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Lie crossed modules and gauge-invariant actions for 2-BF theories

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Abstract

We generalize the BF theory action to the case of a general Lie crossed module $(\partial : H \to G, \triangleright)$, where G and H are non-abelian Lie groups. Our construction requires the existence of G-invariant non-degenerate bilinear forms on the Lie algebras of G and H and we show that there are many examples of such Lie crossed modules by using the construction of crossed modules provided by short chain complexes of vector spaces. We also generalize this construction to an arbitrary chain complex of vector spaces, of finite type. We construct two gauge-invariant actions for 2-flat and fake-flat 2-connections with auxiliary fields. The first action is of the same type as the BFCG action introduced by Girelli, Pfeiffer and Popescu for a special class of Lie crossed modules, where H is abelian. The second

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action is an extended BFCG action which contains an additional auxiliary field. However, these two actions are related by a field redefinition. We also construct a three-parameter deformation of the extended BFCG action, which we believe to be relevant for the construction of non-trivial invariants of knotted surfaces embedded in the four-sphere.

1 Introduction

Crossed modules, or equivalently 2-groups, have become recently an object of intense study in the context of higher gauge theory, since 2-groups offer a natural way to generalize the physics and the geometry of ordinary gauge theories, see [6]. The corresponding concept of a 2-bundle with a Lie crossed module as the fiber, or equivalently, the concept of an abelian or non-abelian gerbe with a connection, was studied in [2, 8, 11, 27, 31]. It was observed in [8, 9, 25] that the vanishing of the fake curvature 2-tensor permits a construction of a non-abelian surface holonomy and a precise construction of the surface holonomy was realized in [20, 21, 34, 35]. It was also observed in [6] that the holonomy of a 2-connection with a non-zero fake curvature tensor can be addressed in the context of 3-connections and the corresponding three-dimensional holonomy; [23]. All these objects can be understood within the framework of higher category theory.

Given a Lie crossed module $(\partial\colon H \to G, \triangleright)$, where $\triangleright$ is a smooth action of the Lie group G on the Lie group H by automorphisms, see [3, 5, 7, 20], an associated differential crossed module $(\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ can be constructed, where $\triangleright$ is now a left action of the Lie algebra $\mathfrak{g}$ of G on the underlying vector space of the Lie algebra $\mathfrak{h}$ of H by derivations. On a manifold M, a (local) 2-connection is given by a $\mathfrak{g}$-valued 1-form $A \in \mathcal{A}^1(M, \mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(M, \mathfrak{h})$. The corresponding fake curvature and 2-curvature tensors can be written as

$$\begin{aligned}
\mathcal{F}_{(A,\beta)} &= \mathrm{F}_A - \partial(\beta), \\
\mathcal{G}_{(A,\beta)} &= d\beta + A \wedge^{\triangleright} \beta,
\end{aligned}$$

where $\mathrm{F_A} = dA + A \wedge A$ is the curvature of A (for notation and conventions see subsection 3.1). A local 2-connection (A, β) will be called fake-flat if the fake curvature tensor vanishes. Similarly, (A, β) is called 2-flat if the 2-curvature tensor vanishes.

We will be only interested in the local aspects of 2-connections, while the interested reader can consult [2, 8, 11, 21] for the global properties. For the case of abelian gerbes and their holonomy see [27, 31].

We will consider two types of gauge transformations for a (local) 2-connection (A,β). These transformations appear in the context of 2-connections on 2-bundles when we pass from one coordinate neighborhood to another. Given a smooth map $\phi\colon M \to G$, let

$$A \mapsto \phi^{-1}A\phi + \phi^{-1}d\phi, \quad \beta \mapsto \phi^{-1} \triangleright \beta. \tag{1.1}$$

Given a 1-form $\eta \in \mathcal{A}^1(M,\mathfrak{h})$, let

$$A \mapsto A + \partial\eta, \quad \beta \mapsto \beta + d\eta + A \wedge^{\triangleright} \eta + \eta \wedge \eta. \tag{1.2}$$

We will call the transformations (1.1) and (1.2) thin and fat, respectively.

The group of gauge transformations will be given by all pairs (ϕ,η) and the group product will be a semidirect product

$$(\phi,\eta)(\phi',\eta') = (\phi\phi', \phi \triangleright \eta' + \eta).$$

The thin and the fat gauge transformations provide a right action of the gauge group on the set of local 2-connections.

The action of the thin gauge transformations on the curvature F_A, the 2-curvature $\mathcal{G}_{(A,\beta)}$ and the fake curvature $\mathcal{F}_{(A,\beta)}$ is given by

$$\mathrm{F}_A \mapsto \phi^{-1}\mathrm{F}_A\phi, \quad \mathcal{G}_{(A,\beta)} \mapsto \phi^{-1} \triangleright \mathcal{G}_{(A,\beta)}, \quad \mathcal{F}_{(A,\beta)} \mapsto \phi^{-1}\mathcal{F}_{(A,\beta)}\phi.$$

The action of the fat gauge transformations on these fields is not straightforward to find, see [2, 9, 11] and below. It is given by

$$\begin{gathered}\mathrm{F}_A \mapsto \mathrm{F}_A + \partial\left(d\eta + A \wedge^{\triangleright} \eta + \eta \wedge \eta\right), \\ \mathcal{F}_{(A,\beta)} \mapsto \mathcal{F}_{(A,\beta)}, \quad \mathcal{G}_{(A,\beta)} \mapsto \mathcal{G}_{(A,\beta)} + \mathcal{F}_{(A,\beta)} \wedge^{\triangleright} \eta.\end{gathered}$$

Constructing an action for a theory of fake-flat and 2-flat 2-connections is important for the quantization of the theory and consequently for constructing new manifold invariants, as well as for quantizing gravity, see [6,26]. In [26] a gauge-invariant action was constructed for a special class of crossed modules $(\partial\colon H \to G, \triangleright)$, such that H is the abelian group associated to the vector space $\mathfrak{g}$ and $\triangleright$ is the adjoint action of G on $\mathfrak{g}$. Also $\partial(X) = 1_G$ for each $X \in \mathfrak{g}$. In the associated differential crossed module $(\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$, $\mathfrak{h}$ is an abelian Lie algebra which is given by the underlying vector space of $\mathfrak{g}$, with trivial bracket. The corresponding action can be written as

$$S_0 = \int_M \langle B \wedge \mathcal{F}_{(A,\beta)} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G}_{(A,\beta)} \rangle_{\mathfrak{g}}, \tag{1.3}$$

where $B \in \mathcal{A}^2(M, \mathfrak{g})$, $C \in \mathcal{A}^1(M, \mathfrak{h})$ and $\langle , \rangle_{\mathfrak{g}}$ is a G-invariant non-degenerate bilinear form in $\mathfrak{g}$.

The action S_0 defines the dynamics of a theory of 2-flat and fake-flat 2-connections, and we will refer to this type of actions as a BFCG action. S_0 is invariant under the thin gauge transformations (1.1) if

$$B \mapsto \phi^{-1} B \phi, \quad C \mapsto \phi^{-1} \triangleright C, \tag{1.4}$$

while the invariance under the fat gauge transformations (1.2) requires

$$B \mapsto B - [C, \eta], \quad C \mapsto C. \tag{1.5}$$

In this article we will generalize the BFCG action S_0 to the case of a general Lie crossed module, i.e., the case when the group H is non-abelian and the morphism $\partial \colon H \to G$ is non-trivial. Our construction requires the existence of a non-degenerate bilinear form on $\mathfrak{h}$ such that it is G-invariant. We will show that there are many examples of such Lie crossed modules by using the construction of crossed modules provided by short chain complexes of vector spaces. We will also extend this construction to the case of an arbitrary finite type chain complex of vector spaces.

