our analysis below goes through for this model just as well, but it has fewer complex structure moduli. (The linear system $D_1 + D_2$ is a pencil (i.e., $h^0 = 2$), and p, p' are its base points. Indeed, a basis for the pencil consists of $D_1 + D_2$ and $D_1' + D_2'$, and their intersection is p and p'. Moreover, these are the only points in the base locus.)

Our spectral surface is defined by the (projectivized) equation:

$$b_0 u^5 + b_2 u^3 v^2 + b_3 u^2 v^3 + b_4 u v^4 + b_5 v^5 = 0. \tag{4.18}$$

The b_{even} are elements of the linear system $D_1 + 3D_2$, which has $h^0 = 4$. An explicit basis consists of:

$$b_{\text{even}}: \qquad D_1 + 3D_2, \quad D_1' + D_2' + 2D_2, \quad D_1 + D_2 + 2D_2', \quad D_1' + 3D_2'. \tag{4.19}$$

The base locus is non-empty: it consists of the points p and p' with multiplicity one. Similarly, the b_{odd} are elements of the linear system $D_1 + 3D_2 + c_1 = D_1' + 3D_2$. An explicit basis consists of

$$b_{\text{odd}}: \qquad D_1' + 3D_2, \quad D_1' + 2D_2' + D_2, \quad D_1 + D_2' + 2D_2, \quad D_1 + 3D_2'. \tag{4.20}$$

Again the base locus is non-empty. It is not hard to see that it consists of the points q, q' with multiplicity one. Thus even though we will choose the b_i to be distinct linear combinations of the generators of these linear systems, there are two special points (namely p, p') where the b_{even} all vanish, and two more special points (namely q, q') where the b_{odd} all vanish.

Since the b_i above have some special properties, one might worry that the linear system $|C|$ has base points. This is indeed the case, it is not hard to see that the base points are given by

$$(x, u, v) \in (S, \mathbf{P}(\mathcal{O} \oplus K_S))|x = p, p' \text{ and } v = 0, x = q, q' \text{ and } u = 0. \tag{4.21}$$

Therefore we cannot apply Bertini to conclude that the generic spectral surface is smooth. Since the formula for the net generation number (2.69) was computed under the assumption that the spectral surface is smooth, it seems that we may have to do more work to write down suitable fluxes and check the generation number. But fortunately we can show that the spectral surface is in fact smooth generically despite the presence of base points in the linear system, and so we can automatically still use the universal flux and apply the formula (2.69) for the net number of generations.

To see this, consider an element in the linear system of C:

$$b_0 u^5 + b_2 u^3 v^2 + b_3 u^2 v^3 + b_4 u v^4 + b_5 v^5 = 0. \tag{4.22}$$

The generic such surface is smooth away from the base locus, so the only special loci we need to worry about correspond to the base points of $|C|$

found above. When $x = p$ or p', we get

$$\frac{\partial}{\partial v}(b_0(p)u^5 + .. + b_5(p)v^5) = 2u^3vb_3(p) + 5b_5(p)v^4. \tag{4.23}$$

Since $b_3(p)$ and $b_5(p)$ are generically non-zero, the only dangerous points also have $v = 0$. However

$$\frac{\partial}{\partial x}(b_0(x)u^5 + \cdots + b_5(x)v^5)|_{x=p,v=0} = \frac{\partial}{\partial x}(b_0(x)u^5)|_{x=p}, \tag{4.24}$$

which is generically non-zero, even at $x = p$ or p' because the base locus of $D_1 + 3D_2$ only has multiplicity one. Similarly one checks that $x = q, q'$ and $u = 0$ are smooth points generically.

Thus we have verified that the generic spectral cover in this linear system is smooth, and hence the application of (2.69) for the number of generations in our Enriques models with 'universal' fluxes is justified.

4.2 Examples with $SO(10)$ gauge group

We can repeat much of the discussion for $SU(5)$ with few changes. Again we consider the Weierstrass equation

$$y^2 = x^3 + x\sum_{i=0}^{4} f_{4c_1+(i-4)t}\, s^i + \sum_{j=0}^{6} g_{6c_1+(j-6)t}\, s^j. \tag{4.25}$$

In order to get an enhanced $SO(10)$ symmetry for $s = 0$, we need to specify a section of the weighted projective bundle $\mathcal{W} \to B_2$ with fiber

$$\mathcal{M}_{SU(4)} = \mathbf{WP}^3_{(1,1,1,1)}. \tag{4.26}$$

That is we need to specify four sections

$$h_{2c_1-t}, \quad q_{3c_1-t}, \quad f_{4c_1-t}, \quad g_{6c_1-t}. \tag{4.27}$$

The f's and g's are recovered as

$$f_{4c_1-2t} \sim h^2_{2c_1-t}, \quad g_{6c_1-3t} \sim h^3_{2c_1-t}, \quad g_{6c_1-2t} = q^2_{3c_1-t} - f_{4c_1-t}h_{2c_1-t}. \tag{4.28}$$

The leading terms in s are

$$\Delta = s^7 h^3_{2c_1-t} q^2_{3c_1-t} + \mathcal{O}(s^8), \quad f \sim s^2 h^2_{2c_1-t}, \quad g \sim s^3 h^3_{2c_1-t}. \tag{4.29}$$

The **16**'s are localized at $h_{2c_1-t}=0$ where the symmetry is enhanced to E_6, and the **10**'s are localized at $q_{3c_1-t}=0$ where the symmetry is enhanced to $SO(12)$.

As an example (not present in the literature as far as we know), let us take the base to be a del Pezzo surface with a -1-curve, denoted by E, and take $\eta = 7c_1(B_2) - 2E$. Then the quantization condition (3.12) is satisfied if λ is integral, and there exist sections (4.27) with the required Chern classes. From the analogue of (2.69) for $SO(10)$ we have

$$N_{\text{gen}} = -\lambda\eta\cdot(\eta - 4c_1(B_2)) = -(21c_1^2 - 24)\lambda. \tag{4.30}$$

Therefore by taking $S = dP_8$ and $\lambda = 1$ we get an explicit three-generation model.

5 Breaking the GUT group to the SM

So far we have discussed how to engineer GUT groups. To get a realistic model however we need some way to break the GUT group to the SM gauge group. As is well known, it is typically hard in string theory to obtain representations that are large enough to achieve this. For instance in the heterotic string let's suppose we would like to get four-dimensional fields in the adjoint representation of the GUT group. These would originate from Wilson lines on the Calabi–Yau. But on manifolds of $SU(3)$ holonomy there are no harmonic one-forms, so in this setting we cannot get any four-dimensional fields in the adjoint of the GUT group.

On the F-theory side, we could get adjoint matter in four dimensions from zero modes of the gauge field or of the adjoint field of the eight-dimensional gauge theory. In duals of the heterotic string, the gauge 7-brane is wrapped on a base B_2 which has $h^{0,1} = h^{2,0} = 0$, hence we get no such zero modes. In order to get adjoint fields we must wrap our gauge brane on a surface of general type or a $K3$ surface. For instance the $K3$ surface can be realized as a quartic in $\mathbf{P}^3$; since the canonical bundle is trivial and the normal bundle has many sections, it is not hard to see that we can easily get an elliptic fibration with I_5 singular fibers along such a surface. However just as in conventional four-dimensional models we would then have to face the doublet–triplet splitting problem. Hence we prefer to look for an alternative mechanism.

