Adv. Theor. Math. Phys. **14** (2010) 1779–1856

On the global structure of the Pomeransky–Senkov black holes

Piotr T. Chruściel[1], Julien Cortier[2] and Alfonso García-Parrado Gómez-Lobo[3]

[1]LMPT, Fédération Denis Poisson, Tours and Mathematical Institute and Hertford College, Oxford, UK
piotr.chrusciel@univie.ac.at
[2]Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier 2, France
jcortier@aei.mpg.de
[3]Centro de Matemática, Universidade do Minho, 4710-057 Braga, Portugal
alfonso@math.uminho.pt

Abstract

We construct analytic extensions of the Pomeransky–Senkov metrics with multiple Killing horizons and asymptotic regions. We show that, in our extensions, the singularities associated to an obstruction to differentiability of the metric lie beyond event horizons. We analyze the topology of the non-empty singular set, which turns out to be parameter-dependent. The resulting global structure is somewhat reminiscent of that of Kerr space-time.

Contents

e-print archive: http://lanl.arXiv.org/abs/0911.0802

1 Introduction

In [17] a family of five-dimensional vacuum black-hole-candidate metrics has been presented:[1]

$$\begin{aligned} ds^2 = {} & \frac{2H(x,y)k^2}{(1-\nu)^2(x-y)^2}\left(\frac{dx^2}{G(x)}-\frac{dy^2}{G(y)}\right) - 2\frac{J(x,y)}{H(y,x)}d\varphi d\psi \\ & - \frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 + \frac{F(y,x)}{H(y,x)}d\varphi^2, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} H(x,y) &= \lambda^2 + 2\nu\left(1-x^2\right)y\lambda + 2x\left(1-\nu^2y^2\right)\lambda - \nu^2 \\ &\quad + \nu\left(-\lambda^2-\nu^2+1\right)x^2y^2+1, \\ F(x,y) &= \frac{2k^2}{(x-y)^2(1-\nu)^2}\Big(\left(1-y^2\right)\left(\left((1-\nu)^2-\lambda^2\right)(\nu+1)\right. \\ &\quad \left. + \; y\lambda\left(-\lambda^2-3\nu^2+2\nu+1\right)\right)G(x) + \Big(-(1-\nu)\nu\left(\lambda^2+\nu^2-1\right)x^4 \\ &\quad + \; \lambda\left(2\nu^3-3\nu^2-\lambda^2+1\right)x^3 + \left((1-\nu)^2-\lambda^2\right)(\nu+1)x^2 \\ &\quad + \; \lambda\left(\lambda^2+(1-\nu)^2\right)x + 2\lambda^2\Big)G(y)\Big), \\ J(x,y) &= 2k^2\left(1-x^2\right)\left(1-y^2\right)\lambda\sqrt{\nu}\,\Big(\lambda^2+2(x+y)\nu\lambda-\nu^2-xy\nu \\ &\quad \times\left(-\lambda^2-\nu^2+1\right)+1\Big)\times\left((x-y)(1-\nu)^2\right)^{-1}, \\ G(x) &= \left(1-x^2\right)\left(\nu x^2+\lambda x+1\right), \end{aligned}$$

and where Ω is a 1-form given by

$$\Omega = M(x,y)d\psi + P(x,y)d\varphi,$$

[1] We use (ψ,φ) where Pomeransky and Senkov use (φ,ψ).

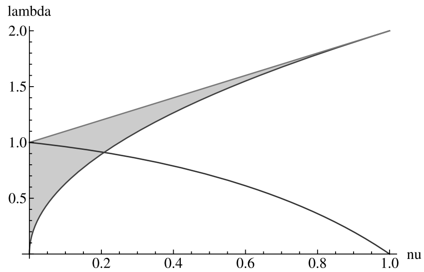

Figure 1.1: The parameters ν and λ belong to the shaded region bounded by the vertical axis and by the two increasing graphs. The decreasing graph is the function $\lambda_k(\nu)$ of the proof of Proposition 3.6.

with

$$
\begin{aligned}
&M(x,y)\\
&= 2k\lambda\sqrt{(\nu+1)^2-\lambda^2}(y+1)(-\lambda+\nu-2\nu x\\
&\quad +\nu x((\lambda+\nu-1)x+2)y-1)\times((1-\lambda+\nu)H(y,x))^{-1}\\
&=: \frac{\sqrt{(\nu+1)^2-\lambda^2}\hat{M}(x,y)}{(1-\lambda+\nu)H(y,x)},\\
P(x,y) &= \frac{2k\lambda\sqrt{\nu}\sqrt{(\nu+1)^2-\lambda^2}\left(x^2-1\right)y}{H(y,x)}\\
&=: \frac{2\sqrt{\nu}\hat{P}(x,y)}{H(y,x)},
\end{aligned}
$$

where $\hat{P}$ and $\hat{M}$ are polynomials in all variables.

The parameter k is assumed to be in $\mathbb{R}^*$, while the parameters λ and ν have been restricted in [17] to belong to the set[2]

$$
\mathcal{U} := \{(\nu,\lambda):\ \nu\in(0,1),\ 2\sqrt{\nu}\le\lambda<1+\nu\}, \tag{1.2}
$$

which is the region between the two increasing curves of figure 1.1. The coordinates x, y, ϕ, ψ, t vary within the ranges $-1\le x\le 1$, $-\infty<y<-1$, $0\le\phi\le 2\pi$, $0\le\psi\le 2\pi$ and $-\infty<t<\infty$.

Studies of the properties of the metric (1.1) are available in the literature: some physical properties of this solution can be found in [9], and a study of

[2]Strictly speaking, $\nu=0$ is allowed in [17]. It is shown there that this corresponds to Emparan–Reall metrics (compare Appendix B), which have already been analyzed elsewhere [5], and so we only consider $\nu>0$.

some of its geodesics has been performed in [8]. However, there remain unanswered questions dealing with global properties of the solution: causality, existence and topology of event horizons, absence of naked singularities, and structure of the singular set.

One thus wishes to understand the global structure of the corresponding space-time, and its possible extensions. In particular, one wishes to analyze the zeros of the denominators of the metric functions, and the zeros of the metric functions themselves, and their consequences for the space-time. The zeros of $G(y)$ are obvious, given by $y = \pm 1$ and

$$y_{\rm h} := -\frac{\lambda - \sqrt{\lambda^2 - 4\nu}}{2\nu}, \quad y_{\rm c} := -\frac{\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu}.$$

These quantities are real for values of λ, ν belonging to $\mathcal{U}$ and we have $y_{\rm c} \le y_{\rm h}$. All these considerations lead naturally to the definition

$$\Omega_0 \equiv \{(x, y, \nu, \lambda) \in \mathbb{R}^4 \; ; \; -1 \le x \le 1, \; y_{\rm c} \le y < -1, \; 0 < \nu < 1, \\ 2\sqrt{\nu} \le \lambda < 1 + \nu\}, \tag{1.3}$$

see figure 1.2. One then wishes to know the following:

1. Do the denominators of the metric functions have zeros in Ω_0?
2. One expects that the hypersurface $\{y = y_{\rm h}\}$ is an event horizon, and that the hypersurface $\{y = y_{\rm c}\}$ is a Cauchy horizon. Is this the case?
3. Are the space-times so defined extendible?
4. Do they represent suitably regular black-hole space-times, as implicit in [17], so that, e.g., the usual classification theory [5, 13, 14] applies?
5. Are the conditions defining the set Ω_0 the only possibility for the existence of regular black hole space-times or is it possible to select different values for the parameters and the coordinates such that regular black holes are present too?

The aim of this paper is to answer some of those questions: We show that the metric is smooth and Lorentzian in the range of coordinates and parameters defined by Ω_0. We show that the non-empty set $\{(x, y, \nu, \lambda) : H(x, y) = 0\}$ does not intersect Ω_0, and that the metric cannot be C^2 continued across this set. We show that there are always singularities of the Kretschmann scalar somewhere on this set, and report numerical studies that suggest blow up of the Kretschmann scalar everywhere there. We construct Kruskal–Szekeres type extensions of the metric across the Killing horizons $y = y_{\rm c}$ and $y = y_{\rm h}$, and we show that the set $\{y = y_{\rm h}\}$ forms the boundary of the domain of outer communications (d.o.c.) in our extensions. We construct an extension of the metric across "the set $\{y = -\infty\}$"

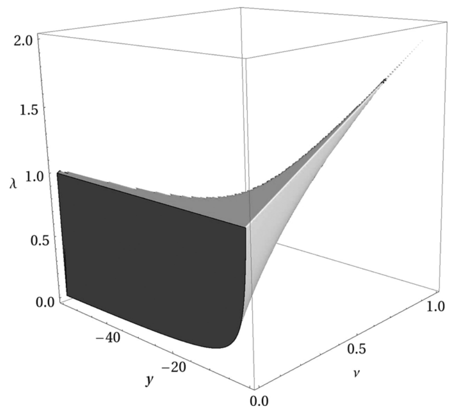

Figure 1.2: Points in Ω_0 with $y_{\rm c} < y < y_{\rm h}$; the x variable has been suppressed.

to a region which contains singularities, causality violations, and another asymptotic end. In the extended space-time the singular set has topology $\mathbb{R} \times \mathbb{T}^3$, or $\mathbb{R} \times S^1 \times S^2$, or a "pinched $\mathbb{R} \times S^1 \times S^2$," depending upon the values of the parameters. Three representative (x, Y) coordinate plots, where $Y = -1/y$ (see Section 6), illustrating the behavior of the metric are presented in figures 1.3 and 1.4. It should be kept in mind that those figures do *not* provide a representation of the global structure of the associated space-time, as the metric is singular at $Y = -1/y_{\rm c}$ and $Y = -1/y_{\rm h}$ in the (x, Y) coordinates: the space-time is constructed from the (x, Y)-coordinates representation by continuation in appropriate new coordinates, across every non-degenerate Killing horizon, to three distinct new regions. The global structure of the resulting analytic extension resembles that of the Kerr space-time, see Section 7.

Similarly to the analysis of the Emparan–Reall black rings in [4], one would like to prove that the d.o.c. is globally hyperbolic, that it is I^+-regular in the sense of [5], and that the extensions described in Section 7 are maximal. One would also like to understand better the nature of the asymptotic end associated with $(x = 1, Y = -1)$. All those questions require further studies.

An essential tool in the research reported on here was the tensor manipulation package xAct [16].

2 Generalities

In this section, we establish some generic properties of the Pomeransky-Senkov familiy of solutions, some of which will be needed in the rest of the

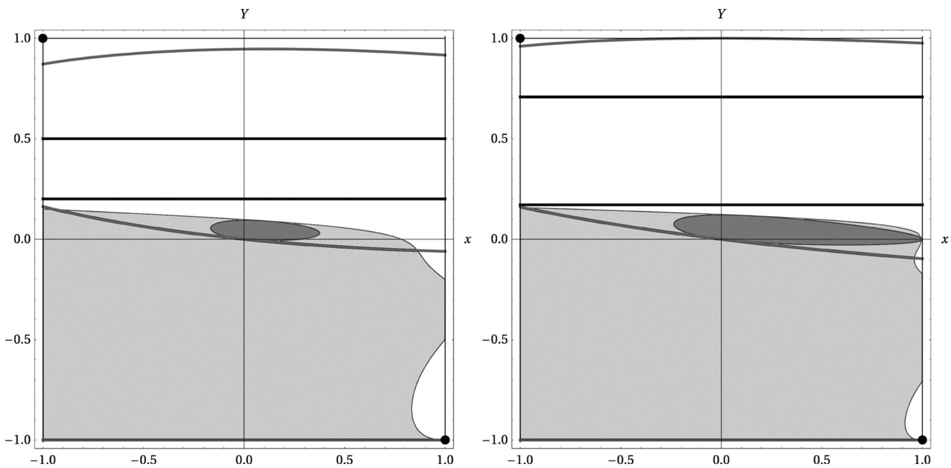

Figure 1.3: Some global features of the solutions in $(x, Y = -1/y)$ coordinates: (a) the $\mathbb{T}^3$ singularity with $\nu = 0.1$ and $\lambda = 0.7$; (b) the borderline case $\lambda = 1 - \nu$ with $\nu = 0.121$. The boundaries $x = \pm 1$ and $Y = \pm 1$ are rotation axes for suitable Killing vectors, with a conical singularity at $Y = -1$ for generic values of k, except for the black dot at $(-1, 1)$ which is the infinity of an asymptotically Minkowskian region, and the black dot at $(1, -1)$, which is the infinity of a second, non-asymptotically Minkowskian, end. The singular set $\{H(x, y) = 0\}$ is the boundary of the darker shaded region, where $H(x, y) < 0$. The strong-causality violations occur in the lightly shaded region, where $\det g_{AB} < 0$ (see Section 5.1). The upper thick horizontal line corresponds to the location of the event horizon, the lower thick horizontal line corresponds to an interior Killing horizon and the not-bounding curves to the ergosurface, $\{H(y, x) = 0\}$. These pictures indicate that when $\lambda + \nu < 1$ the ergosurface consists of two disconnected *rings* $\mathbb{R} \times S^1 \times S^2$ (the *inner* ergosurface, intersecting the causality violating region, and the *outer* ergosurface). When $\lambda + \nu = 1$ the *outer* ergosurface becomes a "pinched" $\mathbb{R} \times S^1 \times S^2$.

paper. First of all, the requirement that the metric (1.1) is real-valued and well-defined enforces

$$0 \le \nu \ne 1 \quad \text{and} \quad -(\nu + 1) \le \lambda < \nu + 1. \tag{2.1}$$

Next, we look for the points in the (x, y) plane in which the signature is Lorentzian. It follows from (A.4) that, away from the sets $\{x = y\}$, $\{H(x, y) = 0\}$, $\{H(y, x) = 0\}$, $\{F(y, x) = 0\}$, $\{G(x) = 0\}$ and $\{G(y) = 0\}$, the metric signature is

$$\begin{aligned}&(-\text{sign}(H(y,x)H(x,y)), -\text{sign}(G(y)H(x,y)),\\ &\quad -\text{sign}(G(x)H(x,y)F(y,x)G(y)),\\ &\quad \text{sign}(F(y,x)H(y,x)), \text{sign}(G(x)H(x,y))).\end{aligned} \tag{2.2}$$

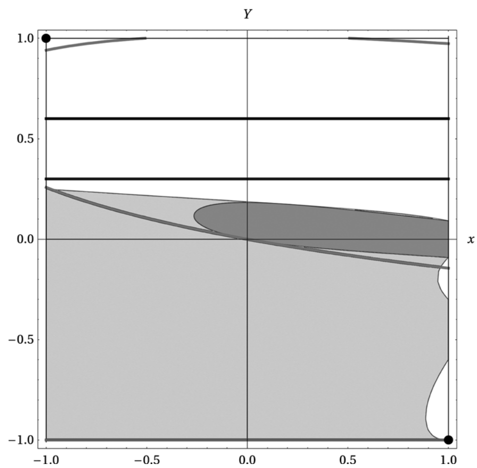

Figure 1.4: The global structure in $(x, Y = -1/y)$ coordinates when $\lambda + \nu > 1$: the $S^1 \times S^2$ singularity with $\nu = 0.18$ and $\lambda = 0.9$ and the ergosurface which now has three connected components. The *outer* ergosurface is the union of two $\mathbb{R} \times S^3$'s and the inner ergosurface is a ring $\mathbb{R} \times S^2 \times S^1$. Shading, etc., as in figure 1.3.

However, it follows from (A.2) and (A.8) that the signature of the metric can change at most at $\{G(x) = 0\}$, $\{G(y) = 0\}$, $\{x = y\}$, $\{H(y, x) = 0\}$ and $\{H(x, y) = 0\}$. It has been pointed out to us by H. Elvang (private communication) that the singularity of the metric at $\{H(y, x) = 0\}$ is an artifact of the parameterization, see Section 3.2 for details, and therefore no signature change can occur across this set. Similarly, zeros of $F(y, x)$ in (2.2) (if any) are an artifact of the choice of frame in (A.4).

A numerical study of (2.2) indicates that, under the conditions of equation (2.1), the signature is never Lorentzian if either $|x| > 1$, $|y| > 1$ or $|x| < 1$ and $|y| < 1$. Therefore, it is likely that the coordinates x, y must vary within the set $\mathcal{D}$ defined by

$$\mathcal{D} \equiv (\{|x| < 1\} \cap \{|y| > 1\}) \cup (\{|x| > 1\} \cap \{|y| < 1\}) \tag{2.3}$$

including possibly parts of its boundaries, which would, e.g., correspond to lower dimensional orbits of the isometry group.

At this stage it is useful to recall the Lichnerowicz theorem, which asserts in space-time dimension four that the only stationary, with one asymptotically flat end, well-behaved solution of the Einstein equations is Minkowski space-time. In retrospect, this theorem can be viewed as a simple consequence of the positive energy theorem, regardless of dimension, on those manifolds, without boundary and with one asymptotically flat end, on which the rigid positivity of mass holds: Indeed, as is well known, the ADM mass equals the Komar mass; but the latter is zero as the divergence of the Komar boundary integrand is zero. Since the positive energy theorem is true for all (suitably regular) manifolds in dimension five [18], there are no non-trivial, five-dimensional, asymptotically flat metrics containing only one asymptotic end.[3] As the existence of another asymptotically flat end leads to an event horizon, any non-trivial such solution must either contain event horizons, or regions with non-asymptotically-flat failure of spatial compactness (compare [1]). If we decide that the latter is undesirable ("naked singularities"), we conclude that the only solutions of interest are those with horizons. It is known that within the current class of metrics the existence of a horizon implies existence of a Killing prehorizon (cf., e.g., [3, 4]); the existence of a Killing horizon in the interesting solutions follows then from [6] under the usual global conditions.

Now, Killing horizons require real zeros of the polynomials $G(x)$ and/or $G(y)$, keeping in mind that the sets $y = \pm 1$ and $x = \pm 1$ are expected to be axes of rotation. One is then led to require that the polynomial

$$p(\xi) \equiv \nu\xi^2 + \lambda\xi + 1, \tag{2.4}$$

where ξ represents either the variable x or y, has real zeros, denoted by $\xi_- \leq \xi_+$; this will be the case if

$$2\sqrt{\nu} \leq |\lambda|$$

holds, which we assume henceforth. We may distinguish now two alternative possibilities: the case with $0 \leq \nu < 1$ and the case with $1 < \nu$. In the first possibility it can be shown that $|\xi_\pm| > 1$ whereas the second possibility leads to $|\xi_\pm| < 1$ (see Proposition A.2 of Appendix A). Combining this with (2.3) one is led to consider the following coordinate ranges:

- $0 < \nu < 1$, $|x| < 1$, $|y| > 1$. This is the case which we are going to study in this paper, see below for further explanations.
- $0 < \nu < 1$, $|x| > 1$, $|y| < 1$. We have not been able to exclude the possibility of existence of well behaved asymptotically flat solutions in this range of parameters and coordinates.
- $1 < \nu$, $|x| < 1$, $|y| > 1$. The transformation of Proposition A.1 below implies that this case is equivalent to the previous one.
- $1 < \nu$, $|x| > 1$, $|y| < 1$. Again, using the transformation (A.7) we deduce that this case is equivalent to the case of the first bullet point and hence the considerations there also apply here.

[3] By "asymptotically flat" we mean a manifold which is the union of a finite number of asymptotically flat ends and of a compact region.

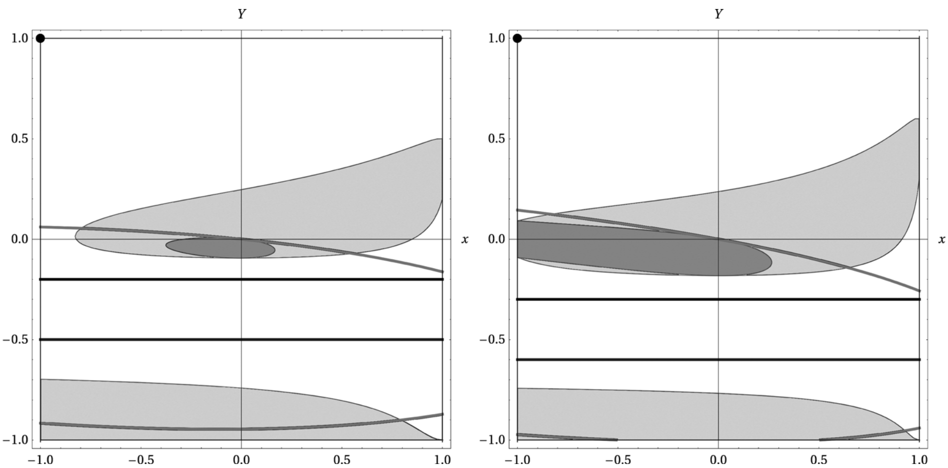

Figure 1.5: Killing horizons, ergosurfaces, causality violating regions, and singularities in space-times with ν having the same value as in the space-times depicted in the left figure 1.3 and in figure 1.4 (shading, etc., as in previous figures), but with λ here equal to $-\lambda$ there. In the current space-times the singularity is surrounded by the causality violating region.

Now, as already pointed out by Pomeransky and Senkov in [17], and analyzed in detail in Section 5.3 below, the point $x = -1$, $y = -1$, corresponds to an asymptotically flat region for any λ, ν such that the PS metric is defined. Anticipating the analysis in Section 6, the introduction of the new variable $Y \equiv -1/y$ leads to a polynomial function $Y^2 H(x, -1/Y, \lambda, \nu)$ that vanishes at the point $x = 0$, $Y = 0$, for any λ, ν; and it turns out that this point corresponds to a naked singularity of the metric for all parameters for which the metric is defined. This, together with the requirement of absence of naked singularities within the domain of outer communications, leads to the condition of existence of a negative root of $G(y)$; equivalently

$$0 < \lambda < 1 + \nu.$$

The presence of naked singularities for a set of representative parameter values with $-1 - \nu < \lambda < 0$ is shown figure 1.5.

We have therefore recovered the set (1.2).

We will say that (ν, λ) are *admissible* if $(\nu, \lambda) \in \mathcal{U}$. Unless otherwise stated, only admissible values of λ, ν will be considered in this paper.