We will construct two gauge invariant actions for 2-flat and fake-flat 2-connections with auxiliary fields. One of them will be a BFCG action, and it will require a generalization of the fat gauge transformations for the B field found in [26], see (1.5). The second action, which will be called extended BFCG action, will require an additional auxiliary field. However, we will show that these two actions are related by a field redefinition.

We also construct a three-parameter deformation of the extended BFCG action, which we believe to be relevant for the construction of non-trivial invariants of knotted surfaces embedded in the four-sphere.

2 Lie crossed modules

In this section we are going to give the necessary definitions and the properties of Lie crossed modules which will be needed for the construction of a generalized BFCG action. All Lie groups and Lie algebras considered here will be finite-dimensional. For a more detailed exposition of Lie crossed modules see [3, 7, 20]. For general facts about crossed modules see [12, 18].

Definition 2.1 (Lie crossed module)**.** A crossed module $\mathcal{X} = (\partial\colon H \to G, \triangleright)$ is given by a group morphism $\partial\colon H \to G$ together with a left action $\triangleright$ of G on H by automorphisms, such that:

(1) $\partial(g \triangleright h) = g\partial(h)g^{-1}$; for each $g \in G$ and $h \in H$,
(2) $\partial(h) \triangleright h' = hh'h^{-1}$; for each $h, h' \in H$.

If both G and H are Lie groups, $\partial\colon H \to G$ is a smooth morphism, and the left action of G on H is smooth then $\mathcal{X}$ will be called a Lie crossed module.

We will call G the base group of $\mathcal{X}$ and we will be mainly interested in the case when the base group G is compact in the real case, or has a compact real form in the complex case. This ensures that G-invariant non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$ always exist; see Definition 2.3 and Lemma 2.1.

A morphism $\mathcal{X} \to \mathcal{X}'$ between the crossed modules $\mathcal{X} = (\partial\colon H \to G, \triangleright)$ and $\mathcal{X}' = (\partial'\colon H' \to G', \triangleright')$ is given by a pair of maps $\phi\colon G \to G'$ and $\psi\colon H \to H'$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\partial} & G \\ \psi\big\downarrow & & \big\downarrow\phi \\ H' & \xrightarrow{\partial'} & G' \end{array}$$

is commutative. In addition, we must have $\psi(g \triangleright h) = \phi(g) \triangleright' \psi(h)$ for each $h \in H$ and each $g \in G$.

Example 2.1. Let G be a Lie group and V a vector space carrying a representation ρ of G. Then $(V \xrightarrow{v \mapsto 1_G} G, \rho)$ is a crossed module.

Example 2.2. Let G be a connected Lie group and $\mathrm{Aut}(G)$ the automorphisms Lie group of G. The group $\mathrm{Aut}(G)$ has a left action in G by automorphisms $f \triangleright g = f(g)$, where $f \in \mathrm{Aut}(G)$ and $g \in G$. Together with the map $g \in G \mapsto \mathrm{Ad}_g$ which sends $g \in G$ to the automorphism $h \mapsto ghg^{-1}$ this defines a crossed module.

2.1 Crossed modules of Lie algebras

Given a Lie crossed module $\mathcal{X} = (\partial\colon H \to G, \triangleright)$, then the induced Lie algebra map $\partial\colon \mathfrak{h} \to \mathfrak{g}$, together with the derived action of $\mathfrak{g}$ on $\mathfrak{h}$ (also denoted by

$\triangleright$) is a differential crossed module, in the sense of the following definition, see [3, 5, 8, 9, 20, 21].

Definition 2.2 (Differential crossed module)**.** A differential crossed module $\mathfrak{X} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ is given by a Lie algebra morphism $\partial\colon \mathfrak{h} \to \mathfrak{g}$ together with a left action of $\mathfrak{g}$ on the underlying vector space of $\mathfrak{h}$, such that:

(1) For any $X \in \mathfrak{g}$ the map $\xi \in \mathfrak{h} \mapsto X \triangleright \xi \in \mathfrak{h}$ is a derivation of $\mathfrak{h}$, which can be written as

$$X \triangleright [\xi, \nu] = [X \triangleright \xi, \nu] + [\xi, X \triangleright \nu], \quad \forall X \in \mathfrak{g}, \forall \xi, \nu \in \mathfrak{h}.$$

(2) The map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ from $\mathfrak{g}$ into the derivation algebra of $\mathfrak{h}$ induced by the action of $\mathfrak{g}$ on $\mathfrak{h}$ is a Lie algebra morphism, which can be written as

$$[X, Y] \triangleright \xi = X \triangleright (Y \triangleright \xi) - Y \triangleright (X \triangleright \xi), \quad \forall X, Y \in \mathfrak{g}, \forall \xi \in \mathfrak{h},$$

(3)
$$\partial(X \triangleright \xi) = [X, \partial(\xi)], \quad \forall X \in \mathfrak{g}, \forall \xi \in \mathfrak{h}, \tag{2.1}$$

(4)
$$\partial(\xi) \triangleright \nu = [\xi, \nu], \quad \forall \xi, \nu \in \mathfrak{h}. \tag{2.2}$$

Therefore, given a differential crossed module $\mathfrak{X} = (\partial\colon \mathfrak{h} \to \gamma, \triangleright)$, there exists a unique crossed module of simply connected Lie groups $\mathcal{X} = (\partial\colon H \to G, \triangleright)$ which corresponds to $\mathfrak{X}$, up to an isomorphism.

A very useful identity satisfied in any differential crossed module is

$$\partial(\xi) \triangleright \nu = [\xi, \nu] = -[\nu, \xi] = -\partial(\nu) \triangleright \xi, \quad \forall \nu, \xi \in \mathfrak{h}.$$

2.1.1 Mixed relations

Let $\mathcal{X} = (\partial\colon H \to G, \triangleright)$ be a Lie crossed module, and let $\mathfrak{X} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module. Therefore G acts on $\mathfrak{g}$ by the adjoint action, and on $\mathfrak{h}$ by the action induced by $\triangleright$. The following mixed relations are satisfied

$$\partial(g \triangleright \xi) = g\partial(\xi)g^{-1}, \quad g \in G, \xi \in \mathfrak{h},$$
$$\partial(X \triangleright h) = X\partial(h) - \partial(h)X, \quad X \in \mathfrak{h}, h \in H,$$
$$\partial(h) \triangleright \xi = h\xi h^{-1}, \quad h \in H, \xi \in H,$$

and

$$\partial(\xi) \triangleright h = \xi h - h\xi, \quad \xi \in \mathfrak{h}, h \in H.$$

An identity that will be used to prove the gauge invariance of the extended BFCG action is

$$g \triangleright (X \triangleright \xi) = \left(gXg^{-1}\right) \triangleright (g \triangleright \xi), \quad \forall g \in G, \xi \in \mathfrak{h}, X \in \mathfrak{g}. \tag{2.3}$$

2.1.2 G-invariant bilinear forms

We will introduce the following G-invariant bilinear forms on the Lie algebras of a Lie crossed module in order to be able to construct a generalized BFCG action.

Definition 2.3 (non-degenerate symmetric G-invariant forms)**.** Let us consider a Lie crossed module $\mathcal{X} = (\partial\colon H \to G, \triangleright)$ and let $\mathfrak{X} = (\partial\colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module. A symmetric non-degenerate G-invariant form in $\mathfrak{X}$ is given by a pair of non-degenerate symmetric bilinear forms $\langle , \rangle_{\mathfrak{g}}$ in $\mathfrak{g}$ and $\langle , \rangle_{\mathfrak{h}}$ in $\mathfrak{h}$ such that

(1) $\langle , \rangle_{\mathfrak{g}}$ is G-invariant, i.e.,

$$\langle gXg^{-1}, gYg^{-1}\rangle = \langle X, Y\rangle, \quad \forall\, g \in G,\ X, Y \in \mathfrak{g},$$

(2) $\langle , \rangle_{\mathfrak{h}}$ is G-invariant, i.e.,

$$\langle g \triangleright \xi, g \triangleright \nu\rangle = \langle \xi, \nu\rangle, \quad \forall\, g \in G,\ \xi, \nu \in \mathfrak{h}.$$

Note that $\langle , \rangle_{\mathfrak{h}}$ is necessarily H-invariant. This is because

$$\langle h\xi h^{-1}, h\nu h^{-1}\rangle_{\mathfrak{h}} = \langle \partial(h) \triangleright \xi, \partial(h) \triangleright \nu\rangle_{\mathfrak{h}} = \langle \xi, \nu\rangle_{\mathfrak{h}}, \quad \forall \xi, \nu \in \mathfrak{h},\ h \in H,$$

where we have used the mixed relations 2.1.1.