Another idea, which was already considered in the early days of heterotic model building (see e.g., [45]), is to turn on certain $U(1)$ fluxes. We have essentially already seen this in the context of coincident branes. For instance in the case of an $SU(5)$ model, we could turn on an internal flux on σ_{B_2} for the gauge field that corresponds to hypercharge. The commutant of this $U(1)$ in $SU(5)$ is clearly $SU(3) \times SU(2) \times U(1)$. However turning on such a flux will typically spoil gauge coupling unification. As we discussed earlier, the $U(1)$ generator whose flux is turned on will swallow an RR axion and become massive. This can be avoided by turning on a $U(1)$ flux in the same cohomology class in the hidden sector. The axion then couples to the sum of these $U(1)$'s, and the difference will remain massless. As discussed in [45], because hypercharge is now a linear combination of the 'original' hypercharge generator and a $U(1)$ in the hidden sector, the model is not truly unified and this mechanism would typically change the relation of the $U(1)$ coupling to the $SU(2)$ and $SU(3)$ couplings at the GUT scale[14] .

A third approach for breaking the GUT group, which does not have the usual baggage of four-dimensional GUTs and has the cleanest phenomenological features, is to use discrete Wilson lines. Namely if B_2 admits a non-trivial fundamental group, then we could turn on a discrete G-flux, or perhaps we could fiber the dP_9 over B_2 in such a way that the GUT group is globally broken to the Standard Model group. If we restrict to the usual models with heterotic duals, then the only allowed B_2 which has non-trivial fundamental group is the Enriques surface, and it does not lead to consistent models due to lack of stability in the hidden sector. However locally it is not hard to construct such models. As an example, consider the three generation $SU(5)$ model from Section 4.1 based on the Enriques surface. It allows a Z_2 Wilson line to break the $SU(5)$ GUT group to $SU(3) \times SU(2) \times U(1)$. Presumably this local model has global embeddings which are not dual to the heterotic string.

However, we may give a stronger argument against such models: any smooth model in F-theory with discrete Wilson lines has light lepto-quarks. To see this, suppose more generally that we break the GUT group through a line bundle L. Then the spectrum of lepto-quarks descending from the eight-dimensional gauge and adjoint fields is determined by the formulae in section 2.3. In particular, the Euler character

$$\chi(B_2, L) = \frac{1}{2}c_1(L)^2 - \frac{1}{2}c_1(L)c_1(K) + \frac{1}{12}(c_2(S) + c_1(K)^2) \qquad (5.1)$$

must vanish. Now let us assume that $c_1(L)$ vanishes as for discrete Wilson lines. Then the above is just equal to $\chi(B_2, \mathcal{O}) = h^{0,0}(B_2) - h^{0,1}(B_2) + h^{0,2}(B_2)$. But we assume that $h^{0,1}(B_2) = 0$ (and actually also $h^{0,2}(B_2) = 0$) in order to avoid massless adjoint fields. It follows that vanishing $c_1(L)$

[14] On the other hand, such a coupling to the hidden sector provides an interesting possibility for mediation of SUSY breaking [69].

implies that $\chi(B_2, L)$ cannot vanish, so models with discrete Wilson lines also necessarily come with light exotic matter, whatever the surface that we wrap the 7-branes on.

The fact that models based on rational surfaces do not seem to allow for discrete Wilson line breaking raises a puzzle though, because they arise in heterotic/F-theory duality. On the heterotic side one may certainly construct elliptically fibered three-folds with a rational base and with a finite fundamental group. These three-folds do not have a section, only a multi-section. However they are quotients by an automorphism of elliptically fibered Calabi–Yaus with a section, so we can construct the F-theory dual of the cover. What does the automorphism get mapped to?

Consider a freely acting involution τ from the elliptically fibered three-fold to itself. Then τ can be decomposed as

$$\tau = t_\xi \circ \alpha, \tag{5.2}$$

where α maps the zero section of the elliptic fibration to itself and t_ξ is translation by a section ξ different from σ_{B_2}. The automorphism α induces an involution α_{B_2} on the base B_2 which necessarily has fixed points. Now t_ξ acts trivially on the Wilson lines on each T^2 fiber, so it does not appear to induce any action on the dual T^2 or the dP_9 surface constructed from the dual T^2 and the Wilson lines of the E_8 bundle. Therefore the action of τ on the heterotic side seems to induce only the action of α_{B_2} on the F-theory side, which has fixed points, and we would have to understand how to deal with the fixed points. Unfortunately, F-theory is currently only understood as a large volume expansion. When the F-theory base B_3 has singularities, there is no small parameter available and no clear way to understand the physics.

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Appendix A Spinors and complex geometry

In this appendix, we would like to review some properties of spinors on complex manifolds. We will not be very rigorous; instead we will use the fastest route available. See [70] for a more thorough treatment.

Suppose we are given a m-dimensional Riemannian manifold M with a spin structure, i.e., a lifting of the structure group $SO(m) \to \mathrm{Spin}(m)$. On such a manifold, we may construct the spin bundle S associated to the spinor representation of $\mathrm{Spin}(m)$. Given a local ortho-normal frame e_a and a set of Γ-matrices satisfying $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$, the Dirac operator is defined to be the first order differential operator given by

$$\not{D} = \Gamma^a \nabla_a, \tag{A.1}$$

where ∇_a is the lift of the Levi–Civita connection.

When M is $2n$-dimensional, the spinor representation of $\mathrm{Spin}(2n)$ is reducible, and S decomposes as $S = S^+ \oplus S^-$. The Dirac operator interchanges these representations, i.e., $\not{D} : S^\pm \to S^\mp$. If M is a Kähler manifold, i.e. if the holonomy can be further reduced to $U(n) \subset SO(2n)$, then we can relate the Dirac operator to certain standard operators appearing in holomorphic geometry.

Let us first consider one-dimensional Kähler spaces. We can then define a spinor to be an object which gets mapped to minus itself under a 2π rotation on every holomorphic tangent plane. This identifies it as a section of the bundle $S = T^{-1/2} \oplus T^{1/2}$ where T is the holomorphic tangent bundle. Under a rotation by π (i.e., a reflection $z \to -z$) spinors transform by $\pm i$. The sign is called its chirality. The bundle T^{-1} is also known as K, the canonical bundle. Moreover if we have a Kähler metric then we can identify T with the bundle of $(0,1)$ forms $\Omega^{(0,1)}$ by mapping sections as $f^z\,\partial_z \to f^z g_{z\bar{z}}\,d\bar{z}$. Therefore we can also write

$$S = K^{1/2} \oplus \Omega^{(0,1)}(K^{1/2}). \tag{A.2}$$

Up to normalization, the Dirac operator $\not{D}$ therefore corresponds to $\bar{\partial}_A + \bar{\partial}_A^\dagger$, where $\bar{\partial}_A$ is the Dolbeault operator coupled to $K^{1/2}$. More explicitly we can write this as

$$\not{D} = \begin{pmatrix} 0 & -\partial_z - A_z \\ \bar{\partial}_{\bar{z}} + A_{\bar{z}} & 0 \end{pmatrix}. \tag{A.3}$$

We can also couple the spinors to various bundles, by adding further gauge fields.

We can generalize this to higher dimensions by using a splitting principle. That is we decompose the holomorphic tangent bundle for a complex n-fold formally into a sum of n line bundles and tensor the corresponding spinor bundles together. For instance on a complex three-fold we would decompose $T = T_1 \oplus T_2 \oplus T_3$ and tensor the $T_i^{-1/2} \oplus T_i^{1/2}$ together. The result, after reconstructing representations of the full $U(n)$ holonomy, is

$$S^+ = \sum_{p\text{even}} \Omega^{(0,p)}(K^{1/2}) \quad S^- = \sum_{p\text{odd}} \Omega^{(0,p)}(K^{1/2}). \tag{A.4}$$

The Dirac operator is then formally thought of as the sum of the Dirac operators associated to each T_i.