We finish this section by noting that the PS metrics are C^2–inextendible across $\{H(x, y) = 0\}$: this can be seen by inspection of the norm $g(\partial_t, \partial_t)$ of the Killing vector ∂_t:

$$g(\partial_t, \partial_t) = g_{tt} = -\frac{H(y,x)}{H(x,y)}.$$

We will see shortly, in Section 3.1, that $H(y,x)$ does not vanish on the set $\{H(x,y)=0\}$. This shows that g_{tt} is unbounded near this set, and standard arguments (see, e.g., [4, Section 4.2]) show C^2-inextendibility of the metric across the zero level set of $H(x,y)$. Evidence of a curvature singularity there will be presented in Section 5.5. As already mentioned, points near $x=y=-1$, with $y \le -1 \le x$ and $(x,y) \neq (-1,-1)$, belong to an asymptotically flat region, where $H(x,y)$ is positive, and where the signature is Lorentzian. For reasons just explained, in the associated domain of outer communications we must thus have $\{H(x,y)>0\}$.

To summarize, we want to understand the geometry of the PS metric in those connected components of the region

$$\{x \in [-1,1],\ y \notin (-1,1),\ H(x,y)>0\} \tag{2.5}$$

which contain $(1,1)$ and $(-1,-1)$ in their closures and assume admissible values of (ν,λ). This is the region which shall be considered in this paper.

3 The function H

The function H appears in the denominators of (1.1) as $H(x,y)$ and $H(y,x)$. We start by eliminating the possibility that both functions vanish simultaneously.

3.1 $H(x,y)=0$, $H(y,x)=0$

First of all note the algebraic property

$$H(x,y) - H(y,x) = 2\lambda(-1+\nu xy)(x-y)(-1+\nu). \tag{3.1}$$

If $H(x,y)=0$, $H(y,x)=0$ then (3.1) entails the alternatives

$$\lambda = 0 \quad \text{or} \quad x=y \quad \text{or} \quad y=\frac{1}{\nu x} \quad \text{or} \quad \nu=1. \tag{3.2}$$

In the region $x \neq y$ only the third alternative is compatible with admissible (λ,ν). If we impose this condition on $H(x,y)=0$, $H(y,x)=0$ we obtain

$$\frac{(\nu-1)\left((\nu+1)^2-\lambda^2\right)}{\nu} = 0,$$

which is again not compatible with the ranges of λ, ν imposed by Pomeransky and Senkov:

$$0 < \nu < 1, \quad 2\sqrt{\nu} \le \lambda < 1 + \nu. \tag{3.3}$$

3.2 $H(y,x) = 0$

It follows from (A.2) that the zeros of $H(y,x)$ only occur in the denominators of the components of the metric induced on the planes Span$\{\partial_\varphi, \partial_\psi\}$. It has been pointed out to us by H. Elvang (private communication) that this is an artifact of the parameterization of the metric, as those components can be rewritten in a way which makes it clear that their regularity is not affected by zeros of $H(y,x)$:

$$\begin{aligned} g_{\varphi\varphi} &= \frac{2k^2(1-x^2)\Theta_{\varphi\varphi}(x,y)}{H(x,y)(-1+\nu)^2(x-y)^2}, \\ g_{\varphi\psi} &= \frac{2k^2\lambda\sqrt{\nu}(x^2-1)(1+y)\Theta_{\varphi\psi}(x,y)}{H(x,y)(-1+\nu)^2(x-y)}, \\ g_{\psi\psi} &= \frac{2k^2(1+y)\Theta_{\psi\psi}(x,y)}{(1-\lambda+\nu)(-1+\nu)^2H(x,y)(x-y)^2}, \end{aligned} \tag{3.4}$$

where $\Theta_{\varphi\varphi}(x,y)$, $\Theta_{\psi\psi}(x,y)$ and $\Theta_{\varphi\psi}(x,y)$ are polynomials in x, y, ν and λ:

$$\begin{aligned} \Theta_{\varphi\varphi}(x,y) &= (-1+\nu)^2(1+\nu x^2y^2)(1+\nu y^2)(1+\nu) + \lambda(-1+\nu)\big((-1+x^2) \\ &\quad \times \nu y(-1+y^2)(-1+\nu) + x(1+\nu y^2)(-3-\nu+\nu y^2(1+3\nu))\big) \\ &\quad - \lambda^3\big(\nu x^2 y(-1+y^2) - x(-1+\nu y^2)^2 + \nu y(1-y^2)\big) \\ &\quad + \lambda^2\big(1+2x^2+\nu\big(-1+y^2(1-3\nu+x^2(-3+\nu)) \\ &\quad + \nu y^4(2\nu+x^2(-1+\nu))\big)\big), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \Theta_{\varphi\psi}(x,y) &= (1-y)\,\big((-1+\nu xy)(-1+\nu^2) + \lambda^2(1+\nu xy)\big) \\ &\quad + 2\lambda(-x(1+\nu y(-2+\nu)) + y(1+\nu(-2+\nu y))). \end{aligned} \tag{3.6}$$

The explicit form of $\Theta_{\psi\psi}(x,y)$, calculated with Mathematica and omitted because of its length, is available upon request. Here we only report that near $x = y = 1$ we have

$$\begin{aligned} \Theta_{\psi\psi}(x,y) &= -(y-1)(\nu-1)^2(\lambda-\nu-1)(\lambda+\nu+1)^3 \\ &\quad + O\big((x-1)^2 + (y-1)^2\big), \end{aligned} \tag{3.7}$$

while on $\{y = 1\}$ it holds that

$$\begin{aligned}\Theta_{\psi\psi}(x,1) &= 4(x-1)^2\lambda^2(\nu-1)^2\left(x^2\nu(\lambda+\nu-1)-\lambda+\nu-1\right)\\ &= -4(x-1)^2\lambda^2(1-\nu)^3(\lambda+\nu+1)+O\left((x-1)^3\right). \qquad (3.8)\end{aligned}$$

Since

$$H(1,1) = (1-\nu)(\lambda+\nu+1)^2 > 0,$$

we see that $g_{\psi\psi}$ is negative for $y = 1 > x$ sufficiently near $x = y = 1$, and hence the periodic Killing vector ∂_ψ is timelike there. This shows existence of causality violations in that region. In fact, the zeros of $\Theta_{\psi\psi}(x,1)$ are $x = \pm 1$ and

$$x = \pm\frac{\sqrt{\lambda-\nu+1}}{\sqrt{\nu(\lambda+\nu-1)}}.$$

This is real only for $\lambda+\nu > 1$, but then always larger than one in absolute value. We will prove that $H(x,1)$ is positive for admissible λ, hence causality is violated throughout a neighborhood of $\{x \in [-1,1],\ y = 1\}$.

3.3 $H(x,y) = 0$: formulation of the problem

The question of zeros of $H(x,y)$ appears to be considerably more difficult. In this section, we wish to prove that the polynomial $H(x,y)$ defined by

$$\begin{aligned}H(x,y) := {}& 1+\lambda^2-\nu^2+2\lambda\nu(1-x^2)y+2x\lambda(1-\nu^2y^2)\\ &+x^2y^2\nu(1-\lambda^2-\nu^2)\end{aligned}$$

does not vanish on the set Ω_0 defined in (1.3). Every slice of Ω_0 at fixed ν and λ corresponds to the region outside the “interior”, presumably Cauchy, horizon of the metric with parameters ν and λ. Equivalently, we want to show that the set

$$\mathcal{H} \equiv \{(x,y,\lambda,\nu) \in \mathbb{R}^4 : (\nu,\lambda) \in \mathcal{U},\ y_{\rm c}(\nu,\lambda) \le y < -1,\ H(x,y) = 0\} \qquad (3.9)$$

does not intersect Ω_0.

We start by writing H as a polynomial in x,

$$\begin{aligned}H(x,y) = {}& (1+\lambda^2-\nu^2+2\lambda\nu y) \ + \ 2\lambda(1-\nu^2y^2)x\\ &+ \ \nu y\Big(y(1-\lambda^2-\nu^2)-2\lambda\Big)x^2,\end{aligned}$$

and note that the coefficient of x^2 does not vanish in the ranges of interest:

Lemma 3.1. *The highest order coefficient* $\nu y\Big(y(1-\lambda^2-\nu^2)-2\lambda\Big)$ *of the polynomial in* x, $H(x,y)$, *does not vanish in the domain of interest* $\{(y,\nu,\lambda):(\nu,\lambda)\in\mathcal{U},\ y\in[y_c(\nu,\lambda),-1)\}$.

Proof. For $\nu y\neq 0$ the condition $\nu y\Big(y(1-\lambda^2-\nu^2)-2\lambda\Big)=0$ is equivalent to the condition

$$y=y_0(\nu,\lambda):=\frac{-2\lambda}{\nu^2+\lambda^2-1}.$$

So, for all $(\nu,\lambda)\in\mathcal{U}$ such that $\nu^2+\lambda^2<1$, $y_0(\nu,\lambda)$ is positive, hence outside the region of interest for the parameter y. The simultaneous equalities $\nu^2+\lambda^2=1$ and $y=y_0$ are also impossible on Ω_0. We claim that in the case $\nu^2+\lambda^2>1$ we have the inequality $y_0<y_c$: Indeed, we first note that

$$y_0(\nu,1+\nu)=-\frac{1}{\nu}=y_c(\nu,1+\nu).$$

Next, the derivatives of y_0 and y_c with respect to λ read

$$\frac{\partial y_0}{\partial\lambda}(\nu,\lambda)=\frac{2(1+\lambda^2-\nu^2)}{(\nu^2+\lambda^2-1)^2},\quad \frac{\partial y_c}{\partial\lambda}(\nu,\lambda)=-\frac{\sqrt{\lambda^2-4\nu}+\lambda}{2\nu\sqrt{\lambda^2-4\nu}}.$$

The first expression is positive for all $\{0<\nu<1,\nu^2+\lambda^2\neq 1\}\subset\mathcal{U}$, thus $\lambda\mapsto y_0(\nu,\lambda)$ is increasing on $(\sqrt{1-\nu^2},1+\nu)$, while the second expression is negative when $\lambda\in(2\sqrt{\nu},1+\nu)$ so $\lambda\mapsto y_c(\nu,\lambda)$ is decreasing on $(2\sqrt{\nu},1+\nu)$. This enables us to conclude that $y_0(\nu,\lambda)$ is always outside the region $(y_c(\nu,\lambda),-1)$, and therefore that the dominant coefficient of $H(x,y)$ is nonzero for all $(\nu,\lambda)\in\mathcal{U}$. □

Then, the discriminant of the second-order polynomial in x, $H(x,y)$, is the function Δ_x, given by

$$\begin{aligned}\Delta_x(y,\nu,\lambda)=4[\lambda^2(1-\nu^2y^2)^2-\nu y(1+\lambda^2-\nu^2\\ +2\lambda\nu y)(y(1-\lambda^2-\nu^2)-2\lambda)].\end{aligned}\tag{3.10}$$

If $\Delta_x(y,\nu,\lambda) \geq 0$ for some values of y, ν, λ, then the equation $H(x,y) = 0$ has two roots (counting multiplicity) $x = x_\pm(y,\nu,\lambda)$ which have the expression:

$$x_\pm(y,\nu,\lambda) = \frac{-\lambda(1-\nu^2y^2) \pm \sqrt{W(y,\nu,\lambda)}}{\nu y(y(1-\lambda^2-\nu^2)-2\lambda)}, \tag{3.11}$$

with

$$W := \frac{1}{4}\Delta_x.$$

Note that the denominator has no zeros in the range of interest by Lemma 3.1.

In what follows, we begin with considering the restriction to the particular cases $y = -1$, and $x = -1$. We then pass to a study of the set $\mathcal{A}$, defined as the collection of all (y,ν,λ), $y_{\rm c} \leq y \leq -1$, $(\nu,\lambda) \in \mathcal{U}$, such that $W(y,\nu,\lambda)$ is non-negative, in particular its connectedness. We eventually conclude, using the continuity of the functions $x_\pm$ on $\mathcal{A}$, that for all such (y,ν,λ), the corresponding roots $x_\pm(y,\nu,\lambda)$ lie outside the required interval $-1 \leq x \leq 1$.

3.4 $H(x,y) = 0$, $y = -1$

Lemma 3.2. *There exist values of $(\nu,\lambda) \in \mathcal{U}$ such that $x_+(-1,\nu,\lambda) < -1$, and thus $x_-(-1,\nu,\lambda) < -1$ as well.*

Proof. Let us try values of ν and λ in the allowed ranges such that $\nu^2 + \lambda^2 = 1$. This is possible for ν small enough, namely $\nu \in \left(0, \sqrt{5}-2\right]$. We first have that

$$W(-1,\nu,\sqrt{1-\nu^2}) = (1-\nu^2)\big((1-\nu^2)^2 + 4\nu(\nu - \sqrt{1-\nu^2})\big)$$

is positive. Then we compute $x_\pm$, and we get:

$$x_\pm(-1,\nu,\sqrt{1-\nu^2}) + 1 = \frac{-1+\nu^2+2\nu \pm \sqrt{(1-\nu^2)^2 + 4\nu(\nu-\sqrt{1-\nu^2})}}{2\nu}. \tag{3.12}$$

The numerator of the right-hand side term of the last equality for x_+ is negative, as can be seen from the following equivalent inequalities:

$$\begin{aligned} &1-\nu^2-2\nu > \sqrt{(1-\nu^2)^2 + 4\nu(\nu-\sqrt{1-\nu^2})} \\ \Leftrightarrow &(1-\nu^2)^2 - 4\nu(1-\nu^2) + 4\nu^2 > (1-\nu^2)^2 + 4\nu^2 - 4\nu\sqrt{1-\nu^2} \end{aligned}$$

$$\Leftrightarrow -4\nu(1-\nu^2) > -4\nu\sqrt{1-\nu^2}$$

the last one being of course true for $\nu \in (0,1)$. □

3.5 $H(x,y) = 0$, $x = -1$

Lemma 3.3. *There is no solution for $H(x,y) = 0$ when $x = -1$ and $y < -1$.*

Proof. We can first write

$$H(-1,y) = (1+\nu-\lambda)\big(\nu(1+\lambda-\nu)y^2 + 1 - \nu - \lambda\big).$$

The first factor is positive from the definition of $\mathcal{U}$. We show that the second factor cannot vanish for any value of the parameters y, ν, λ in the allowed ranges. Indeed, this second factor is quadratic in y, the coefficient $\nu(1+\lambda-\nu)$ is positive, so that the roots are

$$y_\pm = \pm\sqrt{\frac{\nu+\lambda-1}{\nu(1+\lambda-\nu)}},$$

provided that $\nu+\lambda \geq 1$, otherwise there is no root and $H(-1,y)$ is indeed positive. But the required condition $\lambda < 1+\nu$ is equivalent, for $\nu \in (0,1)$, to $\nu+\lambda-1 < \nu(1+\lambda-\nu)$, so that both solutions $y_\pm$ above are larger than -1, thus out of the authorized range for the coordinate y. □

In other words, this lemma expresses that no connected component of the set $\mathcal{H}$ can intersect the hypersurface $\{x=-1\}$ for values of y smaller than -1.

Recall that $\mathcal{A}$ is the set of points (y,ν,λ), $(\nu,\lambda) \in \mathcal{U}$, $y_c \leq y \leq -1$, such that solutions $x \in \mathbb{R}$ of the equation $H(x,y) = 0$ do exist. We will show shortly that $\mathcal{A}$ is connected. Then, since x_+ and x_- are continuous functions we deduce that $x_+(\mathcal{A})$ and $x_-(\mathcal{A})$ are connected subsets of $\mathbb{R}$ and hence they must be intervals. On the other hand, by Lemma 3.3 we have $x_+(\mathcal{A}) \cap \{-1\} = \emptyset$, $x_-(\mathcal{A}) \cap \{-1\} = \emptyset$ and hence either $x_+(\mathcal{A}) \subset (-\infty,-1)$ or $x_+(\mathcal{A}) \subset (-1,\infty)$; and similarly $x_-(\mathcal{A}) \subset (-\infty,-1)$ or $x_-(\mathcal{A}) \subset (-1,\infty)$. The alternatives $x_+(\mathcal{A}) \subset (-1,\infty)$ and $x_-(\mathcal{A}) \subset (-1,\infty)$ can be ruled out because in Lemma 3.2 we have proved that, for the particular case of $y=-1$, $\lambda = \sqrt{1-\nu^2}$, and for ν small enough, both solutions $x_\pm(-1,\nu,\sqrt{1-\nu^2})$ satisfy $x < -1$. Therefore necessarily $x_-(\mathcal{A}) \subset (-\infty,-1)$, $x_+(\mathcal{A}) \subset (-\infty,-1)$

which entails $x_-(\mathcal{A}) \cap (-1,1) = \emptyset$, $x_+(\mathcal{A}) \cap (-1,1) = \emptyset$. From this result we conclude that $\mathcal{H}$ does not intersect Ω_0.

The aim of the next section is to establish the connectedness of $\mathcal{A}$, needed to complete the proof.

3.6 Connectedness of $\mathscr{A}$

Throughout this section, we assume that $(\nu,\lambda) \in \mathcal{U}$. Recall that if $H(x,y) = 0$ then $x = x_\pm$, where

$$x_\pm = \frac{\lambda - \lambda\,\nu^2 y^2 \pm \sqrt{W(y,\nu,\lambda)}}{\nu\, y\, ((\lambda^2+\nu^2-1)y + 2\lambda)},$$

and where

$$\begin{aligned} W(y,\nu,\lambda) := {}& \lambda^2\nu^4 y^4 + 2\,\lambda\,\nu^2\left(\lambda^2+\nu^2-1\right) y^3 \\ & - \nu\left(1 - 2\,\nu\,\lambda^2 - \lambda^4 + \nu^4 - 2\,\nu^2\right) y^2 \\ & + 2\,\nu\,\lambda\left(1+\lambda^2-\nu^2\right) y + \lambda^2. \end{aligned}$$

For large y, whether positive or negative, W is positive so zeros of $H(x,y)$ exist. Our task here is to prove that the region $\{(\nu,\lambda) \in \mathcal{U},\ x \in [-1,1],\ y_{\rm c} \le y \le -1\}$ does not contain any of them. As just explained, this will follow from:

Theorem 3.4. *The set*

$$\mathscr{A} := \{(y,\nu,\lambda):\ W(y,\lambda,\nu) \ge 0,\ (\lambda,\nu) \in \mathcal{U},\ y_{\rm c}(\nu,\lambda) \le y \le -1\} \qquad (3.13)$$

is connected.

Proof. We show in Lemma 3.5 below that $W(-1,\nu,\lambda) > 0$ on $\mathcal{U}$. Next, Proposition 3.6 establishes that for all $(\lambda,\nu) \in \mathcal{U}$, the set $\{y \in [y_{\rm c},-1] : W(y,\lambda,\nu) \ge 0\}$ is connected, which readily implies the result. □

We supply now the details:

Lemma 3.5. $W(-1,\nu,\lambda) > 0$ *on* $\mathcal{U}$.

Proof. We have

$$W(-1,\nu,\lambda) = \underbrace{\left(1+\nu\,\lambda^2-2\,\nu\,\lambda-\nu^2\right)}_{=:P(\nu,\lambda)}\underbrace{\left(\nu^3-\nu-2\,\nu\,\lambda+\lambda^2\right)}_{=:Q(\nu,\lambda)}.$$

The equation $P=0$ is solved by

$$\nu_\pm = \frac{1}{2}\left(\lambda^2-2\lambda\pm\sqrt{(\lambda^2-2\,\lambda)^2+4}\right).$$

Clearly $\nu_-<0$ and $\nu_+>[(\lambda-1)^2+1]/2$. For $\lambda\in[0,2)$ we have

$$\nu_-(\lambda)<\lambda-1<\frac{1}{4}\lambda^2<\nu_+(\lambda).$$

Indeed, the first inequality is equivalent to positivity of $3\lambda^2-4\lambda+4=2\lambda^2+(\lambda-2)^2$. The second one is always true for $\lambda\neq 2$, while the last inequality follows easily from the already indicated inequality $\nu_+>[(\lambda-1)^2+1]/2$. Then, we notice that:

$$(\nu,\lambda)\in\mathcal{U}\Longleftrightarrow\lambda\in(0,2),\quad \lambda-1<\nu\leq\lambda^2/4.$$

Hence, for all $(\nu,\lambda)\in\mathcal{U}$, we have $\nu_-(\lambda)<\nu<\nu_+(\lambda)$, and we conclude that $P>0$ on $\mathcal{U}$.