There are no compatibility conditions between the symmetric bilinear forms $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{\mathfrak{h}}$. From the well-known fact that any representation of G can be made unitary if G is a compact group, it follows that

Lemma 2.1. *Let $\mathcal{X} = (\partial \colon H \to G, \triangleright)$ be a Lie crossed module with the base group G being compact in the real case, or having a compact real form in the complex case. Then one can construct G-invariant symmetric non-degenerate bilinear forms $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{\mathfrak{h}}$ in the associated differential crossed module $\mathfrak{X} = (\partial \colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$. Furthermore these forms can be chosen to be positive definite.*

Given G-invariant symmetric non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$, we can define a bilinear antisymmetric map $\mathcal{T} \colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ by the rule

$$\langle \mathcal{T}(u,v), Z \rangle_{\mathfrak{g}} = -\langle u, Z \triangleright v \rangle_{\mathfrak{h}}, \quad u, v \in \mathfrak{h},\ Z \in \mathfrak{g}.$$

Note that $\langle , \rangle_{\mathfrak{g}}$ is non-degenerate and

$$\langle u, Z \triangleright v \rangle_{\mathfrak{h}} = -\langle Z \triangleright u, v \rangle_{\mathfrak{h}} = -\langle v, Z \triangleright u \rangle_{\mathfrak{h}}.$$

Moreover, given $g \in G$ and $u, v \in \mathfrak{h}$, we have

$$\mathcal{T}(g \triangleright u, g \triangleright v) = g\mathcal{T}(u,v)g^{-1},$$

since for each $X \in \mathfrak{g}$ and $u, v \in \mathfrak{h}$ we have

$$\begin{aligned}\langle X, g^{-1}\mathcal{T}(g \triangleright u, g \triangleright v) g \rangle_{\mathfrak{g}} &= \langle gXg^{-1}, \mathcal{T}(g \triangleright u, g \triangleright v) \rangle_{\mathfrak{g}} \\ &= \langle (gXg^{-1}) \triangleright g \triangleright u, g \triangleright v \rangle_{\mathfrak{h}} \\ &= \langle X \triangleright u, v \rangle_{\mathfrak{h}} = \langle X, \mathcal{T}(u,v) \rangle_{\mathfrak{g}}.\end{aligned}$$

We thus have the following identity:

$$\mathcal{T}(X \triangleright u, v) + \mathcal{T}(u, X \triangleright v) = [X, \mathcal{T}(u,v)].$$

The map $\mathcal{T}$ will play a major role in the construction of the generalized BFCG action and the corresponding gauge transformations.

2.2 Crossed modules from chain complexes of vector spaces

We will now show that there exists a rich class of examples of Lie crossed modules $(\partial : H \to G, \triangleright)$ with the desired properties, i.e., G being compact and H a non-abelian group, by constructing them from chain complexes of vector spaces.

2.2.1 Crossed modules from short chain complexes of vector spaces

The definition of a Lie crossed module from a short chain complex of vector spaces was given in [5, 7]. We are going to use this definition in order to explicitly construct a Lie crossed module with the desired properties.

A minor modification of the definition will yield a Lie crossed module from an arbitrary chain complex of vector spaces; see below.

Let $\mathcal{V} = (V \xrightarrow{\phi} U)$ be a short chain complex of finite-dimensional vector spaces. This means that V and U are vector spaces and $\phi\colon V \to U$ is a linear map. We can define a crossed module of Lie groups

$$\mathrm{GL}(\mathcal{V}) = (\partial\colon \mathrm{GL}_1(\mathcal{V}) \to \mathrm{GL}_0(\mathcal{V}), \triangleright)$$

in the following way:

Let $\mathrm{Hom}_0(\mathcal{V})$ be the algebra of chain maps $f\colon \mathcal{V} \to \mathcal{V}$, such that the composition of maps is the algebra product. A chain map $f\colon \mathcal{V} \to \mathcal{V}$ is defined by a pair of linear maps (f_V, f_U), where $f_U\colon U \to U$ and $f_V\colon V \to V$, such that the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "f_V"'] & U \arrow[d, "f_U"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd}$$

is commutative. This is equivalent to $\phi \circ f_V = f_U \circ \phi$. Note that if $F = (f_V, f_U)$ and $F' = (f'_V, f'_U)$ are chain maps, then their composition $F \circ F' = (f_V \circ f_{V'}, f_U \circ f_{U'})$ is also a chain map.

Consider the set $\mathrm{GL}_0(\mathcal{V})$ of invertible elements of $\mathrm{Hom}_0(\mathcal{V})$, which is the set of pairs $F = (f_V, f_U)$ such that $f_U\colon U \to U$ and $f_V\colon V \to V$ are invertible linear maps. Note that this is an open subset of $\mathrm{Hom}_0(\mathcal{V})$. Then $\mathrm{GL}_0(\mathcal{V})$ is a Lie group under the composition of chain maps. More precisely, $\mathrm{GL}_0(\mathcal{V})$ is a closed subgroup of the general linear group $\mathrm{GL}(V \oplus U)$. The Lie algebra $\mathfrak{gl}_0(\mathcal{V})$ of $\mathrm{GL}_0(\mathcal{V})$ is identical to $\mathrm{Hom}_0(\mathcal{V})$ as a vector space, with the bracket given by the associative algebra structure of $\mathrm{Hom}_0(\mathcal{V})$, so that $[F, F'] = F \circ F' - F' \circ F$, for chain maps $F, F'\colon \mathcal{V} \to \mathcal{V}$.

Consider the semigroup $\mathrm{Hom}_1(\mathcal{V})$ of all linear maps $s\colon U \to V$, with a product defined as

$$s * t = s + t + s\phi t.$$

This is an associative product, with the unit being the null linear map $U \to V$. The map $\partial\colon \mathrm{Hom}_1(\mathcal{V}) \to \mathrm{Hom}_0(\mathcal{V})$ such that

$$\partial(s) = (s\phi + \mathrm{id}, \phi s + \mathrm{id}), \qquad \text{where } s \in \mathrm{Hom}_1(\mathcal{V})$$

preserves the products. The set $\mathrm{GL}_1(\mathcal{V})$ of linear maps $s\colon U \to V$ for which $\partial(s)$ is invertible is certainly open in $\mathrm{Hom}_1(\mathcal{V})$, since $\partial\colon \mathrm{Hom}_1(V) \to \mathrm{Hom}_0(\mathcal{V})$ is continuous, and $\mathrm{GL}_0(\mathcal{V})$ is open in $\mathrm{Hom}_1(\mathcal{V})$. Furthermore, since ∂ preserves the products, if s and t are in $\mathrm{GL}_1(\mathcal{V})$, then so is $s * t$. If $s \in \mathrm{GL}_1(\mathcal{V})$, then the inverse $\overline{s}$ of s with respect to $*$, whose unit is the null map $U \to V$, is

$$\overline{s} = -(\mathrm{id} + s\phi)^{-1}s = -s(\mathrm{id} + \phi s)^{-1}.$$

Notice that $\partial(\overline{s})$ is invertible if $\partial(s)$ is invertible. Therefore $\mathrm{GL}_1(\mathcal{V})$ is a Lie group of dimension $\dim(U) \times \dim(V)$, and $\partial\colon \mathrm{Hom}_1(\mathcal{V}) \to \mathrm{Hom}_0(\mathcal{V})$ is a Lie group morphism.