Appendix B Branes and twisted Yang–Mills–Higgs theory

In this appendix, we briefly review the Yang–Mills theories living on branes in string theory, with an emphasis on curved embeddings of the brane in space-time.

The collective coordinates of Dp-branes are are given by the field content of maximally supersymmetric Yang–Mills theory in $p+1$ dimensions. They may all be obtained by starting with $N = 1$ Yang–Mills theory in ten dimensions and reducing it to $p+1$ dimensions. For applications to F-theory we would like to understand how to reduce ten-dimensional Yang–Mills theory to a complex submanifold denoted B. The ten-dimensional action is of the form

$$\int d^{10}x - \frac{\mathrm{i}}{2g^2}\mathrm{Tr}(\bar{\psi}\, \not{D}\, \psi) - \frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{B.1}$$

Now let us do the reduction, assuming the case of a 7-brane wrapped on a surface S in a Calabi–Yau three-fold. Following Hitchin [71] we write

$$A^{0,1} = A_{\bar{z}_1}(z_1, z_2) + A_{\bar{z}_2}(z_1, z_2) + \Phi_{\bar{3}}(z_1, z_2). \tag{B.2}$$

Here we used the splitting principle to express the tangent bundle as $T = T_1 \oplus T_2 \oplus K_S \oplus T\mathbf{R}^{1,3}$. Thus our data consist of a connection A on a bundle

E on S, together with the complex "Higgs field" Φ, which we may view as a map $\Phi : E \to E \otimes K_S$. Similarly spinors now becomes sections of

$$\begin{aligned}(T_1^{-\frac{1}{2}} \oplus T_1^{\frac{1}{2}}) \otimes (T_2^{-\frac{1}{2}} \oplus T_2^{\frac{1}{2}}) \otimes (K_S^{-\frac{1}{2}} \oplus K_S^{\frac{1}{2}}) \\ = \sum_p \Omega^{(0,p)}(K_B^{1/2}) \otimes (K_S^{-\frac{1}{2}} \oplus K_S^{\frac{1}{2}})\end{aligned} \tag{B.3}$$

tensored with four-dimensional spinors.

The ten-dimensional gaugino variation is of the form

$$\delta\psi \simeq F_{\mu\nu}\Gamma^{\mu\nu}\epsilon. \tag{B.4}$$

Thus requiring a BPS solution means that we have to solve $F_{\mu\nu}\Gamma^{\mu\nu}\epsilon = 0$, or equivalently

$$F^{0,2} = 0, \quad g^{i\bar{j}}F_{i\bar{j}} = 0. \tag{B.5}$$

These are the hermitian Yang–Mills equations. To reduce this to eight dimensions, we now substitute (B.2)

$$\delta\psi \simeq (F_{\mu\nu}\Gamma^{\mu\nu} + 2D_\mu\Phi_a\Gamma^{\mu a} + [\Phi_a, \Phi_b]\Gamma^{ab})\epsilon = 0, \tag{B.6}$$

where we use μ, ν for real indices tangent to the brane, and a, b for real indices normal to the brane. The F-terms and D-terms of the effective four-dimensional gauge theory can be read off from the right-hand side. In particular the F-terms come from the lack of integrability of $\bar{D} = \bar{\partial} + A^{0,1} + \Phi$. Preservation of supersymmetry thus requires $\bar{D}^2 = 0$. By decomposing we get the following equations:

$$F^{0,2} = 0, \quad \bar{\partial}\Phi_{\bar{3}} + [A^{0,1}, \Phi_{\bar{3}}] = 0, \tag{B.7}$$

Similarly the D-terms can be written as[15]

$$g^{i\bar{j}}F_{i\bar{j}} + [\Phi_{\bar{3}}^\dagger, \Phi_{\bar{3}}] = 0. \tag{B.8}$$

Let us take a closer look at these equations. The first equation in (B.7) says that the gauge field is a connection on a holomorphic bundle (i.e., the transition functions may all be chosen holomorphic). As is well known,

[15] We would like to thank J. Heckman for pointing out the term involving Φ, which is crucial for a correct understanding of the D-terms but which we had initially ignored.

we may then apply a complexified gauge transformation so that the anti-holomorphic component $A^{0,1}$ vanishes in holomorphic frames. The $(0,1)$ part of the gauge covariant derivative then reduces to the Dolbeault operator $\bar{\partial}$, and the second equation in (B.7) says that Φ must be a holomorphic section.

The D-terms are not invariant under the complexified gauge transformations, and require us to choose a hermitian metric, or equivalently a reduction of the complexified structure group to a compact subgroup. Recall that given a hermitian metric h, we can pick a canonical connection by requiring the covariant derivative to be compatible with the hermitian metric and with the complex structure. This determines $A^{1,0} \sim -(\partial h)h^{-1}$. Thus assuming we have fixed the F-term data, we see that the D-terms may be viewed as an equation for the hermitian metric h on E. This equation is a highly non-linear PDE, which is virtually impossible to solve explicitly. Nevertheless, existence and uniqueness of a solution can be reformulated as an algebro-geometric criterion.

A subbundle $F \subset E$ is said to be a Higgs subbundle if $\Phi(F) \subset F \otimes K$. A Higgs bundle is said to be J-stable if

$$\mu(F) < \mu(E) \tag{B.9}$$

for every Higgs subbundle, where the slope is defined as usual, $\mu = J$-degree/rank. A Higgs bundle is poly-stable if it is a direct sum of stable Higgs bundles with the same slope. If the Higgs bundle is polystable, then the D-terms should have a unique solution. (For abelian bundles, this requires adding an explicit Fayet–Iliopoulos-like term to the equation). The corresponding hermitian metric is sometimes called the hermitian-Einstein metric.

For smooth F-theory compactifications, we encountered the primitiveness condition $J \wedge G = 0$. It is clearly reminiscent of $i^*J \wedge F = 0$ on the spectral cover, in the gauge where A and Φ are diagonalizable. This might seem to suggest that D-flatness corresponds not to stability, but to the statement that the hermitian connection on the spectral line bundle is J-primitive. However this is not correct. It is actually well known that the push-down of the hermitian connection on the spectral line bundle should not be identified with the solution of the D-terms of the Higgs bundle. To see this, note that if the spectral cover is locally defined by $y^2 - z = 0$ then

$$F_{z\bar{z}} \sim F_{y\bar{y}}/(z\bar{z})^{1/2}, \tag{B.10}$$

which diverges at the branch locus, here given by $z = 0$. Thus the push-down of the hermitian connection on the spectral line bundle is singular at the branch locus. However, Donaldson–Uhlenbeck–Yau provides a smooth solution to the D-term equations on the Higgs bundle. In particular we have $[\Phi^\dagger, \Phi] \neq 0$ for the actual hermitian metric solving the D-terms.[16] We essentially already discussed the source of this apparent discrepancy in Section 2.6. It arises because the supergravity derivation of $J \wedge G = 0$ requires a large and smooth four-fold. This is different from the regime where we get parametrically light $M2$-branes, which is where the $8d$ gauge theory description can be trusted.

In [6] we therefore proposed that when there are parametrically light $M2$-branes, the correct criterion is existence of the hermitian-Einstein metric in the Higgs bundle picture. More generally, we expect that D-flatness in F-theory compactifications is described by a stability condition. This is something we can study in the four-fold picture. Stability can be phrased in terms of Fayet–Iliopoulos terms. Thus a question that we can ask is how the Fayet–Iliopoulos terms are related in the different pictures, and if there are certain classes of interesting compactifications where stability should be automatic. We turn to this in the next section.

