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Compact supersymmetric solutions of the heterotic equations of motion in dimensions 7 and 8

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Abstract

We construct explicit compact solutions with non-zero field strength, non-flat instanton and constant dilaton to the heterotic string equations in dimensions 7 and 8. We present a quadratic condition on the curvature, which is necessary and sufficient the heterotic supersymmetry and the anomaly cancellation to imply the heterotic equations of motion in dimensions 7 and 8. We show that some of our examples are compact supersymmetric solutions of the heterotic equations of motion in dimensions 7 and 8.

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1 Introduction

The bosonic fields of the ten-dimensional supergravity, which arises as the low-energy effective theory of the heterotic string are the spacetime metric g, the NS three-form field strength H, the dilaton ϕ and the gauge connection A with curvature F^A. The bosonic geometry considered in this paper is of the form $R^{1,9-d} \times M^d$ where the bosonic fields are non-trivial only on M^d, $d \leq 8$. We consider the two connections

$$\nabla^{\pm} = \nabla^g \pm \frac{1}{2} H,$$

where ∇^g is the Levi–Civita connection of the Riemannian metric g. Both connections preserve the metric, $\nabla^{\pm} g = 0$, and have totally skew-symmetric torsion $\pm H$, respectively.

The Green–Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an α' correction of the form

$$dH = \frac{\alpha'}{4}(p_1(M^{\mathrm{d}}) - p_1(E)) = 2\pi^2\alpha'\Big(\mathrm{Tr}(R\wedge R) - \mathrm{Tr}(F^A \wedge F^A)\Big), \quad (1.1)$$

where $p_1(M^{\mathrm{d}}), p_1(E)$ are the first Pontrjagin forms of M^{d} with respect to a connection ∇ with curvature R, and that of the vector bundle E with connection A, respectively.

A class of heterotic-string backgrounds for which the Bianchi identity of the three-form H receives a correction of type (1.1) are those with (2,0) world-volume supersymmetry. Such models were considered in [55]. The target-space geometry of (2,0)-supersymmetric sigma models has been extensively investigated in [55, 73, 52]. Recently, there is revived interest in these models [37, 13, 38, 39, 41] as string backgrounds and in connection to heterotic-string compactifications with fluxes [12, 1, 2, 3, 67, 34, 35, 4].

In writing (1.1) there is a subtlety to the choice of connection ∇ on M^{d} since anomalies can be cancelled independently of the choice [53]. Different connections correspond to different regularization schemes in the two-dimensional worldsheet non-linear sigma model. Hence, the background fields given for the particular choice of ∇ must be related to those for a different choice by a field redefinition [70]. Connections on M^{d} proposed to investigate the anomaly cancellation (1.1) are ∇^g [73, 39], ∇^+ [13], ∇^- [5, 12, 41, 56], Chern connection ∇^{c} when $d = 6$ [73, 67, 34, 35, 4].

A heterotic geometry will preserve supersymmetry if and only if, in 10 dimensions, there exists at least one Majorana–Weyl spinor ϵ such that the supersymmetry variations of the fermionic fields vanish, i.e., the following Killing spinor equations hold [73]:

$$\begin{gathered}
\delta_\lambda = \nabla_m \epsilon = \left(\nabla^g_m + \frac{1}{4} H_{mnp}\Gamma^{np}\right)\epsilon = \nabla^+\epsilon = 0, \\
\delta_\Psi = \left(\Gamma^m \partial_m \phi - \frac{1}{12} H_{mnp}\Gamma^{mnp}\right)\epsilon = \left(d\phi - \frac{1}{2}H\right)\cdot\epsilon = 0, \qquad (1.2) \\
\delta_\xi = F^A_{mn}\Gamma^{mn}\epsilon = F^A \cdot \epsilon = 0,
\end{gathered}$$

where λ, Ψ and ξ are the gravitino, the dilatino and the gaugino fields, respectively, and $\cdot$ means the Clifford action of forms on spinors.

The bosonic part of the ten-dimensional supergravity action in the string frame is [5]

$$S = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}\,\mathrm{e}^{-2\phi}\Big[Scal^g + 4(\nabla^g\phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\Big(\mathrm{Tr}|F^A|^2 - \mathrm{Tr}|R|^2\Big)\Big]. \tag{1.3}$$

The string frame field equations (the equations of motion induced from the action (1.3)) of the heterotic string up to two-loops [54] in sigma model perturbation theory are (we use the notations in [41])

$$\begin{gathered} Ric^g_{ij} - \frac{1}{4}H_{imn}H^{mn}_j + 2\nabla^g_i\nabla^g_j\phi - \frac{\alpha'}{4}\Big[(F^A)_{imab}(F^A)^{mab}_j - R_{imnq}R^{mnq}_j\Big] = 0,\\ \nabla^g_i(\mathrm{e}^{-2\phi}H^i_{jk}) = 0, \\ \nabla^+_i(\mathrm{e}^{-2\phi}(F^A)^i_j) = 0. \end{gathered} \tag{1.4}$$

The field equation of the dilaton ϕ is implied from the first two equations above.

We search for a solution to lowest non-trivial order in α' of the equations of motion in dimensions 7 and 8 that follow from the bosonic action, which also preserves at least one supersymmetry.

It is known [19, 38] ([41] for dimension 6) that the equations of motion of type I supergravity (1.4) with $R = 0$ are automatically satisfied if one imposes, in addition to the preserving supersymmetry equations (1.2), the three-form Bianchi identity (1.1) taken with respect to a flat connection ($R = 0$) on TM.

A lot of effort had been done in dimension six and compact torsional solutions for the heterotic/type I string are known to exist [18, 1, 2, 13, 39, 67, 34, 35, 4, 17, 24].

In dimension five compact supersymmetric solutions to the heterotic equations of motion with non-zero fluxes and constant dilaton have been constructed recently in [25].

In dimensions 7 and 8, the only known heterotic/type I solutions with non-zero fluxes to the equations of motion preserving at least one supersymmetry (satisfying (1.2) and (1.1) without the curvature term, $R = 0$) are those constructed in [21, 36, 51] for dimension 8, and those presented in [48] for dimension 7. All these solutions are non-compact and conformal to a flat space. Non-compact solutions to (1.2) and (1.1) in dimensions 7 and 8 are presented also in [56].

The main goal of this paper is to construct explicit compact supersymmetric valid solutions with non-zero field strength, non-flat instanton and constant dilaton to the heterotic equations of motion (1.4) in dimensions 7 and 8.

According to no-go (vanishing) theorems (a consequence of the equations of motion [28, 19]; a consequence of the supersymmetry [60, 59] for $\mathrm{SU}(n)$-case and [39] for the general case) there are no compact solutions with non-zero flux and non-constant dilaton satisfying simultaneously the supersymmetry equations (1.2) and the three-form Bianchi identity (1.1) if one takes flat connection on TM, more precisely a connection with zero first Pontrjagin four-form, $\mathrm{Tr}(R \wedge R) = 0$.

In the compact case one necessarily has to have a non-zero term $\mathrm{Tr}(R \wedge R)$. However, under the presence of a non-zero curvature 4-form $\mathrm{Tr}(R \wedge R)$ the solution of the supersymmetry equations (1.2) and the anomaly cancellation condition (1.1) obeys the second and the third equations of motion (the second and the third equations in (1.4)) but does not always satisfy the Einstein equation of motion (the first equation in (1.4)). We give in Theorem 4.1 a quadratic expression for R, which is necessary and sufficient condition in order that (1.2) and (1.1) imply (1.4) in dimensions 7 and 8 based on the properties of the special geometric structure induced from the first two equations in (1.2). (A similar condition in dimensions five and six we presented in [25, 24], respectively.) In particular, if R is a G_2-instanton (resp. Spin(7)-instanton) the supersymmetry equations together with the anomaly cancellation imply the equations of motion. The latter can also be seen following the considerations in the appendix of [38].

In this article, we present compact nilmanifolds in dimensions seven and eight satisfying the heterotic supersymmetry equations (1.2) with non-zero flux H, non-flat instanton and constant dilaton obeying the three-form Bianchi identity (1.1) with curvature term $R = R^+$, which is of instanton type. According to Theorem 4.1 these nilmanifolds are compact supersymmetric solutions of the heterotic equations of motion (1.4) in dimensions 7 and 8. The solutions in dimension 7 are constructed on the seven-dimensional generalized Heisenberg nilmanifold, which is a circle bundle over a six-torus with curvature inside the Lie algebra $\mathfrak{su}(3)$. The eight-dimensional compact solutions can be described as a circle bundle over the product of a two-torus by the total space of a circle bundle over a four-torus, or alternatively as the total space of a circle bundle with curvature inside the Lie algebra $\mathfrak{g}_2$ over a seven-manifold which is a circle bundle over a six-torus (see Section 5 for details). Based on the examples we present in Section 5, as well as on constructions proposed in [39], we outline in the last section a more general construction of compact manifolds solving the first two equations in (1.2) with non-constant dilaton depending on reduced number of variables.

Our solutions seem to be the first explicit compact valid supersymmetric heterotic solutions with non-zero flux, non-flat instanton and constant dilaton in dimensions 7 and 8 satisfying the equations of motion (1.4).

Our conventions: We rise and lower the indices with the metric and use the summation convention on repeated indices. For example,

$$B_{ijk}C^{ijk} = B_i^{jk}C^i_{jk} = B_{ijk}C_{ijk} = \sum_{i,j,k=1}^{d} B_{ijk}C_{ijk}.$$

The connection one-forms σ_{ji} of a metric connection $\nabla, \nabla g = 0,$ with respect to a local basis $\{E_1, \dots, E_d\}$ are given by

$$\sigma_{ji}(E_k) = g(\nabla_{E_k}E_j, E_i),$$

since we write $\nabla_X E_j = \sigma_j^s(X)\, E_s$.

The curvature two-forms Ω^i_j of ∇ are given in terms of the connection one-forms σ^i_j by

$$\Omega^i_j = d\sigma^i_j + \sigma^i_k \wedge \sigma^k_j, \quad \Omega_{ji} = d\sigma_{ji} + \sigma_{ki} \wedge \sigma_{jk}, \quad R^l_{ijk} = \Omega^l_k(E_i, E_j),$$
$$R_{ijkl} = R^s_{ijk}g_{ls}, \tag{1.5}$$

and the first Pontrjagin class is represented by the four-form

$$p_1(\nabla) = \frac{1}{8\pi^2} \sum_{1\le i<j\le d} \Omega^i_j \wedge \Omega^i_j.$$

2 General properties of G_2 and Spin(7) structures

We recall the basic properties of the geometric structures induced from the gravitino and dilatino Killing spinor equations (the first two equations in (1.2)) in dimensions 7 and 8.

G_2-structures in $d = 7$. Endow $\mathbb{R}^7$ with its standard orientation and inner product. Let $\{E_1, \dots, E_7\}$ be an oriented orthonormal basis and $\{e^1, \dots, e^7\}$

its dual basis. Consider the three-form Θ on $\mathbb{R}^7$ given by

$$\Theta = e^{127} - e^{236} + e^{347} + e^{567} - e^{146} - e^{245} + e^{135}. \tag{2.1}$$

The subgroup of $GL(7,\mathbb{R})$ fixing Θ is the exceptional Lie group G_2. It is a compact, connected, simply connected, simple Lie subgroup of SO(7) of dimension 14 [7]. The Lie algebra is denoted by $\mathfrak{g}_2$, and it is isomorphic to the two-forms satisfying seven linear equations, namely $\mathfrak{g}_2 \cong \Lambda^2_{14}(\mathbb{R}^7) = \{\beta \in \Lambda^2(\mathbb{R}^7) | *(\beta \wedge \Theta) = -\beta\}$. The three-form Θ corresponds to a real spinor ϵ and therefore, G_2 can be identified as the isotropy group of a non-trivial real spinor.

The Hodge star operator supplies the four-form $*\Theta$ given by

$$*\Theta = e^{3456} + e^{1457} + e^{1256} + e^{1234} + e^{2357} + e^{1367} - e^{2467}. \tag{2.2}$$

The space $\Lambda^2_{14}(\mathbb{R}^7)$ can also be described as the subspace of 2-forms β which annihilate $*\Theta$, i.e., $\beta \wedge *\Theta = 0$. A seven-dimensional Riemannian manifold M is called a G_2-manifold if its structure group reduces to the exceptional Lie group G_2. The existence of a G_2-structure is equivalent to the existence of a global non-degenerate three-form, which can be locally written as (2.1). The three-form Θ is called *the fundamental form* of the G_2-manifold [6]. From the purely topological point of view, a seven-dimensional paracompact manifold is a G_2-manifold if and only if it is an oriented spin manifold [66]. We will say that the pair (M, Θ) is a G_2-manifold with G_2*-structure* (determined by) Θ.