The Lie algebra $\mathfrak{gl}_1(\mathcal{V})$ of $\mathrm{GL}_1(\mathcal{V})$ is given by the vector space $\mathrm{Hom}_1(\mathcal{V})$ of all maps $s\colon U \to V$, with the bracket given by

$$[s, t] = s\phi t - t\phi s.$$

The map $\partial\colon \mathfrak{gl}_1(\mathcal{V}) \to \mathfrak{gl}_0(\mathcal{V})$ such that

$$\partial(s) = (s\phi, \phi s)$$

is a morphism of Lie algebras, and it is exactly the derivative of ∂.

A left action of $\mathrm{GL}_0(\mathcal{V})$ on $\mathrm{GL}_1(\mathcal{V})$, by automorphism can be defined as

$$(f_V, f_U) \triangleright s = f_V s f_U^{-1}$$

where $s \in \mathrm{Hom}_1(\mathcal{V})$ and $(f_U, f_V)\colon \mathcal{V} \to \mathcal{V}$ is an invertible chain map. Therefore if $F = (f_V, f_U)$ is invertible and $\partial(s)$ is invertible, then so is:

$$\partial(F \triangleright s) = \left(f_V \partial(s) f_V^{-1}, f_U \partial(s) f_U^{-1}\right) = F\partial(s)F^{-1}.$$

The differential form of this action is the left action of $\mathfrak{gl}_0(\mathcal{V})$ on $\mathfrak{gl}_1(\mathcal{V})$ by derivations such that:

$$(f_V, f_U) \triangleright s = f_V s - s f_U.$$

To finish proving this construction defines a crossed module note that if $s, t \in \mathrm{GL}_1(\mathcal{V})$

$$\partial(s) \triangleright t = (s\phi + \mathrm{id})t(\phi s + \mathrm{id})^{-1},$$

whereas

$$\begin{aligned}(s * t * \overline{s}) &= (s + t + s\phi t) * \overline{s} = s + \overline{s} + s\phi\overline{s} + t + s\phi t + t\phi\overline{s} + s\phi t\phi\overline{s} \\ &= t + s\phi t + t\phi\overline{s} + s\phi t\phi\overline{s},\end{aligned}$$

and therefore, since $\overline{s} = -s(\mathrm{id} + \phi s)^{-1}$:

$$(s * t * \overline{s})(\phi s + \mathrm{id}) = t\phi s + t + s\phi t\phi s + s\phi t - t\phi s - s\phi t\phi s = (s\phi + \mathrm{id})t.$$

We have thus given a vector space map $\mathcal{V} = (V \xrightarrow{\phi} U)$, a short chain complex of vector spaces, defined a Lie crossed module

$$\mathrm{GL}(\mathcal{V}) = (\partial\colon \mathrm{GL}_1(\mathcal{V}) \to \mathrm{GL}_0(\mathcal{V}), \triangleright)\,,$$

whose differential form is the differential crossed module

$$\mathfrak{gl}(\mathcal{V}) = (\partial\colon \mathfrak{gl}_1(\mathcal{V}) \to \mathfrak{gl}_0(\mathcal{V}), \triangleright)\,.$$

The proof that $\mathfrak{gl}(\mathcal{V})$ is a differential crossed module is completely analogous to the case of the crossed module.

Example 2.3. Consider the case when $V = U = \mathbb{C}^2$ and $\phi = \mathrm{id}$. Then $\mathrm{GL}_0(\mathcal{V}) = \mathrm{GL}_1(\mathcal{V}) = \mathrm{GL}(\mathbb{C}^2)$ and ∂ is the identity map.

Example 2.4. Consider the case when $V = U = \mathbb{C}^2$ and $\phi = 0$. Then $\mathrm{GL}_0(\mathcal{V}) = \mathrm{GL}(\mathbb{C}^2) \times \mathrm{GL}(\mathbb{C}^2)$. On the other hand $\mathrm{GL}_1(\mathcal{V}) = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$, the space of all linear maps $\mathbb{C}^2 \to \mathbb{C}^2$, with left action $(A, B) \triangleright f = AfB^{-1}$. In this case $\partial(f) = 1$ for each $f \in \mathrm{GL}_1(\mathcal{V})$, and the (abelian) group structure in $\mathrm{GL}_1(\mathcal{V})$ is the usual sum of linear maps.

Example 2.5. Let X, Y, Z be vector spaces, The most general case of the crossed module defined by a short chain complex of vector spaces is given by the case when $V = X \oplus Y$ and $U = X \oplus Z$, and $\phi(x, y) = (x, 0)$. A map $V \to V$ is given by a matrix $\begin{pmatrix} A\colon X \to X & B\colon Y \to X \\ C\colon X \to Y & D\colon Y \to Y \end{pmatrix}$ of linear maps, and analogously for a map $U \to U$, which is given by a matrix $\begin{pmatrix} A\colon X \to X & B\colon Z \to X \\ C\colon X \to Z & D\colon Z \to Z \end{pmatrix}$ of linear maps. The Lie algebra $\mathfrak{gl}_0(\mathcal{V})$ is given by all pairs of linear maps of the form,

$$(f_U, f_V) = \left(\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \begin{pmatrix} A & B' \\ 0 & D' \end{pmatrix} \right),$$

with commutator being the usual commutator of matrices.

$$\left[\left(\begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_1 & B_1' \\ 0 & D_1' \end{pmatrix}\right), \left(\begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix}, \begin{pmatrix} A_2 & B_2' \\ 0 & D_2' \end{pmatrix}\right)\right]$$
$$= \left(\left[\begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix}\right], \left[\begin{pmatrix} A_1 & B_1' \\ 0 & D_1' \end{pmatrix}, \begin{pmatrix} A_2 & B_2' \\ 0 & D_2' \end{pmatrix}\right]\right).$$

The Lie algebra $\mathfrak{gl}_1(\mathcal{V})$ is given by all matrices of linear maps of the form:

$$s = \begin{pmatrix} A\colon X \to X & B\colon Z \to X \\ C\colon X \to Y & D\colon Z \to Y \end{pmatrix},$$

with boundary

$$\beta'(s) = \left(\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right)$$

and commutator (not coinciding with the commutator of matrices):

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\right] = \begin{pmatrix} [A, A'] & AB' - A'B \\ CA' - C'A & CB' - C'B \end{pmatrix}.$$

The left action of $\mathfrak{gl}_0(\mathcal{V})$ on $\mathfrak{gl}_1(\mathcal{V})$ by derivations is

$$\left(\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \begin{pmatrix} A & B' \\ 0 & D' \end{pmatrix}\right) \triangleright \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$
$$= \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} - \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} A & B' \\ 0 & D' \end{pmatrix}.$$

It is an easy calculation to prove that this indeed yields a crossed module of Lie algebras.

In the previous example note that the kernel $\ker \partial$ of $\partial\colon \mathfrak{gl}_1(\mathcal{V}) \to \mathfrak{gl}_0(\mathcal{V})$ is the vector space of all linear maps $Z \to Y$, with trivial commutator. The cokernel of ∂ is given by all the pairs of maps $(D\colon Y \to Y, D'\colon Z \to Z)$ with pairwise commutator:

$$[(Z_1, Z_1'), (Z_2, Z_2')] = ([Z_1, Z_2], [Z_1', Z_2']).$$

The action of $\mathfrak{gl}_0$ on $\mathfrak{gl}_1$ descends to an action on $\operatorname{coker}(\phi)$ on $\ker \phi$ which has the form:

$$(Z, Z') \triangleright D = ZD - DZ'.$$

2.2.2 The compact case

Given a linear map of vector spaces $\mathcal{V} = (V \xrightarrow{\phi} U)$, the group $\mathrm{GL}_0(\mathcal{V})$ is clearly non-compact. Therefore the Lemma 2.1 cannot be applied to the crossed module $\mathrm{GL}(\mathcal{V}) = (\partial\colon \mathrm{GL}_1(\mathcal{V}) \to \mathrm{GL}_0(\mathcal{V}), \triangleright)$. However we can easily modify the construction of the crossed module $\mathrm{GL}(\mathcal{V})$ in order to get a compact base group.