For further reading on the structures discussed here, see [6, 71–75].

Appendix C Definition of G_γ

In this appendix, we give a detailed definition of the class G_γ on the F-theory space Y_4, obtained via the cylinder map from a $(1,1)$-class $\gamma \in H^{1,1}(C, \mathbf{Z})$ on the heterotic spectral cover C. We also consider Fayet–Iliopoulos terms, which are an important ingredient for defining stability conditions. We will see that the classical expression for the Fayet–Iliopoulos terms in F-theory can be matched at least qualitatively with the tree level and one-loop expressions on the heterotic side.

Let us briefly recall the general set-up. On the heterotic side, the E_8-bundle restricted on each elliptic fiber is determined by its Wilson line, an element

$$f \in \mathrm{Hom}(\Lambda_{E_8}, T^2) \tag{C.1}$$

[16] A second issue in the comparison is that J is not quite a pull-back of a class on the base.

taken modulo $\mathcal{W}_{E_8}$. Here Λ_{E_8} represents the root lattice of E_8, the dual of the coroot lattice. This lattice is actually self-dual, so we do not need to be too careful on this point. Now let us fiber over B_2, and let us consider not modding out by $\mathcal{W}_{E_8}$. Then the fibers $\mathrm{Hom}(\Lambda_{E_8}, T^2)$ fit together in a flat vector bundle of rank eight, with structure group $\mathcal{W}_{E_8}$. In order to represent this data by a spectral cover of finite degree, we have to choose a suitable representation. The physically relevant representation is the adjoint representation of E_8. This is a minuscule representation: its weights are the roots, and these lie in a single Weyl group orbit Roots $\subset \Lambda_{E_8}$. The the image

$$f(\mathrm{Roots}) \subset T^2 \tag{C.2}$$

gives a collection of 240 points on the T^2, one for each root of the adjoint representation of E_8. The Weyl group $\mathcal{W}_{E_8}$ acts as the monodromy group on these 240 points. Fibering over B_2 yields the degree 240 spectral cover of an E_8 bundle. For a generic E_8 bundle, this is the smallest non-trivial permutation representation available, i.e., it is the smallest representation we can use for the monodromy action on a spectral cover. If the monodromy is smaller than E_8, there can be other available permutation representations and thus there will be spectral covers of smaller degree which capture the same information.

On the F-theory side, the same data determines a dP_8 surface. The anti-canonical divisor is identified with the T^2 above, and the 240 lines of the dP_8 intersect the T^2 in the 240 points above. Blowing up, we get a dP_9 with a distinguished zero section. Generically the 9th point is not on any of the lines, and the 240 lines of the dP_8 lift to 240 sections of the dP_9, disjoint from the zero section. When the ninth point does lie on a line, the curve on the dP_9 corresponding to this line is its total transform. This is a numerical section, i.e., an effective curve with self-intersection -1 and intersection number 1 with the elliptic fiber. Its intersection number with the zero section still vanishes. But this is now a reducible curve, consisting of two components: the zero section plus the proper transform of the original line. The Weyl group $\mathcal{W}_{E_8}$ acts as a monodromy group on these 240 lines. Fibering over B_2 yields the cylinder. In [14], this was the variety R' whose fibers over each point in B_2 are the 240 lines in the dP_8's. Since we work here with Y_4, we consider instead the total transform R of R'. This is the subvariety of Y_4 whose fibers over each point in B_2 are the 240 numerical sections in the dP_9's. By the above observation, the intersection of R with the zero section σ_{B_3} is the union of lines (in either R or σ_{B_3}) over the intersection curve $\Sigma = C \cdot \sigma(B_2)$. Explicitly,

$$R \cdot \sigma_{B_3} = p_R^*[\Sigma]_C = \rho^*[\Sigma]_{B_2}. \tag{C.3}$$

Physically it is probably more natural to identify the roots with -2 classes in the dP_8. However they are equivalent to the lines up to a shift by the canonical class of the dP_8, which does not affect the arguments below due to the tracelessness condition of the fluxes. The lines are effective classes and are easier to keep track of.

Let us fix the notation for this appendix:

$$\begin{aligned}
\pi_Y : Y_4 \to B_3 &\quad \text{elliptic fibration} \\
\sigma : B_3 \to Y_4 &\quad \text{the section} \\
\rho : B_3 \to B_2 &\quad P^1 \text{ fibration} \\
\sigma_{B_2} &\quad \text{its section, embedded in either } B_3 rm or Y \\
Z \subset Y_4 &\quad \pi_Y^{-1} \text{ of a section of } \rho \\
i_Z : Z \hookrightarrow Y &\quad \text{the natural inclusion} \\
p : Y_4 \to B_2 &\quad dP_9 \text{ fibration} \\
\pi_C : C \to B_2 &\quad \text{the heterotic spectral cover} \\
\pi_Z : Z \to B_2 &\quad \text{the restriction of } p \\
p_R : R \to C &\quad \text{the ``cylinder'', or union of lines in the } dP_8\text{'s} \\
&\quad \text{(i.e. numerical sections of } dP_9\text{'s, disjoint from } \sigma) \\
&\quad \text{parametrized by points of } C \\
\Sigma = C \cdot \sigma_{B_2} &\quad \text{the matter curve} \\
j : (C = R \cap Z) \subset R &\quad \text{the inclusion ``at infinity''} \\
i_R : R \hookrightarrow Y &\quad \text{the natural inclusion}
\end{aligned} \tag{C.4}$$

The spectral line bundle is mapped to G-flux. Given a class

$$\gamma \in H^2(C, \mathbf{Z}), \quad \pi_{C*}\gamma = 0 \tag{C.5}$$

in [14] the dual G-flux was defined as

$$G \stackrel{?}{=} i_{R*} p_R^* \gamma \in H^4(Y_4, \mathbf{Z}). \tag{C.6}$$

Actually, there is also a shift by $r/2$ in the quantization law on the heterotic side, and by $c_2/2$ on the F-theory side, but one can easily correct for this and we will not mention it any further. In [14], Y_4 was considered to be a dP_8-fibration over B_2. As we discussed in Section 3.2.1, in the case of dP_9-fibrations there is an ambiguity in the mapping from $H^2(C) \to H^4(Y_4)$. On the other hand, not all generators of $H^4(Y, \mathbf{Z})$ can be realized off-shell as G-fluxes in F-theory. By requiring that γ gets mapped to an allowed G-flux, we can fix the ambiguity.

A more conceptual way to arrive at the same conclusion is as follows. As we vary the dP_9 surface over B_2, the 240 lines on the dP_8 (which becomes sections of dP_9 after blow-up) are exchanged by monodromies which take value in the E_8 Weyl group. Although the Weyl group does not act on the spectral cover or its cohomology, we can decompose the cohomologies $R^0\pi_{C*}\mathbf{R}$ and $R^2p_*\mathbf{R}$ into isotypic pieces. On the heterotic side, the 240 dimensional permutation representation of $\mathcal{W}_{E_8}$ on each fiber of the 240-fold spectral cover can be decomposed in the following irreducible representations

$$\mathbf{240} = \mathbf{1} + \mathbf{8} + \mathbf{35} + \mathbf{84} + \mathbf{112}. \tag{C.7}$$

The one-dimensional piece corresponds to the sum over the Weyl group orbit. The eight-dimensional piece corresponds to the action of W on the root lattice of E_8.