Next, we have

$$Q=\left(\lambda-\nu+\sqrt{\nu^2-\nu^3+\nu}\right)\left(\lambda-\nu-\sqrt{\nu^2-\nu^3+\nu}\right),$$

and note that the polynomial $\nu^2-\nu^3+\nu$ vanishes at 0 and at $(1\pm\sqrt{5})/2$. We want to show that Q is positive on $\mathcal{U}$, this proceeds as follows: Straightforward algebra shows that, for $\nu>0$, the inequality

$$2\sqrt{\nu}>\nu+\sqrt{\nu^2-\nu^3+\nu} \tag{3.14}$$

is equivalent to

$$\left((\nu+1)^2+8\right)(\nu-1)^2>0.$$

So (3.14) holds for $\nu\in[0,1)$. But the right-hand side of (3.14) is the larger root of Q, and we conclude that the roots of Q do not intersect the graph of $\nu\mapsto 2\sqrt{\nu}$ in the range of interest. Next, (3.14) also shows that Q is positive on this graph for small positive ν. Since Q does not change sign on $\mathcal{U}$, it is positive on $\mathcal{U}$. □

We continue with:

Proposition 3.6. *For every admissible values of λ and ν, the set $\{y \in [y_c, -1] : W(y,\nu,\lambda) \geq 0,\ (\nu,\lambda) \in \mathcal{U}\}$ is connected.*

Proof. We start by a study of the variations of $y \mapsto W(y,\nu,\lambda)$ on $[y_c(\nu,\lambda), -1]$, for all $(\nu,\lambda) \in \mathcal{U}$. To do so, we first compute the derivatives of W, with respect to y, up to third order. We have

$$\begin{aligned}
\frac{\partial W}{\partial y}(y,\nu,\lambda) &= 2\nu\Big(2\nu^3\lambda^2y^3 + 3\nu\lambda(\nu^2+\lambda^2-1)y^2 \\
&\qquad + \big(\lambda^4-\nu^4-1+2\nu(\nu+\lambda^2)\big)y + \lambda(1+\lambda^2-\nu^2)\Big), \\
\frac{\partial^2 W}{\partial y^2}(y,\nu,\lambda) &= 2\nu\Big(6\nu^3\lambda^2y^2 + 6\nu\lambda(\nu^2+\lambda^2-1)y + \lambda^4-\nu^4-1+2\nu(\nu+\lambda^2)\Big), \\
\frac{\partial^3 W}{\partial y^3}(y,\nu,\lambda) &= 12\nu^2\lambda\big(2\nu^2\lambda y + \nu^2+\lambda^2-1\big).
\end{aligned}$$

Since $\nu\lambda \neq 0$ for all allowed ν and λ, we see that $\frac{\partial^3 W}{\partial y^3}(y,\nu,\lambda)$ vanishes at $y = y_3(\nu,\lambda)$, where

$$y_3(\nu,\lambda) := \frac{1-\nu^2-\lambda^2}{2\nu^2\lambda}, \tag{3.15}$$

and therefore the function $y \mapsto \frac{\partial^2 W}{\partial y^2}(y,\nu,\lambda)$ reaches its minimum there, equal to

$$\min_{y\in\mathbb{R}} \frac{\partial^2 W}{\partial y^2}(y,\nu,\lambda) = (1+\nu+\lambda)(1+\nu-\lambda)\underbrace{\left(\lambda^2(3-2\nu)-(1-\nu)^2(3+2\nu)\right)}_{k(\nu,\lambda)}. \tag{3.16}$$

The sign of the minimum is determined by the sign of the third factor $k(\nu,\lambda)$ in (3.16), since the first two factors are positive for $(\nu,\lambda) \in \mathcal{U}$. We start by supposing that $k(\nu,\lambda) \geq 0$. This corresponds to values of ν and λ such that $\lambda \geq \lambda_k(\nu)$, where

$$\lambda_k(\nu) := (1-\nu)\sqrt{\frac{3+2\nu}{3-2\nu}}, \tag{3.17}$$

see figure 1.1. In this range of parameters the function $y \mapsto W(y,\nu,\lambda)$ is therefore convex. Connectedness of $\{y \in [y_c,-1] : W(y,\nu,\lambda) \geq 0,\ (\nu,\lambda) \in \mathcal{U}\}$ in the case $k(\nu,\lambda) \geq 0$, will be a consequence of the following:

Lemma 3.7. *For all $(\nu,\lambda) \in \mathcal{U}$, $W(y_c(\nu,\lambda),\nu,\lambda)$ is negative.*

This result, together with the convexity of $y \mapsto W(y,\nu,\lambda)$ and with the Lemma 3.5, shows that the function $y \mapsto W(y,\nu,\lambda)$ is negative on $[y_c, y_*)$, and then positive on $(y_*, -1]$ for some $y_* \in (y_c, -1)$, hence Proposition 3.6 is proved for all $(\nu, \lambda) \in \mathcal{U}$ such that $k(\nu, \lambda) \geq 0$.

We now turn to the proof of the Lemma:

Proof. We have

$$\begin{aligned}
&W(y_c(\nu,\lambda),\nu,\lambda) \\
&\quad = -\frac{1-\nu}{2\nu}\Big(\lambda^6 + \lambda^5\sqrt{\lambda^2-4\nu} - 2\lambda^4(1+3\nu) - 2\lambda^3\sqrt{\lambda^2-4\nu}(1+2\nu) \\
&\qquad + \lambda^2(1+7\nu+9\nu^2-\nu^3) + \lambda\sqrt{\lambda^2-4\nu}(1+5\nu+3\nu^2-\nu^3) \\
&\qquad - 2\nu(1-\nu)(1+\nu)^2\Big).
\end{aligned}$$

The occurrence of $\sqrt{\lambda^2 - 4\nu}$ above leads us to introduce a change of variables $(\nu, \lambda) \to (\nu, \eta)$, with $\eta \geq 0$, defined as

$$\lambda = 2\sqrt{\nu}\cosh\eta.$$

Then the expression simplifies remarkably as a polynomial in e^η:

$$W\big(y_c(\nu, 2\sqrt{\nu}\cosh\eta), \nu, 2\sqrt{\nu}\cosh\eta\big) = -(1-\nu)\mathrm{e}^{-2\eta}(\mathrm{e}^{4\eta}-\nu)(1-\mathrm{e}^{2\eta}\nu)^2.$$

The factors are all positive for $\nu \in (0,1)$ and $\eta \geq 0$, except the last factor $(1 - e^{2\eta}\nu)^2$ which can vanish for $\eta = -\ln\nu/2$, which corresponds precisely to $\lambda = 1 + \nu$, hence not for $(\nu, \lambda) \in \mathcal{U}$. This finishes the proof of the lemma. □

We now turn attention to the case $k(\nu, \lambda) < 0$. Equivalently, $\lambda < \lambda_k(\nu)$. Note that

$$\lambda_k'(\nu) = -\frac{3 - 4\nu^2 + 6\nu}{(3 - 2\nu)\sqrt{9 - 4\nu^2}},$$

which is negative for $\nu \in (0, 1)$. Since $\lambda_k(0) = 1$ and $\lambda_k(1) = 0$, the inequality $\lambda < \lambda_k(\nu)$ is compatible with the allowed ranges of the parameters ν and λ only when $\nu \in (0, \nu_0)$, where $\nu_0 \approx 0.207$ is the unique solution in $(0, 1)$ of the equation $2\sqrt{\nu} = \lambda_k(\nu)$; see figure 1.1. In other words, it suffices to consider $0 < \nu \leq \nu_0$. Recall that this case corresponds to a negative minimum for $y \mapsto \frac{\partial^2 W}{\partial y^2}(y, \nu, \lambda)$, obtained for $y = y_3(\nu, \lambda)$ (see equation (3.15)). To analyse the position of y_3 compared to -1 we write:

$$y_3(\nu, \lambda) + 1 = \frac{1 - \nu^2 - \lambda^2 + 2\nu^2\lambda}{2\nu^2\lambda}. \tag{3.18}$$

The numerator is a polynomial in λ of degree-two which has, for all ν, two real roots $\lambda_\pm(\nu) = \nu^2 \pm \sqrt{1 - \nu^2 + \nu^4}$.

Now, for all ν in $(0, \nu_0)$, one has the chain of inequalities

$$\lambda_-(\nu) < 0 < 2\sqrt{\nu} < \lambda_k(\nu) < 1 < \lambda_+(\nu). \tag{3.19}$$

Indeed, to obtain the first inequality we note that for $\nu \in (0, 1)$ we have $1 - \nu^2 + \nu^4 > \nu^4$, and negativity of $\lambda_-(\nu)$, for all $\nu \in (0, \nu_0)$, follows. The third inequality has already been established, ν_0 being precisely the value at which the inequality is saturated. The fourth follows from the fact that λ_k is decreasing. The last inequality can be proved by noting that $1 - \nu^2 + \nu^4 = (1 - \nu^2)^2 + \nu^2 > (1 - \nu^2)^2$.

Since $y_3(\nu, \lambda) + 1$ is positive for λ between $\lambda_-(\nu)$ and $\lambda_+(\nu)$, we obtain that $y_3(\nu, \lambda) + 1$ is positive for all $\nu \in (0, \nu_0)$ and $\lambda \in [2\sqrt{\nu}, \lambda_k(\nu))$. Thus

$$y_3 > -1$$

in the range of parameters of interest. This implies that $y \mapsto \frac{\partial^3 W}{\partial y^3}(y, \nu, \lambda)$ is negative on $(-\infty, y_3(\nu, \lambda))$, and in particular on $[y_c, -1]$. Hence $y \mapsto \frac{\partial W}{\partial y}(y, \nu, \lambda)$ is concave on $[y_c, -1]$, and therefore lies above its arc. So it reaches its minimum on this interval either at $y = y_c(\nu, \lambda)$, or at $y = -1$. But from Lemma 3.8, which will be proved shortly, we get that $\partial_y W(-1, \nu, \lambda)$ is non-negative for $(\nu, \lambda) \in \mathcal{U}$, $\lambda < \lambda_k(\nu)$. Then, we have again two cases:

- If $\partial_y W(y_c, \nu, \lambda)$ is non-negative, then $\partial_y W(y, \nu, \lambda)$ is non-negative for all $y \in [y_c, -1]$, therefore $y \mapsto W(y, \nu, \lambda)$ is increasing on this interval, and the set $\{y \in [y_c, -1] : \ W(y, \nu, \lambda) \geq 0\}$ is connected, and contains -1.
- If $\partial_y W(y_c, \nu, \lambda)$ is negative, then, since it is concave, the function $y \mapsto \partial_y W(y, \nu, \lambda)$ is negative on $[y_c, y_*)$ and non-negative on $[y_*, -1]$ for some $y_*(\nu, \lambda) \in (-1/\nu, -1]$. Thus, $y \mapsto W(y, \nu, \lambda)$ is decreasing on $[y_c, y_*)$, and increasing on $[y_*, -1]$. From Lemma 3.7, this implies that $W(y, \nu, \lambda)$ is negative at least on $[y_c, y_*)$, then increasing on $[y_*, -1]$. We can therefore conclude again that the set $\{y \in [y_c, -1] : W(y, \nu, \lambda) \geq 0\}$ is connected. Thus, in order to finish the proof of the proposition, and hence of the theorem, the following lemma remains to be proved:

Lemma 3.8. *For* $0 < \nu < \nu_0$ *and* $2\sqrt{\nu} \leq \lambda \leq \lambda_k(\nu)$ *the function* $\frac{\partial W}{\partial y}(-1, \nu, \lambda)$ *is non-negative.*

Proof. The result is clear by inspection of the graph in figure 3.1, a possible formal proof proceeds as follows:

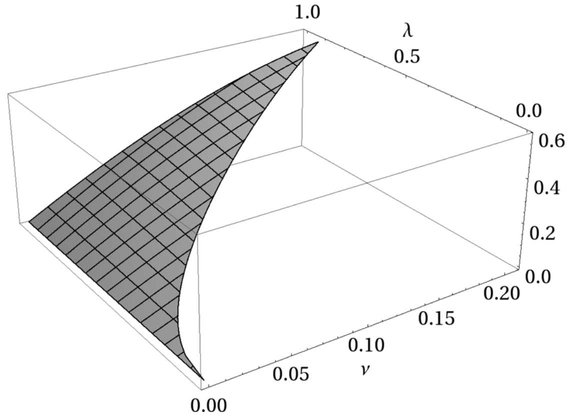

Figure 3.1: The graph of $\partial_y W$ over the set $\{0 < \nu < \nu_0,\ 2\sqrt{\nu} \le \lambda \le \lambda_k(\nu)\}$.

At $\lambda = 2\sqrt{\nu}$ we have

$$\frac{\partial W}{\partial y}(-1, \nu, 2\sqrt{\nu}) = 2\nu(1-\sqrt{\nu})^3(1+5\sqrt{\nu}+12\nu+24\nu^{3/2}+15\nu^2+7\nu^{5/2}), \tag{3.20}$$

which is clearly positive for all $0 < \nu < 1$. We continue by noting that

$$\frac{\partial^2 W}{\partial\lambda\partial y}(-1, \nu, \lambda) = 2\nu\big(1 - 3\nu - \nu^2 + 3\nu^3 - 4\nu(1+\nu^2)\lambda + 3(1+3\nu)\lambda^2 - 4\lambda^3\big), \tag{3.21}$$

$$\frac{\partial^3 W}{\partial\lambda^2\partial y}(-1, \nu, \lambda) = -4\nu\big(6\lambda^2 - 3(1+3\nu)\lambda + 2\nu(1+\nu^2)\big). \tag{3.22}$$

For all $0 < \nu < 1$ the right-hand side of (3.22) has two real roots,

$$\lambda = \frac{1+3\nu}{4} \pm \frac{\sqrt{3(27\,\nu^2 + 2\,\nu + 3 - 16\,\nu^3)}}{12} =: \lambda_{0\pm}(\nu). \tag{3.23}$$

Then, we have the inequalities $\lambda_{0-}(\nu) < 2\sqrt{\nu}$ and $\lambda_{0+}(\nu) < \lambda_k(\nu)$ for $0 < \nu < \nu_0$, whereas the difference $2\sqrt{\nu} - \lambda_{0+}(\nu)$ is positive on $(0, \nu_1)$ and negative on (ν_1, ν_0) for some $\nu_1 \in (0, \nu_0)$. Indeed we have, for $\nu \in (0, 1)$,

$$\begin{aligned}\lambda_{0-}(\nu) &= \frac{1+3\nu}{4} - \frac{\sqrt{3}}{12}\sqrt{3+2\nu+27\nu^2-16\nu^3} < \frac{1+3\nu}{4} - \frac{\sqrt{3}}{12}\sqrt{3} \\ &= \frac{3\nu}{4} < 2\nu < 2\sqrt{\nu}.\end{aligned}$$

Next, we prove that $\nu \mapsto \lambda_{0+}(\nu) - 2\sqrt{\nu}$ is decreasing on $(0, \nu_0)$. Indeed,

$$\frac{d}{d\nu}(\lambda_{0+}(\nu) - 2\sqrt{\nu}) = \frac{1}{4}\left(3 - \frac{4}{\sqrt{\nu}} + \frac{1/3 + 9\nu - 8\nu^2}{\sqrt{1 + 2\nu/3 + 9\nu^2 - 16\nu^3/3}}\right).$$

For $\nu \in (0,1)$, we have

$$\frac{1/3 + 9\nu - 8\nu^2}{\sqrt{1 + 2\nu/3 + 9\nu^2 - 16\nu^3/3}} < 1/3 + 9\nu - 8\nu^2,$$

so that we obtain

$$\frac{d}{d\nu}(\lambda_{0+}(\nu) - 2\sqrt{\nu}) < \frac{1}{4}\left(\frac{10}{3} - \frac{4}{\sqrt{\nu}} + 9\nu - 8\nu^2\right). \tag{3.24}$$

Moreover, $1/3 + 9\nu - 8\nu^2$ is less than 5 for $\nu \in (0, 1/4)$, while $3 - \frac{4}{\sqrt{\nu}}$ is less than -5 for all $\nu \in (0, 1/4) \supset (0, \nu_0)$. Therefore, $\frac{d}{d\nu}(\lambda_{0+}(\nu) - 2\sqrt{\nu})$ is negative for $\nu \in (0, 1/4)$, in particular for $\nu \in (0, \nu_0)$.

We further note that the function $\nu \mapsto \lambda_{0+}(\nu) - 2\sqrt{\nu}$ takes the value $1/2$ at $\nu = 0$, and that it is negative at $\nu = 1/8 < \nu_0$:

$$\lambda_{0+}(1/8) - 2\sqrt{1/8} = \frac{11}{32} - \frac{1}{\sqrt{2}} + \frac{1}{4}\sqrt{\frac{233}{192}} < \frac{1}{8}\left(\frac{11}{4} - 4\sqrt{2} + \sqrt{5}\right) < 0.$$

This proves the existence of $\nu_1 \in (0, \nu_0)$ (and more precisely $\nu_1 \in (0, 1/8)$) such that $\lambda_{0+}(\nu) > 2\sqrt{\nu}$ for $\nu \in (0, \nu_1)$, and $\lambda_{0+}(\nu) < 2\sqrt{\nu}$ for $\nu \in (\nu_1, \nu_0)$. Moreover, integrating the inequality (3.24), we obtain

$$\lambda_{0+}(\nu) < \frac{1}{2} + \frac{5}{6}\nu + \frac{9}{8}\nu^2 - \frac{2}{3}\nu^3,$$

therefore $\lambda_{0+}(\nu) < 5/8$ for all $\nu \in (0, 1/8)$, whereas $\lambda_k(\nu) > 3/4 > 5/8$ for all $\nu \in (0, 1/4)$. Then, since $2\sqrt{\nu} < \lambda_k(\nu)$ for all $\nu \in (0, \nu_0)$, we obtain, combining the previous remarks, that $\lambda_{0+}(\nu) < \lambda_k(\nu)$ for all $\nu \in (0, \nu_0)$. So we have again two cases:

- if $\nu \in (0, \nu_1)$: then $\lambda \mapsto \frac{\partial^2 W}{\partial\lambda\partial y}(-1, \nu, \lambda)$ is increasing on $[2\sqrt{\nu}, \lambda_{0+}(\nu)]$, then decreasing on $[\lambda_{0+}(\nu), \lambda_k(\nu)]$;
- if $\nu \in [\nu_1, \nu_0)$: then $\lambda \mapsto \frac{\partial^2 W}{\partial\lambda\partial y}(-1, \nu, \lambda)$ is decreasing on $[2\sqrt{\nu}, \lambda_k(\nu)]$.

So, to make sure that the function $\lambda \mapsto \frac{\partial^2 W}{\partial\lambda\partial y}(-1,\nu,\lambda)$ is positive on $[2\sqrt{\nu},\lambda_k(\nu)]$, we only need to show that it is positive for $\lambda = 2\sqrt{\nu}$ and for $\lambda = \lambda_k(\nu)$, this for all $\nu \in (0,\nu_0)$. We have in fact:

$$\frac{\partial^2 W}{\partial\lambda\partial y}(-1,\nu,2\sqrt{\nu}) = 2\nu(1-\sqrt{\nu})^2(1+2\sqrt{\nu}+12\nu-18\nu^{3/2}-13\nu^2-8\nu^{5/2}),$$

and we can write this, with $\nu = s^2$:

$$\begin{aligned}\frac{\partial^2 W}{\partial\lambda\partial y}(-1,s^2,2s) &= 2s^2(1-s)^2\\ &\times\Big((1-8s^3)+(2s-8s^3)+(s^2-2s^3)+(4s^2-16s^4)+7s^2+3s^4-8s^5\Big),\end{aligned}$$

with each term positive for $s \in (0,1/2)$, i.e., for $\nu \in (0,1/4)$. Then we have

$$\begin{aligned}\frac{\partial^2 W}{\partial\lambda\partial y}(-1,\nu,\lambda_k(\nu)) = \frac{8\nu}{3-2\nu}\Big(&(1-\nu)(3+4\nu-5\nu^2-3\nu^3)\\ &+\lambda_k(\nu)(-3+\nu+3\nu^2-5\nu^3+2\nu^4)\Big).\end{aligned}$$

Since $0 < \lambda_k(\nu) < 1$ on $(0,1/4)$, and since the factor $3-\nu-3\nu^2+5\nu^3-2\nu^4$ is positive in this range of ν, we have

$$\frac{\partial^2 W}{\partial\lambda\partial y}(-1,\nu,\lambda_k(\nu)) > \frac{8\nu}{3-2\nu}(2\nu-6\nu^2-3\nu^3+5\nu^4),$$

still positive for $\nu \in (0,1/4)$. The plot of $\frac{\partial^2 W}{\partial\lambda\partial y}(-1,\nu,\lambda_k(\nu))$ can be found in figure 3.2.

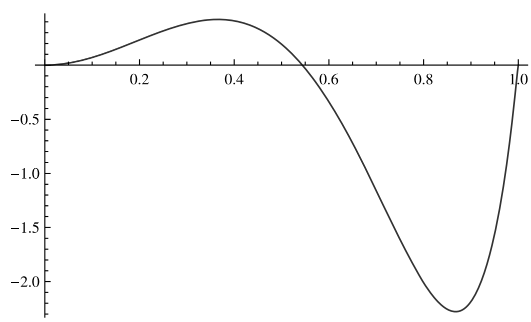

Figure 3.2: The function $\nu \mapsto \frac{\partial^2 W}{\partial\lambda\partial y}(-1,\nu,\lambda_k(\nu))$; the non-trivial zero is at, approximately, 0.544.

It follows that the function $\lambda \mapsto \frac{\partial W}{\partial y}(-1,\nu,\lambda)$ is increasing on $[2\sqrt{\nu},\lambda_k(\nu)]$ for all $\nu \in (0,\nu_0)$. This, together with (3.20) above, finishes the proof of Lemma 3.8. □

4 Extensions across Killing horizons

In this section, we write H, F, J, etc. to mean, respectively, $H(x,y)$, $F(x,y)$, $J(x,y)$, whereas $\hat{H}$, $\hat{F}$ refer, respectively, to $H(y,x)$, $F(y,x)$.

We write

$$G(y) = \nu(1-y^2)(y-y_{\rm h})(y-y_{\rm c}),$$

and we assume $y_{\rm h} \neq y_{\rm c}$. In the calculations that follow we suspend the convention $y_{\rm h} > y_{\rm c}$, used elsewhere in this paper, so that the analysis below applies both to the smaller and to the larger roots of G.

We want to construct explicitly a Kruskal–Szekeres-type extension of the metric at $y = y_{\rm h}$ and $y = y_{\rm c}$. An identical calculation applies at both values of y, so in the calculations that follow the reader can think of the symbol $y_{\rm h}$ as representing either $y_{\rm h}$ or $y_{\rm c}$.