The fundamental form of a G_2-manifold determines a Riemannian metric *implicitly* through $g_{ij} = \frac{1}{6}\sum_{kl} \Theta_{ikl}\Theta_{jkl}$ [47]. This is referred to as the metric induced by Θ.

In [23], Fernández and Gray divide G_2-manifolds into 16 classes according to how the covariant derivative of the fundamental three-form behaves with respect to its decomposition into G_2 irreducible components (see also [14, 37]). If the fundamental form is parallel with respect to the Levi–Civita connection, $\nabla^g\Theta = 0$, then the Riemannian holonomy group is contained in G_2. In this case the induced metric on the G_2-manifold is Ricci-flat, a fact first observed by Bonan [6]. It was shown by Gray [47] (see also [23, 7, 71]) that a G_2-manifold is parallel precisely when the fundamental form is harmonic, i.e., $d\Theta = d * \Theta = 0$. The first examples of complete parallel G_2-manifolds were constructed by Bryant and Salamon [9] and Gibbons *et al.* [40]. Compact examples of parallel G_2-manifolds were obtained first by Joyce [61, 62, 63] and recently by Kovalev [65].

The Lee form θ^7 is defined by [11]

$$\theta^7 = -\tfrac{1}{3} * (*d\Theta \wedge \Theta) = \tfrac{1}{3} * (*d * \Theta \wedge *\Theta). \tag{2.3}$$

If the Lee form vanishes, $\theta^7 = 0$, then the G_2-structure is said to be *balanced*. If the Lee form is closed, $d\theta^7 = 0$, then the G_2-structure is locally conformally equivalent to a balanced one [32]. If the G_2-structure satisfies the condition $d * \Theta = \theta^7 \wedge *\Theta$ then it is called *integrable* and an analog of the Dolbeault cohomology is investigated in [27]. A *cocalibrated* G_2-structure is a balanced G_2-structure which is also integrable.

Spin(7)-structures in $d = 8$. Consider $\mathbb{R}^8$ endowed with an orientation and its standard inner product. Let $\{E_1, \dots, E_8\}$ be an oriented orthonormal basis and $\{e^1, \dots, e^8\}$ its dual basis. Consider the four-form Φ on $\mathbb{R}^8$ given by

$$\begin{aligned}\Phi = {}& e^{1238} - e^{1347} + e^{1458} + e^{1678} - e^{1257} - e^{1356} + e^{1246} \\ & + e^{4567} + e^{2568} + e^{2367} + e^{2345} + e^{3468} + e^{2478} - e^{3578}.\end{aligned} \tag{2.4}$$

The four-form Φ is self-dual $*\Phi = \Phi$ and the 8-form $\Phi \wedge \Phi$ coincides with the volume form of $\mathbb{R}^8$. The subgroup of $GL(8, \mathbb{R})$, which fixes Φ is isomorphic to the double covering Spin(7) of SO(7) [50]. Moreover, Spin(7) is a compact simply-connected Lie group of dimension 21 [7]. The Lie algebra of Spin(7) is denoted by $\mathfrak{spin}(7)$ and it is isomorphic to the two-forms satisfying 7 linear equations, namely $\mathfrak{spin}(7) \cong \{\beta \in \Lambda^2(\mathbb{R}^8) | * (\beta \wedge \Phi) = -\beta\}$. The four-form Φ corresponds to a real spinor ϕ and therefore, Spin(7) can be identified as the isotropy group of a non-trivial real spinor.

A *Spin(7)-structure* on an eight-manifold M is by definition a reduction of the structure group of the tangent bundle to Spin(7); we shall also say that M is a *Spin(7)-manifold*. This can be described geometrically by saying that there exists a nowhere vanishing global differential four-form Φ on M, which can be locally written as (2.4). The four-form Φ is called the *fundamental form* of the Spin(7)-manifold M [6].

The fundamental form of a Spin(7)-manifold determines a Riemannian metric *implicitly* through $g_{ij} = \frac{1}{24} \sum_{klm} \Phi_{iklm} \Phi_{jklm}$ [47]. This is referred to as the metric induced by Φ.

In general, not every 8-dimensional Riemannian spin manifold M admits a Spin(7)-structure. We explain the precise condition [66]. Denote by $p_2(M)$, $\mathbb{X}(M), \mathbb{X}(S_\pm)$ the second Pontrjagin class, the Euler characteristic of M and the Euler characteristic of the positive and the negative spinor bundles, respectively. It is well known [66] that a spin eight-manifold admits a

Spin(7)-structure if and only if $\mathbb{X}(S_+) = 0$ or $\mathbb{X}(S_-) = 0$. The latter conditions are equivalent to $p_1^2(M) - 4\,p_2(M) + 8\,\mathbb{X}(M) = 0$, for an appropriate choice of the orientation [66].

Let us recall that a Spin(7)-manifold (M, g, Φ) is said to be parallel (torsion-free [62]) if the holonomy of the metric $Hol(g)$ is a subgroup of Spin(7). This is equivalent to saying that the fundamental form Φ is parallel with respect to the Levi–Civita connection ∇^g of the metric g. Moreover, $Hol(g) \subset Spin(7)$ if and only if $d\Phi = 0$ [22] (see also [7, 71]) and any parallel Spin(7)-manifold is Ricci flat [6]. The first known explicit example of complete parallel Spin(7)-manifold with $Hol(g) = Spin(7)$ was constructed by Bryant and Salamon [9, 40]. The first compact examples of parallel Spin(7)-manifolds with $Hol(g) = Spin(7)$ were constructed by Joyce [61, 62].

There are four classes of Spin(7)-manifolds according to the Fernández classification [22] obtained as irreducible representations of Spin(7) of the space $\nabla^g\Phi$.

The Lee form θ^8 is defined by [10]

$$\theta^8 = -\frac{1}{7} * (*d\Phi \wedge \Phi) = \frac{1}{7} * (\delta\Phi \wedge \Phi). \tag{2.5}$$

The four classes of Fernández classification can be described in terms of the Lee form as follows [10]: $W_0 : d\Phi = 0$; $W_1 : \theta^8 = 0$; $W_2 : d\Phi = \theta^8 \wedge \Phi$; $W : W = W_1 \oplus W_2$.

A Spin(7)-structure of the class W_1 (i.e., Spin(7)-structure with zero Lee form) is called a *balanced* Spin(7)-structure. If the Lee form is closed, $d\theta^8 = 0$, then the Spin(7)-structure is locally conformally equivalent to a balanced one [58]. It is shown in [10] that the Lee form of a Spin(7)-structure in the class W_2 is closed and therefore such a manifold is locally conformally equivalent to a parallel Spin(7)-manifold. The compact spaces with closed but not exact Lee form (i.e., the structure is not globally conformally parallel) have different topology than the parallel ones [58].

Coeffective cohomology and coeffective numbers of Riemannian manifolds with Spin(7)-structure are studied in [74].

3 The supersymmetry equations

Geometrically, the vanishing of the gravitino variation is equivalent to the existence of a non-trivial real spinor parallel with respect to the metric connection ∇^+ with totally skew-symmetric torsion $T = H$. The presence of

∇^+-parallel spinor leads to restriction of the holonomy group $Hol(\nabla^+)$ of the torsion connection ∇^+. Namely, $Hol(\nabla^+)$ has to be contained in $\mathrm{SU}(3), d = 6$ [73, 60, 59, 39, 49, 13, 1, 2], the exceptional group $G_2, d = 7$ [31, 37, 39, 32], the Lie group $Spin(7), d = 8$ [37, 58, 39]. A detailed analysis of the induced geometries is carried out in [39] and all possible geometries (including non-compact stabilizers) are investigated in [43, 45, 44, 46].

Dimension $d = 7$.

The precise conditions to have a solution to the gravitino Killing spinor equation in dimension 7 were found in [31]. Namely, there exists a non-trivial parallel spinor with respect to a G_2-connection with torsion 3-form T if and only if there exists an integrable G_2-structure (Θ, g), i.e. $d * \Theta = \theta^7 \wedge *\Theta$. In this case, the torsion connection ∇^+ is unique and the torsion three-form T is given by

$$H = T = \tfrac{1}{6}(d\Theta, *\Theta)\, \Theta - *d\Theta + *(\theta^7 \wedge \Theta).$$

The Riemannian scalar curvature is [32] ([8] for the general case) $s^g = \frac{1}{18}(d\Theta, *\Theta) + ||\theta^7||^2 - \frac{1}{12}||T||^2 + 3\,\delta\theta^7$.

The necessary conditions to have a solution to the system of dilatino and gravitino Killing spinor equations were derived in [37, 31, 32], and the sufficiency was proved in [31, 32]. The general existence result [31, 32] states that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations in dimension 7 if and only if there exists a G_2-structure (Θ, g) satisfying the equations

$$d * \Theta = \theta^7 \wedge *\Theta, \quad d\Theta \wedge \Theta = 0, \quad \theta^7 = 2d\phi. \tag{3.1}$$

Consequently, the torsion three-form (the flux H) is given by

$$H = T = - * d\Theta + 2 * (d\phi \wedge \Theta). \tag{3.2}$$

The Riemannian scalar curvature satisfies $s^g = 8||d\phi||^2 - \frac{1}{12}||T||^2 + 6\,\delta d\phi$.

The equations (3.1) hold exactly when the G_2-structure $(\bar\Theta = \mathrm{e}^{-\frac{3}{2}\phi}\Theta$, $\bar g = \mathrm{e}^{-\phi} g)$ obeys the equations

$$d \bar{*} \bar\Theta = d\bar\Theta \wedge \bar\Theta = 0,$$

i.e., it is cocalibrated of pure type.

Dimension $d = 8$.

It is shown in [58] that the gravitino Killing spinor equation always has a solution in dimension 8. Namely, any Spin(7)-structure admits a unique Spin(7)-connection with totally skew-symmetric torsion T satisfying

$$T = *d\Phi - \tfrac{7}{6} * (\theta^8 \wedge \Phi).$$

(In fact, the converse is also true, namely if there are no obstructions to exist a solution to the gravitino Killing spinor equation then dimension is 8 and the structure is Spin(7) [29, 68].)

The necessary conditions to have a solution to the system of dilatino and gravitino Killing spinor equations were derived in [37, 58], and the sufficiency was proved in [58]. The general existence result [58] states that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations in dimension 8 if and only if there exists a Spin(7)-structure (Φ, g) with an exact Lee form which is equivalent to the statement that the Spin(7)-structure is conformally balanced, i.e., the Spin(7) structure $(\bar\Phi = e^{-\frac{12}{7}\phi}\Phi, \bar g = e^{-\frac{6}{7}\phi} g)$ satisfies $\bar{*} d\bar\Phi \wedge \bar\Phi = 0$.

The torsion three-form (the flux H) and the Lee form are given by

$$H = T = *d\Phi - 2 * (d\phi \wedge \Phi), \quad \theta^8 = \frac{12}{7} d\phi. \tag{3.3}$$

The Riemannian scalar curvature satisfies $s^g = 8||d\phi||^2 - \frac{1}{12}||T||^2 + 6\,\delta d\phi$.

In addition to these equations, the vanishing of the gaugino variation requires the two-form F^A to be of instanton type [16, 73, 51, 69, 20, 39]:

Case $d = 7$: a G_2-instanton, i.e., the gauge field A is a G_2-connection and its curvature 2-form $F^A \in \mathfrak{g}_2$. The latter can be expressed in any of the following two equivalent ways:

$$F^A_{mn}\Theta^{mn}{}_p = 0 \quad \Leftrightarrow \quad F^A_{mn} = -\tfrac{1}{2} F^A_{pq} (*\Theta)^{pq}{}_{mn}; \tag{3.4}$$

Case $d = 8$: a Spin(7)-instanton, i.e., the gauge field A is a Spin(7)-connection and its curvature 2-form $F^A \in \mathfrak{spin}(7)$. The latter is equivalent to

$$F^A_{mn} = -\tfrac{1}{2} F^A_{pq} \Phi^{pq}{}_{mn}. \tag{3.5}$$

4 Heterotic supersymmetry and equations of motion

It is known [19, 38] ([41] for dimension 6) that the equations of motion of type-I supergravity (1.4) with $R = 0$ are automatically satisfied if one imposes, in addition to the preserving supersymmetry equations (1.2), the three-form Bianchi identity (1.1) taken with respect to a flat connection on $TM, R = 0$. However, the no-go theorem [28, 19, 60, 59, 39] states that if even $\mathrm{Tr}(R \wedge R) = 0$ there are no compact solutions with non-zero flux H and non-constant dilaton.