Similarly, on the F-theory side, the second cohomology of the dP_9 decomposes in the following irreducible representations

$$\mathbf{10} = \mathbf{1_b} + \mathbf{1_f} + \mathbf{8}. \tag{C.8}$$

Now the one-dimensional pieces come from the base and fiber, respectively, while the eight-dimensional piece is the part of the second cohomology of the dP_9 orthogonal to the base and fiber. This was denoted by H^2_Λ in [14].

Now the cylinder map in (C.6) does not exactly identify the eight-dimensional representations on both sides. To get the map that does this, we need to subtract a singlet on the F-theory side. This is the "projected cylinder map" of [14].

An allowed G-flux has three indices along B_3 and one index along the elliptic fiber. More precisely, allowed G-fluxes must be orthogonal to

(i) classes in $\sigma_{B_3*}H_4(B_3,\mathbf{Z})$;
(ii) classes in $\pi_Y^*H^4(B_3,\mathbf{Z})$.

However, with the above definition of G we find that

$$G \cdot_Y \sigma_{B_2} = \gamma \cdot_C \Sigma \neq 0, \tag{C.9}$$

where $\Sigma = C \cdot \sigma_{B_2}$. Therefore the flux we defined above is not allowed in F-theory, and the map from γ to G-flux has to be modified by projecting on $\sigma_{B_3*}H_4(B_3)^\perp$. We claim that the correct definition of the G-flux dual to

γ is[17]

$$G_\gamma \equiv \mathrm{i}_{R*} p_R^* \gamma - n_\gamma [dP_9], \quad n_\gamma = \gamma \cdot_C \Sigma, \tag{C.10}$$

where $[dP_9] = p^{-1}(pt)$ denotes the class of a dP_9 fiber of Y_4.

Our first task is to show that this is an allowed G-flux that is orthogonal to all classes of type (i) and (ii). Let us make a list of these classes. Since B_3 is a $\mathbf{P}^1$-fibration, $H_4(B_3, \mathbf{Z})$ is spanned by

$$H_4(B_3, \mathbf{Z}) = \mathrm{span}\{\sigma_{B_2}, \rho^*(w)\}, \quad \text{where } w \in H_2(B_2, \mathbf{Z}). \tag{C.11}$$

Similarly,

$$H_2(B_3, \mathbf{Z}) = \mathrm{span}\{\sigma_{B_2 *} w, \rho^*(pt)\}, \quad \text{where } w \in H_2(B_2, \mathbf{Z}) \tag{C.12}$$

and therefore,

$$\pi_Y^* H^4(B_3, \mathbf{Z}) = \mathrm{span}\{i_{Z*} \pi_Z^* w, p^*(pt)\}, \quad \text{where } w \in H_2(B_2, \mathbf{Z}). \tag{C.13}$$

Now we simply proceed by computing the intersections with all these classes. We have

$$\begin{aligned} i_{R*} p_R^* \gamma \cdot i_{Z*} \pi_Z^* w &= i_{C*} \gamma \cdot_Z \pi_Z^* w \\ &= \gamma \cdot_C \pi_C^* w \\ &= 0. \end{aligned} \tag{C.14}$$

Similarly

$$i_{R*} p_R^* \gamma \cdot p^*(pt) = p_R^* \gamma \cdot_R p_R^* \pi_C^*(pt) = 0. \tag{C.15}$$

It is also easy to show that

$$[dP_9] \cdot i_{Z*} \pi_Z^* w = 0, \qquad [dP_9] \cdot p^*(pt) = 0 \tag{C.16}$$

since we may choose the support of these classes to be disjoint. Therefore G_γ is indeed orthogonal to classes of type (ii).

[17] We are indebted to Taizan Watari for pointing out that the expression (C.6) for the G-flux in a previous version of this paper needed to be projected as in [14] and (C.10) in order to produce an allowed flux.

As for orthogonality against classes of type *(i)*, we use (C.3) and compute:

$$i_{R*}p_R^*\gamma \cdot \sigma_{B_3*}\rho^* w = \rho^*(\gamma \cdot \Sigma) \cdot \rho^* w = 0. \tag{C.17}$$

We have already pointed out in C.9 that $i_{R*}p_R^*\gamma \cdot \sigma_{B_2} = \gamma \cdot \Sigma$, and it is not hard to check that

$$[dP_9] \cdot \sigma_{B_3*}\rho^* w = 0, \quad [dP_9] \cdot \sigma_{B_2} = 1. \tag{C.18}$$

Therefore G_γ is also orthogonal to classes of type *(i)*. It is therefore an allowed flux in F-theory, as claimed.

Next we discuss the wedge product $G_\gamma \wedge J$, where J is the Kähler class. Any Kähler class J on B_3 can be decomposed as

$$J = p^* J_{B_2} + a\, \pi_Y^* J_0, \tag{C.19}$$

where $J_{B_2} \in H^2(B_2, \mathbf{R})$, a is a real number, and J_0 is any one class in $H^2(B_3)$ not in the image of $\rho^* : H^2(B_2) \to H^2(B_3)$. We will take J_0 to be the class in $H^2(B_3)$ of the divisor σ_{B_2}. We could also take the section at infinity J_∞, which is related to J_0 by $J_0 - J_\infty = \rho^* c_1(N_{B_2/B_3})$. Note that $\pi_Y^* J_\infty$ is the class in $H^2(Y)$ of the divisor Z. We will see that $G_\gamma \wedge J$ is closely related to certain moment maps associated to $U(1)$ gauge symmetries, known as Fayet–Iliopoulos parameters.

Since we are working with harmonic forms, we can work modulo torsion, and calculating $G_\gamma \wedge J$ is equivalent to finding all the intersection products

$$\int_{Y_4} G_\gamma \wedge J \wedge D \tag{C.20}$$

for any $D \in H^2(Y_4)$. Let us make a list of such classes. To identify all the divisors in Y_4, we use the Leray spectral sequence associated to the fibration $Y_4 \to B_2$, following [14]. The divisors in Y_4 are spanned by

$$H^2(Y_4) \sim H^2(B_2, R^0 p_*) \oplus H^0(B_2, R^2 p_*). \tag{C.21}$$

Some of these generators may be lifted by higher order differentials in the spectral sequence, but we are interested in the span and this level of analysis suffices. Generators of $H^2(B_2, R^0 p_*)$ correspond to pull-backs of divisors in B_2 by p^*. The second cohomology of the dP_9 can be split up into three pieces:

$$R^2 p_* = [\mathbf{P}^1]_{\text{base}} + [T^2]_{\text{fibre}} + \Lambda_{E_8}. \tag{C.22}$$

This decomposition makes sense globally over B_2 because $Y^4 \to B_3$ has a section. The rank eight lattice Λ_{E_8} is denoted by H^2_Λ in [14]. Using this decomposition, we see that the divisors in Y_4 are spanned by

$$H^2(Y_4, \mathbf{Z}) = \text{span}\{\sigma_{B_3}, \pi_Y^* H^2(B_3, \mathbf{Z}), \Lambda\}, \tag{C.23}$$

where $\Lambda = H^0(B_2, \Lambda_{E_8})$ is the coroot lattice defined in (2.5).

