Stable causality of the Pomeransky–Senkov black holes

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Abstract

We show stable causality of the Pomeransky–Senkov black rings.

1   Introduction

Five-dimensional black rings and their generalizations, have attracted a lot of attention in recent literature (see, e.g., [2]). In [1] it has been shown that the Pomeransky–Senkov [4] metrics, with appropriate values of parameters, do not contain naked singularities in their domains of outer communications (d.o.c.). In that reference the question of causality violations within the d.o.c. has been left open, except for reporting some numerical evidence. The object of this note is to point out that the Pomeransky–Senkov (PS) black holes are stably causal.

Ideally one would like to show that the d.o.c.'s of the PS metrics are globally hyperbolic, but such a result lies outside of the scope of this work.

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2 Stable causality

We use the conventions and notations of [1], except that we write $G(x, \lambda, \nu)$ for $G(x)$ from [1], etc. In that reference it has been shown that

$$
\begin{aligned}
g(\nabla t, \nabla t) = g^{tt} &= \frac{g_{xx}g_{yy}}{\det g_{\mu\nu}} \det \begin{pmatrix} g_{\psi\psi} & g_{\psi\varphi} \\ g_{\psi\varphi} & g_{\varphi\varphi} \end{pmatrix} \\
&= \frac{(\nu-1)^2(x-y)^4}{4k^4 G(x,\lambda,\nu)G(y,\lambda,\nu)} \det \begin{pmatrix} g_{\psi\psi} & g_{\psi\varphi} \\ g_{\psi\varphi} & g_{\varphi\varphi} \end{pmatrix} \\
&=: \frac{(1+y)(1-x^2)\Theta(x,y,\lambda,\nu)}{(1-\lambda+\nu)H(x,y,\lambda,\nu)G(x,\lambda,\nu)G(y,\lambda,\nu)},
\end{aligned}
\tag{2.1}
$$

where Θ is a *polynomial* in the coordinates x, y, and in the parameters λ and ν, whose exact form it too complicated to be usefully displayed here. On the d.o.c. of the PS metrics, we have

$$
x \in [-1, 1], \quad y \in (y_h, -1], \quad \nu \in (0, 1), \ 2\sqrt{\nu} \leq \lambda < 1+\nu,
$$

where

$$
y_h := -\frac{\lambda - \sqrt{\lambda^2 - 4\nu}}{2\nu} > -\frac{\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu} =: y_c.
$$

Stable causality of the d.o.c. will follow if one can prove that g^{tt} is strictly negative there. Away from the boundaries $y = -1$ and $x = \pm 1$, this is equivalent to strict negativity of Θ. This remains true on those boundaries because

$$
G(y, \lambda, \nu) = \left(1 - y^2\right)\left(\nu y^2 + \lambda y + 1\right).
$$

This shows that the multiplicative factor $(1 + y)$ in the numerator of g^{tt} is cancelled by the first order zero of $G(y, \lambda, \nu)$, so ∇t will again be timelike at $y = -1$ if Θ is strictly negative there. An identical argument applies to $x = \pm 1$.

The following change of variables can be used to show that Θ has a sign: let $a \in [0, \infty)$ and $d \in (0, \infty)$, the redefinition

$$
x = -1 + \frac{2}{1+a}, \quad \nu = \frac{1}{(1+d)^2},
$$

leads to the right ranges of x and ν, except for $x = -1$ which will be considered later. Setting

$$
\lambda = 2\frac{2d^2 + 2(2+c)d + (2+c)^2}{(2+c)(1+d)(2+c+2d)},
$$

where $c \in (0, \infty)$ covers the range of allowed λ's, except for the borderline case $\lambda = 2\sqrt{\nu}$ (to which we will return shortly); to check this it is useful to note that

$$\partial_c \lambda = -\frac{8d^2(2 + c + d)}{(2 + c)^2(1 + d)(2 + c + 2d)^2} < 0.$$

Finally, the formula

$$y = -1 - \frac{d(4 + c + 2d)}{(1 + b)(2 + c)}$$

leads to y in the range $[y_c, -1)$ if $b \in [0, \infty)$, which is more than needed for $(y_h, -1)$, but note that $y = -1$ is missing.

Inserting the above into Θ one obtains a rational function with denominator

$$(1 + a)^4(1 + b)^5(2 + c)^9(1 + d)^{12}(2 + c + 2d)^5.$$

and with numerator which is a polynomial, say P, in (a, b, c, d). A MATHEMATICA calculation shows that all coefficients are *negative integers* in

$$[-61382522306560, -1] \cap \mathbb{Z}.$$

One also finds

$$P < -2048c^4d^{23},$$

which proves *strict* negativity of Θ away from the boundaries $x = -1$ and $y = -1$, for non-extreme configurations $\lambda > 2\sqrt{\nu}$.

Consider now the case $y = -1$. We proceed as before, except that we first set $y = -1$ in Θ, and then replace (x, λ, ν) by (a, c, d). The end result is a rational function with denominator

$$(1 + a)^4(2 + c)^5(1 + d)^{12}(2 + c + 2d)^5,$$

with a numerator, say R, a polynomial with strictly negative coefficients belonging to

$$[-19763036160, -2] \cap \mathbb{Z},$$

satisfying

$$R < -2c^{10}d^{12},$$

hence strictly negative.

The case $x = -1$ is analysed in a similar way.

When $\lambda = 2\sqrt{\nu}$ strict negativity of Θ is established by using instead

$$y = -1 - \frac{d}{(1 + b)}$$

in the arguments above.

Acknowledgments

References

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