In the presence of a curvature term $\mathrm{Tr}(R \wedge R) \neq 0$, a solution of the supersymmetry equations (1.2) and the anomaly cancellation condition (1.1) obeys the second and the third equations in (1.4) but does not always satisfy the Einstein equation of motion (the first equation in (1.4)). However, if the curvature R is of instanton type (1.2) and (1.1) imply (1.4). This can be seen following the considerations in the appendix of [38]. We shall give below an independent proof for the Einstein equation of motion (the first equation in (1.4)) based on the properties of the special geometric structure induced from the first two equations in (1.2).

A consequence of the gravitino and dilatino Killing spinor equations is an expression of the Ricci tensor $Ric^+_{mn} = R^+_{imnj} g^{ij}$ of the (+)-connection, and therefore an expression of the Ricci tensor Ric^g of the Levi–Civita connection, in terms of the suitable trace of the torsion three-form $T = H$ (the Lee form) and the exterior derivative of the torsion form $dT = dH$ (see [31] in dimension 7 and [58] in dimension 8). We outline a unified proof for dimensions 7 and 8.

Indeed, the two Ricci tensors are connected by (see, e.g., [31])

$$\begin{aligned} Ric^g_{mn} &= Ric^+_{mn} + \tfrac{1}{4} T_{mpq} T^{pq}_n - \tfrac{1}{2} \nabla^+_s T^s_{mn}, \\ Ric^+_{mn} - Ric^+_{nm} &= \nabla^+_s T^s_{mn} = \nabla^g_s T^s_{mn}, \qquad (4.1) \\ Ric^g_{mn} &= \tfrac{1}{2}(Ric^+_{mn} + Ric^+_{nm}) + \tfrac{1}{4} T_{mpq} T^{pq}_n. \qquad (4.2) \end{aligned}$$

Denote by Ψ the 4-form $- * \Theta$ in dimension 7 or the Spin(7)-form $-\Phi$ in dimension 8. Since $Hol(\nabla^+) \subset G_2, \mathrm{Spin}(7)$, we have the next sequence of identities

$$2Ric^+_{mn} = R^+_{mjkl} \Psi_{jkln} = \frac{1}{3}(R^+_{mjkl} + R^+_{mklj} + R^+_{mljk}) \Psi_{jkln}. \qquad (4.3)$$

We apply the following identity established in [31]

$$\begin{aligned} R^+_{jklm} + R^+_{kljm} + R^+_{ljkm} - R^+_{mjkl} - R^+_{mklj} - R^+_{mljk} \\ = \frac{3}{2} dT_{jklm} - T_{jks}T_{lms} - T_{kls}T_{jms} - T_{ljs}T_{kms}. \end{aligned} \tag{4.4}$$

The first Bianchi identity for ∇^+ reads (see e.g.[31])

$$\begin{aligned} R^+_{jklm} + R^+_{kljm} + R^+_{ljkm} = dT_{jklm} - T_{jks}T_{lms} - T_{kls}T_{jms} - T_{ljs}T_{kms} \\ + \nabla^+_m T_{jkl}. \end{aligned} \tag{4.5}$$

Now, (4.5), (4.4) and (4.3) yield

$$Ric^+_{mn} = \tfrac{1}{12} dT_{mjkl}\Psi_{jkln} + \tfrac{1}{6}\nabla^+_m T_{jkl}\Psi_{jkln}. \tag{4.6}$$

Using the special expression of the torsion (3.2) and (3.3) for dimensions 7 and 8, respectively, the equation (4.6) takes the form

$$Ric^+_{mn} = \frac{1}{12} dT_{mjkl}\Psi_{jkln} - 2\nabla^+_m d\phi_n = \frac{1}{12} dT_{mjkl}\Psi_{jkln} - 2\nabla^g_m d\phi_n + d\phi_s T^s_{mn}. \tag{4.7}$$

Substitute (4.7) into (4.2), insert the result into the first equation of (1.4) and use the anomaly cancellation (1.1) to conclude

Theorem 4.1. *The Einstein equation of motion (the first equation in* (1.4)*) in dimensions 7 and 8 is a consequence of the heterotic Killing spinor equations* (1.2) *and the anomaly cancellation* (1.1) *if and only if the next identity holds*

$$\frac{1}{6}\Big[R_{mjab}R_{klab} + R_{mkab}R_{ljab} + R_{mlab}R_{jkab}\Big]\Psi_{jkln} = R_{mpqr}R^{pqr}_n, \tag{4.8}$$

where the 4-form Ψ *is equal to* $-*\Theta$ *in dimension 7 and to the Spin(7)-form* $-\Phi$ *in dimension 8.*

In particular, if R *is an instanton then* (4.8) *holds.*

It is shown in [57] that the curvature of R^+ satisfies the identity $R^+_{ijkl} = R^+_{klij}$ if and only if $\nabla^+_i T_{jkl}$ is a four-form. Now Theorem 4.1 yields

Corollary 4.1. *Suppose the torsion three-form is* ∇^+*-parallel,* $\nabla^+_i T_{jkl} = 0$*. The equations of motion* (1.4) *with respect to the curvature* R^+ *of the (+)-connection are consequences of the heterotic Killing spinor equations* (1.2) *and the anomaly cancellation* (1.1).

Manifolds with parallel torsion three-form are studied in detail in dimension 6 [72] and dimension 7 [30].

4.1 Heterotic supersymmetry equations of motion with constant dilaton

In the case when the dilaton is constant we arrive to the following problems:

Dimension 7

We look for a compact G_2-manifold (M,Θ), which satisfies the following conditions:

(a) Gravitino and dilatino Killing spinor equations with constant dilaton: search for a cocalibrated G_2-manifold of pure type, i.e., $d*\Theta = d\Theta\wedge\Theta = 0$.

(b) Gaugino equation: look for a vector bundle E of rank r over M equipped with a G_2-instanton, i.e., a connection A with curvature 2-form Ω^A satisfying

$$(\Omega^A)_{E_i,E_j}(E_k,E_l)(*\Theta)(E_m,E_n,E_k,E_l) = -2(\Omega^A)_{E_i,E_j}(E_m,E_n), \qquad (4.9)$$

where $\{E_1,\dots,E_7\}$ is a G_2-adapted basis on M.

(c) Anomaly cancellation condition:

$$dH = dT = -d*d\Theta = 2\pi^2\alpha'\Big(p_1(M) - p_1(A)\Big), \qquad \alpha' > 0. \qquad (4.10)$$

(d) The first Pontrjagin form $p_1(M)$ satisfies equation (4.8).

Dimension 8

We look for a compact Spin(7)-manifold (M,Φ) satisfying the following conditions:

(a) Gravitino and dilatino Killing spinor equations with constant dilaton: (M,Φ) is balanced, i.e., $*d\Phi\wedge\Phi = 0$.

(b) Gaugino equation: look for a vector bundle E of rank r over M equipped with a Spin(7)-instanton, i.e., a connection A with curvature 2-form Ω^A satisfying

$$(\Omega^A)_{E_i,E_j}(E_k,E_l)(\Phi)(E_m,E_n,E_k,E_l) = -2(\Omega^A)_{E_i,E_j}(E_m,E_n), \qquad (4.11)$$

where $\{E_1,\dots,E_8\}$ is a Spin(7)-adapted basis on M.

(c) Anomaly cancellation condition:

$$dH = dT = d * d\Phi = 2\pi^2\alpha'\Big(p_1(M) - p_1(A)\Big), \qquad \alpha' > 0. \tag{4.12}$$

(d) The first Pontrjagin form $p_1(M)$ satisfies equation (4.8).

5 The Lie group setup

Let us suppose that g is a left invariant Riemannian metric on a Lie group G of dimension m, and let $\{e^1, \dots, e^m\}$ be an orthonormal basis of left invariant one-forms, so that $g = e^1 \otimes e^1 + \cdots + e^m \otimes e^m$. Let

$$de^k = \sum_{1\le i<j\le m} a^k_{ij}\, e^i \wedge e^j, \quad k = 1, \dots, m$$

be the structure equations in the basis $\{e^k\}$.

Let us denote by $\{E_1, \dots, E_m\}$ the dual basis. Since $de^k(E_i, E_j) = -e^k([E_i, E_j])$, the Levi–Civita connection 1-forms $(\sigma^g)^i_j$ are

$$\begin{aligned}(\sigma^g)^i_j(E_k) &= -\frac{1}{2}(g(E_i, [E_j, E_k]) - g(E_k, [E_i, E_j]) + g(E_j, [E_k, E_i]))\\ &= \frac{1}{2}(a^i_{jk} - a^k_{ij} + a^j_{ki}).\end{aligned} \tag{5.1}$$

The connection one-forms $(\sigma^+)^i_j$ for the torsion connection ∇^+ are given by

$$(\sigma^+)^i_j(E_k) = (\sigma^g)^i_j(E_k) - \tfrac{1}{2}T^i_j(E_k), \qquad T^i_j(E_k) = T(E_i, E_j, E_k). \tag{5.2}$$

We shall focus on seven and eight-dimensional nilmanifolds $M = \Gamma\backslash G$ endowed with an invariant special structure.

5.1 Explicit solutions in dimension 7

We consider cocalibrated G_2-structures of pure type. From (3.2) we have that the torsion 3-form in this case is given by

$$\nabla^+ = \nabla^g + \tfrac{1}{2}T, \quad H = T = - * d\Theta. \tag{5.3}$$

Starting from a balanced SU(3)-structure (F, Ψ_+, Ψ_-) on a manifold M^6 it is easy to see that the G_2-structure given by $\Theta = F \wedge e^7 + \Psi_+$ on the product $M^7 = M^6 \times S^1$ is cocalibrated of pure type, where e^7 denotes the standard one-form on the circle S^1. Moreover, following the argument given in [56, Theorem 4.6] we conclude that the natural extension of an SU(3)-instanton on M^6 gives rise to a G_2-instanton on M^7, and if the torsion connection of the SU(3)-structure satisfies the modified Bianchi identity then the corresponding ∇^+ given in (5.3) also satisfies (4.10). We can apply this to the compact six-dimensional explicit solutions given in [24] to get compact solutions in dimension 7:

Corollary 5.1. *Let (M^6, F, Ψ_+, Ψ_-) be a compact balanced SU(3)-nilmanifold with an SU(3)-instanton solving the modified Bianchi identity for $\nabla = \nabla^+$ or ∇^g. Then, the G_2-manifold $M^7 = M^6 \times S^1$ with the structure $\Theta = F \wedge e^7 + \Psi_+$, the G_2-instanton obtained as an extension of the SU(3)-instanton and ∇ being the Levi–Civita connection ∇^g or the torsion connection ∇^+ given in* (5.3)*, provides a compact valid solution to the supersymmetry equations in dimension 7.*

Our goal next is to find more compact G_2-solutions to the supersymmetry equations with non-zero flux and constant dilaton on non-trivial extensions of the balanced Hermitian structures on the Lie algebra $\mathfrak{h}_3$ given in [24]. We also provide a new solution to the equations of motion based on the seven-dimensional generalized Heisenberg compact nilmanifold.

Seven-dimensional extensions of $\mathfrak{h}_3$: For any $t \neq 0$, the structure equations

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = 0, \\ de^6 = -2t\, e^{12} + 2t\, e^{34}, \end{cases} \tag{5.4}$$

correspond to the nilpotent Lie algebra $\mathfrak{h}_3 = (0, 0, 0, 0, 0, 12 + 34)$. As it is shown in [24], the SU(3)-structure given by

$$F = e^{12} + e^{34} + e^{56}, \quad \Psi = \Psi_+ + i\,\Psi_- = (e^1 + \mathrm{i}\, e^2) \wedge (e^3 + \mathrm{i}\, e^4) \wedge (e^5 + \mathrm{i}\, e^6),$$

is balanced for all the values of the parameter t. Consider any nilpotent seven-dimensional extension $\mathfrak{h}^7 = \mathfrak{h}_3 \oplus \langle E_7 \rangle$ such that the G_2-structure

$$\Theta = F \wedge e^7 + \Psi_+ \tag{5.5}$$

is cocalibrated of pure type on $\mathfrak{h}^7$, where e^7 denotes the dual of E_7. Using (2.1) and (2.2) it is easy to check that $de^7 = c_0(e^{12} - e^{34}) + c_1(e^{13} + e^{24}) + c_2(e^{14} - e^{23})$, where $c_0, c_1, c_2 \in \mathbb{R}$. Since $t \neq 0$ in (5.4), we can consider

$c_0 = 0$. Therefore, the nilpotent Lie algebra $\mathfrak{h}^7$ must be given by the structure equations

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = 0, \\ de^6 = -2t(e^{12} - e^{34}), \\ de^7 = c_1(e^{13} + e^{24}) + c_2(e^{14} - e^{23}), \end{cases} \tag{5.6}$$

where $c_1, c_2 \in \mathbb{R}$. Moreover, a direct calculation shows that the torsion is given by

$$T = - * d\Theta = -2t(e^{12} - e^{34}) \wedge e^6 + c_1(e^{13} + e^{24}) \wedge e^7 + c_2(e^{14} - e^{23}) \wedge e^7.$$

Hence, $dT = -2(4t^2 + c_1^2 + c_2^2)e^{1234}$. It is easy to prove that T is parallel with respect to the torsion connection ∇^+ if and only if $c_1 = c_2 = 0$, which corresponds to the situation described in Theorem 5.1.