We claim that $G_\gamma \wedge J$ is automatically orthogonal to divisors of the first two types in (C.23). This is easy to see: J itself is a class in $\pi_Y^* H^2(B_3, \mathbf{Z})$, and its intersection with a divisor in the span of σ_{B_3} and $\pi_Y^* H^2(B_3, \mathbf{Z})$ gives a four-cycle of type (i) or (ii). But G_γ is always orthogonal to such four-cycles, and hence the claim follows. The remaining intersections are of the form

$$\int_Y G_\gamma \wedge J \wedge \omega, \quad \omega \in \Lambda. \tag{C.24}$$

Although for a generic E_8 bundle the spectral cover will be irreducible (since the 240 sheets form a single orbit of $\mathcal{W}_{E_8}$), physically we are typically interested in a situation where the monodromy group is smaller and the cover decomposes into several pieces. For instance suppose that we restrict the monodromies to lie in the Weyl group of some $SU(5)_{\text{H}}$ subgroup of E_8. The adjoint representation of E_8 decomposes as

$$\mathbf{248} = (\mathbf{24},\mathbf{1}) + (\mathbf{1},\mathbf{24}) + (\mathbf{5},\mathbf{10}) + (\bar{\mathbf{5}},\overline{\mathbf{10}}) + (\mathbf{10},\bar{\mathbf{5}}) + (\overline{\mathbf{10}},\mathbf{5}) \tag{C.25}$$

under $SU(5)_{\text{H}} \times SU(5)_{\text{GUT}}$. Thus in this example, the degree 240 spectral cover splits up into pieces of degree 20, 1, 5 and 10, with various multiplicities (determined by the dimension of the corresponding $SU(5)_{\text{GUT}}$ representation). If we additionally restrict the monodromy so as to get extra abelian gauge symmetries, the spectral cover would split up further.

In order to proceed, we further subdivide the divisors in Λ. The lattice Λ_{E_8} may be split into two pieces (up to torsion, which is irrelevant for us): a varying piece, also known as the Mordell–Weil lattice; and a fixed piece which is locally constant, also known as the vertical component:

$$\Lambda_{E_8} = \Lambda_{\mathrm{MW}} \oplus \Lambda_{\mathrm{vert}} \tag{C.26}$$

The vertical piece is generated by the exceptional divisors of the ADE singularity of the generic dP_9. Although Λ_{vert} is locally constant, as we go from patch to patch there may be some non-trivial monodromies. Whether such monodromies are present should correspond to the "split" versus "non-split" distinction in Tate's algorithm.

Let us denote divisors in $H^0(B_2, \Lambda_{\mathrm{vert}})$ by E. By definition, the remaining generators in $H^0(B_2, \Lambda_{\mathrm{MW}})$ can be identified with certain linear combinations of the non-vanishing lines in dP_9 varying over B_2, fitting together into a divisor of Y_4. That is, they correspond to certain linear combinations of the irreducible components of the cylinder. They are interpreted as "extra" $U(1)$'s, i.e., generators of $\mathbf{g}/[\mathbf{g}, \mathbf{g}]$ where $\mathbf{g}$ is the Lie algebra of the $4d$ gauge group. Let us denote generators of $H^0(B_2, \Lambda_{\mathrm{MW}})$ by ω_X.

Now recall that we are supposed to compute (C.32) for any generator of $\Lambda = H^0(B_2, \Lambda_{E_8})$. Since Λ is spanned by E and ω_X, we will compute these intersections. We first note that

$$i_* p_R^* \gamma \cdot p^* J_{B_2} = \sum_k i_* p_R^* (\gamma_k \cdot \pi_{C_k}^* J_{B_2}). \tag{C.27}$$

Geometrically, this represents a collection of lines sitting over a finite number of points in B_2. Let us concentrate on one such dP_9 fiber. Let us also assume that Λ_{vert} is actually constant, so there are no monodromies that can further break the gauge group (the simply laced cases), which is the case usually relevant for phenomenological applications. Now all the lines that lie in a single orbit of the Weyl group have the same intersection number with a given exceptional cycle of the dP_9. Furthermore, the intersection pairing is preserved by the monodromies. Therefore we get

$$i_* p_R^* \gamma \cdot p^* J_{B_2} \cdot E = \sum_k n_k (\gamma_k \cdot \pi_{C_k}^* J_{B_2}), \tag{C.28}$$

where n_k is the intersection number of a line in R_k with E. For our purpose, the only thing that matters is that this number is the same for all the sheets in a single Weyl orbit. But then by tracelessness of γ, the sum over all k belonging to a single Weyl orbit vanishes. Hence

$$i_* p_R^* \gamma \cdot p^* J_{B_2} \cdot E = 0. \tag{C.29}$$

The other intersections are easier to check. We have

$$[dP_9] \cdot p^* J_{B_2} \cdot E = 0, \tag{C.30}$$

since we can take the support of the dP_9 fiber and $p^*J_{B_2}$ to be disjoint. Further

$$G_\gamma \cdot \pi_Y^* J_0 \cdot E = 0 \tag{C.31}$$

follows because because $\pi_Y^* J_\infty$ and E have disjoint support, and the difference between $\pi_Y^* J_0$ and $\pi_Y^* J_\infty$ has zero intersection by the calculation above. (Recall we took the singularities to be along the zero section of B_3, and we defined J_∞ to be the section at infinity, which is disjoint from it). Therefore $G_\gamma \wedge J$ is automatically orthogonal to E.

The only remaining intersections are $G \wedge J \wedge \omega^X$. In particular, if there are no $\omega^X \in H^0(B_2, \Lambda_{\mathrm{MW}})$ then the G-flux should be J-primitive. These remaining intersections correspond to the Fayet–Iliopoulos terms in F-theory:

$$\xi_F^X \simeq m_{10}^4 \int_Y G_\gamma \wedge J \wedge \omega^X, \quad \omega^X \in H^0(B_2, \Lambda_{\mathrm{MW}}). \tag{C.32}$$

For a derivation, see Section 5.2 of [76]. They must vanish in the $11d$ supergravity regime, but they may be non-vanishing in F-theory. In order to compare with the heterotic string, it is better to use $4d$ quantities, so we rewrite the Fayet–Iliopoulos term as

$$\xi_F^X \simeq M_{Pl}^2 \frac{1}{\mathcal{V}_F} \int_Y G_\gamma \wedge J_F \wedge \omega^X. \tag{C.33}$$

Here $\mathcal{V}_F$ is the volume of B_3 in $10d$ Planck units, and we also absorbed a factor of m_{10}^2 in J_F, so that the Kähler moduli correspond to volumes measured in Planck units. By supersymmetry, this expression is also related to the matrices Π_M^X in [76] which describe couplings between $U(1)_X$ gauge fields and RR axions. As discussed in [6], we expect that the Fayet–Iliopoulos parameters can be non-vanishing away from the $11d$ supergravity limit, and that they can be used to define stability conditions on an F-theory compactification.

On the heterotic side, the Fayet–Iliopoulos terms are given by

$$\xi_{Het}^X \sim M_{Pl}^2 \frac{1}{\mathcal{V}_h} \int_Z c_1(L^X) \wedge J_h \wedge J_h, \tag{C.34}$$

where $\mathcal{V}_h = \mathrm{vol}(Z)$, and all volumes are measured in string units. Here L^X is a fractional power of the determinant of a sub-bundle $V' \subset V$ against which we are testing stability. The heterotic Green–Schwarz terms also give a loop correction to this expression of the form [77, 78]

$$\delta\xi^X_{het} \sim M^2_{Pl}\frac{1}{S}\int c_1(L^X)\wedge(c_2(V)-\frac{1}{2}c_2(T)), \tag{C.35}$$

which describes one-loop corrections related to $U(1)$ anomalies. For spectral cover bundles, we can write this suggestively as

$$\delta\xi^X_{het} \sim \int_{B_2} c_1(L)\cdot(\eta-6c_1) = \int_{B_2} c_1(L)\wedge c_1(N_{B_2/B_3}), \tag{C.36}$$

where $L^X = \pi_Z^* L$, and we used $\pi_{Z*}(c_2(V)-\frac{1}{2}c_2(T)) = \eta - 6c_1$.