We define new coordinates (inspired from the extension of the Kerr metrics, see [2], and of the Emparan-Reall metrics, see [4]) via the equations

$$\begin{aligned} du &= dt + \frac{\sigma}{y-y_{\rm h}}dy, \\ dv &= dt - \frac{\sigma}{y-y_{\rm h}}dy, \end{aligned} \tag{4.1}$$

and new angular coordinates:

$$\begin{aligned} d\hat{\psi} &= d\psi - a dt, \\ d\hat{\varphi} &= d\varphi - b dt, \end{aligned} \tag{4.2}$$

where a, b, σ are ($\nu-$ and λ-dependent) constants, to be chosen shortly. In terms of the new coordinates $(u,v,\hat{\psi},\hat{\varphi},x)$ the original coordinate differentials (t,y,ψ,φ,x) read:

$$\begin{aligned} dt &= \frac{du+dv}{2}, \\ dy &= \frac{y-y_{\rm h}}{2\sigma}(du-dv), \end{aligned}$$

$$
\begin{aligned}
d\psi &= d\hat{\psi} + a\frac{du+dv}{2}, \\
d\varphi &= d\hat{\varphi} + b\frac{du+dv}{2}.
\end{aligned}
$$

It is convenient to write

$$
\Omega_\psi = \frac{\hat{\Omega}_\psi}{\hat{H}}, \quad \Omega_\varphi = \frac{\hat{\Omega}_\varphi}{\hat{H}},
$$

where $\hat{\Omega}_\psi$ and $\hat{\Omega}_\varphi$ are polynomials in x and y, with coefficients which are rational functions of ν and λ. One can now write the coefficients of the metric expressed in the coordinates $(u, v, \hat{\psi}, \hat{\varphi}, x)$:

$$
\begin{aligned}
g_{uu} = g_{vv} &= \frac{1}{4}\Bigg(-\frac{\hat{H}}{H}\big(1 + a\Omega_\psi + b\Omega_\varphi\big)^2 - a^2\frac{F}{\hat{H}} - 2ab\frac{J}{\hat{H}} + b^2\frac{\hat{F}}{\hat{H}} \\
&\quad - \frac{2k^2 H(y-y_{\rm h})}{\sigma^2\nu(1-\nu)^2(1-y^2)(x-y)^2(y-y_{\rm c})}\Bigg), \\
&= \frac{1}{4}\Bigg(-\frac{\hat{H} + 2a\hat{\Omega}_\psi + 2b\hat{\Omega}_\varphi}{H} + a^2 g_{\psi\psi} + 2ab g_{\varphi\psi} + b^2 g_{\varphi\varphi} \\
&\quad - \frac{2k^2 H(y-y_{\rm h})}{\sigma^2\nu(1-\nu)^2(1-y^2)(x-y)^2(y-y_{\rm c})}\Bigg), \\
g_{uv} &= \frac{1}{4}\Bigg(-\frac{\hat{H} + 2a\hat{\Omega}_\psi + 2b\hat{\Omega}_\varphi}{H} + a^2 g_{\psi\psi} + 2ab g_{\varphi\psi} + b^2 g_{\varphi\varphi} \\
&\quad + \frac{2k^2 H(y-y_{\rm h})}{\sigma^2\nu(1-\nu)^2(1-y^2)(x-y)^2(y-y_{\rm c})}\Bigg), \\
g_{u\hat{\psi}} = g_{v\hat{\psi}} &= -\frac{1}{2}\left(\frac{\hat{H}}{H}\big(1 + a\Omega_\psi + b\Omega_\varphi\big)\Omega_\psi + a\frac{F}{\hat{H}} + b\frac{J}{\hat{H}} \right) \\
&= \frac{1}{2}\left(a g_{\psi\psi} + b g_{\psi\varphi} - \frac{\hat{\Omega}_\psi}{H} \right), \\
g_{u\hat{\varphi}} = g_{v\hat{\varphi}} &= -\frac{1}{2}\left(\frac{\hat{H}}{H}\big(1 + a\Omega_\psi + b\Omega_\varphi\big)\Omega_\varphi + a\frac{J}{\hat{H}} - b\frac{\hat{F}}{\hat{H}} \right) \\
&= \frac{1}{2}\left(a g_{\varphi\psi} + b g_{\varphi\varphi} - \frac{\hat{\Omega}_\varphi}{H} \right),
\end{aligned}
$$

$$
\begin{aligned}
g_{\hat{\psi}\hat{\psi}} &= -\frac{\hat{H}}{H}\Omega_\psi^2 - \frac{F}{\hat{H}} = g_{\psi\psi},\\
g_{\hat{\varphi}\hat{\varphi}} &= -\frac{\hat{H}}{H}\Omega_\varphi^2 + \frac{\hat{F}}{\hat{H}} = g_{\varphi\varphi},\\
g_{\hat{\psi}\hat{\varphi}} &= -\frac{\hat{H}}{H}\Omega_\psi\Omega_\varphi - \frac{J}{\hat{H}} = g_{\psi\varphi},\\
g_{xx} &= \frac{2k^2 H}{(1-\nu)^2(x-y)^2 G(x)},
\end{aligned}
$$

whereas the other components vanish. Recall that the potential singularity of the coefficients above at zeros of $\hat{H}$ is an artifact of the parameterization of the metric, as explained in Section 3.2, regardless of the values of a and b, and that we have proved that there are no zeros of H in the region of interest. It should now be clear that in the new coordinate system the metric coefficients are analytic functions of all their arguments near $y = y_{\mathrm{h}}$.

The Jacobian of the transformation relating the two coordinate systems reads

$$
\frac{\partial(u,v,\hat{\psi},\hat{\varphi},x)}{\partial(t,y,\psi,\varphi,x)} = -\frac{2\sigma}{y-y_{\mathrm{h}}},
$$

so that the determinant of the metric in the coordinates $(u,v,\hat{\psi},\hat{\varphi},x)$ is

$$
\det\left(g_{(u,v,\hat{\psi},\hat{\varphi},x)}\right) = -\frac{4k^8 H^2 (y-y_{\mathrm{h}})^2}{\sigma^2(1-\nu)^6(x-y)^8}.
$$

To get rid of the zero of the determinant at $y = y_{\mathrm{h}}$, the usual calculation is to introduce exponential coordinates

$$
\hat{u} := \mathrm{e}^{\gamma u}, \quad \hat{v} := \mathrm{e}^{-\gamma v},
$$

hence

$$
d\hat{u} = \gamma\, \hat{u}\, du, \quad d\hat{v} = -\gamma\, \hat{v}\, dv.
$$

We then express the metric coefficients in the coordinates $(\hat{u},\hat{v},\hat{\psi},\hat{\varphi},x)$. We first notice that, for $y > y_{\mathrm{h}}$,

$$
\hat{u}\hat{v} = \mathrm{e}^{\gamma(u-v)} = \exp\left(2\gamma\int^y \frac{\sigma}{y-y_{\mathrm{h}}}dy\right) = \mathrm{e}^{2\gamma\sigma\ln(y-y_{\mathrm{h}})} = (y-y_{\mathrm{h}})^{2\gamma\sigma},
$$

with an appropriate choice of the integration constant for the second equality above. The choice

$$
2\gamma\sigma = 1
$$

leads to

$$\hat{u}\hat{v} = y - y_{\rm h}, \tag{4.3}$$

which can be used to define y as a function of the exponential coordinates $\hat{u}$, $\hat{v}$. The Jacobian of the last coordinate transformation is

$$\frac{\partial(\hat{u},\hat{v},\hat{\psi},\hat{\varphi},x)}{\partial(u,v,\hat{\psi},\hat{\varphi},x)} = -\gamma^2\hat{u}\hat{v} = -\frac{y-y_{\rm h}}{4\sigma^2},$$

and the determinant of the metric in the exponential coordinates has no zeros near $y = y_{\rm h}$:

$$\det\left(g_{(\hat{u},\hat{v},\hat{\psi},\hat{\varphi},x)}\right) = -\frac{64k^8\sigma^2H^2}{(1-\nu)^6(x-y)^8}.$$

The metric coefficient in the exponential coordinates read:

$$g_{\hat{u}\hat{u}} = \frac{g_{uu}\hat{v}^2}{\gamma^2\hat{u}^2\hat{v}^2}, \quad g_{\hat{v}\hat{v}} = \frac{g_{vv}\hat{v}^2}{\gamma^2\hat{u}^2\hat{v}^2}, \quad g_{\hat{u}\hat{v}} = \frac{g_{uv}}{\gamma^2\hat{u}\hat{v}}, \quad g_{\hat{u}\hat{\psi}} = \frac{g_{u\hat{\psi}}\,\hat{v}}{\gamma\hat{u}\hat{v}}, \quad \text{etc.}$$

If we replace $\hat{u}$, $\hat{v}$ by their values in terms of the original coordinates in the first three expressions we get

$$g_{\hat{u}\hat{u}} = \frac{{\rm e}^{-2\gamma t}g_{uu}}{\gamma^2(-y_{\rm h}+y)} = g_{\hat{v}\hat{v}}, \quad g_{\hat{u}\hat{v}} = \frac{g_{uv}}{(-y_{\rm h}+y)\gamma^2}.$$

So, from (4.3), to establish regularity of the new metric coefficients we need to check that

- $g_{uu} = g_{vv}$ have a zero of order 2 at $y = y_{\rm h}$, and that
- g_{uv}, $g_{u\hat{\psi}} = g_{v\hat{\psi}}$, and $g_{u\hat{\varphi}} = g_{v\hat{\varphi}}$ all vanish at $y = y_{\rm h}$.

(If we just seek an extension through the future event horizon, then the conditions are

- $g_{uu} = g_{vv}$ and g_{uv} all vanish at $y = y_{\rm h}$, and that
- $g_{u\hat{\psi}} = g_{v\hat{\psi}}$, and $g_{u\hat{\varphi}} = g_{v\hat{\varphi}}$ all vanish at $y = y_{\rm h}$.)

More precisely, we wish to determine the parameters a, b and σ so that the conditions required above are fulfilled. We start by solving the linear system in a and b:

$$\begin{aligned} g_{u\hat{\psi}}|_{y=y_{\rm h}} &= 0, \\ g_{u\hat{\varphi}}|_{y=y_{\rm h}} &= 0, \end{aligned}$$

which we write as (all functions evaluated at $y = y_{\rm h}$):

$$\begin{aligned} a\left(-\frac{\hat{H}}{H}\Omega_\psi^2 - \frac{F}{\hat{H}}\right) + b\left(-\frac{\hat{H}}{H}\Omega_\psi\Omega_\varphi - \frac{J}{\hat{H}}\right) &= \frac{\hat{\Omega}_\psi}{H}, \\ a\left(-\frac{\hat{H}}{H}\Omega_\psi\Omega_\varphi - \frac{J}{\hat{H}}\right) + b\left(-\frac{\hat{H}}{H}\Omega_\varphi^2 + \frac{\hat{F}}{\hat{H}}\right) &= \frac{\hat{\Omega}_\varphi}{H}. \end{aligned}$$

The determinant of this system reads

$$\begin{aligned} \Delta_{a,b} &= \left(-\frac{\hat{H}}{H}\Omega_\psi^2 - \frac{F}{\hat{H}}\right)\left(-\frac{\hat{H}}{H}\Omega_\varphi^2 + \frac{\hat{F}}{\hat{H}}\right) - \left(-\frac{\hat{H}}{H}\Omega_\psi\Omega_\varphi - \frac{J}{\hat{H}}\right)^2 \\ &= \frac{1}{H}\left(F\Omega_\varphi^2 - \hat{F}\Omega_\psi^2 - 2J\Omega_\varphi\Omega_\psi\right) - \frac{F\hat{F} + J^2}{\hat{H}^2}, \end{aligned}$$

Then, from the identity (A.3), at $y = y_{\rm h}$ we have

$$F\hat{F} = -J^2, \tag{4.4}$$

and so the last term $-(F\hat{F} + J^2)/\hat{H}^2$ vanishes. Next, if we view the remaining part of $\Delta_{a,b}$ as a second-order polynomial in Ω_φ, it has discriminant $4\Omega_\psi^2(J^2 + \hat{F}F)$, which vanishes again for $y = y_{\rm h}$. This leads to the simpler expression for the determinant of the system in a, b:

$$\Delta_{a,b} = \frac{F}{H}\left(\Omega_\varphi - \frac{J\Omega_\psi}{F}\right)^2.$$

Next, assuming that $\Delta_{a,b}$ does not vanish at $y = y_{\rm h}$,[4] we can write the expressions of a and b solving the system:

$$\begin{aligned} a &= \frac{\hat{F}\Omega_\psi + J\Omega_\varphi}{H\Delta_{a,b}}, \\ b &= \frac{J\Omega_\psi - F\Omega_\varphi}{H\Delta_{a,b}}. \end{aligned}$$

[4] In fact, the vanishing or not of this determinant is irrelevant, insofar as we check that the values of a and b that are calculated below give the answer we need.

Using (4.4), this can be rewritten as

$$a = \frac{J}{F\Omega_\varphi - J\Omega_\psi},$$
$$b = -\frac{F}{F\Omega_\varphi - J\Omega_\psi}.$$

We insert in this expression the explicit values of Ω_ψ and Ω_φ:

$$a = \frac{H(y_{\rm h},x)J(x,y_{\rm h})(-1+\lambda-\nu)}{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}} \times \Big(y_{\rm h}\sqrt{\nu}F(x,y_{\rm h})(-1+x^2)(-1+\lambda-\nu)+J(x,y_{\rm h}) \times (1+y_{\rm h})\big(-1-\lambda+\nu+2\nu x(-1+y_{\rm h})+y_{\rm h}\nu x^2(-1+\lambda+\nu)\big)\Big)^{-1},$$

$$b = \frac{-F(x,y_{\rm h})H(y_{\rm h},x)(-1+\lambda-\nu)}{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}} \times \Big(y_{\rm h}\sqrt{\nu}F(x,y_{\rm h})(-1+x^2)(-1+\lambda-\nu)+J(x,y_{\rm h}) \times (1+y_{\rm h})\big(-1-\lambda+\nu+2\nu x(-1+y_{\rm h})+y_{\rm h}\nu x^2(-1+\lambda+\nu)\big)\Big)^{-1}.$$

We need to check that a and b are x-independent. For this, we found it convenient to replace λ by a parameter $t \in \mathbb{R}$ defined as (we hope that a conflict of notation with the time coordinate t will not confuse the reader)

$$\lambda = 2\sqrt{\nu}\cosh t. \tag{4.5}$$

With this redefinition we have

$$y_{\rm h} = -\frac{{\rm e}^{-t}}{\sqrt{\nu}}, \quad y_{\rm c} = -\frac{{\rm e}^{t}}{\sqrt{\nu}}.$$

Thus, the transition from $y_{\rm h}$ to $y_{\rm c}$ is obtained by changing t to its negative.

Using MATHEMATICA, the expressions above are indeed x-independent as desired, and take the form

$$a = \frac{(1-\sqrt{\nu}{\rm e}^{-t})(1-\sqrt{\nu}{\rm e}^{t})}{2k\sqrt{1+\nu^2-2\nu\cosh(2t)}},$$
$$b = \frac{(\sqrt{\nu}-{\rm e}^{-t})(\sqrt{\nu}-{\rm e}^{t})(1+{\rm e}^{2t}\nu)}{2k\sqrt{\nu}(1+{\rm e}^{2t})\sqrt{1+\nu^2-2\nu\cosh(2t)}}.$$

Note that the value of a is the same both for both horizons, but that of b is not.

We continue by checking that the remaining metric coefficients vanish with these values of a and b. First, one finds directly that $1 + a\Omega_\psi + b\Omega_\varphi$ vanishes at $y = y_\mathrm{h}$. Next, the last term in $g_{uu}|_{y=y_\mathrm{h}}$ vanishes, while the remaining term in the expression of $g_{uu}|_{y=y_\mathrm{h}}$ reads

$$-\frac{a^2 F + 2abJ - b^2 \hat{F}}{4\hat{H}} = \frac{F(J^2 + F\hat{F})}{4\hat{H}(F\Omega_\varphi - J\Omega_\psi)^2}.$$

This vanishes at $y = y_\mathrm{h}$ by (4.4). The calculation also shows that both g_{uu} and g_{uv} vanish at $y = y_\mathrm{h}$.

Next, a MATHEMATICA calculation shows that g_{uu} will have a second order zero at y_h if and only if the constant σ equals

$$\sigma = \pm\frac{2k\sqrt{\nu}(\mathrm{e}^t + \sqrt{\nu})\coth(t)}{(1-\nu)(1-\mathrm{e}^t\sqrt{\nu})}.$$

Reexpressing everything in terms of the original parameters, our calculations in this section can be summarized as follows: The metric functions g_{uu} and g_{uv} vanish at $y = y_\mathrm{h}$ when the parameters a and b take the values

$$a = \frac{1-\lambda+\nu}{2k\sqrt{(1+\nu)^2-\lambda^2}}, \tag{4.6}$$

$$b = \frac{\left((\nu-1)\sqrt{\lambda^2-4\nu} + \lambda(1+\nu)\right)(1+\nu-\lambda)}{4k\lambda\sqrt{\nu((1+\nu)^2-\lambda^2)}}. \tag{4.7}$$

With this choice, g_{uu} has a second order zero at $y = y_\mathrm{h}$ if and only if σ takes the value

$$\sigma^2 = \frac{2k^2\lambda^2\left(\lambda^2\left(\nu^2+1\right) - \lambda\left(\nu^2-1\right)\sqrt{\lambda^2-4\nu} - 2\nu(\nu+1)^2\right)}{(\nu-1)^2\left(\lambda^2-4\nu\right)(-\lambda+\nu+1)^2}. \tag{4.8}$$

At $y = y_\mathrm{c}$ the analogous analysis leads to the same value of a, while b and σ are now

$$b = \frac{\left((-\nu+1)\sqrt{\lambda^2-4\nu} + \lambda(1+\nu)\right)(1+\nu-\lambda)}{4k\lambda\sqrt{\nu((1+\nu)^2-\lambda^2)}}, \tag{4.9}$$

$$\sigma^2 = \frac{2k^2\lambda^2\left(\lambda^2\left(\nu^2+1\right) + \lambda\left(\nu^2-1\right)\sqrt{\lambda^2-4\nu} - 2\nu(\nu+1)^2\right)}{(\nu-1)^2\left(\lambda^2-4\nu\right)(-\lambda+\nu+1)^2}. \tag{4.10}$$

We note that the values obtained for the constants a, b and σ are all well defined under the assumptions of equation (2.1) together with $\lambda^2 - 4\nu > 0$ ($y_{\rm h}$ and $y_{\rm c}$ real and distinct) and hence the extension through the Killing horizons will remain valid for any member of the PS family of solutions whose parameters meet these requirements.

4.1 Degenerate case

It was shown in [15] that the near-horizon limit of the degenerate PS solutions admits smooth extensions. Here we check that the method presented there for extending across a degenerate horizon applies to the PS metrics, and not only their near-horizons limits.

In the original coordinates (t, y, ψ, x, φ), the PS metrics do not satisfy the requirement that the coefficients $g_{t\psi}$, $g_{t\varphi}$ and g_{tt} vanish at the degenerate horizon $\{y = y_0 := -\frac{1}{\sqrt{\nu}}\}$, with cancelation at order two for g_{tt}. So the first step is to define new coordinates $\hat{t}, \hat{\psi}, \hat{\varphi}$, not to be confused with the hatted coordinates defined previously in this section in the non-degenerate case:

$$\hat{t} = t, \quad \hat{\psi} = \psi - a_\psi t, \quad \hat{\varphi} = \varphi - a_\varphi t,$$

where a_ψ and a_φ are constants. Written in these coordinates, the conditions for the metric coefficients $g_{\hat{t}\hat{\psi}}$ and $g_{\hat{t}\hat{\varphi}}$ to vanish at y_0 read again:

$$a_\psi \left(-\frac{\hat{H}}{H}\Omega_\psi^2 - \frac{F}{\hat{H}} \right) + a_\varphi \left(-\frac{\hat{H}}{H}\Omega_\psi\Omega_\varphi - \frac{J}{\hat{H}} \right) = \frac{\hat{\Omega}_\psi}{H},$$
$$a_\psi \left(-\frac{\hat{H}}{H}\Omega_\psi\Omega_\varphi - \frac{J}{\hat{H}} \right) + a_\varphi \left(-\frac{\hat{H}}{H}\Omega_\varphi^2 + \frac{\hat{F}}{\hat{H}} \right) = \frac{\hat{\Omega}_\varphi}{H},$$

where all the functions are evaluated at $y = y_0 = -\frac{1}{\sqrt{\nu}}$. Here, since $\lambda = 2\sqrt{\nu}$, we obtain that

$$a_\psi = \frac{1-\nu}{2k(1+\sqrt{\nu})^2},$$
$$a_\varphi = \frac{1-\nu^2}{4k\sqrt{\nu}(1+\sqrt{\nu})^2}$$

are solutions, i.e. make the coefficients $g_{\hat{t}\hat{\psi}}$ and $g_{\hat{t}\hat{\varphi}}$ vanish at y_0. The metric is then of the form of equation (41) in [15], where R is replaced by $y - y_0$, the indices $i, j = 1, 2$ refer to the angular coordinates $\hat{\psi}$ and $\hat{\varphi}$, and t is replaced by $\hat{t}$. Explicitely, we have

$$g = g_{\hat{t}\hat{t}} d\hat{t}^2 + 2g_{\hat{t}i} d\hat{t} d\hat{\varphi}^i + g_{ij} d\hat{\varphi}^i d\hat{\varphi}^j + g_{yy} dy^2 + g_{xx} dx^2, \tag{4.11}$$

with

$$g_{\hat{t}\hat{t}} = f_t(y, x)(y - y_0)^2, \quad g_{\hat{t}i} = f_i(y, x)(y - y_0), \quad g_{yy} = \frac{h(y, x)}{(y - y_0)^2}, \tag{4.12}$$

for some functions f_t, f_ψ, f_φ and h, bounded at $y = y_0$ in their first variable.