From (1.5), (5.1) and (5.2), it follows that the non-zero curvature forms $(\Omega^+)^i_j$ of the torsion connection ∇^+ are

$$\begin{aligned} &(\Omega^+)^1_2 = -(\Omega^+)^3_4 = -4t^2(e^{12} - e^{34}), \\ &(\Omega^+)^1_3 = (\Omega^+)^2_4 = -c_1^2(e^{13} + e^{24}) - c_1c_2(e^{14} - e^{23}) + 4tc_2\, e^{67}, \\ &(\Omega^+)^1_4 = -(\Omega^+)^2_3 = -c_1c_2(e^{13} + e^{24}) - c_2^2(e^{14} - e^{23}) - 4tc_1\, e^{67}, \end{aligned}$$

which implies that the first Pontrjagin form of ∇^+ is

$$p_1(\nabla^+) = -\frac{1}{2\pi^2}\left(16t^4 + (c_1^2 + c_2^2)^2\right) e^{1234}.$$

Let us consider $(c_1, c_2) \in \mathbb{Q}^2 - \{(0,0)\}$. The well-known Malcev theorem asserts that the simply connected nilpotent Lie group H^7 corresponding to the Lie algebra $\mathfrak{h}^7$ has a lattice Γ of maximal rank. We denote by M^7 the compact nilmanifold $\Gamma\backslash H^7$.

Lemma 5.1. *Let A_λ be the linear connection preserving the metric on M^7 defined by the connection forms*

$$(\sigma^{A_\lambda})^i_j = \lambda\, e^7, \qquad (\sigma^{A_\lambda})^6_7 = e^1 + e^2 + e^3 + e^4 + e^5 + \lambda\, e^6 + \lambda\, e^7,$$

for $(i,j) = (1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,3)$, $(2,4)$, $(2,5)$, $(3,4)$, $(3,5)$, $(4,5)$, where $\lambda \in \mathbb{R}$. Then, A_λ is a G_2-instanton with respect to the structure (5.5)*, and*

$$p_1(A_\lambda) = -\frac{\lambda^2}{4\pi^2}\left(4t^2 + 11(c_1^2 + c_2^2)\right) e^{1234}.$$

Proof. A direct calculation shows that the non-zero curvature forms $(\Omega^{A_\lambda})^i_j$ of the connection A_λ are given by:

$$
\begin{aligned}
(\Omega^{A_\lambda})^i_j &= \lambda c_1(e^{13}+e^{24}) + \lambda c_2(e^{14}-e^{23}),\\
(\Omega^{A_\lambda})^6_7 &= -2\lambda t(e^{12}-e^{34}) + \lambda c_1(e^{13}+e^{24}) + \lambda c_2(e^{14}-e^{23}).
\end{aligned}
$$

for $(i,j) = (1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)$. On the other hand, since the Lie algebra of G_2 can be identified with the subspace of 2-forms which annihilate $*\Theta$ and $(e^{12}-e^{34})\wedge *\Theta = (e^{13}+e^{24})\wedge *\Theta = (e^{14}-e^{23})\wedge *\Theta = 0$, the connection A_λ is a G_2-instanton. ☐

As a consequence we get the following compact seven-dimensional solutions.

Theorem 5.1. *Let A_λ be the G_2-instanton on M^7 given above. If $\lambda^2 < \min\{8t^2, 2(c_1^2+c_2^2)/11\}$, then*

$$dT = 2\pi^2\alpha'\,(p_1(\nabla^+) - p_1(A_\lambda)),$$

with $\alpha' > 0$ and $(M^7,\Theta,\nabla^+,A_\lambda)$ is a compact solution to the heterotic Killing spinor equations (1.2) *satisfying the anomaly cancellation condition* (1.1).

Proof. Note that $p_1(\nabla^+) - p_1(A_\lambda) = \frac{1}{4\pi^2}[4t^2(\lambda^2-8t^2) + (c_1^2+c_2^2)(11\lambda^2 - 2(c_1^2+c_2^2))]e^{1234}$. Therefore, if $\lambda^2 < \min\{8t^2, 2(c_1^2+c_2^2)/11\}$ then $p_1(\nabla^+) - p_1(A_\lambda)$ is a negative multiple of e^{1234}. Since $dT = -2(4t^2+c_1^2+c_2^2)e^{1234}$, the result follows. ☐

Remark 5.1. The first Pontrjagin form of the Levi–Civita connection is given by

$$p_1(\nabla^g) = -\frac{1}{16\pi^2}\left[3(4t^2-c_1^2-c_2^2)^2 + 16t^2(c_1^2+c_2^2)\right]e^{1234},$$

so there is $\lambda \neq 0$ sufficiently small such that $dT = 2\pi^2\alpha'\,(p_1(\nabla^g) - p_1(A_\lambda))$, with $\alpha' > 0$.

The seven-dimensional generalized Heisenberg group: Next we construct a seven-dimensional compact solution to the equations of motion which is not an extension of the six-dimensional nilmanifolds given in [24].

Let $H(3,1)$ be the seven-dimensional generalized Heisenberg group, i.e., the nilpotent Lie group consisting of the matrices of real numbers of the form

$$H(3,1)=\left\{\begin{pmatrix}1 & x_1 & x_2 & x_3 & z\\ 0 & 1 & 0 & 0 & y_1\\ 0 & 0 & 1 & 0 & y_2\\ 0 & 0 & 0 & 1 & y_3\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \mid x_i, y_i, z\in\mathbb{R}, 1\leq i\leq 3\right\}.$$

We consider the basis for the left invariant one-forms on $H(3,1)$ given by

$$e^1=\frac{1}{a}dx_1,\quad e^2=dy_1,\quad e^3=\frac{1}{b}dx_2,\quad e^4=dy_2,\quad e^5=\frac{1}{c}dx_3,\quad e^6=dy_3,$$
$$e^7=x_1dy_1+x_2dy_2+x_3dy_3-dz,$$

where $a,b,c\in\mathbb{R}-\{0\}$, so that the structure equations become

$$\begin{cases} de^1=de^2=de^3=de^4=de^5=de^6=0,\\ de^7=a\,e^{12}+b\,e^{34}+c\,e^{56}. \end{cases} \tag{5.7}$$

Lemma 5.2. *The G_2-structure given by*

$$\Theta=(e^{12}+e^{34}+e^{56})\wedge e^7+e^{135}-e^{146}-e^{236}-e^{245}$$

is cocalibrated for each $a,b,c\in\mathbb{R}-\{0\}$. Moreover, Θ is of pure type if and only if $c=-a-b$ or, equivalently, $de^7\in\mathfrak{su}(3)$.

Proof. A direct simple calculation shows that $d*\Theta=0$ and $\Theta\wedge d\Theta=2(a+b+c)e^{1234567}$. ☐

From now on, let us consider $c=-a-b\neq 0$ in equations (5.7). The torsion three-form for the cocalibrated G_2-structure of pure type is given by

$$T=-*d\Theta=(de^7)\wedge e^7=a\,e^{127}+b\,e^{347}-(a+b)e^{567}.$$

Hence

$$dT=2ab\,e^{1234}-2a(a+b)e^{1256}-2b(a+b)e^{3456}. \tag{5.8}$$

Moreover, it is forward to check that T is parallel with respect to the torsion connection ∇^+, i.e.,

Lemma 5.3. *For any $a,b\in\mathbb{R}-\{0\}$ such that $b\neq -a$, we have $\nabla^+T=0$.*

On the other hand, by (1.5), (5.1) and (5.2) we have that the non-zero curvature forms $(\Omega^+)^i_j$ of the torsion connection ∇^+ are

$$\begin{aligned}
&(\Omega^+)^1_2 = -a\left(a\, e^{12} + b\, e^{34} - (a+b)e^{56}\right),\\
&(\Omega^+)^3_4 = -b\left(a\, e^{12} + b\, e^{34} - (a+b)e^{56}\right),\\
&(\Omega^+)^5_6 = -(\Omega^+)^1_2 - (\Omega^+)^3_4 = (a+b)\left(a\, e^{12} + b\, e^{34} - (a+b)e^{56}\right). \qquad (5.9)
\end{aligned}$$

Let $\Gamma(3,1)$ denote the subgroup of matrices of $H(3,1)$ with integer entries and consider the compact nilmanifold $N(3,1) = \Gamma(3,1)\backslash H(3,1)$. We can describe $N(3,1)$ as a principal circle bundle over a 6-torus

$$S^1 \hookrightarrow N(3,1) \to \mathbb{T}^6,$$

whose connection 1-form $\eta = e^7$ has curvature $d\eta = a(e^{12} - e^{56}) + b(e^{34} - e^{56})$ in $\mathfrak{su}(3)$.

Next we show a three-parametric family of G_2-instantons on the nilmanifold $N(3,1)$.

Proposition 5.1. *Let $A_{\lambda,\mu,\tau}$ be the linear connection on $N(3,1)$ defined by the connection forms*

$$\begin{aligned}
&(\sigma^{A_{\lambda,\mu,\tau}})^1_2 = -(\sigma^{A_{\lambda,\mu,\tau}})^2_1 = \lambda\, e^7, \quad (\sigma^{A_{\lambda,\mu,\tau}})^3_4 = -(\sigma^{A_{\lambda,\mu,\tau}})^4_3 = \mu\, e^7,\\
&(\sigma^{A_{\lambda,\mu,\tau}})^5_6 = -(\sigma^{A_{\lambda,\mu,\tau}})^6_5 = \tau\, e^7,
\end{aligned}$$

and $(\sigma^{A_{\lambda,\mu,\tau}})^i_j = 0$ for the remaining (i,j), where $\lambda, \mu, \tau \in \mathbb{R}$. Then, $A_{\lambda,\mu,\tau}$ is a G_2-instanton with respect to the cocalibrated G_2-structure of pure type given in Lemma 5.2 for any a, b, $A_{\lambda,\mu,\tau}$ preserves the metric, and its first Pontrjagin form is given by

$$p_1(A_{\lambda,\mu,\tau}) = \frac{\lambda^2 + \mu^2 + \tau^2}{4\pi^2}\left(ab\, e^{1234} - a(a+b)\, e^{1256} - b(a+b)\, e^{3456}\right).$$

Proof. A direct calculation shows that the non-zero curvature forms $(\Omega^{A_{\lambda,\mu,\tau}})^i_j$ of the connection $A_{\lambda,\mu,\tau}$ are:

$$\begin{aligned}
&(\Omega^{A_{\lambda,\mu,\tau}})^1_2 = \lambda\left(a\, e^{12} + b\, e^{34} - (a+b)e^{56}\right),\\
&(\Omega^{A_{\lambda,\mu,\tau}})^3_4 = \mu\left(a\, e^{12} + b\, e^{34} - (a+b)e^{56}\right),\\
&(\Omega^{A_{\lambda,\mu,\tau}})^5_6 = \tau\left(a\, e^{12} + b\, e^{34} - (a+b)e^{56}\right).
\end{aligned}$$

On the other hand, the Lie algebra of G_2 can be identified with the subspace of 2-forms which annihilate $*\Theta$. Since $(a\,e^{12} + b\,e^{34} - (a+b)e^{56}) \wedge *\Theta = 0$, the connection $A_{\lambda,\mu,\tau}$ is a G_2-instanton. □

The next result gives explicit compact valid solutions on $N(3,1)$ to the heterotic supersymmetry equations with non-zero flux and constant dilaton satisfying the anomaly cancellation condition which also solve the equations of motion (1.4) due to Lemma 5.3 and Theorem 4.1.

Theorem 5.2. *Let $N(3,1)$ be the compact cocalibrated of pure type G_2-nilmanifold, ∇^+ the torsion connection and $A_{\lambda,\mu,\tau}$ the G_2-instanton given in Proposition* 5.1. *If $(\lambda,\mu,\tau) \neq (0,0,0)$ are small enough so that $\lambda^2+\mu^2+\tau^2 < 2(a^2+ab+b^2)$, then*

$$dT = 2\pi^2\alpha'\,(p_1(\nabla^+) - p_1(A_{\lambda,\mu,\tau})),$$

where $\alpha' = 4(2(a^2+ab+b^2) - \lambda^2-\mu^2-\tau^2)^{-1} > 0$.

Therefore, the manifold $(N(3,1), \Theta, \nabla^+, A_{\lambda,\mu,\tau})$ is a compact solution to the supersymmetry equations (1.2) *obeying the anomaly cancellation* (1.1) *and solving the equations of motion* (1.4).