To match the expressions qualitatively, we assume for simplicity that ω^X is completely localized on some singularity, and we take $B_3 = B_2 \times \mathbf{P}^1$. For simplicity we also take the size of the elliptic fiber on the heterotic side of order one. The heterotic and F-theory quantities are related as

$$\mathcal{V}_F \sim \mathrm{vol}(B_2)\times\mathrm{vol}(\mathbf{P}^1) \sim \mathcal{V}_h\,\lambda_h^{-1} \tag{C.37}$$

and also

$$\lambda_h J_{B_2,F} \sim J_{B_2,h}, \quad \lambda_h^2 \sim \mathcal{V}_h/S. \tag{C.38}$$

Then we find

$$M^2_{Pl}\frac{1}{\mathcal{V}_F}\int_Y G_\gamma\wedge J_{B_2,F}\wedge\omega^X \sim M^2_{Pl}\frac{1}{\mathcal{V}_h}\int_{B_2} c_1(L)\wedge J_{B_2,h} \tag{C.39}$$

and similarly

$$M^2_{Pl}\frac{1}{\mathcal{V}_F}\int_Y G_\gamma\wedge J_0\wedge\omega^X \sim M^2_{Pl}\frac{1}{S}\int_{B_2} c_1(L)\wedge(\eta-6\,c_1(S)). \tag{C.40}$$

In other words, the two pieces of the classical F-theoretic expression match qualitatively with the tree level and the one-loop contribution in the heterotic string respectively. If the $U(1)$ anomalies vanish, as in the MSSM, then the one-loop contribution will vanish. In addition, there could be non-perturbative corrections on both sides

We can relate both heterotic and F-theoretic expressions to spectral cover data. The calculation is slightly intricate. We did not match the expressions from F-theory and the heterotic string precisely, but they should likely match. It would be interesting to check this more precisely.

It is interesting to note that since relative size of the $4d$ string coupling S and the Kähler moduli T is varied as we extrapolate from the heterotic string to F-theory, the loop-corrected slope also varies and we could write down models which are stable in the heterotic regime and unstable in the F-theory regime, or vice versa. This appears to be one of the few settings where we can study stability issues as a function of the string coupling.

References

[1] V. Bouchard and R. Donagi, *An SU(5) heterotic standard model*, Phys. Lett. B **633** (2006), 783, [arXiv:hep-th/0512149].

[2] M. Wijnholt, *Geometry of particle physics*, arXiv:hep-th/0703047.

[3] H. Verlinde and M. Wijnholt, *Building the standard model on a D3-brane*, J. High Energy Phys. **01** (2007), 106, [arXiv:hep-th/0508089].

[4] C. Vafa, *Evidence for F-theory*, Nucl. Phys. B **469** (1996), 403, [arXiv:hep-th/9602022].

[5] R. Donagi and M. Wijnholt, *Higgs bundles and UV completion in F-theory*, arXiv:0904.1218 [hep-th].

[6] R. Donagi, M. Wijnholt, *Gluing branes, I*, [arXiv:1104.2610 [hep-th]].

[7] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - I*, arXiv:0802.3391 [hep-th].

[8] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G-flux*, J. High Energy Phys. **08** (1999), 023, [arXiv:hep-th/9908088].

[9] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi–Yau four-folds*, Nucl. Phys. B **584** (2000), 69; Erratum-S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi–Yau four-folds*, Nucl. Phys. B **608** (2001), 477, [arXiv:hep-th/9906070].

[10] B.R. Greene, A.D. Shapere, C. Vafa and S.T. Yau, *Stringy cosmic strings and noncompact Calabi–Yau manifolds*, Nucl. Phys. B **337** (1990), 1.

[11] K. Becker and M. Becker, *M-theory on eight-manifolds*, Nucl. Phys. B **477** (1996), 155, [arXiv:hep-th/9605053].

[12] E. Witten, *On flux quantization in M-theory and the effective action*, J. Geom. Phys. **22** (1997), 1, [arXiv:hep-th/9609122].

[13] S. Gukov and J. Sparks, *M-theory on Spin(7) manifolds. I*, Nucl. Phys. B **625** (2002), 3, [arXiv:hep-th/0109025].

[14] G. Curio and R.Y. Donagi, *Moduli in N = 1 heterotic/F-theory duality*, Nucl. Phys. B **518** (1998), 603, [arXiv:hep-th/9801057].

[15] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, arXiv:hep-th/9907189.

[16] B. Andreas, G. Curio and D. Lust, $N = 1$ *dual string pairs and their massless spectra*, Nucl. Phys. B **507** (1997), 175, [arXiv:hep-th/9705174].

[17] B. Andreas, $N = 1$ *heterotic/F-theory duality*, Fortsch. Phys. **47** (1999), 587, [arXiv:hep-th/9808159].

[18] A. Johansen, *A comment on BPS states in F-theory in 8 dimensions*, Phys. Lett. B **395** (1997), 36, [arXiv:hep-th/9608186].

[19] M.R. Gaberdiel and B. Zwiebach, *Exceptional groups from open strings*, Nucl. Phys. B **518** (1998), 151, [arXiv:hep-th/9709013].

[20] O. DeWolfe and B. Zwiebach, *String junctions for arbitrary Lie algebra representations*, Nucl. Phys. B **541** (1999), 509, [arXiv:hep-th/9804210].

[21] M. Bershadsky, K.A. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, *Geometric singularities and enhanced gauge symmetries*, Nucl. Phys. B **481** (1996), 215, [arXiv:hep-th/9605200].

[22] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi–Yau three-folds — II*, Nucl. Phys. B **476** (1996), 437, [arXiv:hep-th/9603161].

[23] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F theory*, Commun. Math. Phys. **187** (1997), 679, [arXiv:hep-th/9701162].

[24] R. Donagi, *Principal bundles on elliptic fibrations*, Asian J. Math. **1** (1997), 214, [arXiv:alg-geom/9702002].

[25] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F-theory*, Nucl. Phys. B **505** (1997), 165, [arXiv:hep-th/9701165].

[26] R. Donagi, *ICMP lecture on heterotic/F-theory duality*, Proceedings International Congress of Math. Phys, Brisbane, 1997, [arXiv:hep-th/9802093].

[27] R. Donagi, *Taniguchi Lecture on Principal Bundles on Elliptic Fibrations*, Integrable Systems and Algebraic Geometry, Proc. Taniguchi Symposium **33** 1997, [arXiv:hep-th/9802094].

[28] P. Candelas and A. Font, *Duality between the webs of heterotic and type II vacua*, Nucl. Phys. B **511** (1998), 295, [arXiv:hep-th/9603170].

[29] E. Perevalov and H. Skarke, *Enhanced gauge symmetry in type II and F-theory compactifications: Dynkin diagrams from polyhedra*, Nucl. Phys. B **505** (1997), 679, [arXiv:hep-th/9704129].

[30] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4D $N = 2$ gauge theories. I*, Adv. Theor. Math. Phys. **1** (1998), 53, [arXiv:hep-th/9706110].

[31] P. Berglund and P. Mayr, *Heterotic string/F-theory duality from mirror symmetry*, Adv. Theor. Math. Phys. **2** (1999), 1307, [arXiv:hep-th/9811217].

[32] S.H. Katz and C. Vafa, *Matter from geometry*, Nucl. Phys. B **497** (1997), 146, [arXiv:hep-th/9606086].

[33] M. Wijnholt, *Parameter space of quiver gauge theories*, arXiv:hep-th/0512122.

[34] R. Minasian and G.W. Moore, *K-theory and Ramond–Ramond charge*, J. High Energy Phys. **9711** (1997), 002, [arXiv:hep-th/9710230].