Then, in order to remove the singularity at the horizon $y = y_0$, we define new coordinates (v, z, Ψ, Φ), such that:

$$\begin{aligned}
\hat{t} &= v - \frac{a_0}{z - y_0} + f(z, x), \\
y &= z, \\
\hat{\psi} &= \Psi + b^\psi \ln(z - y_0), \\
\hat{\varphi} &= \Phi + b^\varphi \ln(z - y_0),
\end{aligned}$$

where a_0, b^ψ and b^φ are constants. Our aim is to find values of those constants for which the PS metrics, written in the coordinates (v, z, Ψ, Φ) , are regular at $z = y_0$. In fact, we choose $f(z, x) = a_1 \ln(z - y_0)$, where a_1 is a constant, as in [15]. Thus, the coordinate transformations read

$$\begin{aligned}
d\hat{t} &= dv + \left(\frac{a_0}{(z - y_0)^2} + \frac{a_1}{z - y_0} \right) dz, \\
dy &= dz, \\
d\hat{\psi} &= d\Psi + \frac{b^\psi}{z - y_0} dz, \\
d\hat{\varphi} &= d\Phi + \frac{b^\varphi}{z - y_0} dz.
\end{aligned}$$

Using these coordinate transformations along with the equalities (4.11) and (4.12), we compute the metric coefficients in the new coordinate system (v, z, Ψ, x, Φ):

$$\begin{aligned}
&g_{vv} = g_{\hat{t}\hat{t}}, \quad g_{vz} = (a_0 + (z - y_0)a_1) f_t + b^i f_i, \quad g_{vi} = g_{\hat{t}i}, \\
&g_{zi} = \Big((a_0 + (z - y_0)a_1) f_i + g_{ij} b^j \Big) \frac{1}{z - y_0},
\end{aligned}$$

for $i = 1, 2$ referring to the variables Ψ and Φ, and

$$g_{zz} = \Big((a_0 + (z - y_0)a_1)^2 f_t + 2(a_0 + (z - y_0)a_1)b^i f_i + g_{ij}b^i b^j + h\Big)\frac{1}{(z - y_0)^2}.$$

This shows that the metric coefficients, to be smooth at $z = y_0$, must satisfy:

$$(z - y_0)^2 g_{zz}|_{z=y_0} = 0, \quad \frac{\partial}{\partial z}((z - y_0)^2 g_{zz})|_{z=y_0} = 0, \quad (z - y_0)g_{zi}|_{z=y_0} = 0$$

for $i = 1, 2$. Therefore we can derive the conditions that the constants a_0, a_1, b^ψ and b^φ should satisfy to yield a smooth metric at $z = y_0$. Those read:

$$\begin{aligned} a_0 f_\psi + g_{\psi i}b^i &= 0, \\ a_0 f_\varphi + g_{\varphi i}b^i &= 0, \\ h + a_0^2 f_t + 2a_0 b^i f_i + g_{ij}b^i b^j &= 0, \\ \partial_z h + a_0^2 \partial_z f_t + 2a_0 b^i \partial_z f_i + \partial_z g_{ij} b^i b^j + 2a_1 a_0 f_t + 2a_1 b^i f_i &= 0. \end{aligned}$$

Before making any attempt to solve this system, note that we can slightly simplify it:

$$\begin{aligned} a_0 f_\psi + g_{\psi i}b^i &= 0, \\ a_0 f_\varphi + g_{\varphi i}b^i &= 0, \\ h + a_0^2 f_t + a_0 b^i f_i &= 0, \\ \partial_z h + a_0^2 \partial_z f_t + 2a_0 b^i \partial_z f_i + \partial_z g_{ij} b^i b^j - 2\frac{a_1}{a_0}h &= 0. \end{aligned}$$

Then, we start by looking for solutions in (a_0, b^ψ, b^φ) of the first three equations of the system (4.13) above, and we begin with the special case $x = 0$. The calculations are then tractable using Mathematica, and we obtain the following two triplets of solutions:

$$(a_0, b^\psi, b^\varphi) \in \left\{\left(\frac{4k}{(1-\sqrt{\nu})^2}, 0, -1\right), \left(-\frac{4k}{(1-\sqrt{\nu})^2}, 0, 1\right)\right\}.$$

Next, we insert these values in the left-hand side of the three first equalities in (4.13) (this time for any value of x), and we still obtain zero at the corresponding right-hand sides. Moreover, the value of a_1 is imposed by the fourth equation and the values of a_0, b^ψ and b^φ. Finally, we obtain two solutions:

$$(a_0, b^\psi, b^\varphi, a_1) \in \left\{ \left(\frac{4k}{(1-\sqrt{\nu})^2}, 0, -1, -\frac{4k\sqrt{\nu}(1+\nu)}{(1-\sqrt{\nu})^2(1-\nu)} \right), \left(-\frac{4k}{(1-\sqrt{\nu})^2}, 0, 1, \frac{4k\sqrt{\nu}(1+\nu)}{(1-\sqrt{\nu})^2(1-\nu)} \right) \right\}.$$

It is important to note that these are *not* functions of x but constants, as desired. We may ask whether other solutions exist for this system. But the determinant of the linear system of the first two equations in the variables b^ψ and b^φ, at fixed a_0, is

$$\det\left((g_{ij})_{1\le i,j\le 2}\right)|_{y=y_0} = \frac{32k^4\nu(1+\sqrt{\nu})(1-x^2)}{(1-\sqrt{\nu})^3(1+\nu+4x\sqrt{\nu}+x^2(1+\nu))},$$

and this last expression is always positive for any allowed values of ν and x, except at the axis $x = \pm 1$ where it vanishes. Hence, one can obtain each of b^ψ and b^φ in terms of a_0 from the two first equations. Next, the third equation, when replacing the b^i's, becomes a quadratic equation in a_0, hence admitting no more than two solutions. Finally, a_1 is uniquely determined by the fourth equation from a_0 and the b^i's.

We conclude that, when performing a transformation of the coordinates the way described above, these values of the parameters a_0, a_1, b^ψ and b^φ give two coordinate systems in which the metric is smooth at the degenerate horizon $y = y_0$, and we can therefore locally analytically extend the PS metric across the horizon in the degenerate case. One choice corresponds to an extension through the future event horizon, the other through the past event horizon. Note also that the values of the parameters a_0, a_1, b^ψ and b^φ are all well-defined away from $\nu = 1$ and hence the computations of this subsection remain valid for any member of the PS family with a degenerate horizon and $\nu \neq 1$.

5 Some local and global properties

5.1 Causal stability of the domain of outer communications

There is a well-developed theory of black hole uniqueness [5, 13, 14] which requires various global regularity conditions. In particular, the domain of outer communications should be globally hyperbolic, and the orbits of the group generated by the periodic Killing vectors should be spacelike or trivial. We do not know whether global hyperbolicity holds for the solutions at hand, and we note that the proof of global hyperbolicity for Emparan–Reall metrics [4] required a considerable amount of work, including a detailed understanding of causal geodesics. In this section we discuss shortly the causality properties of the solutions.

We start by noting that the causal character of the orbits of the Killing vectors ∂_φ and ∂_ψ is closely related to the question of causal properties of the solution: should a linear combination of those vectors become null, one would immediately obtain violation of strong causality, or even causality. To analyze this, one needs to know whether the determinant of the two by two matrix obtained by taking the scalar products of the periodic Killing vectors has a sign on the region of interest.

In fact, this problem turns out to be essentially equivalent to the question of stable causality of the d.o.c. Indeed, from the form of the metric together with (A.8) one finds

$$\begin{aligned} g(\nabla t, \nabla t) = g^{tt} &= \frac{g_{xx}g_{yy}}{\det g_{\mu\nu}} \det \begin{pmatrix} g_{\psi\psi} & g_{\psi\varphi} \\ g_{\psi\varphi} & g_{\varphi\varphi} \end{pmatrix} \\ &= \frac{(\nu-1)^2(x-y)^4}{4k^4 G(x)G(y)} \det \begin{pmatrix} g_{\psi\psi} & g_{\psi\varphi} \\ g_{\psi\varphi} & g_{\varphi\varphi} \end{pmatrix} \\ &= \frac{(1+y)(1-x^2)\Theta(x,y,\lambda,\nu)}{(1-\lambda+\nu)H(x,y)G(x)G(y)}, \end{aligned} \tag{5.1}$$

where Θ is the *polynomial* defined in (A.13). Recall that $G(y)$ is negative for $y \in (y_{\rm h}, -1)$ while $G(x)$ is positive for $x \in (-1, 1)$, and note that the zeros of G at $x = \pm 1$ and $y = -1$ are canceled by factors in the numerator. We conclude that t will be a time function on the region $y > y_{\rm h}$, and thus the d.o.c. will be stably causal, if the polynomial Θ is *strictly negative*. This property of Θ has been established in [7].

The second line of (5.1) shows that the principal orbits of the action of the group generated by any linear combination of ∂_φ and ∂_ψ are spacelike within the d.o.c., as desired. On the other hand, there are always causality violations near $y = 1$, as follows from the explicit computation

$$\det \begin{pmatrix} g_{\psi\psi} & g_{\psi\varphi} \\ g_{\psi\varphi} & g_{\varphi\varphi} \end{pmatrix}\Bigg|_{y=1} = \frac{32k^4\lambda^2(1+\lambda+\nu)G(x)}{(\lambda-\nu-1)(1-x)^2H(x,1)}. \tag{5.2}$$

The function $H(x, 1)$ is positive for admissible λ, ν (see Remark 5.6, p. 1834) and hence this determinant is again negative for admissible values of the parameters. In our extensions below the set $y = 1$ is part of the extended manifold, and so there are always causality violations behind y_c.

Similarly, the two-by-two determinant from the first line of (5.1) is *strictly negative* at points arbitrarily close to $x = 0$, $Y = 0$, when Y is negative, where Y is the coordinate of Section 6. This can be seen for instance by computing the explicit value of the two-by-two determinant. For $x = 0$ and for small Y we find

$$\det \begin{pmatrix} g_{\psi\psi} & g_{\psi\varphi} \\ g_{\psi\varphi} & g_{\varphi\varphi} \end{pmatrix}\Bigg|_{x=0} = -\frac{8k^4\lambda\nu(\lambda+1+\nu)}{Y(\lambda-\nu-1)(\nu-1)^2} + o(Y^{-1}) \tag{5.3}$$

The first, dominant, term of the right-hand side is manifestly negative for admissible λ and ν when $Y < 0$. We note the formula

$$Y^2 H(0, -1/Y) = Y\big(Y(\lambda^2 - \nu^2 + 1) - 2\lambda\nu\big). \tag{5.4}$$

When $Y < 0$ this is a manifestly positive quantity for admissible λ, ν because then $\lambda\nu > 0$ and $\lambda^2 - \nu^2 + 1 = \lambda^2 + (1-\nu)(1+\nu) > 0$. This, together with what is said elsewhere in this paper, shows that the associated causality violations occur in our candidate maximal extensions of the metric.

We conclude that causality violations are a typical feature of the solutions in the region $y < y_c$; this is illustrated in Figures 1.3 and 1.4.

5.2 No struts

In this section, we verify the regularity of the metric at the rotation axes.

5.2.1 ψ axis: $y = -1$

First of all, note that

$$g_{\psi\psi}|_{y=-1} = 0.$$

The ψ-y part of the metric can be cast in the form

$$ds^2 = -\frac{2k^2 H(x,y)(y+1)}{(\nu-1)^2 G(y)(x-y)^2} \times \left(\frac{(\nu-1)^2 G(y)(x-y)^2 \left(\frac{F(x,y)}{H(y,x)} + \frac{H(y,x)M(x,y)^2}{H(x,y)}\right)}{2k^2(y+1)H(x,y)} d\psi^2 + \frac{dy^2}{y+1} \right).$$

The conformal factor is regular, bounded away from zero, provided that $x \neq y$ and $y_{\rm h} < y \neq y_{\rm c}$. On the other hand,

$$\lim_{y\to -1} \frac{(\nu-1)^2 G(y)(x-y)^2 \left(\frac{F(x,y)}{H(y,x)} + \frac{H(y,x)M(x,y)^2}{H(x,y)}\right)}{2k^2(y+1)^2 H(x,y)} = 4.$$

This shows that the usual quadratic change of variables, $y+1=\rho^2$, leads to a smooth metric near a rotation axis $\rho=0$ provided that ψ is a 2π-periodic angular coordinate.

5.2.2 φ axis: $x = \pm 1$

Here we are interested in the behaviour of the metric near $x=\pm 1$, where again

$$g_{\varphi\varphi}|_{x=\pm 1} = 0.$$

Similarly to the analysis in Section 5.2.1, we write

$$ds^2 = \frac{2k^2(x\pm 1)H(x,y)}{(\nu-1)^2 G(x)(x-y)^2} \times \left(\frac{dx^2}{x\pm 1} - \frac{(\nu-1)^2 G(x)(x-y)^2 \left(\frac{H(y,x)P(x,y)^2}{H(x,y)} - \frac{F(y,x)}{H(y,x)}\right)}{2k^2(x\pm 1)H(x,y)} d\varphi^2 \right).$$

One finds again a well-behaved conformal factor on Ω_0 away from $\{x=y\}$, and

$$\lim_{x\to\pm 1} \frac{(\nu-1)^2 G(x)(x-y)^2 \left(\frac{H(y,x)P(x,y)^2}{H(x,y)} - \frac{F(y,x)}{H(y,x)}\right)}{2k^2(x\pm 1)^2 H(x,y)} = 4.$$

Imposing 2π-periodicity on φ, we conclude that, as long as one stays away from the set $y\in[-1,1]$, the coordinates (x,φ) are coordinates on two-spheres.

5.2.3 $\hat{\psi}$ axis: $y = 1$

The Killing vector

$$\hat{\xi} := \frac{\partial}{\partial t} + \underbrace{\frac{\sqrt{(1+\nu)^2-\lambda^2}}{4k\lambda}}_{=:\alpha} \frac{\partial}{\partial \psi}. \tag{5.5}$$

is spacelike near $\{y=1\}$, and lies in the kernel of g at $y=1$, in the sense that

$$\lim_{y\to 1} g(\hat{\xi},\cdot) = 0. \tag{5.6}$$

If we use a new coordinate system $(\hat{t},\hat{x},\hat{y},\hat{\psi},\hat{\varphi})$, where

$$\hat{t}=t,\quad \hat{x}=x,\quad \hat{y}=y,\quad \hat{\psi}=\psi-\alpha t,\quad \hat{\varphi}=\varphi,$$

then (5.6) implies existence of functions $f_{\hat{\mu}}$, smooth near $y = 1$, such that

$$g_{\hat{t}\hat{\mu}} = (y-1)f_{\hat{\mu}},$$

in particular $g_{\hat{t}\hat{t}}$ vanishes at $y = 1$. As in the last two sections, a conical singularity at $y = 1$ will be avoided if and only if

$$\frac{(1+\nu)^2 - \lambda^2}{4k^2\lambda^2} = \lim_{y\to 1} \frac{g_{\hat{t}\hat{t}}}{g_{yy}(y-1)^2} = 4,$$

the first equality above resulting from the calculation of the limit. So, there will be a conical singularity unless k is chosen to be equal to

$$k = \frac{\sqrt{(1+\nu)^2 - \lambda^2}}{4\lambda}. \tag{5.7}$$

It will become clear in Section 6 that the axis $y = 1$ lies beyond event horizons, in a region where both causality violations and naked singularities are present anyway, and therefore there does not seem to be any significant reason for imposing (5.7).

5.3 Asymptotics of the Pomeransky and Senkov solution

In this section we verify asymptotic flatness. To that end we need to write down the line element in a suitable coordinate system. In [11] a coordinate system leading to manifest asymptotic flatness was proposed, related to the *ring coordinates* x, y as follows

$$r_1 := L\frac{\sqrt{1-x^2}}{x-y}, \quad r_2 := L\frac{\sqrt{y^2-1}}{x-y}, \tag{5.8}$$

where L is a nonzero real constant. If L is positive these relations establish a diffeomorphism between the open region of $\mathbb{R}^2$ defined by the conditions $-1 < x < 1$, $-\infty < y < -1$, and the open positive quadrant of $\mathbb{R}^2$ defined as $0 < r_1 < \infty$, $0 < r_2 < \infty$. Indeed the inverse of (5.8) is

$$x = \frac{L^2 - (r_1^2 + r_2^2)}{\Sigma}, \quad y = -\frac{L^2 + r_1^2 + r_2^2}{\Sigma}, \tag{5.9}$$

where

$$\Sigma := \sqrt{L^4 + 2L^2(r_1^2 - r_2^2) + (r_1^2 + r_2^2)^2}.$$

Similarly, for L negative one obtains a diffeomorphism with the region $-1 < x < 1$ and $y > 1$ by changing both signs above

$$x = -\frac{L^2 - (r_1^2 + r_2^2)}{\Sigma}, \quad y = \frac{L^2 + r_1^2 + r_2^2}{\Sigma}, \tag{5.10}$$

with the same function Σ.

Equation (5.9) adopts a simpler form if we make the transformation

$$r_1 = r \sin\theta, \quad r_2 = r \cos\theta,$$

where $0 < r < \infty$ and $0 < \theta < \pi/2$. In this case we have

$$x = \frac{L^2 - r^2}{\Sigma}, \quad y = -\frac{L^2 + r^2}{\Sigma}, \quad \Sigma = \sqrt{L^4 - 2L^2 r^2 \cos 2\theta + r^4}. \tag{5.11}$$

The Jacobian of this transformation is

$$-\frac{8L^4 r^3 \sin(2\theta)}{\Sigma^4},$$

which vanishes at the axes $\theta = 0, \pi/2$, and therefore some care is required there.

We perform the coordinate change (5.11) in (1.1) and study the resulting expression for large values of r. To understand the asymptotic behavior of the metric it is convenient to choose L as

$$L := \sqrt{\frac{2k^2(1-\lambda+\nu)}{1-\nu}}, \quad \text{or} \quad L := -\sqrt{\frac{2k^2(1+\lambda+\nu)}{1-\nu}},$$

and this choice will be made in what follows. Choosing the positive value (which corresponds to points near $(x = -1, y = -1)$) one then obtains

$$g_{tt} = -1 + \frac{8k^2\lambda}{(1-\lambda+\nu)r^2} + O\left(r^{-4}\right),$$
$$g_{rr} = 1 - \frac{4k^2\lambda\big((\lambda - 4\nu + \lambda\nu)\cos(2\theta) - (-1+\nu)^2\big)}{(1-\lambda+\nu)(-1+\nu)^2 r^2} + O\left(r^{-4}\right),$$

$$
\begin{aligned}
g_{\theta\theta} &= r^2 \left(1 - \frac{4k^2 \cos(2\theta) \left(-3\lambda\nu^2 + 2((\lambda-1)\lambda - 1)\nu + \lambda + 2\nu^3\right)}{(\nu-1)^2(-\lambda+\nu+1)r^2} \right. \\
&\quad \left. + \frac{4k^2\lambda}{(-\lambda+\nu+1)r^2} \right) + O\left(r^{-2}\right), \\
g_{\varphi\varphi} &= r^2 \sin^2\theta \left(1 + \frac{2k^2((\nu-1)\cos(2\theta)(\lambda-2\nu) + 3\lambda\nu + \lambda - 2(\nu-1)\nu)}{r^2(\nu-1)^2} \right. \\
&\quad \left. + O\left(r^{-4}\right) \right), \\
g_{t\varphi} &= \frac{\sin^2\theta}{r^2} \left(\frac{16k^3\lambda\sqrt{\nu}\sqrt{(\nu+1)^2 - \lambda^2}}{(\nu-1)^2(\lambda-\nu-1)} + O\left(r^{-2}\right) \right), \\
g_{t\psi} &= \frac{\cos^2\theta}{r^2} \left(- \frac{8k^3\lambda\left(\lambda\nu + \lambda + \nu^2 - 6\nu + 1\right)\sqrt{(\nu+1)^2 - \lambda^2}}{(\nu-1)^2(-\lambda+\nu+1)^2} + O\left(r^{-2}\right) \right), \\
g_{\varphi\psi} &= \frac{\sin^2\theta\ \cos^2\theta}{r^2} \left(- \frac{32k^4\lambda\sqrt{\nu}\left(\lambda^2(\nu+1) - 4\lambda\nu + (\nu-1)^2(\nu+1)\right)}{(\nu-1)^4(\lambda-\nu-1)} \right. \\
&\quad \left. + O\left(r^{-2}\right) \right), \\
g_{\psi\psi} &= r^2 \cos^2\theta \left(1 + \frac{2k^2\left(\lambda^2(3\nu+1) - \lambda(\nu(\nu+10) - 3) + 2\nu\left(\nu^2 - 1\right)\right)}{r^2(\nu-1)^2(-\lambda+\nu+1)} \right. \\
&\quad \left. + \frac{2k^2\cos(2\theta)(\lambda-2\nu)}{r^2(\nu-1)} + O\left(r^{-4}\right) \right). \qquad (5.12)
\end{aligned}
$$

The leading powers of r in the diagonal terms in (5.12) correspond to the metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\psi^2), \qquad (5.13)$$

which is just the 5-dimensional Minkowski space-time, as one checks by making the coordinate transformation $x^0 = t$ and

$$
\begin{aligned}
x^1 &= r\cos\theta\cos\psi, \quad x^2 = r\cos\theta\sin\psi, \\
x^3 &= r\sin\theta\sin\varphi, \quad x^4 = r\sin\theta\cos\varphi, \qquad (5.14)
\end{aligned}
$$

which leads to

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2.$$

To avoid ambiguities, we will write $\bar{g}_{x^\mu x^\nu}$ for the components of the metric tensor in the manifestly asymptotically flat coordinates (5.14). One can check that the angular prefactors in $g_{\psi\psi}$, etc., have the right structure so that in the coordinates above we have

$$\bar{g}_{x^\mu x^\nu} - \eta_{\mu\nu} = O(r^{-2}), \quad \partial_{x^\sigma}\bar{g}_{x^\mu x^\nu} = O(r^{-3}). \tag{5.15}$$

For instance, let us define the functions $h_{\mu\nu}(r,\theta)$ by the equations

$$\begin{aligned}
g_{tt} &= -1 + \frac{h_{tt}}{r^2}, \quad g_{rr} = 1 + \frac{h_{rr}}{r^2}, \quad g_{\psi\psi} = \left(1 + \frac{h_{\psi\psi}}{r^2}\right) r^2 \cos^2\theta,\\
g_{t\phi} &= \frac{h_{t\phi}\sin^2\theta}{r^2}, \quad g_{t\psi} = \frac{h_{t\psi}\cos^2\theta}{r^2}, \quad g_{\varphi\psi} = \frac{h_{\varphi\psi}\sin^2\theta\,\cos^2\theta}{r^2},\\
g_{\theta\theta} &= r^2\left(1 + \frac{h_{\theta\theta}}{r^2}\right), \quad g_{\varphi\varphi} = r^2\sin^2\theta\left(1 + \frac{h_{\varphi\varphi}}{r^2}\right),
\end{aligned}$$

The prefactors have been chosen so that the functions $h_{\mu\nu}$ are rational functions of x and y, smooth near $\{x = y = -1\}$. One finds

$$\begin{aligned}
&\bar{g}_{x^1x^1}\\
&= 1 + \frac{(x^2)^2 r^2\, h_{\psi\psi} + (x^1)^2\big((x^1)^2 + (x^2)^2\big)\, h_{rr} + (x^1)^2\big((x^3)^2 + (x^4)^2\big)\, h_{\theta\theta}}{r^4\left((x^1)^2 + (x^2)^2\right)}\\
&\qquad = 1 + \frac{\sin^2\psi\, h_{\psi\psi} + \cos^2\theta\cos^2\psi\, h_{rr} + \sin^2\theta\cos^2\psi\, h_{\theta\theta}}{r^2},\\
&\bar{g}_{x^1x^2} = \frac{x^1x^2\Big(\big((x^1)^2 + (x^2)^2\big)\, h_{rr} + \big((x^3)^2 + (x^4)^2\big)h_{\theta\theta} - r^2\, h_{\psi\psi}\Big)}{r^4\left((x^1)^2 + (x^2)^2\right)}\\
&\qquad = \frac{\cos\psi\,\sin\psi\,\big(\cos^2\theta\, h_{rr} + \sin^2\theta\, h_{\theta\theta} - h_{\psi\psi}\big)}{r^2}.
\end{aligned}$$

Continuity of $\bar{g}_{x^1x^1}$ and $\bar{g}_{x^1x^2}$ at the rotation axis $\theta = \pi/2$ requires

$$h_{\theta\theta}(r,\pi/2) = h_{\psi\psi}(r,\pi/2) \iff g_{\theta\theta}(r,\pi/2) = \lim_{\theta\to\pi/2}\frac{g_{\psi\psi}(r,\theta)}{\cos^2\theta}, \tag{5.16}$$

which can be checked by direct calculations. To obtain differentiability one writes

$$\begin{aligned}
\bar{g}_{x^1x^1} = 1 &+ \frac{(x^2)^2\, h_{\psi\psi} + (x^1)^2 h_{rr} + \big((x^3)^2 + (x^4)^2\big)h_{\theta\theta}}{r^4}\\
&+ \frac{(x^2)^2\big((x^3)^2 + (x^4)^2\big)\big(h_{\psi\psi} - h_{\theta\theta}\big)}{r^4\left((x^1)^2 + (x^2)^2\right)}.
\end{aligned} \tag{5.17}$$

A Mathematica calculation shows that

$$h_{\psi\psi} - h_{\theta\theta} = \frac{1+y}{x-y} W(x, y, \nu, \lambda),$$

where W is a rational function of its arguments, smooth near $\{x = y = -1\}$, which is precisely what is needed to cancel the factor $(x^1)^2 + (x^2)^2$ in the denominator of the second line of (5.17), and make this term uniformly well behaved as claimed in (5.15); here it is useful to observe that

$$\frac{\partial x}{\partial x^i} = O(x^i r^{-2}), \quad \frac{\partial y}{\partial x^i} = O(x^i r^{-2}).$$

The equality (5.16) similarly guarantees uniform derivative estimates for $\bar{g}_{x^1x^2}$ and $\bar{g}_{x^2x^2}$ at the rotation axis $\theta = \pi/2$.