Definition 2.4. Given a map $\mathcal{V} = (V \xrightarrow{\phi} U)$ of vector spaces, an inner product for $\mathcal{V}$ is simply given by two non-degenerate positive definite bilinear forms $\langle,\rangle_U$ and $\langle,\rangle_V$ in U and V, with no further compatibility conditions.

Given $\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$ as above, define a Lie group $\mathrm{U}_0\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$ as being given by all pairs $F = \big(f_V, f_U\big) \in \mathrm{GL}_0(\mathcal{V})$ for which $f_V\colon V \to V$ and $f_U\colon U \to U$ each are unitary. Clearly $\mathrm{U}_0\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$ is a closed (in fact compact) Lie subgroup of $\mathrm{GL}_0(\mathcal{V})$. In addition, since $\partial\colon \mathrm{GL}_1(\mathcal{V}) \to \mathrm{GL}_0(\mathcal{V})$ is a Lie group morphism,

$$\mathrm{U}_1\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big) \doteq \partial^{-1}\Big(\mathrm{U}_0\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)\Big)$$

is a Lie subgroup of $\mathrm{GL}_1(\mathcal{V})$.

Given $s \in \mathrm{U}_1\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$ and $F = \big(f_V, f_U\big) \in \mathrm{U}_0\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$ let us see that $F \triangleright s$ is still in $\mathrm{U}_1\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$; the pair $\partial(F \triangleright s)$ satisfies:

$$\begin{aligned}
\partial(F \triangleright s) &= \big(f_V s f_U^{-1}\phi + \mathrm{id}, \phi f_V s f_U^{-1} + \mathrm{id}\big) \\
&= \big(f_V s\phi f_V^{-1} + \mathrm{id}, f_U \phi s f_U^{-1} + \mathrm{id}\big) \\
&= \big(f_V(s\phi + \mathrm{id}) f_V^{-1}, f_U(\phi s + \mathrm{id}) f_U^{-1}\big)
\end{aligned}$$

thus $F \triangleright s \in \mathrm{U}_1\big(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U\big)$.

We have therefore defined a Lie crossed module with compact base group

$$U(\mathcal{V}, \langle,\rangle_U, \langle,\rangle_V) = \Big(\mathrm{U}_1(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U) \xrightarrow{\partial} \mathrm{U}_0(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U), \triangleright\Big)$$

given a $(\mathcal{V}, \langle,\rangle_V, , \langle,\rangle_U)$ as in Definition 2.4. The differential form of this is given by the differential crossed module

$$\mathfrak{u}(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U) = \Big(\mathfrak{u}_1(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U) \xrightarrow{\partial} \mathfrak{u}_0(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U), \triangleright\Big),$$

defined in a completely analogous way. Therefore $\mathfrak{u}_0(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U)$ is given by all pairs $F = (f_V, f_U)$ of linear maps such that $f_U + f_U^* = 0$, $f_V + f_V^* = 0$, and also $\phi f_V = f_U \phi$. The commutators are given by

$$[(f_V, f_U), (f'_V, f'_U)] = ([f_V, f'_V], [f_U, f'_U]).$$

On the other hand $\mathfrak{u}_1(\mathcal{V}, \langle,\rangle_V, \langle,\rangle_U)$ is given by all linear maps $s\colon U \to V$ such that $(\phi s)^* + \phi s = 0$ and also $(s\phi)^* + s\phi = 0$, with commutator $[s,t] = s\phi t - t\phi s$. The left action of $\mathfrak{u}_0$ on $\mathfrak{u}_1$ by derivations is, as before

$$(f_V, f_U) \triangleright s = f_V s - s f_U.$$

Thus if $s \in \mathfrak{u}_1$ and $(f_V, f_U) \in \mathfrak{u}_0$ then $(f_V, f_U) \triangleright s \in \mathfrak{u}_1$.

Example 2.6. If $U = V = \mathbb{C}^2$ with standard inner product, and $\phi\colon V \to U$ is the identity map then $\mathrm{U}_0(\mathcal{V})$ and $\mathrm{U}_1(\mathcal{V})$ each are equal to $U(\mathbb{C}^2)$ and $\partial\colon \mathrm{U}_1 \to \mathrm{U}_0$ is the identity map. The action of U_0 on U_1 is the action by conjugation.

Example 2.7. Consider a vector space W with a positive-definite inner product. Let $U = V = W \oplus W$, and let the map $\phi\colon U \to V$ be such that $\phi(w, w') = (w, 0)$. Then $\mathfrak{u}_0(\mathcal{V}) = \mathfrak{u}(W) \oplus \mathfrak{u}(W) \oplus \mathfrak{u}(W)$ and $\mathfrak{u}_1(\mathcal{V}) = \mathfrak{u}(W) \oplus \mathfrak{gl}(W)$, with $\partial(w_1, w_2, w_3) = (w_1, 0, 0)$.

In the previous example, to get genuinely new crossed modules, we should consider the case when U and V have orthogonal decompositions $U = X \oplus Y$ and $V = X' \oplus Z$, and where $\phi(x, y) = (x, 0)$. Here X' is X as a vector space, but with a different inner product.

2.2.3 The differential 2-crossed module given by an arbitrary chain complex of vector spaces

Any chain complex of finite-dimensional vector spaces

$$\mathcal{V} = (\dots \xrightarrow{\phi} V_{n+1} \xrightarrow{\phi} V_n \xrightarrow{\phi} V_{n-1} \xrightarrow{\phi} \dots),$$

with arbitrary, albeit finite, length, also gives a differential crossed module

$$\mathfrak{gl}(\mathcal{V}) = (\partial\colon \mathfrak{gl}_1(\mathcal{V}) \to \mathfrak{gl}_0(\mathcal{V}), \triangleright),$$

thus a crossed module of Lie groups $\mathrm{GL}(\mathcal{V})$. In the case of 2-crossed modules this construction appeared in [28], for more details see [23].

The Lie algebra $\mathfrak{gl}_0(V)$ is given by all chain maps $f\colon \mathcal{V} \to \mathcal{V}$, with the usual commutator. A degree n map $h\colon \mathcal{V} \to \mathcal{V}$ is given by a sequence of linear maps $h_i\colon V_i \to V_{i+n}$, without any compatibility relations with the boundary maps ϕ. Two degree 1-maps $s, t\colon \mathcal{V} \to \mathcal{V}$ (homotopies) are (2-fold) homotopic if there exists a degree-2 map $h\colon \mathcal{V} \to \mathcal{V}$ (a 2-fold homotopy) such that

$$s_i(v) - t_i(v) = \phi h(v) - h\phi(v), \text{ for each } v \in V_i.$$

The Lie algebra $\mathfrak{gl}_1(\mathcal{V})$ is defined as being the vector space of all degree 1 maps $s\colon \mathcal{V} \to \mathcal{V}$ up to (2-fold) homotopy. The boundary map $\partial\colon \mathfrak{gl}_1(\mathcal{V}) \to \mathfrak{gl}_0(\mathcal{V})$ is

$$\partial(s) = s\phi + \phi s,$$

and the bracket is

$$[s,t] = s\phi t - t\phi s + st\phi - ts\phi.$$

(This is antisymmetric and satisfies the Jacobi relation on the nose, i.e., before passing to the quotient.) It is easy to see that ∂ is a Lie algebra morphism.