[35] Y.K. Cheung and Z. Yin, *Anomalies, branes, and currents*, Nucl. Phys. B **517** (1998), 69, [arXiv:hep-th/9710206].

[36] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*, Cambridge, UK, Cambridge University Press, 1998 p. 531

[37] A. Grassi and D.R. Morrison, *Group representations and the Euler characteristic of elliptically fibered Calabi–Yau threefolds*, arXiv:math/0005196.

[38] N. Arkani-Hamed and M. Schmaltz, *Hierarchies without symmetries from extra dimensions*, Phys. Rev. D **61** (2000), 033005, [arXiv:hep-ph/9903417].

[39] D.E. Kaplan, G.D. Kribs and M. Schmaltz, *Supersymmetry breaking through transparent extra dimensions*, Phys. Rev. D **62** (2000), 035010, [arXiv:hep-ph/9911293].

[40] Z. Chacko, M.A. Luty, A.E. Nelson and E. Ponton, *Gaugino mediated supersymmetry breaking*, J. High Energy Phys. **0001** (2000), 003, [arXiv:hep-ph/9911323].

[41] C. Csaki, *TASI lectures on extra dimensions and branes*, arXiv:hep-ph/0404096.

[42] R. Bousso and J. Polchinski, *Quantization of four form fluxes and dynamical neutralization of the cosmological constant*, J. High Energy Phys. **0006** (2000), 006, [arXiv:hep-th/0004134].

[43] F. Denef and M.R. Douglas, *Distributions of flux vacua*, J. High Energy Phys. **0405** (2004), 072, [arXiv:hep-th/0404116].

[44] E. Witten, *Phase transitions in M theory and F theory*, Nucl. Phys. B **471** (1996), 195, [arXiv:hep-th/9603150].

[45] E. Witten, *New issues in manifolds of SU(3) holonomy*, Nucl. Phys. B **268** (1986), 79.

[46] J.X. Fu and S.T. Yau, *Existence of supersymmetric Hermitian metrics with torsion on non-Kaehler manifolds*, arXiv:hep-th/0509028.

[47] G. Curio, *Chiral matter and transitions in heterotic string models*, Phys. Lett. B **435** (1998), 39, [arXiv:hep-th/9803224].

[48] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, arXiv:alg-geom/9709029.

[49] B. Andreas and D. Hernandez-Ruiperez, *Comments on N = 1 heterotic string vacua*, Adv. Theor. Math. Phys. **7** (2004), 751, [arXiv:hep-th/0305123].

[50] D.E. Diaconescu and G. Ionesei, *Spectral covers, charged matter and bundle cohomology*, J. High Energy Phys. **9812** (1998), 001, [arXiv:hep-th/9811129].

[51] K. Becker, M. Becker, K. Dasgupta and S. Prokushkin, *Properties of heterotic vacua from superpotentials*, Nucl. Phys. B **666** (2003), 144, [arXiv:hep-th/0304001].

[52] G. Lopes Cardoso, G. Curio, G. Dall'Agata and D. Lust, *BPS action and superpotential for heterotic string compactifications with fluxes*, J. High Energy Phys. **0310** (2003), 004, [arXiv:hep-th/0306088].

[53] E. Witten, *Non-perturbative superpotentials in string theory*, Nucl. Phys. B **474** (1996), 343, [arXiv:hep-th/9604030].

[54] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, Nucl. Phys. B **456** (1995), 130, [arXiv:hep-th/9507158].

[55] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, arXiv:hep-th/9907026.

[56] E. Witten, *World-sheet corrections via D-instantons*, J. High Energy Phys. **0002** (2000), 030, [arXiv:hep-th/9907041].

[57] O.J. Ganor, *A note on zeroes of superpotentials in F-theory*, Nucl. Phys. B **499** (1997), 55, [arXiv:hep-th/9612077].

[58] B. Florea, S. Kachru, J. McGreevy and N. Saulina, *Stringy instantons and quiver gauge theories*, J. High Energy Phys. **0705** (2007), 024, [arXiv:hep-th/0610003].

[59] L.E. Ibanez, A.N. Schellekens and A.M. Uranga, *Instanton induced neutrino Majorana masses in CFT orientifolds with MSSM-like spectra*, J. High Energy Phys. **06** (2007), 011, [arXiv:0704.1079 [hep-th]].

[60] R. Blumenhagen, M. Cvetic and T. Weigand, *Spacetime instanton corrections in 4D string vacua — the seesaw mechanism for D-brane models*, Nucl. Phys. B **771** (2007), 113, [arXiv:hep-th/0609191].

[61] R. Blumenhagen, A. Collinucci and B. Jurke, *On instanton effects in F-theory*, J. High Energy Phys. **08** (2010), 079, [arXiv:1002.1894 [hep-th]].

[62] R. Donagi and M. Wijnholt, *MSW instantons*, arXiv:1005.5391 [hep-th].

[63] E. Witten, *Five-brane effective action in M-theory*, J. Geom. Phys. **22** (1997), 103, [arXiv:hep-th/9610234].

[64] E.I. Buchbinder, R. Donagi and B.A. Ovrut, *Superpotentials for vector bundle moduli*, Nucl. Phys. B **653** (2003), 400, [arXiv:hep-th/0205190].

[65] S. Kachru, R. Kallosh, A.D. Linde and S.P. Trivedi, *De Sitter vacua in string theory*, Phys. Rev. D **68** (2003), 046005, [arXiv:hep-th/0301240].

[66] M. Dine, R. Rohm, N. Seiberg and E. Witten, *Gluino condensation in superstring Models*, Phys. Lett. B **156** (1985), 55.

[67] D.E. Diaconescu, B. Florea, S. Kachru and P. Svrcek, *Gauge —mediated supersymmetry breaking in string compactifications*, J. High Energy Phys. **02** (2006), 020, [arXiv:hep-th/0512170].

[68] W. Barth, C. Peters, A. van de Ven, *Compact complex surfaces*, Springer Verlag, 1984.

[69] H. Verlinde, L.T. Wang, M. Wijnholt and I. Yavin, *A higher form (of) mediation*, arXiv:0711.3214 [hep-th].

[70] H.B. Lawson, M.-L. Michelsohn, *Spin geometry*, Princeton University Press, 1989.

[71] N.J. Hitchin, *The selfduality equations on a Riemann surface*, Proc. Lond. Math. Soc. **55** (1987), 59.

[72] C.T. Simpson, *Higgs bundles and local systems*, Publications Mathématiques de l'IHES, **75** (1992), 5–95.

[73] R. Donagi, *Spectral covers*, arXiv:alg-geom/9505009.

[74] S.H. Katz and E. Sharpe, *D-branes, open string vertex operators, and Ext groups*, Adv. Theor. Math. Phys. **6** (2003), 979, [arXiv:hep-th/0208104].

[75] R. Donagi, S. Katz and E. Sharpe, *Spectra of D-branes with Higgs vevs*, Adv. Theor. Math. Phys. **8** (2005), 813, [arXiv:hep-th/0309270].

[76] R. Donagi and M. Wijnholt, *Breaking GUT groups in F-theory*, arXiv:0808.2223 [hep-th].

[77] M. Dine, N. Seiberg and E. Witten, *Fayet–Iliopoulos terms in string theory*, Nucl. Phys. B **289** (1987), 589.

[78] R. Blumenhagen, G. Honecker and T. Weigand, *Loop-corrected compactifications of the heterotic string with line bundles*, J. High Energy Phys. **06** (2005), 020, [arXiv:hep-th/0504232].