We continue with

$$\begin{aligned} &\bar{g}_{x^3x^3} \\ &= 1 + \frac{h_{rr}(x^3)^2\left((x^3)^2 + (x^4)^2\right) + h_{\theta\theta}(x^3)^2\left((x^1)^2 + (x^2)^2\right) + h_{\varphi\varphi}(x^4)^2 r^2}{r^4\left((x^3)^2 + (x^4)^2\right)} \\ &= 1 + \frac{h_{rr}(x^3)^2 + h_{\theta\theta}(r^2 - (x^3)^2)}{r^4} + \frac{(h_{\varphi\varphi} - h_{\theta\theta})(x^4)^2}{r^4\left((x^3)^2 + (x^4)^2\right)}. \end{aligned}$$

For continuity of $\bar{g}_{x^3x^3}$ one thus obtains the condition

$$h_{\varphi\varphi}(r,0) = h_{\theta\theta}(r,0) \Longleftrightarrow g_{\theta\theta}(r,0) = \lim_{\theta\to 0} \frac{g_{\varphi\varphi}(r,\theta)}{\sin^2\theta}, \tag{5.18}$$

while uniform differentiability is equivalent to uniform differentiability of

$$\frac{(x^4)^2(h_{\varphi\varphi} - h_{\theta\theta})}{r^4\big((x^3)^2 + (x^4)^2\big)}.$$

This, in turn, requires a factorization of $h_{\varphi\varphi} - h_{\theta\theta}$ by $(x^3)^2 + (x^4)^2$. The required regularity ensues from the identity

$$h_{\varphi\varphi} - h_{\theta\theta} = \frac{(1-x^2)}{x-y} \hat{W}(x, y),$$

where $\hat{W}$ is a rational function regular near $\{x = y = -1\}$. The same formula takes care of the regularity of $\bar{g}_{x^3x^4}$ and $\bar{g}_{x^4x^4}$.

The remaining components of the metric are manifestly asymptotically flat, e.g.,

$$\begin{aligned} \bar{g}_{x^0x^0} &= -1 + \frac{h_{tt}}{r^2}, \\ \bar{g}_{x^0x^1} &= -\frac{x^2\, h_{t\psi}}{r^4} = -\frac{\cos\theta\, \sin\psi\, h_{t\psi}}{r^3}, \\ \bar{g}_{x^1x^3} &= \frac{(h_{rr} - h_{\theta\theta})r^2x^1x^3 - h_{\varphi\psi}x^2x^4}{r^6}, \end{aligned} \tag{5.19}$$

with similar expressions for those non-zero $\bar{g}_{x^\mu x^\nu}$'s that have not been listed so far.

Uniform decay estimates on higher order derivatives follow from (5.15) using elliptic estimates applied to the stationary Einstein equations, which establishes asymptotic flatness of the solutions.

It is well known that the ADM mass m of a stationary solution equals its Komar mass, independently of dimension. One can therefore read the ADM mass from the $1/r^2$ term in g_{tt} and so, perhaps up to normalization-dependent factors, the mass is

$$\frac{4k^2\lambda}{(1-\lambda+\nu)}.$$

For positive λ, positivity of the total mass is then equivalent to

$$\lambda < 1 + \nu. \tag{5.20}$$

This proves that, for λ's that do not satisfy that constraint, the domain of outer communications associated to this asymptotically flat end contains naked singularities, in the sense that the hypotheses of the positive energy theorem with horizon boundaries are violated. However, this does not necessarily prove that the solutions are nakedly singular for all domains of outer communications for λ's that do not satisfy (5.20), as the asymptotically flat end obtained as above near $\{x = y = -1\}$ could be shielded from other such ends by a horizon. Equivalently, to show that (5.20) is necessary for regularity, one would need to locate all asymptotically flat regions of all maximal extensions of the metric (1.1), and analyse the associated domains of outer communication.

5.3.1 $(\boldsymbol{x = 1, y = 1})$

Near $(x = 1, y = 1)$, the asymptotics of those components of the metric which do not carry a ψ index can be obtained by replacing λ by $-\lambda$ in (5.12).

This is due to the fact that the transformation

$$(x, y, \lambda, \nu) \mapsto (-x, -y, -\lambda, \nu)$$

maps all the metric functions into themselves, except for M, and this last function only affects $g(\partial_\psi, \cdot)$. Those last components of the metric read

$$\begin{aligned} g_{\psi\psi} &= r^2\cos^2\theta + \frac{k^2}{2(\nu-1)^2\left((\nu+1)^2-\lambda^2\right)} \\ &\times \Big((\lambda-\nu-1)\Big((\nu-1)\cos(4\theta)(\lambda+\nu+1)(\lambda+2\nu) \\ &- 4\lambda\cos(2\theta)\left((\lambda+6)\nu+\lambda-\nu^2-1\right)\Big) \\ &+ \lambda^3(-(5\nu+3)) - 2\lambda^2(\nu-1)(13\nu-12) \\ &- \lambda(\nu+1)(3(\nu-8)\nu+5) + 2(\nu-1)\nu(\nu+1)^2\Big) + O\left(r^{-1}\right), \quad (5.21) \\ g_{t\psi} &= \frac{4k\lambda}{\sqrt{(\nu+1)^2-\lambda^2}} + O\left(\cos^2\theta r^{-1}\right), \quad (5.22) \\ g_{\psi\phi} &= -\frac{16k^4\lambda\sqrt{\nu}\sin^2(\theta)}{r^2(\nu-1)^4(\lambda+\nu+1)} \\ &\times \Bigg(\cos(2\theta)\left(\lambda^2(\nu+1)+4\lambda\nu+(\nu-1)^2(\nu+1)\right) \\ &+ \lambda^2(\nu+1)+4\lambda\left(\nu^2-\nu+1\right)+(\nu-1)^2(\nu+1)\Bigg) \\ &+ O\left(\cos^2\theta\sin^2\theta r^{-3}\right). \quad (5.23) \end{aligned}$$

5.4 The singular set $\{H(x, y) = 0\}$

Throughout this section we restrict attention to admissible pairs (ν, λ).

In order to understand the geometry of the singular set

$$\text{Sing} := \{H(x, y) = 0\},$$

we proceed as follows, keeping in mind the analysis of Section 3, which concerned the region $\{x \in [-1, 1],\ y_c < y < -1\}$: The equation $H(x, y) = 0$

can also be solved as

$$y_\pm(x) := \frac{\lambda\nu(1-x^2) \pm \sqrt{\tilde W(x)}}{x\nu\left(2\lambda\nu + x\left(\lambda^2+\nu^2-1\right)\right)}, \tag{5.24}$$

where

$$\tilde W := \nu\left(\left(x^2-1\right)^2\nu\lambda^2 + x\left(\lambda^2+2x\lambda-\nu^2+1\right)\left(2\lambda\nu + x\left(\lambda^2+\nu^2-1\right)\right)\right), \tag{5.25}$$

provided the denominator $x\nu\left(2\lambda\nu + x\left(\lambda^2+\nu^2-1\right)\right)$ of (5.24) is non-zero. So, if this condition holds, at each $x \in \mathbb{R}$ there are either two real values (counting multiplicity) of y for which a solution exists, or none. Each branch $y_\pm$ is a smooth function of x near any given point x_0 if and only if the other one is, except at the zeros of denominator.

We are interested in the topology of Sing in the region $x \in [-1, 1]$, and Section 3 tells us that the graphs $y_\pm$ do *not* meet the region $\{y_{\rm c} \le y < -1\}$ there. For reasons that will become clear in Section 6 we need to understand both the positive and negative branches of $y_\pm$.

The denominator in (5.24) vanishes at $x = 0$, and at $x = x_*$ if $\nu^2 + \lambda^2 \neq 1$, where

$$x_* := -\frac{2\lambda\nu}{\lambda^2+\nu^2-1}. \tag{5.26}$$

The pole x_* belongs to $(0, 1)$ for allowed ν, λ such that $\nu + \lambda < 1$; we have $x_* > 1$ for $\nu + \lambda > 1$ and $\nu^2 + \lambda^2 < 1$, with $x_* \to \infty$ when (ν, λ) approaches from inside the unit circle centered at the origin of the (ν, λ)-plane; and finally $x_* < -1$ for allowed ν, λ such that $\nu^2 + \lambda^2 > 1$, see figure 5.1. At each zero of the denominator in (5.24) the graphs of $y_\pm$ split into two components, except if the numerator vanishes there as well. Given that there are at most two zeros, we conclude that Sing can have up to five connected components.

The identity, in obvious notation,

$$y_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2c}{-b \mp \sqrt{b^2-4ac}}$$

shows that the numerator of one of the $y_\pm$'s necessarily vanishes at a zero of the denominator. At $x = 0$ the expression under the square root equals $\lambda^2\nu^2 > 0$, so there are always precisely two real valued solutions for $x \neq 0$ small. Near $x = 0$ we have

$$y_- = -\frac{(2x\lambda + \lambda^2 - \nu^2 + 1)}{\lambda\nu(1-x^2) + \sqrt{\tilde W}} = -\frac{\lambda^2-\nu^2+1}{2\lambda\nu} + O(x), \quad y_+ = \frac{1}{x\nu} + O(1),$$

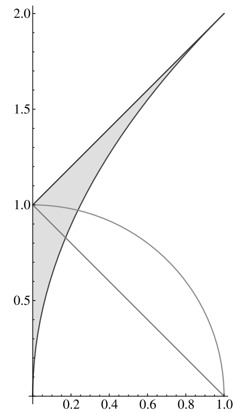

Figure 5.1: The regions of distinct behavior of x_*; the set of admissible (ν, λ) is the shaded region.

with y_- smaller than y_c for $|x|$ small and for admissible ν and λ; this follows from the analysis in Section 3 in any case, but can also be seen by a direct calculation. So, near $x = 0$, the graph of y_+ splits into two branches, tending to minus infinity at the left, and plus infinity at the right, while the graph of y_- continues smoothly across $x = 0$.

In particular the singular set Sing is never empty.

In the case $\nu^2 + \lambda^2 = 1$, the denominator of (5.24) vanishes only at $x = 0$, with the behavior just described being independent of those particular values of the parameters.

Assuming $\nu^2 + \lambda^2 \neq 1$, we have

$$\tilde{W}(x_*) = \left(\frac{\lambda\nu(\lambda - \nu - 1)(\lambda - \nu + 1)(\lambda + \nu - 1)(\lambda + \nu + 1)}{(\lambda^2 + \nu^2 - 1)^2} \right)^2 ,$$

which is positive for all allowed ν and λ, except when $\nu + \lambda = 1$, where $\tilde{W}(x_*)$ vanishes. It follows that for $\nu + \lambda \neq 1$, near x_* the branches behave as

$$y_+ = \frac{(1 - \nu - \lambda)(1 + \nu + \lambda)(1 + \lambda - \nu)(1 + \nu - \lambda)}{\nu(x - x_*)(1 - \nu^2 - \lambda^2)^2} + O(1) \ , \ y_- = O(1)$$

so that y_- extends smoothly across $x = x_*$, whereas y_+ blows up. The case $\nu + \lambda = 1$ requires separate attention, and will be analyzed shortly.

Next, in order to understand the behavior of the components of the set Sing, it is useful to study the two branches $y_\pm$ for all $x \in \mathbb{R}$. Now, the top order term of the polynomial $\tilde{W}(x)$ is $\nu^2\lambda^2x^4$, and so $\tilde{W}$ is strictly positive for large positive or negative x. This implies that at fixed admissible ν and λ the sets

$$\tilde{\Omega}_{\nu,\lambda} := \{x : \tilde{W}(x) \geq 0\} \tag{5.27}$$

have one, two, or three connected components. Moreover, each branch $y_\pm$, which exists for $|x|$ large enough, has a finite limit at infinity provided that $\nu^2 + \lambda^2 \neq 1$:

$$y_+ \to 0, \quad y_- \to \frac{2\lambda}{1-\nu^2-\lambda^2},$$

when $x \to \pm\infty$. In the case $\nu^2 + \lambda^2 = 1$, we obtain instead

$$y_+ \to 0\ ,\ y_- = -\frac{x}{\nu} + O(1) \to \mp\infty$$

when $x \to \pm\infty$.

We now turn our attention to the roots of $\tilde{W}$: at those, both branches $y_\pm$ meet at that side of the root where $\tilde{W}$ is positive, and stop existing nearby on the side where $\tilde{W}$ is negative, which happens if the order of the root is odd. Such points will be referred to as *turning points*. Note that there are at most four such turning points.

Recall that we have $\tilde{W}(0) = \lambda^2\nu^2 > 0$. Further

$$\tilde{W}(1) = \nu(\lambda - \nu + 1)(\lambda + \nu - 1)(\lambda + \nu + 1)^2,$$

so the sign of $\tilde{W}$ at one is determined by the sign of $\lambda + \nu - 1$. Next,

$$\begin{aligned}\tilde{W}(-1) &= 4\nu(\lambda - \nu - 1)^2 \left(\lambda + (1-\nu)\right)\left(\lambda - (1-\nu)\right)\\ &= 4\nu(\lambda - \nu - 1)^2 \left(\lambda^2 - (1-\nu)^2\right),\end{aligned}$$

so the sign at minus one is the same as that at plus one, both vanishing for admissible values of parameters if and only if $\lambda = 1 - \nu$:

$$\tilde{W}(-1)\tilde{W}(1) = \nu^2(\lambda - \nu - 1)^2(\lambda - \nu + 1)^2(\lambda + \nu - 1)^2(\lambda + \nu + 1)^2 \geq 0.$$

When $\lambda + \nu \neq 1$, the equations $H(\pm 1, y) = 0$ have the following four solutions:

$$x = 1: \quad y_{\uparrow,\pm} = \pm\sqrt{\frac{1+\lambda-\nu}{\nu(\lambda+\nu-1)}}; \quad x = -1: \quad y_{\downarrow,\pm} = \pm\sqrt{\frac{\lambda+\nu-1}{\nu(1+\lambda-\nu)}}.$$

The function under the square root in $y_{\uparrow,\pm}$ is positive for admissible (λ, ν) if and only if $\lambda + \nu > 1$, and then it is larger than one. On the other hand, the function under the square root in $y_{\downarrow,\pm}$ is always smaller than 1 for the parameters of interest, non-negative if and only if

$$\lambda > 1 - \nu.$$

We continue with a lemma about the number of roots of $\tilde{W}$:

Lemma 5.1. 1. *There exists a smooth curve γ, separating the set $\mathcal{U}$ of admissible $(\nu,\ \lambda)$ into two components, which is a graph of a function $\chi : [0, \nu_*] \to [0, 1]$, satisfying*

$$1 - \nu \le \chi \le 1 - \nu^2, \quad \chi(\nu_*) = 2\sqrt{\nu_*},$$

such that $\tilde{W}$ has a multiple root, for admissible (ν, λ), if and only if

$$(\nu, \lambda) \in \gamma \iff \lambda = \chi(\nu).$$

Moreover,

(a) *In the connected component of $\mathcal{U} \setminus \gamma$ where $\lambda < \chi(\nu)$ the polynomial $\tilde{W}$ has four distinct real roots, and at least one of them is bigger than x_*.*

(b) *In the remaining connected component the polynomial $\tilde{W}$ has two distinct real roots and two distinct roots in $\mathbb{C} \setminus \mathbb{R}$.*

2. *$\tilde{W}$ has no third-or fourth-order zeros for $(\nu, \lambda) \in \mathcal{U}$.*

Proof. 1. A necessary condition for existence of a second-order zero of $\tilde{W}$ is the existence of a joint zero for $\tilde{W}$ and its first derivative. This, in turn, is equivalent to the vanishing of the resultant of the polynomials $x \mapsto \tilde{W}$ and $x \mapsto \partial_x \tilde{W}$. This resultant is

$$-2^{18}\lambda^6(\nu-1)^2\nu^9\left(\lambda^2-(\nu+1)^2\right)^4 f(\nu,\lambda),$$

where

$$\begin{aligned} f(\nu,\lambda) := \lambda^8 &+ 2\lambda^6\left(1-4\nu+\nu^2\right) + 15\lambda^4(-1+\nu)^2\nu \\ &- 2\lambda^2(-1+\nu)^4\left(1+4\nu+\nu^2\right) - (-1+\nu)^6(1+\nu)^2. \end{aligned}$$

Since f is a polynomial of degree four in λ^2, we can define a polynomial $q(\nu, L) = f(\nu, \lambda)$ where $L = \lambda^2$, and study q. We have

$$\frac{\partial^2 q}{\partial L^2} = 6\left(2L^2 + 2(1-4\nu+\nu^2)L + 5\nu(1-\nu)^2\right),$$

which is obviously positive for $\nu \in [0, 2-\sqrt{3} \approx 0.26]$ and all L, and so q is convex there. For $\lambda \geq 2\sqrt{\nu}$ we find

$$2L^2 + 2(1-4\nu+\nu^2)L \geq 32\nu^2 + 8(1-4\nu+\nu^2)\nu = 8(1+\nu^2)\nu > 0,$$

and so q is always convex in L above the graph of $2\sqrt{\nu}$. Now

$$\begin{aligned} q(\nu,0) &= -(\nu-1)^6(\nu+1)^2 < 0, \\ \partial_L q(\nu,0) &= -2(\nu-1)^4\left(\nu^2+4\nu+1\right) < 0, \\ q(\nu,(1+\nu)^2) &= 27\nu(1-\nu)^2(1+\nu)^4 > 0\,. \end{aligned}$$

Convexity of $L \mapsto q$ implies that, for $\nu \in [0, 2-\sqrt{3}]$, the zero-level set of q is a smooth graph.

For $\nu > 1/4$ one can instead argue as follows: We have

$$\partial_L q(\nu, 4\nu) = 2\left(-\nu^2+10\nu-1\right)\left(\nu^4+10\nu^3-18\nu^2+10\nu+1\right).$$

The second factor is positive for $\nu \in [5-2\sqrt{6}, 5+2\sqrt{6}] \supset [0.102, 1]$. For the last factor, we write

$$10\nu^3 - 18\nu^2 + 10\nu \geq \nu(10\nu^2 - 20\nu + 10) = 10\nu(1-\nu)^2 > 0$$

and the positivity of $\partial_L q(\nu, 4\nu)$ follows. Convexity gives positivity of q in the admissible region.

To finish the proof of the point 1, we note first that the region

$$\mathcal{U}_1 := \mathcal{U} \cap \{\lambda < \chi(\nu)\}$$

contains the line $\lambda = 1-\nu$, where all roots are simple and real. Indeed, in this special case the function H equals

$$H(x,y)|_{\lambda=1-\nu} = -2(\nu-1)(x(y\nu-1)((x-1)y\nu-1)+y\nu+1),$$

and the zeros are

$$\begin{aligned} \nu y_\pm &= \frac{x^2-1\pm\sqrt{\tilde{W}}}{2(x-1)x}, \\ \tilde{W} &= \nu^2(1-\nu)^2(x-1)(x+1)\left(x-\sqrt{5}-2\right)\quad\left(x+\sqrt{5}-2\right), \end{aligned}$$

see figure 5.2.

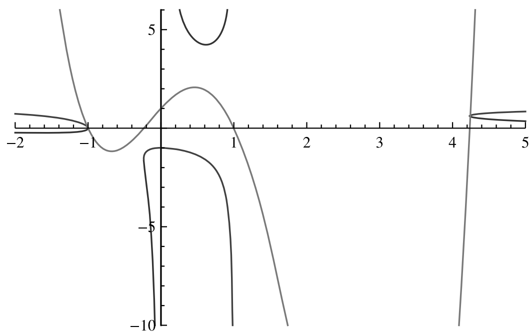

Figure 5.2: The polynomial $\tilde{W}$ (grey) and the set $\{H(x,y)=0\}$ (black) when $\lambda = 1-\nu$. The vertical axis is νy.

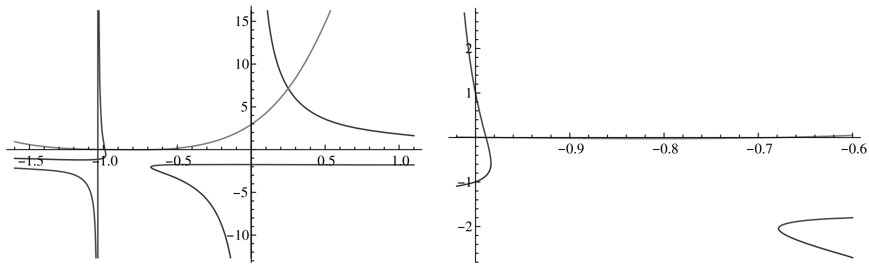

Figure 5.3: The polynomial $\tilde{W}$ (grey) and the set $\{H(x,y)=0\}$ (black) when $\nu = 9/16$ and $\lambda = 49/32$, with a zoom to the region where $\tilde{W}$ is negative.