The action of $\mathfrak{gl}_0(\mathcal{V})$ on $\mathfrak{gl}_1(\mathcal{V})$ is

$$f \triangleright s = fs - sf.$$

This is an action by derivations, before passing to the quotient. We trivially have $\partial(f \triangleright s) = [f, \partial(s)]$. Moreover $\partial(s) \triangleright t - [s,t] = \phi \circ (st - ts) - (st - ts) \circ \phi$, therefore by considering the quotient of the space of degree-1 maps with respect to (2-fold) homotopy defines a differential crossed module $\mathfrak{gl}(\mathcal{V})$.

This construction can be adapted in the obvious way to give a crossed module with a compact base group as in 2.2.2. This can be done by picking inner products $\langle , \rangle_n$ is each V_n, without any compactibility relations with the boundary maps ϕ.

2.3 Lie crossed modules and differential forms

In order to perform the calculations more efficiently, we will introduce the component notation. Let T_m be a basis in $\mathfrak{g}$ and τ_μ a basis in $\mathfrak{h}$, such that

$$[T_m, T_n] = f_{mn}{}^{r}\, T_r, \quad [\tau_\mu, \tau_\nu] = \phi_{\mu\nu}{}^{\rho}\, \tau_\rho. \tag{2.4}$$

Then

$$\partial\tau_\mu = \partial_\mu{}^{m}\, T_m, \quad T_m \triangleright \tau_\mu = \triangleright_{m\mu}{}^{\nu}\, \tau_\nu, \tag{2.5}$$

and the relations (2.1) and (2.2) take the following form:

$$\triangleright_{m\mu}^{\nu}\,\partial_{\nu}^{r} = \partial_{\mu}^{n}\,f_{mn}^{r}, \tag{2.6}$$

$$\partial_{\mu}^{m}\,\triangleright_{m\nu}^{\rho} = \phi_{\mu\nu}^{\rho}. \tag{2.7}$$

The structure constants satisfy the Jacobi identities

$$f_{mt}^{s}\,f_{nr}^{t} = f_{m[n|}^{t}\,f_{t|r]}^{s}, \quad \phi_{\mu\epsilon}^{\sigma}\,\phi_{\nu\rho}^{\epsilon} = \phi_{\mu[\nu|}^{\epsilon}\,\phi_{\epsilon|\rho]}^{\sigma}, \tag{2.8}$$

where $X_{[IJ]} = X_{IJ} - X_{JI}$.

A $\mathfrak{g}$-valued form X on M will be defined as $X = X^m\,T_m$, where X^m are forms on M, all of the same degree, and T_m are taken to be the adjoint representation matrices. Let X and Y be two $\mathfrak{g}$-valued forms on M. We define

$$X \wedge Y = (X^m \wedge Y^n)\,T_m T_n.$$

This matrix will not be in general an adjoint matrix for $\mathfrak{g}$. However, if X is an odd form, then

$$X \wedge X = X^m \wedge X^n\,T_m T_n = \frac{1}{2}\,X^m \wedge X^n\,[T_m, T_n] = \frac{1}{2}\,f_{mn}^{r}\,X^m \wedge X^n\,T_r \in \mathfrak{g},$$

and similarly for an odd form ξ in $\mathfrak{h}$.

Note that $X \wedge X \notin \mathfrak{g}$ if X is an even form and also $X \wedge Y \notin \mathfrak{g}$ for $X \neq Y$. However, if X is a p-form and Y a q-form, then

$$X \wedge Y + (-1)^l\,Y \wedge X \in \mathfrak{g},$$

if $l + pq$ is an odd integer.

If ξ is a $\mathfrak{h}$-valued form on M and X a $\mathfrak{g}$-valued form on M, then we define

$$X \wedge^{\triangleright} \xi = X^m \wedge \xi^\mu\,T_m \triangleright \tau_\mu = X^m \wedge \xi^\mu\,\triangleright_{m\mu}^{\nu}\,\tau_\nu \in \mathfrak{h}.$$

One also has

$$\partial\xi = \xi^\mu \partial\tau_\mu = \xi^\mu \partial_{\mu}^{m}\,T_m.$$

Let us choose a non-degenerate bilinear G-invariant form in $\mathfrak{g}$, $\langle , \rangle_{\mathfrak{g}}$, and a non-degenerate bilinear G-invariant form $\langle , \rangle_{\mathfrak{h}}$ in $\mathfrak{h}$, see Definition 2.3. If

X and Y are two $\mathfrak{g}$-valued forms on M, then we define

$$\langle X \wedge Y \rangle_{\mathfrak{g}} = X^m \wedge Y^n \langle T_m, T_n \rangle_{\mathfrak{g}} = X^m \wedge Y^n Q_{mn}.$$

Similarly, for two $\mathfrak{h}$-valued forms

$$\langle \xi \wedge \eta \rangle_{\mathfrak{h}} = \xi^\mu \wedge \eta^\nu \langle \tau_\mu, \tau_\nu \rangle_{\mathfrak{h}} = \xi^\mu \wedge \eta^\nu q_{\mu\nu}.$$

The matrices Q and q correspond to G-invariant metrics on $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

As far as the bilinear antisymmetric map $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ (see 2.1.2) is concerned, one can write

$$\mathcal{T}(\tau_\mu, \tau_\nu) = \mathcal{T}^m_{\mu\nu} T_m,$$

so that the defining relation for $\mathcal{T}$ becomes

$$\mathcal{T}^n_{\mu\nu} Q_{nm} = \triangleright^{\rho}_{m[\mu|} q_{\rho|\nu]}.$$

Given two $\mathfrak{h}$-valued forms ξ and η, we can define a $\mathfrak{g}$-valued form

$$\xi \wedge^{\mathcal{T}} \eta = \mathcal{T}^m_{\mu\nu} \xi^\mu \wedge \eta^\nu T_m.$$

3 Actions for a 2-BF theory

Our goal now is to construct a gauge invariant action for the theory of 2-flat and fake-flat 2-connections associated to a general Lie crossed module such that this action is a generalization of a BF theory action [4]. We will call such a theory a 2-BF theory, since it can be considered as a BF theory for a 2-group.

3.1 Preliminaries

Let us consider a Lie crossed module $\mathcal{X} = (\partial \colon H \to G, \triangleright)$ and let $\mathfrak{X} = (\partial \colon \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be the associated differential crossed module. Let M be a 4-manifold. Our initial space of fields is given by the forms

- $A \in \mathcal{A}^1(M, \mathfrak{g})$
- $\beta \in \mathcal{A}^2(M, \mathfrak{h})$.

As we have explained in Section 1, the pair (A, β) will be called a local 2-connection.

Let us introduce the curvature 2-form of A

$$\mathrm{F}_A = dA + A \wedge A,$$

and the 2-curvature 3-form of the pair (A, β)

$$\mathcal{G}_{(A,\beta)} = d\beta + A \wedge^{\triangleright} \beta.$$

The corresponding fake curvature tensor is given by

$$\mathcal{F}_{(A,\beta)} = \mathrm{F}_A - \partial\beta.$$

Given that (A, β) is a local 2-connections on a 2-bundle, we will consider the following transformations of the pair (A, β), see Section 1:

- *Thin*: For a smooth map $\phi\colon M \to G$

$$A \mapsto \phi^{-1}A\phi + \phi^{-1}d\phi, \quad \beta \mapsto \phi^{-1} \triangleright \beta. \tag{3.1}$$

- *Fat*: For a 1-form $\eta \in \mathcal{A}^1(M, \mathfrak{h})$

$$A \mapsto A + \partial\eta, \quad \beta \mapsto \beta + d\eta + A \wedge^{\triangleright} \eta + \eta \wedge \eta. \tag{3.2}$$