The Riemannian metric is locally given by

$$g = \frac{1}{a^2}dx_1^2 + dy_1^2 + \frac{1}{b^2}dx_2^2 + dy_2^2 + \frac{1}{a^2+b^2}dx_3^2 + dy_3^2 + (x_1dy_1 + x_2dy_2 + x_3dy_3 - dz)^2.$$

Proof. The non-zero curvature forms of the torsion connection $\nabla^+_{a,b}$ are given by (5.9), which implies that its first Pontrjagin form is

$$p_1(\nabla^+) = \frac{a^2+ab+b^2}{2\pi^2}\left(ab\,e^{1234} - a(a+b)e^{1256} - b(a+b)e^{3456}\right).$$

Now the proof follows directly from (5.8) and Proposition 5.1. The final assertion in the theorem follows from Lemma 5.3 and Theorem 4.1. □

Remark 5.2. The first Pontrjagin form of the Levi–Civita connection is given by

$$p_1(\nabla^g) = \frac{1}{32\pi^2}[ab(5a^2+4ab+5b^2)e^{1234} - a(a+b)(6a^2+6ab+5b^2)e^{1256} - b(a+b)(5a^2+6ab+6b^2)e^{3456}].$$

It is easy to see that there is no solution to the heterotic supersymmetry equations for $\nabla = \nabla^g$ using the instantons of Lemma 5.1.

5.2 Explicit solutions in dimension 8

We consider balanced Spin(7)-structures, i.e., $\theta^8 = 0$. From (3.3) we have that the torsion three-form in this case is given by

$$\nabla^+ = \nabla^g + \tfrac{1}{2}T, \quad H = T = *^8 d\Phi. \tag{5.10}$$

Starting from a cocalibrated G_2-structure of pure type Θ on a seven-manifold M^7 it is easy to see that the Spin(7)-structure given by $\Phi = e^1 \wedge \Theta + *^7\Theta$ on the product $M^8 = M^7 \times S^1$ is balanced, where e^1 denotes the standard 1-form on the circle S^1. Moreover, following the argument given in [56, Theorem 5.1] we conclude that the natural extension of a G_2-instanton on M^7 gives rise to a Spin(7)-instanton on M^8, and if the torsion connection of the G_2-structure satisfies the Bianchi identity then the corresponding ∇^+ given in (5.10) also satisfies (4.12). We can apply this to the compact seven-dimensional explicit solutions given in the preceding section to get compact solutions in dimension 8:

Corollary 5.2. *Let* (M^7, Θ) *be a compact cocalibrated* G_2*-nilmanifold of pure type with a* G_2*-instanton solving the modified Bianchi identity for* $\nabla = \nabla^+$ *or* ∇^g*. Then, the Spin(7)-manifold* $M^8 = M^7 \times S^1$ *with the structure* $\Phi = e^1 \wedge \Theta + *^7\Theta$*, the Spin(7)-instanton obtained as an extension of the* G_2*-instanton and* ∇ *being the Levi–Civita connection* ∇^g *or the torsion connection* ∇^+ *given in* (5.10)*, provides a compact valid solution to the supersymmetry equations in dimension 8. In particular, starting with the solutions on the generalized Heisenberg compact nilmanifold* $N(3,1)$ *given in Theorem* 5.2 *one obtains solutions to the equations of motion in dimension 8 for* $\nabla = \nabla^+$*.*

Next, we find more compact Spin(7)-solutions to the supersymmetry equations with non-zero flux and constant dilaton on non-trivial extensions of the cocalibrated G_2-structures of pure type given on the seven-dimensional generalized Heisenberg group. Moreover, we also provide new eight-dimensional solutions to the equations of motion on some of these non-trivial Spin(7)-extensions.

Non-trivial Spin(7) extension of the seven-dimensional generalized Heisenberg group: Let us consider the eight-dimensional extension of (5.7) given by:

$$\begin{cases} de^1 = c\,(e^{24} + e^{25} - e^{34} + e^{35}), \\ de^2 = de^3 = de^4 = de^5 = de^6 = de^7 = 0, \\ de^8 = a\,e^{23} + b\,e^{45} - (a+b)e^{67}. \end{cases} \tag{5.11}$$

These equations correspond to the structure equations of an eight-dimensional nilpotent Lie algebra, which we denote by $\mathfrak{h}^8$. Let us consider the Spin(7)-structure defined by (2.4). A direct calculation shows that the torsion is given by

$$T = *d\Phi = c\,e^{124} + c\,e^{125} - c\,e^{134} + c\,e^{135} + a\,e^{238} + b\,e^{458} - (a+b)\,e^{678}.$$

The torsion satisfies $T \wedge \Phi = 0$ and

$$dT = 2(ab - 2c^2)\,e^{2345} - 2a(a+b)\,e^{2367} - 2b(a+b)\,e^{4567}. \tag{5.12}$$

There are some special cases for which T is parallel with respect to the torsion connection, more concretely:

Lemma 5.4. $\nabla^+ T = 0$ *if and only if* $(a-b)c = 0$.

Using again (1.5), (5.1) and (5.2), the non-zero curvature forms $(\Omega^+)^i_j$ of the torsion connection ∇^+ are given by

$$\begin{aligned}
&(\Omega^+)^2_3 = -a^2\,e^{23} - ab\,e^{45} + a(a+b)\,e^{67},\\
&(\Omega^+)^2_4 = (\Omega^+)^3_5 = (a-b)c\,e^{18} - c^2\,e^{24} - c^2\,e^{25} + c^2\,e^{34} - c^2\,e^{35},\\
&(\Omega^+)^2_5 = -(\Omega^+)^3_4 = -(a-b)c\,e^{18} - c^2\,e^{24} - c^2\,e^{25} + c^2\,e^{34} - c^2\,e^{35},\\
&(\Omega^+)^4_5 = -ab\,e^{23} - b^2\,e^{45} + b(a+b)\,e^{67},\\
&(\Omega^+)^6_7 = a(a+b)\,e^{23} + b(a+b)\,e^{45} - (a+b)^2\,e^{67},
\end{aligned}$$

which implies that the first Pontrjagin form $p_1(\nabla^+)$ is given by

$$\begin{aligned}
2\pi^2\,p_1(\nabla^+) = {}&(ab(a^2+ab+b^2) - 4c^4)\,e^{2345} - a(a+b)(a^2+ab+b^2)\,e^{2367}\\
&- b(a+b)(a^2+ab+b^2)\,e^{4567}.
\end{aligned}$$

Let us denote by H^8 the simply connected nilpotent Lie group corresponding to the Lie algebra $\mathfrak{h}^8$. From the explicit description of the Lie group $H(3,1)$ and from (2.4), it follows that the left invariant metric g on H^8 determined by the Spin(7)-structure Φ can be expressed globally as

$$\begin{aligned} g = &\left(dw + \frac{c}{b}\left(\frac{x_1}{a} - y_1\right)dx_2 + c\left(\frac{x_1}{a} + y_1\right)dy_2\right)^2 + \left(\frac{1}{a}dx_1\right)^2 + (dy_1)^2 \\ &+ \left(\frac{1}{b}dx_2\right)^2 + (dy_2)^2 + \left(\frac{-1}{a+b}dx_3\right)^2 + (dy_3)^2 \\ &+ (x_1dy_1 + x_2dy_2 + x_3dy_3 - dz)^2, \end{aligned} \tag{5.13}$$

where $(w, x_1, y_1, x_2, y_2, x_3, y_3, z)$ denote the (global) coordinates of H^8, and the w-coordinate of the left translation $L_{(w^0,x_1^0,y_1^0,x_2^0,y_2^0,x_3^0,y_3^0,z^0)}$ by an element $(w^0, x_1^0, y_1^0, x_2^0, y_2^0, x_3^0, y_3^0, z^0)$ of H^8 is given by

$$w \circ L_{(w^0,x_1^0,y_1^0,x_2^0,y_2^0,x_3^0,y_3^0,z^0)} = w - \frac{c}{b}\left(\frac{x_1^0}{a} - y_1^0\right)x_2 - c\left(\frac{x_1^0}{a} + y_1^0\right)y_2 + w^0.$$

Notice that the remaining coordinates of $L_{(w^0,x_1^0,y_1^0,x_2^0,y_2^0,x_3^0,y_3^0,z^0)}$ come easily from the matrix description of $H(3,1)$.

Let Γ be a lattice of maximal rank of H^8 and denote by M^8 the compact nilmanifold $\Gamma\backslash H^8$. Clearly, M^8 can be described as a circle bundle over the compact seven-manifold $N(3,1)$ (defined by (5.7))

$$S^1 \hookrightarrow M^8 \to N(3,1),$$

with connection 1-form $\eta = e^1$ such that the curvature form $d\eta = c\,(e^{24} + e^{25} - e^{34} + e^{35}) \in \mathfrak{g}_2$.

Alternatively, the manifold M^8 may be viewed as the total space of a circle bundle over the product of a two-torus by a five-manifold M^5, which is also the total space of a principal circle bundle over a four-torus, i.e., $S^1 \hookrightarrow M^5 \to \mathbb{T}^4$. In fact, let $\{e^2, \dots, e^5\}$ be a basis for the closed one-forms on $\mathbb{T}^4$. Then, M^5 is the circle bundle over $\mathbb{T}^4$ with connection one-form $\eta = e^1$ such that the curvature form is $d\eta = c\,(e^{24} + e^{25} - e^{34} + e^{35})$. Now, let e^6 and e^7 be a basis for the closed one-forms on $\mathbb{T}^2$. Take the product manifold $M^5 \times \mathbb{T}^2$. Then, M^8 is the circle bundle over $M^5 \times \mathbb{T}^2$

$$S^1 \hookrightarrow M^8 \to M^5 \times \mathbb{T}^2,$$

with connection form $\nu = e^8$ such that $d\nu = a\,e^{23} + b\,e^{45} - (a+b)\,e^{67}$.

Proposition 5.2. *For each* $\lambda, \mu \in \mathbb{R}$, *let* $A_{\lambda,\mu}$ *be the linear connection on* M^8 *defined by the connection forms:*

$$\begin{aligned}
&(\sigma^{A_{\lambda,\mu}})^2_3 = -(\sigma^{A_{\lambda,\mu}})^3_2 = (\sigma^{A_{\lambda,\mu}})^4_5 = -(\sigma^{A_{\lambda,\mu}})^5_4 = \lambda\, e^8,\\
&(\sigma^{A_{\lambda,\mu}})^2_4 = (\sigma^{A_{\lambda,\mu}})^2_5 = (\sigma^{A_{\lambda,\mu}})^3_5 = (\sigma^{A_{\lambda,\mu}})^4_3 = -\mu\, e^1,\\
&(\sigma^{A_{\lambda,\mu}})^3_4 = (\sigma^{A_{\lambda,\mu}})^4_2 = (\sigma^{A_{\lambda,\mu}})^5_2 = (\sigma^{A_{\lambda,\mu}})^5_3 = \mu\, e^1,\\
&(\sigma^{A_{\lambda,\mu}})^6_7 = -(\sigma^{A_{\lambda,\mu}})^7_6 = -2\lambda\, e^8,
\end{aligned}$$

and $(\sigma^{A_{\lambda,\mu}})^i_j = 0$ *for the remaining* (i, j). *Then,* $A_{\lambda,\mu}$ *is a Spin(7)-instanton with respect to the Spin(7)-structure* (2.4) *for any* a, b, c, $A_{\lambda,\mu}$ *preserves the metric, and its first Pontrjagin form is given by*

$$2\pi^2\, p_1(A_{\lambda,\mu}) = (3ab\lambda^2 - 4c^2\mu^2)e^{2345} - 3a(a+b)\lambda^2\, e^{2367} - 3b(a+b)\lambda^2\, e^{4567}.$$

Proof. The non-zero curvature forms $(\Omega^{A_{\lambda,\mu}})^i_j$ of the connection $A_{\lambda,\mu}$ are:

$$\begin{aligned}
&(\Omega^{A_{\lambda,\mu}})^2_3 = (\Omega^{A_{\lambda,\mu}})^4_5 = \lambda\left(a\, e^{23} + b\, e^{45} - (a+b)e^{67}\right),\\
&(\Omega^{A_{\lambda,\mu}})^2_4 = (\Omega^{A_{\lambda,\mu}})^2_5 = -(\Omega^{A_{\lambda,\mu}})^3_4 = (\Omega^{A_{\lambda,\mu}})^3_5 = -\mu\, c(e^{24} + e^{25} - e^{34} + e^{35}),\\
&(\Omega^{A_{\lambda,\mu}})^6_7 = -(\Omega^{A_{\lambda,\mu}})^2_3 - (\Omega^{A_{\lambda,\mu}})^4_5 = -2\lambda\left(a\, e^{23} + b\, e^{45} - (a+b)e^{67}\right).
\end{aligned}$$

Since the Lie algebra of Spin(7) can be identified with the subspace Λ^2_{21} of two-forms β such that $*(\beta \wedge \Phi) = -\beta$, and since $a\, e^{23} + b\, e^{45} - (a+b)e^{67}$, $e^{24} + e^{25} - e^{34} + e^{35} \in \Lambda^2_{21}$ the connection $A_{\lambda,\mu}$ is a Spin(7)-instanton for any λ, μ. ☐

Theorem 5.3. *Let* (M^8, Φ) *be the compact balanced Spin(7)-nilmanifold,* ∇^+ *the torsion connection and* $A_{\lambda,\mu}$ *the Spin(7)-instanton given in Proposition* 5.1. *If* $(\lambda, \mu) \neq (0,0)$ *satisfy* $3\lambda^2 < a^2 + ab + b^2$ *and* $3\lambda^2 - 2\mu^2 = a^2 + ab + b^2 - 2c^2$, *then*

$$dT = 2\pi^2\alpha'\,(p_1(\nabla^+) - p_1(A_{\lambda,\mu})),$$

where $\alpha' = 2(a^2 + ab + b^2 - 3\lambda^2)^{-1} > 0$.