In this case we have four simple real roots, independently of ν. Continuity implies that all roots of $\tilde{W}$ are simple and real in $\mathcal{U}_1$. Moreover, still in the case $\lambda = 1-\nu$, we have $x_* = 1$, and the biggest root of $\tilde{W}$ equals $2+\sqrt{5} > 1$. One can conclude the first point again by a continuity argument: both the function $(\nu,\lambda) \mapsto x_*$ (compare (5.26)) and the function which to (ν,λ) assigns the largest root of $\tilde{W}$ are continuous on $\mathcal{U}_1$. Since $\tilde{W}(x_*) > 0$ for $(\nu,\lambda) \in \mathcal{U}_1$, the largest root of $\tilde{W}$ cannot become smaller than x_* when moving along paths contained in $\mathcal{U}_1$.

Another continuity argument, using the fact that for $\nu = 9/16$ and $\lambda = 7/4$ the zeros of $\tilde{W}$ are, approximately,

$$-1.09746 \pm 0.541984\mathrm{i} \in \mathbb{C} \setminus \mathbb{R}, \quad -0.983642, \quad -0.678586,$$

finishes the proof (exact formulae for the roots can be given, but they are not very enlightening). The resulting $\tilde{W}$ and singular set $\{H(x,y)=0\}$ are plotted in figure 5.3.

2. A necessary condition for existence of a fourth-order zero is the existence of a joint zero for the second and third derivatives. This, in turn, is equivalent to the vanishing of the resultant of the polynomials $x \mapsto \partial_x^2 \tilde{W}$ and $x \mapsto \partial_x^3 \tilde{W}$. This resultant is

$$2^{12} 3^2 \lambda^4 \nu^4 \left(\lambda^4 (2\nu - 3) - 2\lambda^2 \left(\nu^2 - 3\right) - (2\nu + 3)\left(\nu^2 - 1\right)^2\right).$$

The roots of the last factor are $\lambda = \pm(1+\nu)$ and

$$\lambda_\pm = \pm(1-\nu)\sqrt{\frac{2\nu+3}{3-2\nu}},$$

and so λ_+ is the only value of interest. When substituted into f, we obtain

$$f(\nu, \lambda_+) = \frac{27(\nu-1)^6 \nu \left(16\nu^4 - 40\nu^2 + 17\right)}{(3-2\nu)^4},$$

which is clearly positive for $\nu \in (0, \sqrt{17/40}] \approx 0.65] \supset (0, 1/4]$, and so there are no fourth-order roots on γ.

To exclude the possibility of third-order zeros on γ, we calculate likewise the resultant of $\partial_x \tilde{W}$ and $\partial_x^2 \tilde{W}$, which is

$$2^{16} \lambda^4 \nu^6 \left(\lambda^2 - (\nu+1)^2\right)^2 \hat{f}(\nu, \lambda),$$

where

$$\begin{aligned} \hat{f}(\nu,\lambda) \quad = \quad & \lambda^8(8\nu - 9) + 4\lambda^6 \nu(4(\nu-4)\nu + 13) + 6\lambda^4 (\nu-1)^2 \left(5\nu^2 + 3\right) \\ & -4\lambda^2 (\nu-1)^4 \nu (4\nu(\nu+4) + 13) - (\nu-1)^6 (\nu+1)^2 (8\nu+9). \end{aligned}$$

A necessary condition for a third order zero is the vanishing of the resultant of the polynomials $\lambda \mapsto f$ and $\lambda \mapsto \hat{f}$. That resultant is

$$3^{24} (\nu-1)^{40} \nu^8 (\nu+1)^8 \left(256\nu^4 - 864\nu^2 + 513\right)^2,$$

with the last factor vanishing at

$$\pm\frac{1}{4}\sqrt{27 \pm 6\sqrt{6}}.$$

The only value in $(0,1)$ is $\frac{1}{4}\sqrt{27 - 6\sqrt{6}} \approx 0.88$.

Another necessary condition for a third-order zero is the vanishing of the resultant of the polynomials $\nu \mapsto f$ and $\nu \mapsto \hat{f}$. That resultant is

$$-3^{24}(\lambda-2)^2(\lambda-1)^2\lambda^{44}(\lambda+1)^2(\lambda+2)^2\left(\lambda^2+1\right)^2 \\ \times\left(65536\lambda^8-327680\lambda^6+526848\lambda^4-279296\lambda^2+9025\right),$$

with the last factor vanishing at approximately

$$\pm 1.06033, \quad \pm 1.51468, \quad \pm 0.185775, \quad \pm 1.24376,$$

the exact values of the positive solutions being

$$\frac{1}{4}\sqrt{20+3\sqrt{6}-\sqrt{3\left(39-4\sqrt{6}\right)}}, \quad \frac{1}{4}\sqrt{20-3\sqrt{6}+\sqrt{3\left(39+4\sqrt{6}\right)}},$$

$$\frac{1}{4}\sqrt{20+3\sqrt{6}+\sqrt{3\left(39-4\sqrt{6}\right)}}, \quad \frac{1}{4}\sqrt{20-3\sqrt{6}-\sqrt{3\left(39+4\sqrt{6}\right)}}.$$

One checks that f does not vanish at the above values of ν and λ, except at

$$\left(\frac{1}{4}\sqrt{27-6\sqrt{6}}, \frac{1}{4}\sqrt{20-3\sqrt{6}-\sqrt{3\left(39+4\sqrt{6}\right)}}\right) \approx (0.88, 0.18).$$

However, $0.19 \approx \lambda < 2\sqrt{\nu} \approx 1.87$ there, which is therefore not admissible. □

We wish to show, next, that all zeros of $\tilde{W}$ in $[-1,1]$ are simple. The proof of this requires understanding of the behavior of the branches $y_\pm$ in $[-1,1]$; this is the purpose of the next lemma:

Lemma 5.2. *In the region $\{-1 \le x \le 0,\ y < y_c\}$, the two branches $y_\pm$ which exist for small negative values of x meet smoothly at some $\bar{x} \in (-1,0)$, where $\bar{x}$ is a simple root of $\tilde{W}$.*

Proof. The existence of both $y_\pm$ at $x = -\varepsilon$, for small enough positive ε, comes from the facts that $\tilde{W}(0)$ is positive, and that their denominator can vanish only at $x = 0$, or $x = x_*$, and x_* lies always outside $[-1,0]$, for any $(\nu,\lambda) \in \mathcal{U}$. Moreover, the asymptotics of these branches studied above shows that they are both below the $\{y = y_c\}$-level set. One should recall from Section 3 that these branches can neither enter the region $\{-1 \le x \le 1,\ y_c \le y \le -1\}$, nor cross the $\{x = -1\}$-axis below y_c. But the functions $x \mapsto y_\pm(x)$ are continuous on each connected component of the set $\tilde{\Omega}_{\nu,\lambda} \cap (-1,0)$, since this set does not intersect the lines $x = 0$ and $x = x_*$. As a consequence, there exists $\bar{x} \in (-1, 0)$ at which $\tilde{W}$ has a change of sign, that is to say an odd-order zero of $\tilde{W}$. Since $\tilde{W}$ has degree four, the only possibilities for the order are 1 and 3. But the existence of triple-zeros has already been excluded in Lemma 5.1. □

Remark 5.3. *From the proof of the lemma above, $\bar{x}$ can be defined as the largest negative root of $\tilde{W}$, which is also the second lowest root of $\tilde{W}$. Indeed, if $\bar{x}$ were the biggest (of the four) root of $\tilde{W}$ for some $(\nu_1, \lambda_1) \in U_1$, then by connectedness of the region $\mathcal{U}_1$ defined earlier, it would the case for $\lambda = 1 - \nu$. But the the biggest root of $\tilde{W}$ in this case is positive (see the proof of Lemma* 5.1*). Moreover, since $\bar{x}(\nu, \lambda)$ is always a simple root for (ν, λ) in $\mathcal{U}$, $\bar{x}$ is a smooth function of the coefficients of $\tilde{W}(x)$, and therefore the map $(\nu, \lambda) \mapsto \bar{x}(\nu, \lambda)$ defined in $\mathcal{U}$ is continuous.*

We now have:

Proposition 5.4. *For all admissible (ν, λ), those roots of $\tilde{W}$ which belong to $[-1, 1]$ are simple.*

Proof. We start by a proof based on Mathematica plots, an alternative analytic argument will also be given. Another necessary condition for a double zero of $\tilde{W}$ is the vanishing of the resultant of the polynomials $\lambda \mapsto \tilde{W}$ and $\lambda \mapsto \partial_x \tilde{W}$. This resultant is

$$2^{20} x^4 \left(x^2 - 1\right)^4 \nu^{10} \left(\nu^2 - 1\right)^5 \left(x^8 - 4x^6 + 4x^2\nu^2 - \nu^2\right). \tag{5.28}$$

So, zeros of this resultant, with a λ which is a zero of f, provide the only candidates for solutions of the two equations $\tilde{W} = \partial_x \tilde{W} = 0$. The relevant zeros are of course those of the last factor; it is a quadratic polynomial in x^2, so explicit formulae can be given. Mathematica plots show that, in the relevant range of ν's, only two out of the eight possible roots are real, and lie outside of the range of interest, as can be seen on the graph in figure 5.4.

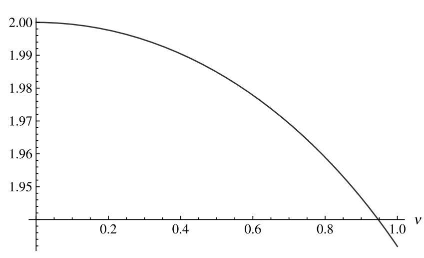

Figure 5.4: One of the real zeros of the last factor in (5.28) as a function of ν; the other one is the negative of the first.

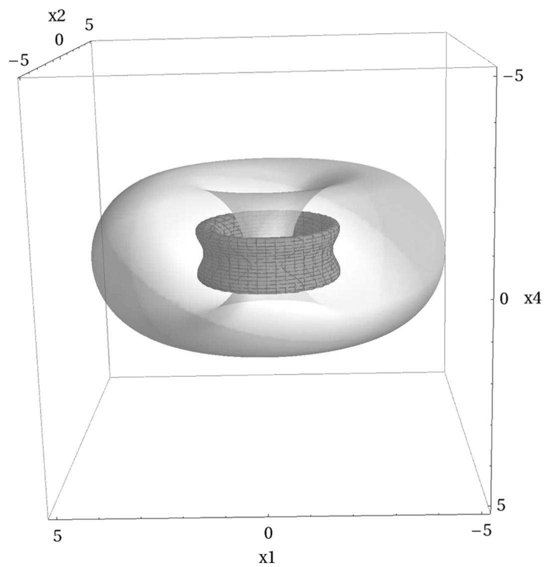

Figure 5.5: Three-dimensional cut of the four-dimensional set $\{t = const.,$ $H(x, y) = 0\}$ (meshed surface) for $\lambda = 1.27$, $\nu = 0.38$, $k = 2.75$ in the asymptotically Euclidean coordinates (x^1, x^2, x^3, x^4). The corresponding event horizon (in translucid grey) has also been added to the picture.

The proof that does not appeal to graphs proceeds as follows:

The study of the resultants in the proof of the Lemma 5.1 shows that a non-simple root x_d of $\tilde{W}$ occurs if and only if $f(\nu, \lambda) = 0$, and that in this case, it is exactly a double root. Then, we saw that the admissible values of ν and λ which satisfy $f(\nu, \lambda) = 0$ are such that $\nu + \lambda > 1$. Hence we are in the case $\tilde{W}(-1) > 0$, with $y_\pm(-1) \in (-1, 1)$; moreover, the pole $x = x_*$ lies in $(1, +\infty)$. From Lemma 5.2, $\tilde{W}$ has to change sign at some $\hat{x} \in (-1, \bar{x})$, since the branches $y_\pm$ cannot enter the region $\{-1 \le x \le 1,\ y_c \le y \le -1\}$; more precisely $\hat{x}$ is another simple root of $\tilde{W}$ in $(-1, \bar{x})$. So we already have two simple roots of $\tilde{W}$ in $(-1, 0)$. But we know from point 1. of Lemma 5.1 that one of the roots of $\tilde{W}$ must be greater than x_* whenever $f(\nu, \lambda) < 0$: this still holds by continuity for $f(\nu, \lambda) \le 0$. Thus so is the double root x_d, which finishes the proof. □

Now, as announced previously, we show that $H(x, 1)$ does not vanish for any $x \in [-1, 1]$:

Lemma 5.5. *No component of the singular set* Sing *can cross the segment* $[-1, 1] \times \{1\}$ *in the* $(x, y)-$*plane.*

Proof. Let us go back to the expressions of $y_\pm(x)$ in (5.24). If we examine the numerator of $y_\pm(x)$, we always have

$$|y_+(x)| = \frac{\lambda\nu(1-x^2)+\sqrt{\tilde{W}(x)}}{\nu|x|\,|2\nu\lambda x(\lambda^2+\nu^2-1)|} \geq \frac{\left|\lambda\nu(1-x^2)-\sqrt{\tilde{W}(x)}\right|}{\nu|x|\,|2\nu\lambda x(\lambda^2+\nu^2-1)|} = |y_-(x)|.$$

Then, from Section 3, we know that if $x \in [-1,1]$ is in the connected component of $\tilde{\Omega}_{\nu,\lambda}$ which contains 0, then we have $y_-(x) < y_\mathrm{c} < -1$. Hence, we have $|y_\pm(x)| > 1$ for $x \in [-1,1]$ in that connected component of $\tilde{\Omega}_{\nu,\lambda}$. But from the analysis of the singular set above, we know that for admissible values of the parameters ν, λ such that $\nu + \lambda \geq 1$, we have another connected component of $\tilde{\Omega}_{\nu,\lambda}$ in $[-1,1]$, which contains -1, and is contained in $[-1,0)$. Then, the denominator $\nu x\left(2\nu\lambda + x(\nu^2+\lambda^2-1)\right)$ of $y_\pm$ is always negative for $x \in [-1,0)$. Indeed,

- if $\nu^2+\lambda^2 \leq 1$, then $2\nu\lambda + x(\nu^2+\lambda^2-1)$ is obviously positive since $x < 0$, hence the denominator is negative;
- if $\nu^2+\lambda^2 > 1$, then one has the set of inequalities:

$$\begin{aligned} 2\nu\lambda - (\nu^2+\lambda^2-1) &= (1+\nu-\lambda)(1+\lambda-\nu) \\ &\leq 2\nu\lambda + x(\nu^2+\lambda^2-1) \\ &\leq 2\nu\lambda, \end{aligned}$$

  for any x in $[-1,0)$. Since the term at the far left is positive for allowed ν and λ, the negativity of the denominator follows.

Therefore, we obtain the inequality $y_+(x) \leq y_-(x)$ for any $x \in [-1,0)$ as long as they exist. Moreover, at $x=-1$, we already noticed that $y_\pm(-1)$ exist and are in $(-1,1)$. Hence, again from Section 3 and by continuity, $y_+(x)$ has to be above -1 for all x in $[-1,0)$ and in the connected component of $\tilde{\Omega}_{\nu,\lambda}$ which contains -1. In conclusion, and from the fact that $|y_+(x)| \geq |y_-(x)|$, we have $-1 < y_+(x) \leq y_-(x) \leq -y_+(x) < 1$ for such x, and the lemma follows. □

Remark 5.6. *A straightforward consequence of this lemma is that $H(x,1)$ has a constant sign for all $x \in [-1,1]$ and for all admissible λ, ν. To find this sign it is enough to compute the sign of this quantity at $x=0$; for instance*

$$H(0,1) = \lambda(\lambda+2\nu) - \nu^2 + 1 > 0. \tag{5.29}$$

Hence we conclude that $H(x,1) > 0$, $\forall x \in [-1,1]$ and admissible parameters.

From what has been said so far we conclude:

Theorem 5.7. *For all admissible* (ν, λ) *let* Sing *denote the set*

$$\{H(x,y) = 0,\ x \in [-1,1]\}.$$

Then

$$\mathrm{Sing} \cap \{y \notin (-1,1)\} = \gamma_+ \cup \gamma_- .$$

where $\gamma_\pm$ *are two connected differentiable curves, with* γ_- *included within the region* $\{y \in (-\infty, y_c)\}$*, and* γ_+ *included in the region* $\{y \in (1,\infty)\}$*, separating each of those regions in two connected components, such that:*

1. *for* $\lambda + \nu < 1$ *the curves* $\gamma_\pm$ *stay away from the axes* $x = \pm 1$ *and asymptote, both at plus and minus infinity, to the vertical lines* $x = 0$ *and* $x = x_*$*;*
2. *for* $\lambda + \nu > 1$ *each of the curves* $\gamma_\pm$ *intersects the vertical line* $\{x = 1\}$ *precisely once, stays away from the vertical line* $x = -1$*, and asymptotes to the axis* $x = 0$ *as* $|y|$ *tends to infinity.*
3. *for* $\lambda + \nu = 1$ *each of the curves* $\gamma_\pm$ *stays away from the vertical line* $x = -1$*, asymptotes to the vertical lines* $x = 0$ *and* $x = 1$ *as* $|y|$ *tends to infinity, without intersecting* $\{x = 1\}$*.*

A three-dimensional representation of the set $H(x,y) = 0$ for certain values of the parameters is presented in figure 5.6.

We finish this section by a short discussion of the special case $\lambda^2 + \nu^2 = 1$, where we set

$$\nu = \cos\alpha, \quad \alpha \in (0, \pi/2).$$

We then have

$$H(x, z/\nu) = -\left(x^2 - 1\right) y \sin(2\alpha) - 2\cos^2(\alpha)\left(xy^2\sin(\alpha) + 1\right) + 2x\sin(\alpha) + 2,$$
$$\tilde{W}(x) = \sin^2(2\alpha)\left(\left(x^2+1\right)^2 + 4x\sin(\alpha)\right),$$

and

$$\nu y_\pm = \frac{\pm\sqrt{\left(x^2+1\right)^2 + 4x\sin(\alpha)} - x^2 + 1}{2x}.$$

We have already seen that two of the roots are imaginary when $\lambda^2 + \nu^2 = 1$. The two remaining ones are graphed as functions of ν in figure 5.6 (see also figure 5.7).

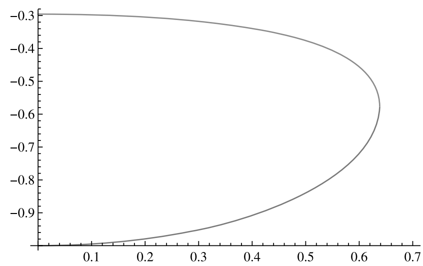

Figure 5.6: Graph representing the two real roots of $\tilde{W}$ as functions of ν when $\lambda^2 + \nu^2 = 1$, whenever they exist. Admissible ν's belong to the interval $[0, \sqrt{5} - 2 \approx 0.236]$, the upper bound being determined by the intersection of the circle with the lower limit $\lambda = 2\sqrt{\nu}$, therefore the double root at $\nu = \sqrt{11/27} \approx 0.638$ corresponds to a non-admissible value of (ν, λ).

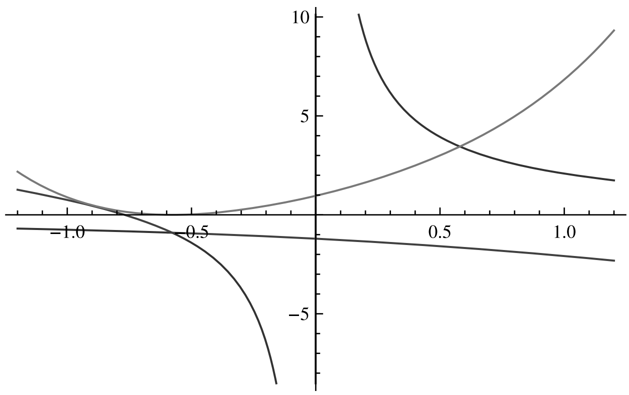

Figure 5.7: The polynomial $\tilde{W}$ (grey) and the set $\{H(x, y) = 0\}$ (the black curves), illustrating the behavior of the singular set at a (non-admissible) double root solution with $\nu = \sqrt{11/27}$ and $\lambda = \sqrt{16/27}$.

5.5 The Kretschmann scalar

Now, one expects existence of a curvature singularity on $\{H(x, y) = 0\}$. In order to test this, we consider the Kretschmann scalar

$$K = R_{abcd}R^{abcd}.$$

An xAct [16] calculation gives[5]

$$K = \frac{3\lambda^2(\nu - 1)^4(x - y)^4\Pi(x, y, \lambda, \nu)}{2k^4 H(x, y)^6}, \tag{5.30}$$

[5] We are grateful to José M. Martín–García for his assistance in this computation.

where $\Pi(x,y,\lambda,\nu)$ is a huge polynomial, still tractable by computer algebra manipulations. Indeed, we can write this polynomial in a shorter form if we introduce the quantities

$$\tilde{G}(x) := \frac{G(x)}{1-x^2}, \quad \tilde{J}(x,y) := \frac{(x-y)(1-\nu)^2 J(x,y)}{2k^2\,(1-x^2)\,(1-y^2)\,\lambda\sqrt{\nu}},$$
$$\tilde{F}(x,y) := \frac{(-1+\nu xy)(-1+\nu)^2(x-y)^2 F(x,y)}{2k^2(-1+y^2)}. \tag{5.31}$$

These quantities are polynomials in x, y, λ, ν as is easily checked and their explicit expressions are

$$\begin{aligned}
\tilde{G}(x) &\equiv \nu x^2 + \lambda x + 1,\\
\tilde{J}(x,y) &\equiv \lambda^2 + 2(x+y)\nu\lambda - \nu^2 - xy\nu(-\lambda^2 - \nu^2 + 1) + 1,\\
\tilde{F}(x,y) &\equiv \lambda\nu x^2(-1+y^2)(-x+y)(-\lambda^2 + (1+\nu)^2)(-1+\nu) + H(x,y)\\
&\quad \times \Bigg(1 + \lambda y + \nu\Big(-1 + \lambda x(-1+\nu x^2 - xy)\\
&\quad + x(-1+\nu)\big(y + x(-1+\nu xy)\big)\Big)\Bigg).
\end{aligned} \tag{5.32}$$

To shorten the final form of $\Pi(x,y,\lambda,\nu)$ one computes a Gröbner basis from $\tilde{G}$, H, $\tilde{J}$, and $\tilde{F}$ and then one uses this basis to show that $\Pi(x,y,\lambda,\nu)$ must take the form

$$\begin{aligned}
\Pi(x,y,\lambda,\nu) &= P_1(x,y,\lambda,\nu)H(x,y) + P_2(x,y,\lambda,\nu)\tilde{G}(x)\\
&\quad + P_3(x,y,\lambda,\nu)\tilde{J}(x,y) + P_4(x,y,\lambda,\nu)\tilde{F}(x,y)
\end{aligned} \tag{5.33}$$

for some polynomials P_1, P_2, P_3, P_4 in x, y, λ, and ν. The Kretschmann scalar was computed by simplifying an expression of 120 Mb size for about 12 h on a desktop computer.