Under the thin gauge transformations, the curvature, the 2-curvature and the fake curvature change as

$$\mathrm{F}_A \mapsto \phi^{-1}\mathrm{F}_A\phi, \quad \mathcal{G}_{(A,\beta)} \mapsto \phi^{-1} \triangleright \mathcal{G}_{(A,\beta)}, \quad \mathcal{F}_{(A,\beta)} \mapsto \phi^{-1}\mathcal{F}_{(A,\beta)}\phi.$$

The action of the fat gauge transformations on F_A is given by

$$\mathrm{F}_A \mapsto \mathrm{F}_A + d(\partial\eta) + A \wedge \partial\eta + \partial\eta \wedge A + \partial\eta \wedge \partial\eta.$$

Since

$$A \wedge \partial\eta + \partial\eta \wedge A = \partial\left(A \wedge^{\triangleright} \eta\right), \quad \partial\eta \wedge \partial\eta = \partial(\eta \wedge \eta),$$

due to (2.6) and (2.7), one obtains

$$\mathrm{F}_A \mapsto \mathrm{F}_A + \partial\left(d\eta + A \wedge^{\triangleright} \eta + \eta \wedge \eta\right).$$

Therefore the fake curvature is invariant under the fat gauge transformations

$$\mathcal{F}_{(A,\beta)} \mapsto \mathcal{F}_{(A,\beta)}.$$

The 2-curvature $\mathcal{G}_{(A,B)}$ transforms under the fat gauge transformations as

$$\begin{aligned}\mathcal{G} \to\ & d\beta + d(A \wedge^{\triangleright} \eta) + d(\eta \wedge \eta) \\ & + A \wedge^{\triangleright} \beta + A \wedge^{\triangleright} d\eta + A \wedge^{\triangleright} (A \wedge^{\triangleright} \eta) + A \wedge^{\triangleright} (\eta \wedge \eta) \\ & + \partial\eta \wedge^{\triangleright} \beta + \partial\eta \wedge^{\triangleright} d\eta + \partial\eta \wedge^{\triangleright} (A \wedge^{\triangleright} \eta) \\ & + \partial\eta \wedge^{\triangleright} (\eta \wedge \eta)\,. \end{aligned} \tag{3.3}$$

By using the following identities

$$\begin{gathered} d(\eta \wedge \eta) + \partial\eta \wedge^{\triangleright} d\eta = 0, \\ d(A \wedge^{\triangleright} \eta) = dA \wedge^{\triangleright} \eta - A \wedge^{\triangleright} d\eta, \\ A \wedge^{\triangleright} (A \wedge^{\triangleright} \eta) = (A \wedge A) \wedge^{\triangleright} \eta, \\ \partial\eta \wedge^{\triangleright} \beta = -\partial\beta \wedge^{\triangleright} \eta, \\ A \wedge^{\triangleright} (\eta \wedge \eta) + \partial\eta \wedge^{\triangleright} (A \wedge^{\triangleright} \eta) = 0, \\ \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) = 0, \end{gathered}$$

which follow from (2.6), (2.7) and (2.8), the transformation (3.3) becomes

$$\mathcal{G}_{(A,\beta)} \mapsto \mathcal{G}_{(A,\beta)} + \mathcal{F}_{(A,\beta)} \wedge^{\triangleright} \eta. \tag{3.4}$$

3.2 BFCG action

Given the 2-group gauge fields (A, β), one would like to find an action invariant under the gauge transformations (3.1) and (3.2), such that the corresponding equations of motion imply the vanishing of the fake and the 2-curvature tensors, i.e.,

$$\mathcal{F}_{(A,\beta)} = \mathrm{F}_A - \partial\beta = 0, \quad \mathcal{G}_{(A,\beta)} = d\beta + A \wedge^{\triangleright} \beta = 0. \tag{3.5}$$

The simplest way to obtain such an action is to consider equations (3.5) as the dynamical constraints and therefore to enforce them by using the Lagrange multiplier terms. Since this was also the way of obtaining the action for a BF theory, where $\mathrm{F}_A = 0$, we will then obtain a generalization of the BF-theory action. Therefore consider

$$S_1 = \int_M \langle B \wedge \mathcal{F}_{(A,\beta)} \rangle_{\mathfrak{g}} + \int_M \langle C \wedge \mathcal{G}_{(A,\beta)} \rangle_{\mathfrak{h}}, \tag{3.6}$$

where $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{\mathfrak{h}}$ are G-invariant, bilinear non-degenerate and symmetric forms in the differential crossed module $(\mathfrak{h} \to \mathfrak{g}, \triangleright)$, see Definition 5. The

Lagrange multiplier field B is a $\mathfrak{g}$-valued two-form, while the Lagrange multiplier field C is a $\mathfrak{h}$-valued one-form. We will refer to this action as the BFCG action.

The action S_1 will be invariant under the thin gauge transformations if

$$C \to \phi^{-1} \triangleright C, \quad B \to \phi^{-1} B \phi. \tag{3.7}$$

This is ensured by the G-invariance of the bilinear forms $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{\mathfrak{h}}$, see Definition 2.3.

In order to make the action (3.6) invariant under the fat gauge transformations, the fields B and C have to transform as

$$B \mapsto B + C \wedge^{\mathcal{T}} \eta, \quad C \mapsto C, \tag{3.8}$$

where the antisymmetric map $\mathcal{T} \colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ is defined in 2.1.2. Note that $C \wedge^{\mathcal{T}} \eta$ is the antisymmetrization of $\mathcal{T}(C, \eta)$.

The BFCG action S_1 reduces to the BFCG action found in [26] in the special case of a Lie crossed module of the form $\left(\partial \colon \mathfrak{g} \overset{v \mapsto 0}{\mapsto} G, \triangleright\right)$, where $\triangleright$ denotes the adjoint action of G on $\mathfrak{g}$, and the abelian Lie group structure on $\mathfrak{g}$ is given by the sum of vectors.

Proposition 3.1. *The BFCG action* (3.6) *is invariant under the gauge transformations* (3.1), (3.2), (3.7) *and* (3.8). *The equations of motion are given by*

$$\mathcal{F}_{(A,\beta)} = 0, \quad \mathcal{G}_{(A,\beta)} = 0, \tag{3.9}$$

$$\partial^*(B) + dC + A \wedge^{\triangleright} C = 0, \tag{3.10}$$

$$dB + A \wedge B + \beta \wedge^{\mathcal{T}} C = 0. \tag{3.11}$$

These equations of motion are obtained by calculating the variational derivatives of S_1 with respect to the fields appearing in it. Here $\partial^* \colon \mathfrak{g} \to \mathfrak{h}$ is obtained from the adjoint $\partial^\dagger \colon \mathfrak{g}^* \to \mathfrak{h}^*$, by using the isomorphisms $\mathfrak{g}^* \cong \mathfrak{g}$ and $\mathfrak{h}^* \cong \mathfrak{h}^*$ provided by the non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$. In components

$$\partial^*(B) = B^m \partial^*(T_m) = B^m \partial^{*\mu}_m \tau_\mu,$$

where $\partial^{*\mu}_m = Q_{mn} \partial^n_\nu q^{\nu\mu}$ and $q^{\nu\mu} = q^{\mu\nu}$ is the inverse matrix of $q_{\mu\nu}$.

3.3 Extended BFCG action

Another way to obtain a 2-BF action is to introduce an additional auxiliary field $\alpha \in \mathcal{A}^1(M, \mathfrak{h})$ beside the Lagrange multiplier fields, such that

$$S_2 = \int_M \langle B' \wedge \mathcal{F}_{(A,\beta)} \rangle_{\mathfrak{g}} + \int_M \langle C \wedge \big(\mathcal{G}_{(A,\beta)} + \mathcal{F}_{(A,\beta)} \wedge^{\triangleright} \alpha\big) \rangle_{\mathfrak{h}}. \tag{3.12}$$

We have written B' instead of B because B' will be invariant under the fat gauge transformations.