Therefore, the manifold $(M^8, \Phi, \nabla^+, A_{\lambda,\mu})$ *is a compact solution to the supersymmetry equations* (1.2) *satisfying the anomaly cancellation* (1.1).

If $a = b$ *then the manifold* $(M^8, \Phi, \nabla^+, A_{\lambda,\mu})$ *with* $(\lambda, \mu) \neq (0, 0)$ *satisfying*

$$\lambda^2 < a^2, \qquad 3\lambda^2 - 2\mu^2 = 3a^2 - 2c^2$$

is a compact supersymmetric solution to the heterotic equations of motion (1.4) *in dimension 8.*

The Riemannian metric is locally given by (5.13) *with* $a = b$.

Proof. The proof follows directly from (5.12), the expression of the first Pontrjagin form of ∇^+ and Proposition 5.2. The final assertion in the theorem follows from Lemma 5.4 and Theorem 4.1. □

Remark 5.3. There are also solutions on M^8 to the supersymmetry equations taking ∇ as the Levi–Civita connection ∇^g. For example, if $a = b = c = 1$ in (5.11) then a direct computation shows that the first Pontrjagin form of ∇^g is given by

$$16\pi^2\, p_1(\nabla^g) = -5\, e^{2345} - 19\, e^{2367} - 19\, e^{4567}.$$

From Proposition 5.2 for $a = b = c = 1$ we get

$$2\pi^2\, p_1(A_{\lambda,\mu}) = (3\lambda^2 - 4\mu^2) e^{2345} - 6\lambda^2\, e^{2367} - 6\lambda^2\, e^{4567}.$$

Since $dT = -2\, e^{2345} - 4\, e^{2367} - 4\, e^{4567}$, if we choose the Spin(7)-instanton $A_{\lambda,\mu}$ such that $48\lambda^2 < 19$ and $64\mu^2 = 96\lambda^2 - 9$, then

$$dT = 2\pi^2\, \alpha' \,(p_1(\nabla^g) - p_1(A_{\lambda,\mu})),$$

where $\alpha' = 32(19 - 48\lambda^2)^{-1} > 0$.

6 Geometric models

The structure of the examples that we have presented as well as constructions proposed in [39] suggest a more general construction. In this section, we describe how to derive compact solutions to the system of gravitino and dilatino Killing spinor equations (the first two equations in (1.2)) in dimensions seven and eight starting with a solution of these equations in low dimensions. The construction is a $\mathbb{T}^k$-bundle with curvature of instanton type over a compact low-dimensional solution. The benefit of this construction is the obtained reduction of the dilaton variables, i.e., the non-constant dilaton depend on reduced number of variables.

First, we recall the dimensions 5 and 6.

D=5 The gravitino and dilatino Killing spinor equations in dimension 5 define a reduction of the structure group SO(5) to SU(2) which is described in terms of differential forms by Conti and Salamon in [15] as follows: an SU(2)-structure on a five-dimensional manifold M is the quadruplet $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a one-form with a dual vector field ξ and $\omega_i, i = 1, 2, 3$, are two-forms on M satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,$$

for some 4-form v, and $X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0$, where $\lrcorner$ denotes the interior multiplication.

Let $\mathbb{H} = Ker \eta$. The two-forms $\omega_i, i = 1, 2, 3$, can be chosen to form a basis of the $\mathbb{H}$-self-dual two-forms [15], i.e., $*_{\mathbb{H}} \omega_i = \omega_i$, where $*_{\mathbb{H}}$ denotes the Hodge operator on the four-dimensional distribution $\mathbb{H}$.

Based on analysis done in [31, 33] it is shown in [25] that:

The first two equations in (1.2) admit a solution in dimension five: exactly when there exists a five dimensional manifold M endowed with an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ satisfying the structure equations:

$$d\omega_i = 2df \wedge \omega_i, \quad *_{\mathbb{H}} d\eta = -d\eta \tag{6.1}$$

where f is a smooth function, which does not depend on ξ, $df(\xi) = 0$.

The flux H is given by $H = T = \eta \wedge d\eta - 2 *_4 df$ and the dilaton ϕ is equal to $\phi = f + cons$.

Therefore, if the dilaton is constant then the structure equations are

$$d\omega_i = 0, \quad *_{\mathbb{H}} d\eta = -d\eta \tag{6.2}$$

and the flux H is given by $H = T = \eta \wedge d\eta$.

If the SU(2) structure is regular, i.e., the orbit space $N = M/\xi$ is a smooth manifold then M is an S^1-bundle over a Calabi-Yau 4-fold (flat torus or K3 surface) with $\mathbb{H}$-anti-self-dual curvature form equal to $d\eta$. The metric has the form

$$g = e^{2f} g_{cy} + \eta \otimes \eta,$$

where g_{cy} is the metric on the Calabi–Yau base and f is a smooth function on the base.

We do not know whether there exist non-regular SU(2)-structures (the integral curves of ξ are not closed) on a compact 5-manifold.

D=6 The gravitino and dilatino Killing spinor equations in dimension 6 define a reduction of the structure group SO(6) to SU(3) which is described in terms of forms by Chiossi and Salamon in [14] as follows: an SU(3)-structure is $(F, \Psi = \Psi^+ + \mathrm{i}\Psi^-)$ with Kähler form F and complex volume form Ψ which satisfy the compatibility relations

$$F \wedge \Psi^{\pm} = 0, \quad \Psi^+ \wedge \Psi^- = \frac{2}{3} F \wedge F \wedge F.$$

The necessary and sufficient condition for the existence of solutions to the first two equations in (1.2) in dimension 6 was derived by Strominger [73], namely the manifold should be complex conformally balanced with non-vanishing holomorphic volume form Ψ satisfying additional condition. In terms of the five torsion classes described in [14], the Strominger condition is interpreted in [13] as follows (see [56] for a slightly different expression):

$$2F \lrcorner dF + \Psi^+ \lrcorner d\Psi^+ = 0.$$

If the dilaton is constant then the Strominger conditions read

$$dF \wedge F = d\Psi^+ = d\Psi^- = 0. \tag{6.3}$$

Examples of the latter via evolution equations were presented recently in [26].

A very promising geometric model in dimension 6 was proposed in [42] to be a certain $\mathbb{T}^2$-bundle over a Calabi-Yau surface (see [42] and references therein). Starting with an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on (a regular) 5-manifold M satisfying (6.2) one considers an S^1-bundle over M with curvature an exact $\mathbb{H}$-anti-self-dual two-form, $d\alpha$ and the SU(3)-structure $(F, \Psi = \Psi^+ + \mathrm{i}\Psi^-)$ defined by

$$\begin{aligned} &F = \omega_1 + \eta \wedge \alpha; \qquad \Psi^+ = \omega_2 \wedge \eta - \omega_3 \wedge \alpha; \\ &\Psi^- = \omega_3 \wedge \eta + \omega_2 \wedge \alpha. \end{aligned} \tag{6.4}$$

Using (6.2) and the fact that $d\alpha$ is $\mathbb{H}$-anti-self-dual it can be shown following Goldstein and Prokushkin [42] that (6.3) holds as a consequence of (6.4). When M is regular, i.e., it is an S^1-bundle over a Calabi–Yau four-manifold one gets a holomorphic $\mathbb{T}^2$-bundle over a Calabi–Yau surface with anti-self-dual integral curvature two-forms, which solves the first two equations in (1.2) with constant dilaton [42]. It also follows from considerations in [42] that if the starting SU(2)-structure solves the equations with non-constant dilaton, i.e., (6.1) hold, then the SU(3)-structure on the circle bundle also solves the first two Killing spinor equations with non-constant dilaton in dimension 6. The $\mathbb{T}^2$-bundle over a K3 surface construction was used in [67, 34, 35, 4] to produce the first compact examples in dimension 6 solving the heterotic supersymmetry equations (1.2) with non-zero flux and non-constant dilaton together with the anomaly cancellation (1.1) with respect to the Chern connection.

6.1 $\mathbb{T}^3$-bundles over a Calabi–Yau surface

The structure of the example Γ/H^7, where H^7 is the nilpotent Lie group defined by (5.6), is generalized in the following:

Theorem 6.1. *Let* Γ_i, $1 \le i \le 3$, *be three closed anti-self-dual* 2*-forms on a Calabi-Yau surface* M^4, *which represent integral cohomology classes. Denote by* ω_1 *and by* $\omega_2 + \mathrm{i}\omega_3$ *the (closed) Kähler form and the holomorphic volume form on* M^4, *respectively. Then, there is a compact 7-dimensional manifold* $M^{1,1,1}$, *which is the total space of a* $\mathbb{T}^3$*-bundle over* M^4, *and it has a* G_2*-structure*

$$\Theta = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 + \eta_1 \wedge \eta_2 \wedge \eta_3, \tag{6.5}$$

solving the first two Killing spinor equations in (1.2) *with constant dilaton in dimension* 7, *where* η_i, $1 \le i \le 3$, *is a* 1*-form on* $M^{1,1,1}$ *such that* $d\eta_i = \Gamma_i$, $1 \le i \le 3$.

For any smooth function f *on* M^4, *the* G_2*-structure on* $M^{1,1,1}$ *given by*

$$\Theta_f = e^{2f}\Big[\omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3\Big] + \eta_1 \wedge \eta_2 \wedge \eta_3 \tag{6.6}$$

solves the first two Killing spinor equations in (1.2) *with non-constant dilaton* $\phi = 2f$ *(in dimension 7). The metric has the form*

$$g_f = e^{2f} g_{cy} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3.$$

Proof. Since $[\Gamma_i]$, $1 \le i \le 3$, define integral cohomology classes on M^4, the well-known result of Kobayashi [64] implies that there exists a circle bundle $S^1 \hookrightarrow M^5 \to M^4$, with connection one-form η_1 on M^5 whose curvature form is $d\eta_1 = \Gamma_1$. (From now on, we write with the same symbol the two-form Γ_i on M^4 and its lifting to M^5 via the projection $M^5 \to M^4$.) Since Γ_i $(i = 2, 3)$ defines an integral cohomology class on M^5, there exists a principal circle bundle $S^1 \hookrightarrow M^6 \to M^5$ corresponding to $[\Gamma_2]$ and a connection one-form η_2 on M^6 such that Γ_2 is the curvature form of η_2. Using again the result of Kobayashi, there exists a principal circle bundle $S^1 \hookrightarrow M^{1,1,1} \to M^6$

with connection one-form η_3 such that $d\eta_3 = \Gamma_3$ since Γ_3 defines an integral cohomology class on M^6. The actions of S^1 on each one of the manifolds M^5, M^6 and $M^{1,1,1}$ define an action of the three-torus on $M^{1,1,1}$ doing $M^{1,1,1}$ a $\mathbb{T}^3$-bundle over M^4.

We have to show that (6.6) implies (3.1). We calculate using (6.6) that

$$\begin{aligned}
*\Theta_f &= e^{2f}\Big[\omega_1\wedge\eta_2\wedge\eta_3 + \omega_2\wedge\eta_3\wedge\eta_1 - \omega_3\wedge\eta_1\wedge\eta_2 + \frac{e^{2f}}{2}\omega_1\wedge\omega_1\Big];\\
d\Theta_f &= 2df\wedge\Theta_f - 2df\wedge\eta_1\wedge\eta_2\wedge\eta_3 + d\eta_1\wedge\eta_2\wedge\eta_3 - \eta_1\wedge d\eta_2\wedge\eta_3\\
&\qquad + \eta_1\wedge\eta_2\wedge d\eta_3.
\end{aligned}$$

From the last two equalities we derive

$$d*\Theta_f = 2df\wedge *\Theta_f, \qquad d\Theta_f\wedge\Theta_f = 0,$$

where we have used the equalities $d\omega_i = 0$, $\omega_i\wedge d\eta_j = 0$ $(i,j = 1,2,3)$ since $d\eta_j = \Gamma_j$ are anti-self-dual two-forms on M^4, and $df\wedge\omega_i\wedge\omega_i = 0$ as a five-form on a four-dimensional Calabi–Yau manifold. □

Notice that in the previous theorem, if we start with a four-torus, we have essentially three possibilities:

(1) Only one of the three two-forms Γ_i is independent. In this case, we get (5.6) with $c_1 = c_2 = 0$. The resulting compact nilmanifold satisfies the equations of motion.
(2) Two of the three two-forms Γ_i are independent. Then, we get (5.6) with $(c_1, c_2) \neq (0,0)$. The resulting compact nilmanifold satisfies the supersymmetry equations but not the equations of motion.
(3) The three two-forms Γ_i are independent. In this case, essentially we get the quaternionic Heisenberg nilmanifold. We did not get any instanton satisfying the supersymmetry equations, but at least the first two Killing spinor equations are satisfied as the previous theorem asserts.

Remark 6.1. Clearly the conclusions of the above theorem are valid also if we start with a compact non-regular M^5 with an SU(2)-structure satisfying (6.1). In this case, we take two anti-self-dual two-forms Γ_2 and Γ_3 on M^5, and we consider $M^{1,1,1}$ the principal circle bundle over M^6 corresponding to $[\Gamma_3]$, which in turn is a principal circle bundle over M^5 corresponding to $[\Gamma_2]$. Now, $M^{1,1,1}$ is a $\mathbb{T}^2$-bundle over M^5, and the G_2-structure defined by (6.5) solves the first two Killing spinor equations.

Suppose that M has a G_2-structure defined by a three-form Θ. Let us recall that a three-dimensional submanifold X of M is called *associative*, with respect to Θ, if the restriction to X of Θ coincides with the Riemannian volume form on X induced by the G_2-metric determined by Θ. (Here we do not assume that Θ is closed.) We do not know whether $M^{1,1,1}$ has a G_2-structure, defined by a three-form Θ, such that the fibers are associative with respect to Θ.

In [42], it is proved that certain non-trivial $\mathbb{T}^2$-bundles M over a Calabi–Yau surface have a natural complex structure not admitting Kähler metric. The key of their proof is that the fibers are complex submanifolds of M. For the previous construction of $\mathbb{T}^3$-bundles $M^{1,1,1}$ over a Calabi–Yau surface we have

Lemma 6.1. *In the conditions of Theorem* 6.1*, suppose that one of the integral cohomology classes represented by* Γ_i *is non-trivial on* M^4*. Let* Θ *be a three-form defining a* G_2*-structure on* $M^{1,1,1}$*, such that there is a fiber* $\mathbb{T}^3$*, which is associative with respect to* Θ*. Then* Θ *is not closed. Therefore, the* G_2*-structure on* $M^{1,1,1}$ *is non-parallel.*

Proof. We know that one of the circle bundles considered in the construction of $M^{1,1,1}$ is non-trivial since one of the forms Γ_i defines a non-zero cohomology class on M^4. Then, one can check that the homology class in $H_3(M^{1,1,1},\mathbb{R})$ defined by the fibers is trivial. Therefore, if some $\mathbb{T}^3$ fiber is associative, then Θ cannot be closed. Otherwise, there is a well-defined cohomology class $[\Theta]$ in $H^3(M^{1,1,1},\mathbb{R})$ and it evaluates on $[\mathbb{T}^3]$ to give a positive number, i.e., the volume of $\mathbb{T}^3$, which is a contradiction with the triviality of $[\mathbb{T}^3]$. □

6.2 $\mathbb{S}^1$-bundles over a manifold with a balanced SU(3)-structure

Next result generalizes the structure of the example $N(3,1)$ defined by (5.7).

Theorem 6.2. *Let* M^6 *be a compact complex six-manifold solving the first two Killing spinor equations with constant dilaton in dimension 6, i.e., there exists an* $\mathrm{SU}(3)$*-structure* (F,Ψ^+,Ψ^-) *satisfying* (6.3)*. Let* Γ *be a closed integral two-form which is an* $\mathrm{SU}(3)$*-instanton,* $\Gamma\in\mathfrak{su}(3)$*, i.e.* $\Gamma_{\alpha\beta}=\Gamma_{\bar\alpha\bar\beta}=\Gamma_{\alpha\bar\beta}F^{\alpha\bar\beta}=0$ *in local holomorphic coordinates. Then, there is a principal circle bundle* $\pi: M^7\longrightarrow M^6$ *with a connection form* η *such that* $\Gamma=d\eta$ *is the curvature of* η *and the* G_2*-structure*

$$\Theta=F\wedge\eta+\Psi^+,\qquad *\Theta=\frac{1}{2}F\wedge F+\Psi^-\wedge\eta \tag{6.7}$$

solves the first two Killing spinor equations in (1.2) *with constant dilaton.*

Proof. The exterior derivative of (6.7), with the help of (6.3), yields

$$d * \Theta = \frac{1}{2} d(F \wedge F) + d\Psi^- \wedge \eta - \Psi^- \wedge d\eta = 0,$$

and

$$d\Theta \wedge \Theta = F^2 \wedge d\eta \wedge \eta + (F \wedge \eta + \Psi^+) \wedge d\Psi^+ - dF \wedge \Psi^+ \wedge \eta = 0,$$

because of the algebraic facts $\Psi^- \wedge d\eta = 0$, $F^2 \wedge d\eta = 0$ since $d\eta \in \mathfrak{su}(3)$, and because $dF \wedge \Psi^+ = 0$ on a complex manifold (see, e.g., [14]). Hence, equations (3.1) hold with $\theta^7 = 0$.

The existence of a principal circle bundle in the conditions above follows again from [64]. □

6.3 $\mathbb{S}^1$-bundles over a cocalibrated G_2-manifold of pure type

We describe a more general situation inspired by the structure of the example Γ/H^8 defined by (5.11) and by considerations in [39].

Theorem 6.3. *Let M^7 be a compact G_2-manifold solving the first two equations of* (1.2) *with constant dilaton in dimension 7, i.e., there exists a G_2-structure Θ satisfying $d * \Theta = d\Theta \wedge \Theta = 0$. Let f be a smooth function on M^7, and let Γ_4 be a closed integral two-form on M^7, which is a G_2-instanton, $\Gamma_4 \in \mathfrak{g}_2$, i.e., it satisfies* (3.4)*. Then, we have*

(i) *There is a principal circle bundle $\pi : M^8 \longrightarrow M^7$ corresponding to $[\Gamma_4]$ and a connection one-form η_4 on M^8 whose curvature form is Γ_4, such that the Spin(7)-structure*

$$\Phi_f = e^{3f} \Theta \wedge \eta_4 + e^{4f} *_7 \Theta, \tag{6.8}$$

solves the first two Killing spinor equations in (1.2) *with non-constant dilaton $\phi = 2f$ in dimension 8, where $*_7$ denotes the Hodge star operator on M^7. The Spin(7)-metric has the form*

$$g_f = e^{2f} g_7 + \eta_4 \otimes \eta_4.$$

(ii) *If M^7 is a circle bundle over a compact 6-manifold (M^6, F, Ψ^+, Ψ^-) as in Theorem 6.2, f is a smooth function on M^6 and the form Γ_4*

of the part i) is such that $\Gamma_4 \in \mathfrak{su}(3)$*, then there is a compact eight-dimensional manifold* $M^{1,1}$ *with a free structure preserving* $\mathbb{T}^2$*-action and a fibration* $\pi : M^{1,1}/\mathbb{T}^2 \cong M^6$ *with the Spin(7)-structure*

$$\Phi_f = e^{3f}\Big[F \wedge \eta + \Psi^+\Big] \wedge \eta_4 + e^{4f}\Big[\frac{1}{2}F \wedge F + \Psi_- \wedge \eta\Big], \qquad (6.9)$$

solving the first two Killing spinor equations in (1.2) *with non-constant dilaton* $\phi = 2f$ *in dimension 8, where* η *is the connection one-form on the circle bundle over* M^6 *corresponding to* Γ*. The metric has the form*

$$g_f = e^{2f}(g_6 + \eta \otimes \eta) + \eta_4 \otimes \eta_4.$$

Proof. To prove (i) first we show that the Lee form $7\theta^8_f = - * (*d\Phi \wedge \Phi)$ is an exact one-form. The exterior derivative of (6.8) yields

$$d\Phi_f = 3e^{3f} df \wedge \Theta \wedge \eta_4 + e^{3f} d\Theta \wedge \eta_4 + 4e^{4f} df \wedge *_7\Theta - e^{3f}\Theta \wedge d\eta_4.$$

The latter leads to

$$\begin{aligned} *d\Phi_f = &-3e^{4f} *_7 (df \wedge \Theta) + e^{4f} *_7 d\Theta + 4e^{3f} *_7 (df \wedge *_7\Theta) \wedge \eta_4 \\ &+ 2e^{4f} d\eta_4 \wedge \eta_4, \end{aligned}$$

where we have used the well-known fact that $*_7(\Theta \wedge d\eta_4) = -2d\eta_4$, since $d\eta \in \mathfrak{g}_2$.

Consequently, we claim

$$\begin{aligned} * d\Phi_f \wedge \Phi_f &= -3e^{7f} *_7 (df \wedge \Theta) \wedge \Theta \wedge \eta_4 + 4e^{7f} *_7 (df \wedge *_7\Theta) \wedge *_7\Theta \wedge \eta_4 \\ &\quad + 3e^{8f} *_7 (df \wedge \Theta) \wedge *_7\Theta + e^{7f} *_7 d\Theta \wedge \Theta \wedge \eta_4 \\ &\quad + e^{8f} *_7 d\Theta \wedge *_7\Theta + 2e^{8f} *_7 \Theta \wedge d\eta_4 \wedge \eta_4 \\ &= 24e^{7f} *_7 df \wedge \eta_4. \end{aligned} \qquad (6.10)$$

Indeed, the second and third lines in (6.10) give no contribution since the first term vanishes because it is a general algebraic identity valid on any G_2-manifold, the second term is zero due to the second equality in (2.3), the third term is zero because of the following chain of equalities

$$*_7 d\Theta \wedge *_7\Theta = g(*_7 d\Theta, \Theta) vol._7 = g(d\Theta, *_7\Theta) vol._7 = d\Theta \wedge \Theta = 0$$

and the fourth term is zero because $*_7\Theta \wedge d\eta_4 = 0$, since $d\eta_4 \in \mathfrak{g}_2$.

The terms in the first line are subject to the following well-known algebraic G_2-identities

$$*_7(df \wedge \Theta) \wedge \Theta = -4 *_7 df, \qquad *_7(df \wedge *_7\Theta) \wedge *_7\Theta = 3 *_7 df.$$

Hence, we obtain from (2.5) and (6.10) that $\theta^8_f = \frac{24}{7} df$, i.e., the Lee form is an exact form, which completes the proof of i). The existence of the principal circle bundle $S^1 \hookrightarrow M^8 \to M^7$ in the conditions above follows from the result of Kobayashi [64].

Now, let us suppose that Γ and Γ_4 are closed integral 2-forms on M^6, such that Γ and $\Gamma_4 \in \mathfrak{su}(3)$. Let M^7 be the principal circle bundle over M^6 corresponding to $[\Gamma]$ as in Theorem 6.2. Since $[\Gamma_4]$ defines an integral cohomology class on M^7, Kobayashi theorem implies that there exists a principal circle bundle $S^1 \hookrightarrow M^{1,1} \to M^7$ corresponding to $[\Gamma_4]$ and a connection one-form η_4 whose curvature is Γ_4. The actions of S^1 on each one of the manifolds M^7 and $M^{1,1}$ define an action of the two-torus on $M^{1,1}$ and $M^{1,1}$ can be considered a $\mathbb{T}^2$-bundle over M^6. Substituting (6.7) in (6.8), and using Theorem 6.2 and the part (i), we conclude (ii). ☐

Remark 6.2. In Theorem 6.3, if M^7 is a $\mathbb{T}^2$-bundle over a compact non-regular M^5 as in Remark 6.2, such that M^5 has an SU(2)-structure $(\eta_1, \omega_1, \omega_2, \omega_3)$ satisfying (6.1), and there exist three closed anti-self-dual 2-forms Γ_2, Γ_3 and Γ_4 on M^5 representing integral cohomology classes, then the S^1-bundle over M^7, constructed in Theorem 6.3, is a $\mathbb{T}^3$-bundle over M^5 with Spin(7)-structure

$$\Phi_f = e^{3f}\Theta_f \wedge \eta_4 + e^{4f} *_7 \Theta_f,$$

solving the first two equations in (1.2) with non-constant dilaton, where the G_2-form Θ_f on M^7 is given by (6.6). The Spin(7)-metric is

$$g_f = e^{2f}(g_5 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3) + \eta_4 \otimes \eta_4,$$

where f and g_5 denote a smooth function and the metric on M^5, respectively.