It is easy to see that the action S_2 will be invariant under the thin gauge transformations if

$$\alpha \to \phi^{-1} \triangleright \alpha, \quad B' \mapsto \phi^{-1} B' \phi, \quad C \mapsto \phi^{-1} \triangleright C, \tag{3.13}$$

while the invariance under the fat gauge transformations can be achieved if

$$\alpha \to \alpha - \eta, \quad B' \mapsto B', \quad C \mapsto C. \tag{3.14}$$

Note that both terms of the S_2 action are invariant under the gauge transformations, which does not happen in the case of the action S_1. As before, the invariance of the S_2 action under the thin gauge transformations is ensured by the G-invariance of the bilinear forms $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{\mathfrak{h}}$, and by the identity (2.3). Because S_2 contains an additional auxiliary field α, we will refer to S_2 as an extended BFCG action.

It is not difficult to see that the actions S_1 and S_2 are related by the following field redefinition:

$$B = B' - C \wedge^{\mathcal{T}} \alpha. \tag{3.15}$$

The transformation

$$(A, \beta, C, B', \alpha) \overset{\Phi}{\mapsto} (A, \beta, C, B' - C \wedge^{\mathcal{T}} \alpha)$$

is not invertible because S_2 has more fields. However, it can be shown that the dynamics of the theory defined by the action S_2 is determined by the dynamics of the S_1 theory.

The equations of motion for the action S_2 can be obtained by calculating the variational derivatives with respect to the fields appearing in the action.

The variation with respect to B' and C gives

$$\mathcal{F}_{(A,\beta)} = 0, \quad \mathcal{G}_{(A,\beta)} + \mathcal{F}_{(A,\beta)} \wedge^{\triangleright} \alpha = 0, \tag{3.16}$$

respectively. The variation with respect to β and A can be obtained by substituting (3.15) into the equations of motion (3.10) and (3.11) for S_1. There will be one more equation, corresponding to the variation of S_2 with respect to α, and this one is given by

$$C \wedge^{\triangleright} \mathcal{F}_{(A,\beta)} = 0. \tag{3.17}$$

Equations (3.16) and (3.17) do not determine α so that α is determined only by equations (3.10) and (3.11) where $B = B' - C \wedge^{\mathcal{T}} \alpha$. Therefore, given a solution (A, β, B, C) of the S_1 equations of motion, then $(A, \beta, B + C \wedge^{\mathcal{T}} \alpha, C, \alpha)$ is a solution of the S_2 equations of motion, where the components of α are arbitrary functions on M.

Note that the extended BFCG action can be easily modified by introducing powers of B' and C fields, in the following way:

$$\begin{aligned} S_2' = S_2 + \int_M \lambda_1 \langle B' \wedge B' \rangle_{\mathfrak{g}} + \lambda_2 \langle \big(B' \wedge^{\triangleright} C\big) \wedge C \rangle_{\mathfrak{h}} \\ + \lambda_3 \langle \big(C \wedge^{\mathcal{T}} C\big) \wedge \big(C \wedge^{\mathcal{T}} C\big) \rangle_{\mathfrak{g}}, \end{aligned} \tag{3.18}$$

where λ_1, λ_2 and λ_3 are arbitrary constants and we have written only the non-trivial terms. This modified action is clearly invariant under thin and fat gauge transformations.

The action (3.18) is a generalization of the cubic term action in the case of three-dimensional BF-theory, as well as a generalization of the quadratic term BF-action in the four-dimensional case, see [14].

4 Conclusions

The BFCG action (3.6) defines the dynamics of fake-flat and 2-flat 2-connections (A, β), as well as the dynamics of the auxiliary fields B and C. One can now use the action (3.6) to quantize the corresponding 2-BF theory.

A perturbative quantization of a BFCG theory can be obtained by using Feynman diagrams, in the same way as the perturbative quantization of a BF theory can be obtained by using the corresponding Feynman diagrams, see [15, 16]. In this way one can obtain perturbative invariants of knotted surfaces, by using the (non-abelian) 2-holonomies (Wilson surface observables) defined in [20, 21, 35].

Due to the fact that these observables are trivial for the case of 2-flat and fake-flat 2-connections and 2-spheres embedded in S^4 [20, 21], and given that the BFCG path-integral gives a delta function on the space of fake-flat and 2-flat 2-connections, it is very likely that the expectation values of these Wilson surface observables will be trivial invariants for embedded spheres in S^4. The three-parameter deformation of the extended BFCG action (3.18) is a good starting point for solving this problem. In this case an appropriate modification of the observables will be needed, and this may yield non-trivial perturbative invariants of knotted surfaces. This belief is based on the analogy with the three-dimensional BF theory case where the corresponding knot invariants can be modified in such a way, yielding Witten–Reshetikhin–Turaev invariants of links, see [14, 22]. A perturbative expansion of such a set of extended BFCG invariants could give a categorified version of Vassiliev knot invariants [1, 29].

For the case of 4-manifolds without Wilson surfaces, a non-perturbative quantization of a BFCG theory can be obtained by using a generalization of the spin foam quantization method, see [26]. Given a closed 4-manifold M and a Lie crossed module $(\partial : H \to G, \triangleright)$, the path-integral

$$Z(M) = \int \mathcal{D}A\, \mathcal{D}\beta\, \mathcal{D}B\, \mathcal{D}C\, e^{i\int_M \langle B\wedge \mathcal{F}_{(A,\beta)}\rangle_{\mathfrak{g}} + \langle C\wedge \mathcal{G}_{(A,\beta)}\rangle_{\mathfrak{h}}},$$

when properly defined, should coincide with the volume of the space of fake-flat and 2-flat 2-connections over M, up to gauge transformations. The $Z(M)$ could be defined by discretizing M and by using the associated one and two-dimensional holonomies. This was the approach used in [26], where a formal discretized expression $Z_{\mathrm{GPP}}(M)$ for $Z(M)$ was given. In the case of a finite crossed module, $Z_{\mathrm{GPP}}(M)$ coincides with Yetter's manifold invariant $Y(M)$, see [32, 36]. The invariant $Y(M)$ was originally defined for three-dimensional closed manifolds, but the expression for $Y(M)$ can be proven to be an invariant of closed manifolds of arbitrary dimension [24, 33]. In fact, $Y(M)$ depends only on the homotopy type of a manifold, and can be extended to general CW-complexes [18, 19, 24].

Although $Y(M)$ is a homotopy invariant for a finite crossed module, there is no immediate reason to believe that $Y(M)$ will be a homotopy invariant for a general Lie crossed module. It is an important open problem how to define $Y(M)$ in this case, and a possible approach is to generalize the construction of the Ponzano–Regge model path integral done in [13] to the case of a Lie crossed module. Such an approach will require an extension of the Peter–Weyl Theorem to categorical representations of crossed modules, see [10, 17], which appears not to have been addressed in the literature.

We expect that it will be possible to obtain a quantum group invariant related with $Z(M)$ by making a quantum-group regularization of the dual complex discretized expression of $Z_{\mathrm{GPP}}(M)$. This expectation comes from the analogy with the BF-theory case, where the quantum group regularization of the dual complex state-sum representation of $Z(M)$ gives the Turaev–Viro invariant in the 3-manifold case, while in the 4-manifold case it gives the Crane-Yetter invariant. Such an approach will require a generalization of Lie crossed modules for the case of quantum groups, i.e., a quantum 2-group, see [30], as well as a representation theory of quantum 2-groups at a root of unity. This quantum 2-group should be relevant for the quantization of the modified extended BFCG action (3.18).

Note that the construction of Lie crossed modules in Section 2 by using chain complexes of vector spaces provides a lot of non-trivial examples of Lie crossed modules where the group H is non-abelian and the morphism $\partial : H \to G$ is non-trivial, such that the group G is compact and non-abelian. It also provides a matrix representation of Lie crossed modules, which is important for classical and quantum field theory considerations.

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