Moreover, we must notice that in Theorem 6.3, if M^7 is a $\mathbb{T}^3$-bundle over a Calabi–Yau surface as in Theorem 6.1, and the form Γ_4 considered in Theorem 6.3 is such that $\Gamma_4 \in \mathfrak{su}(2)$, i.e., anti-self-dual two-form on M^4, then the S^1-bundle constructed in Theorem 6.3 is a $\mathbb{T}^4$-bundle over the

Calabi–Yau M^4 with a Spin(7)-structure given by

$$\Phi_f = \Theta_f \wedge \eta_4 + *_7\Theta_f,$$

which solves the first two equations in (1.2) with non-constant dilaton, where the G_2-form Θ_f is given by (6.6). The metric is given by

$$g_f = e^{2f} g_{cy} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3 + \eta_4 \otimes \eta_4.$$

Suppose that one of the integral cohomology classes represented by Γ_i is non-trivial on M^4. Let Φ be a four-form defining a Spin(7)-structure on the total space of the S^1-bundle over M^7, such that there is a fiber $\mathbb{T}^4$ which is associative with respect to Φ. Then we conclude that Φ is not closed similarly as in the proof of Lemma 6.1. Therefore, the Spin(7)-structure on the total space of the S^1-bundle over M^7 is non-parallel.

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References

[1] K. Becker, M. Becker, K. Dasgupta and P.S. Green, *Compactifications of heterotic theory on non-Kähler complex manifolds: I*, J. High Energy Phys. **04** (2003), 007.

[2] K. Becker, M. Becker, K. Dasgupta, P.S. Green and E. Sharpe, *Compactifications of heterotic strings on non-Kähler complex manifolds: II*, Nucl. Phys. **B678** (2004), 19–100.

[3] K. Becker, M. Becker, K. Dasgupta and S. Prokushkin, *Properties of heterotic vacua from superpotentials*, Nuclear Phys. **B666** (2003), 144–174.

[4] K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng and S.-T. Yau, *Anomaly cancellation and smooth non-Kahler solutions in heterotic string theory*, Nucl. Phys. **B751** (2006), 108–128.

[5] E.A. Bergshoeff and M. de Roo, *The quartic effective action of the heterotic string and supersymmetry*, Nucl. Phys. **B328** (1989), 439.

[6] E. Bonan, *Sur le variétés riemanniennes a groupe d'holonomie G_2 ou Spin(7)*, C. R. Acad. Sci. Paris **262** (1966), 127–129.

[7] R. Bryant, *Metrics with exeptional holonomy*, Ann. Math. **126** (1987), 525–576.

[8] R. Bryant, *Some remarks on G_2 structures*, Proceeding of Gokova Geometry-topology Conf. 2005, ed S. Akbulut, T. Onder and R.J. Stern (2006), International Press, 75–109.

[9] R. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850.

[10] F. Cabrera, *On Riemannian manifolds with Spin(7)-structure*, Publ. Math. Debrecen **46** (3–4) (1995), 271–283.

[11] F. Cabrera, *On Riemannian manifolds with G_2-structure*, Bolletino UMI **A10** (7) (1996), 98–112.

[12] G.L. Cardoso, G. Curio, G. Dall'Agata and D. Lust, *BPS Action and superpotential for heterotic string compactifications with fluxes*, J. High Energy Phys. **10** (2003), 004.

[13] G.L. Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, *Non-Kähler string back-grounds and their five torsion classes*, Nucl. Phys. **B652** (2003), 5–34.

[14] S. Chiossi and S. Salamon, *The intrinsic torsion of SU(3) and G_2-structures*, Differential Geometry, Valencia 2001, World Scientific Publishing, 2002, pp. 115–133.

[15] D. Conti and S. Salamon, *Generalized Killing spinors in dimension* 5, Trans. Amer. Math. Soc. **359** (2007), 5319–5343.

[16] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, *First-order equations for gauge fields in spaces of dimension greater than four*, Nucl. Phys. **B214**(3) (1983), 452–464.

[17] K. Dasgupta, H. Firouzjahi and R. Gwyn, *On the warped heterotic axion*, J. High Energy Phys. **06** (2008), 056.

[18] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G-flux*, J. High Energy Phys. **11** (2002), 006.

[19] B. de Wit, D.J. Smit and N.D. Hari Dass, *Residual supersimmetry of compactified $D = 10$ supergravity*, Nucl. Phys. **B283** (1987), 165.

[20] S.K. Donaldson and R.P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), 31–47, Oxford University Press, Oxford, 1998.

[21] D.B. Fairlie and J. Nuyts, *Spherically symmetric solutions of gauge theories in eight dimensions*, J. Phys. **A17** (1984), 2867.

[22] M. Fernández, *A classification of Riemannian manifolds with structure group Spin(7)*, Ann. Math. Pura Appl. **143** (1982), 101–122.

[23] M. Fernández and A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Math. Pura Appl. (4) **32** (1982), 19–45.

[24] M. Fernández, S. Ivanov, L. Ugarte and R. Villacampa, *Non-Kaehler heterotic-string compactifications with non-zero fluxes and constant dilaton*, Commun. Math. Phys. **288** (2009), 677–697.

[25] M. Fernández, S. Ivanov, L. Ugarte and R. Villacampa, *Compact supersymmetric solutions of the heterotic equations of motion in dimension 5*, Nuclear Phys. **B820** (2009), 483–502.

[26] M. Fernández, A. Tomassini, L. Ugarte and R. Villacampa, *Balanced Hermitian metrics from SU(2)-structures*, J. Math. Phys. **50** (2009), 033507, 15 pp.

[27] M. Fernández and L. Ugarte, *Dolbeault cohomology for* G_2*-manifolds*, Geom. Dedicata **70**(1) (1998), 57–86.

[28] D.Z. Freedman, G.W. Gibbons and P.C. West, *Ten Into Four Won't Go*, Phys. Lett. **B124** (1983), 491.

[29] Th. Friedrich, *On types of non-integrable geometries*, Rend. Circ. Mat. Palermo (2) Suppl. No. **71** (2003), 99–113.

[30] Th. Friedrich, G_2*-Manifolds with parallel characteristic torsion*, Diff. Geom. Appl. **25**(6) (2007), 632–648.

[31] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336.

[32] Th. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable* G_2 *manifolds*, J. Geom. Phys. **48** (2003), 1–11.

[33] Th. Friedrich and S. Ivanov, *Almost contact manifolds, connections with torsion, parallel spinors*, J. Reine Angew Math. **559** (2003), 217–236.

[34] J.-X. Fu and S.-T. Yau, *Existence of supersymmetric hermitian metrics with torsion on non-Kaehler manifolds*, `arXiv:hep-th/0509028`.

[35] J.-X. Fu and S.-T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge–Ampere equation*, J. Diff. Geom. **78**(3) (2008), 369–428.

[36] S. Fubini and H. Nikolai, *The octonionic instanton*, Phys. Let. **B155** (1985), 369.

[37] J. Gauntlett, N. Kim, D. Martelli and D. Waldram, *Fivebranes wrapped on SLAG three-cycles and related geometry*, J. High Energy Phys. **11** (2001), 018.

[38] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-Structures and Wrapped NS5-Branes*, Commun. Math. Phys. **247** (2004), 421–445.

[39] J. Gauntlett, D. Martelli and D. Waldram, *Superstrings with Intrinsic torsion*, Phys. Rev. **D69** (2004), 086002.

[40] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on* $S^3, \mathbb{R}^3$, *and* $\mathbb{R}^4$ *bundles*, Commun. Math. Phys. **127** (1990), 529–553.

[41] J. Gillard, G. Papadopoulos and D. Tsimpis, *Anomaly, fluxes and (2,0) heterotic-string compactifications*, J. High Energy Phys. **06** (2003), 035.

[42] E. Goldstein and S. Prokushkin, *Geometric model for complex non-Käehler manifolds with SU(3) structure*, Commun. Math. Phys. **251** (2004), 65–78.

[43] U. Gran, P. Lohrmann and G. Papadopoulos, *The spinorial geometry of supersymmetric heterotic string backgrounds*, J. High Energy Phys. **02** (2006), 063.

[44] U. Gran, G. Papadopoulos and D. Roest, *Supersymmetric heterotic string backgrounds*, Phys. Lett. **B656** (2007), 119.

[45] U. Gran, G. Papadopoulos, D. Roest and P. Sloane, *Geometry of all supersymmetric type I backgrounds*, J. High Energy Phys. **08** (2007), 074.

[46] U. Gran and G. Papadopoulos, *Solution of heterotic Killing spinor equations and special geometry*, Special metrics and supersymmetry, 144–161, AIP Conf. Proc., 1093, Amer. Inst. Phys., Melville, NY, 2009.

[47] A. Gray, *Vector cross product on manifolds*, Trans. Am. Math. Soc. **141** (1969), 463–504; Correction **148** (1970), 625.

[48] M. Günaydin and H. Nikolai, *Seven-dimensional octonionic Yang–Mills instanton and its extension to an heterotic string soliton*, Phys. Lett. **B353** (1991), 169.

[49] J. Gutowski, S. Ivanov and G. Papadopoulos, *Deformations of generalized calibrations and compact non-Kahler manifolds with vanishing first Chern class*, Asian J. Math. **7** (2003), 39–80.

[50] R. Harvey and H.B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[51] J.A. Harvey and A. Strominger, *Octonionic superstring solitons*, Phys. Rev. Let. **66**(5) (1991), 549.

[52] P.S. Howe and G. Papadopoulos, *Ultraviolet behavior of two-dimensional supersymmetric non-linear sigma models*, Nucl. Phys. **B289** (1987), 264.

[53] C.M. Hull, *Anomalies, ambiquities and superstrings*, Phys. Lett. **B167** (1986), 51.

[54] C.M. Hull and P.K. Townsend, *The two loop beta function for sigma models with torsion*, Phys. Lett. **B191** (1987), 115.

[55] C.M. Hull and E. Witten, *Supersymmetric sigma models and the heterotic string*, Phys. Lett. **B160** (1985), 398.

[56] P. Ivanov and S. Ivanov, *SU(3)-instantons and $G_2, Spin(7)$-heterotic string solitons*, Comm. Math. Phys. **259** (2005), 79–102.

[57] S. Ivanov, *Geometry of quaternionic Kähler connections with torsion*, J. Geom. Phys. **41**(3) (2002), 235–257.

[58] S. Ivanov, *Connection with torsion, parallel spinors and geometry of Spin(7) manifolds*, Math. Res. Lett. **11**(2–3) (2004), 171–186.

[59] S. Ivanov and G. Papadopoulos, *A no-go theorem for string warped compactifications*, Phys. Lett. **B497** (2001), 309–316.

[60] S. Ivanov and G. Papadopoulos, *Vanishing theorems and string backgrounds*, Class. Quant. Grav. **18** (2001), 1089-1110.

[61] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. I*, J. Diff. Geom. **43** (1996), 291–328.

[62] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2. II*, J. Diff. Geom. **43** (1996), 329–375.

[63] D. Joyce, Compact Riemannian manifolds with special holonomy, Oxford University Press, 2000.

[64] S. Kobayashi, *Principal fiber bundles with 1-dimensional toroidal group*, Tohoku Math. J. (56) **8** (1956), 29–45.

[65] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew Math. **565** (2003), 125–160.

[66] B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, 1989.

[67] J. Li and S.-T. Yau, *The existence of supersymmetric string theory with torsion*, J. Diff. Geom. **70**(1) (2005), 143–181.

[68] P.-A. Nagy, *Skew-symmetric prolongations of the Lie Albegras and applications*, `arXiv:0712.1398`, To appear in J. Lie Theory.

[69] R. Reyes Carrión, *A generalization of the notion of instanton*, Diff. Geom. Appl. **8**(1) (1998), 1–20.

[70] A. Sen, *(2,0) supersymmetry and space-time supersymmetry in the heterotic strin theory*, Nucl. Phys. **B167** (1986), 289.

[71] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes Math. Ser. **201** (1989).

[72] N. Schoemann, *Almost Hermitian structures with parallel torsion*, J. Geom. Phys. **57**(11) (2007), 2187–2212.

[73] A. Strominger, *Superstrings with torsion*, Nucl. Phys. **B274** (1986), 253.

[74] L. Ugarte, *Coeffective numbers of Riemannian 8-manifold with holonomy in* Spin(7), Ann. Glob. Anal. Geom. **19** (2001), 35–53.