The formula for the Kretschmann scalar shows that a curvature singularity is present in those points where the polynomial $H(x,y)$ vanishes (the set of these points is studied in detail in subsection 5.4) and $\Pi(x,y,\lambda,\nu)$ is different from zero. We give below necessary and sufficient conditions for this to happen. First of all, we define the polynomial

$$\Phi(x,y,\lambda,\nu) := H(x,y)^2 + \Pi(x,y,\lambda,\nu)^2.$$

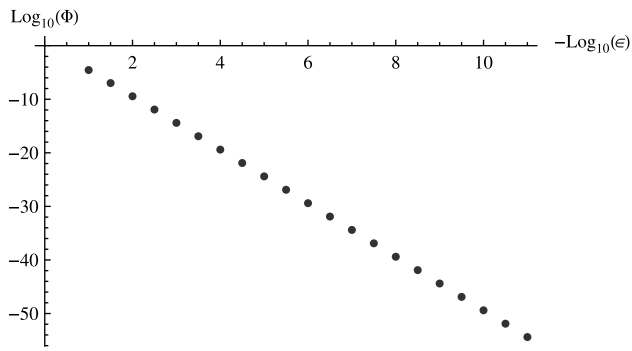

Figure 5.8: Logarithmic representation of the numerically calculated minimum of $\Phi(x, y, \lambda, \nu)$ on $\mathcal{N}_\epsilon$ (denoted by $\Phi_{\min}$) against the parameter ϵ. The graph strongly suggests that the Kretschmann scalar is singular everywhere on the set $\{H(x, y) = 0\}$.

Clearly, $\Phi(x, y, \lambda, \nu)$ is non-negative at any point. Next we compute numerically the minimum of $\Phi(x, y, \lambda, \nu)$ in the set

$$\mathcal{N}_\epsilon := \{-1 \le x \le 1, 2\sqrt{\nu} \le \lambda < 1 + \nu - \epsilon, 0 \le \nu \le 1, |y| \ge 1\}$$

for different values of ϵ. The results of these computations are represented in figure 5.8. The graph shown in this picture suggests that the polynomials $H(x, y)$ and $\Pi(x, y, \lambda, \nu)$ do not have common zeros in the set

$$\mathcal{N}_0 := \{-1 \le x \le 1, 2\sqrt{\nu} \le \lambda < 1 + \nu, 0 \le \nu \le 1, |y| \ge 1\},$$

and when we consider the closure of this set, the polynomials $H(x, y)$ and $\Pi(x, y, \lambda, \nu)$ will have a common set of zeros if and only if $\lambda = 1 + \nu$ holds. We have not been able to obtain an analytical proof of the necessity of this condition and we only rely on the numerical evidence shown in figure 5.8 for the claim. However, the sufficiency of the condition $\lambda = 1 + \nu$ is easily proven analytically and to that end we just set $\lambda = 1 + \nu$ in the definition of $H(x, y)$, obtaining

$$H(x, y, 1 + \nu, \nu) = -2(x + 1)(\nu + 1)(y\nu + 1)(xy\nu - 1). \tag{5.34}$$

Therefore, if $H(x, y, 1 + \nu, \nu) = 0$ at some point in $\overline{\mathcal{N}_0}$, then, either $x = -1$, $y = -1/\nu$, or $y = 1/(x\nu)$. We report the formulae

$$\Pi(x, -1/\nu, 1 + \nu, \nu) = 0, \quad \Pi(-1, y, 1 + \nu, \nu) = 0, \tag{5.35}$$

$$\Pi(x, 1/(x\nu), 1 + \nu, \nu) = \frac{64(x + 1)^{10}(\nu + 1)^4 \left(x^3\nu^2 + x^2\nu - x\nu - 1\right)^2}{x^8\nu^2}. \tag{5.36}$$

The last expression vanishes at some point in $\overline{\mathcal{N}}_0$ when either $x = -1$ or

$$x^3\nu^2 + x^2\nu - x\nu - 1 = 0 \Rightarrow x = -\frac{1}{\nu}, \quad x = \frac{1}{\sqrt{\nu}}, \quad x = -\frac{1}{\sqrt{\nu}}.$$

These values of x are in $\overline{\mathcal{N}}_0$ only if $\nu = 1$ (and thus $x = \pm 1$).

As a summary, we conclude that the polynomials $\Pi(x, y, \lambda, \nu)$ and $H(x, y)$ both vanish at the following points of $\overline{\mathcal{N}}_0$:

$$(x, -1/\nu, 1+\nu, \nu), \quad (-1, y, 1+\nu, \nu), \quad (1, 1, 2, 1),$$

and these are probably the only such points.

We conclude this section by a discussion of the behavior of the Kretschmann scalar for large $|y|$. Both the numerator and the denominator of the Kretschmann scalar are polynomials of order 12 in y. The limit $|y| \to \infty$ of the Kretschmann scalar is a rational function with denominator

$$2k^4x^6\nu^3\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)^6. \tag{5.37}$$

The value of the numerator at $x = 0$ is

$$192\lambda^4(\nu-1)^4\nu^3(\nu+1)^2.$$

The other zero of the denominator is located at

$$x = -\frac{2\lambda\nu}{\lambda^2+\nu^2-1},$$

and the value of the numerator there reads

$$192\lambda^4(\nu-1)^6\nu^3(\lambda-\nu+1)^2(-\lambda+\nu+1)^4(\lambda+\nu-1)^2(\lambda+\nu+1)^4$$
$$\left(\lambda^2(2\nu+1)+\nu^2-1\right)^2 \times \left(\left(\lambda^2+\nu^2-1\right)^8\right)^{-1}.$$

Continuity implies that the Kretschmann scalar is singular on $\{H(x, y) = 0\}$ for all y sufficiently large positive or negative, except possibly at the zeros of the last factor above, which for admissible parameters occur when

$$\lambda = \frac{\sqrt{1-\nu^2}}{\sqrt{2\nu+1}}. \tag{5.38}$$

The limit when $|y|$ goes to infinity of the Kretschmann scalar for this value of λ is a rational expression whose denominator zero-set coincides with the zero-set of the polynomial shown in (5.37) also when λ is set to the value of (5.38). Therefore the Kretschmann scalar will be singular too for all y sufficiently large positive or negative when λ takes the special value (5.38).

5.6 The event horizon has $S^2 \times S^1 \times \mathbb{R}$ topology

In this section we wish to prove that the set $\{y = y_{\rm h}\}$ forms the boundary of the d.o.c., both in the original domain of definition of the metric of [17], and in our extension here. The arguments are a (succinct) adaptation to the metric at hand of those in [4, Section 4.1], the reader is referred to that last reference for more detailed arguments.

We start by noting that

$$g(\nabla y, \nabla y) = g^{yy} = -\frac{(\nu-1)^2 G(y)(x-y)^2}{2k^2 H(x,y)}$$

is negative for $y_{\rm h} < y < y_{\rm c}$, hence y is a time function there. This implies that y is monotonous along causal curves through this region, and hence points for which $y_{\rm h} < y < y_{\rm c}$ lie within a black hole or a while hole region, unless some topological identifications are introduced (for example, consider the manifold consisting of the union of the closures of the blocs *I* to *VII* in figure 7.1, in which blocs *IV* and *VII* are identified; in this space-time there is no black hole region).

Next, consider the determinant of the three-by-three matrix of scalar products of Killing vectors (compare (A.9)–(A.11))

$$\det(g_{ij}) = \frac{4k^4 G(x) G(y)}{(\nu-1)^2 (x-y)^4}. \tag{5.39}$$

This is negative in the region $\{y > y_{\rm h}\}$, which implies that neither the black hole event horizon $\partial J^-(M_{\rm ext})$, nor the white hole event horizon $\partial J^-(M_{\rm ext})$ can intersect this region. We conclude that $\{y = y_{\rm h}\}$ forms the boundary of the d.o.c., as claimed.

Keeping in mind that x and φ are coordinates on S^2, and ψ is a coordinate on S^1 as long as one stays away from the rotation axes $y = \pm 1$, we conclude that the topology of cross-sections of the event horizon $\{y = y_{\rm h}\}$, as well as that of the Killing horizon $\{y = y_{\rm c}\}$, is $S^2 \times S^1$.

6 An extension across $y = -\infty$

We have seen in Section 5.5 that there always exist one or two intervals of x's, say I_a, included in the region for which $H(x,y)$ is positive, such that the metric is defined for all $y \in (-\infty, -1]$. The metric is Lorentzian throughout this region, which follows from (A.8). It turns out that the metric can be analytically extended on those intervals across "the set $\{y = -\infty\}$" to a Lorentzian metric by introducing a new variable

$$Y = -1/y.$$

To see that this is the case, we start by noting that

$$g_{yy}dy^2 = \frac{1}{Y^4} g_{yy} dY^2 = y^4 g_{yy} dY^2.$$

Since

$$\lim_{y\to-\infty} y^4 g_{yy} = -\frac{2k^2 x\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)}{(\nu-1)^2},$$

we see that the function

$$g_{YY}(x,Y) := \left(y^4 g_{yy}\right)(x, y = 1/Y)$$

defined for $x \in I_a$ and for $Y < 1$ is a rational function of Y which analytically extends to negative values across the set $Y = 0$.

Likewise, the remaining metric functions analytically extend across $Y = 0$, except possibly at $x = 0$ and $x = x_*$ defined by (5.26), which follows immediately from

$$\begin{aligned}
\lim_{y\to-\infty} g_{tt} &= \frac{2\lambda(\nu-1)}{x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu} - 1,\\
\lim_{y\to-\infty} g_{t\psi} &= -\frac{2k\lambda\sqrt{(\nu+1)^2-\lambda^2}\,(x(\lambda+\nu-1)+2)}{(\lambda-\nu-1)\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)},\\
\lim_{y\to-\infty} g_{t\phi} &= 0,\\
\lim_{y\to-\infty} g_{xx} &= \frac{2k^2 x\nu\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)}{\left(x^2-1\right)(\nu-1)^2(x(x\nu+\lambda)+1)},\\
\lim_{y\to-\infty} g_{yy} &= 0,\\
\lim_{y\to-\infty} g_{\psi\psi} &= -\frac{2k^2}{x(\nu-1)^2(-\lambda+\nu+1)\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)}\\
&\quad\times\Bigg(x^4(\nu-1)\nu(-\lambda+\nu+1)(\lambda^2+\nu^2-1)
\end{aligned}$$

$$
\begin{aligned}
&+ x^3\lambda(\lambda-\nu-1)(\lambda^2-(\nu-1)^2(4\nu+1)) \\
&+ x^2\Big(\lambda^3(\nu(2\nu-1)+1)+\lambda^2(3\nu-1)(\nu(2\nu-5)+1) \\
&- \lambda(\nu-1)^2(\nu+1)+\nu^4-2\nu^2+1\Big) \\
&+ x\lambda\left(-\lambda^3+\lambda^2(\nu+1)+5\lambda(\nu-1)^2\right. \\
&\left.+3(\nu-1)^2(\nu+1)\right)+2\lambda^2\nu(-\lambda+\nu+1)\Bigg),
\end{aligned}
$$

$$
\lim_{y\to-\infty} g_{\psi\phi} = -\frac{2k^2\left(x^2-1\right)\lambda\sqrt{\nu}}{x(\nu-1)^2},
$$

$$
\lim_{y\to-\infty} g_{\phi\phi} = \frac{2k^2\left(x^2-1\right)\nu(x(\nu-1)+\lambda)}{x(\nu-1)^2},
$$

the remaining components of the metric being identically zero.

The signature remains Lorentzian, which can be seen by calculating the limit of the determinant of the metric in the new coordinates, equal to $y^4 \det g_{\mu\nu}$, where $g_{\mu\nu}$ refers to the original coordinates (t, x, y, ψ, φ):

$$
\lim_{y\to-\infty} y^4 \det g_{\mu\nu} = -\frac{16k^8x^2\nu^2\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)^2}{(\nu-1)^6}.
$$

We also note the limit

$$
\lim_{y\to-\infty} y g_{t\phi} = \frac{2k\left(x^2-1\right)\lambda\sqrt{(\nu+1)^2-\lambda^2}}{x\sqrt{\nu}\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right)},
$$

which shows that $g_{t\phi}$ has a first order zero at $Y=0$.

In the region $Y<0$ one can introduce a new y variable by the formula

$$
y = -1/Y,
$$

which brings the metric back to the original form (1.1), except that y is positive now.

We have $H(x,-1/Y) = Y^{-2}\hat{H}(x,Y)$, where

$$
\begin{aligned}
\hat{H}(x,Y) = 2\left(x^2-1\right)Y\lambda\nu + Y^2\left(2x\lambda+\lambda^2-\nu^2+1\right) \\
- x\nu\left(x\left(\lambda^2+\nu^2-1\right)+2\lambda\nu\right).
\end{aligned}
$$

It follows that the singular set $\{H(x,y)=0\}$ meets the hypersurface $Y=0$ at

$$x = 0 \quad \text{and} \quad x_* = -\frac{2\lambda\nu}{\lambda^2+\nu^2-1}.$$

We have

$$\partial_Y \hat{H}(0,0) = -2\lambda\nu, \quad \partial_x \hat{H}(x_*,0) = 2\lambda\nu^2,$$

which shows that both branches of the singular set $\{H(x,y)=0\}$ form a manifold when crossing $\{Y=0\}$.

The set $\{H(x,y)=0\}$ can be thought as being timelike, in the following sense: For any level set $\{H(x,y)=\epsilon\}$ the norm of the gradient of H is

$$g^{\mu\nu}\partial_\mu H \partial_\nu H = \frac{(\nu-1)^2(x-y)^2}{2k^2 H(x,y)}\big(G(x)(\partial_x H)^2 - G(y)(\partial_y H)^2\big).$$

Both for $y < y_{\rm c}$ and for $y > 1$ the function $G(y)$ is negative, which shows that the normal to the level sets of H is spacelike in that region. Note, however, that this discussion leaves open the possibility of a null limiting hypersurface; we have not attempt to quantify this any further.

A topological space will be called a *pinched* $S^1 \times S^2$ if it is homeomorphic to the set obtained by rotating, around the z-axis in $\mathbb{R}^3$, a disc lying in the (x,z) plane and tangent to the z axis. What has been said so far can now be summarized as follows:

Theorem 6.1. *Suppose that*

$$0 < \nu < 1, \quad 2\sqrt{\nu} \le \lambda < 1+\nu.$$

Consider the set Coords *obtained by replacing the coordinate $y \le y_{\rm c}$ by $Y \in (-1, -1/y_{\rm c}]$ and smoothly adjoining the hypersurface $Y=0$, with the remaining coordinates $x \in [-1,1]$, $t \in \mathbb{R}$, $\varphi, \psi \in [0,2\pi]$, $(x,Y) \ne (-1,1)$. Set*

$$\hat{H}(x,Y) = Y^2 H(x,-1/Y).$$

Then the Pomeransky–Senkov metric extends analytically from the region $0 < Y < -1/y_{\rm c}$ to an analytic Lorentzian vacuum metric on

$$\text{Coords}\backslash\{\hat{H}(x,Y) \le 0\},$$

with an asymptotically flat region near $(x=-1, Y=1)$, and with strong causality violation near $Y=-1$. The boundaries $Y=-1$ and $x=\pm 1$ are rotation axes for suitable Killing vectors with, however, a conical singularity *at* $Y = -1$ *unless*

$$k = \frac{\sqrt{(1+\nu)^2 - \lambda^2}}{4\lambda}. \tag{6.1}$$

The metric has a C^2*-singularity at* $\{\hat{H}(x,Y) = 0\}$*, and, when viewed as a subset in space-time, the singular set* $\{\hat{H}(x,Y) = 0\}$ *has precisely one component homeomorphic to*

1. $\mathbb{R} \times S^1 \times S^1 \times S^1$ *when* $\lambda + \nu < 1$*;*
2. *a pinched* $\mathbb{R} \times S^1 \times S^2$ *when* $\lambda + \nu = 1$*;*
3. $\mathbb{R} \times S^1 \times S^2$ *when* $\lambda + \nu > 1$*.*

7 The global structure

To pass from Theorem 6.1 to an analytic extension of an asymptotically flat region of the PS solution, one needs to keep in mind that every extension across a bifurcate Killing horizon $y = y_{\rm h}$ or $y = y_{\rm c}$ as in Section 4 leads

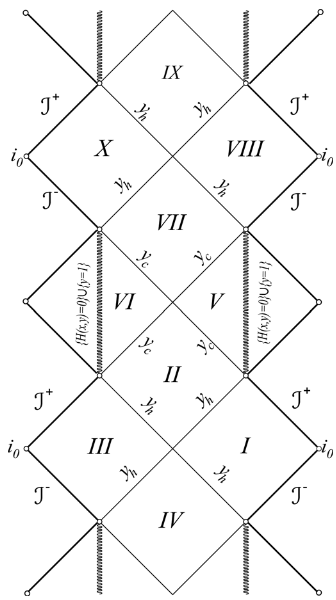

Figure 7.1: A visualization of the global structure of an extension obtained by iterating our procedure when $\lambda \neq 2\sqrt{\nu}$, very similar to that of the non-extreme Kerr space-time. The singular set in the (isometric) regions V and VI does *not* separate this region in two. This is *neither* a conformal diagram, *nor* is the manifold a topological product of the diagram with some three dimensional manifold. However, the picture depicts correctly the causal relations between various regions, when the light-cones are thought to have 45° slopes. We are grateful to M. Eckstein for providing the figure.

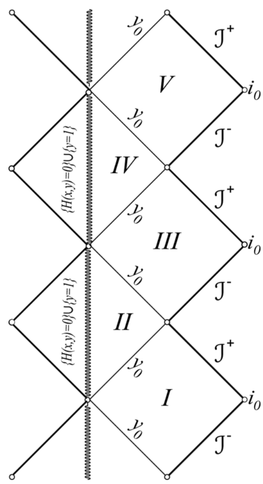

Figure 7.2: Graphical representation of the global structure of the extension for the case $\lambda = 2\sqrt{\nu}$ (extremal case) obtained by iteration of the procedure explained in subsection 4.1 (recall that $y_0 = -1/\sqrt{\nu}$). Similar considerations as in figure 7.1 apply.

to three distinct new regions near that horizon. For $\lambda \neq 2\sqrt{\nu}$ one is then led to a space-time with global structure resembling somewhat that of the usual maximal extension of a non-extreme Kerr black hole (cf., e.g., [2, 12]), with the following notable differences: whereas the extended Kerr space-time contains asymptotically Minkowskian regions with naked singularities, in our case the corresponding asymptotic regions are *not* asymptotically Minkowskian, and their exact nature has yet to be analyzed. Further, except when k is appropriately chosen, in our case the singular set has two components, corresponding to $\hat{H}(x, Y) = 0$, and to the conical singularity at $Y = -1$, with the topology of the former depending upon the parameters (see figures 7.1 and 7.2).

Acknowledgments

The authors are grateful to the Mittag-Leffler Institute, Djursholm, Sweden, for hospitality and financial support during part of work on this paper. P.T.C. was supported in part by the Polish Ministry of Science and Higher Education grant No. N N201 372736, and by the EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1). A.G.P. is supported by the Research Centre of Mathematics of the University of Minho (Portugal) through the "Fundação para a Ciência e a Tecnologia" (FCT) Pluriannual Funding Program. Finally, we thank the "Albert Einstein Institut für Gravitationsphysik" in Golm (Germany), for hospitality during the completion of some parts of this work and financial support.

Appendix A The Pomeransky–Senkov metric

We give here some formulae needed for explicit calculations with the Pomeransky–Senkov (PS) metric (1.1). The expression for $F(x,y)$ can be shortened when written in terms of $H(x,y)$:

$$F(x,y) = \frac{2k^2(-1+y^2)}{(-1+\nu xy)(-1+\nu)^2(x-y)^2} \times \Bigg(\lambda\nu x^2(-1+y^2)(-x+y)(-\lambda^2+(1+\nu)^2)(-1+\nu) + H(x,y)\Big(1+\lambda y+\nu\big(-1+\lambda x(-1+\nu x^2-xy) + x(-1+\nu)(y+x(-1+\nu xy))\big)\Big)\Bigg). \tag{A.1}$$

The non-zero components of the metric tensor are

$$\begin{aligned}
g_{tt} &= -\frac{H(y,x)}{H(x,y)},\\
g_{t\psi} &= -\frac{M(x,y)H(y,x)}{H(x,y)},\\
g_{t\varphi} &= -\frac{P(x,y)H(y,x)}{H(x,y)},\\
g_{xx} &= \frac{2k^2H(x,y)}{(\nu-1)^2G(x)(x-y)^2},\\
g_{yy} &= -\frac{2k^2H(x,y)}{(\nu-1)^2G(y)(x-y)^2},\\
g_{\psi\psi} &= -\frac{H(y,x)^2M(x,y)^2-H(x,y)F(x,y)}{H(x,y)H(y,x)},\\
g_{\varphi\psi} &= -\frac{H(x,y)J(x,y)-H(y,x)^2M(x,y)P(x,y)}{H(x,y)H(y,x)},\\
g_{\varphi\varphi} &= -\frac{H(y,x)^2P(x,y)^2+H(x,y)F(y,x)}{H(x,y)H(y,x)}.
\end{aligned} \tag{A.2}$$

We have the identity

$$-\frac{F(x,y)F(y,x)+J(x,y)^2}{H(x,y)H(y,x)} = -\frac{4k^4G(x)G(y)}{(\nu-1)^2(x-y)^4}, \tag{A.3}$$

which allows us to rewrite the metric (1.1) as

$$ds^2 = \frac{(F(y,x)d\varphi - J(x,y)d\psi)^2}{H(y,x)F(y,x)} - \frac{4k^4 G(x)G(y)H(x,y)d\psi^2}{(\nu-1)^2 F(y,x)(x-y)^4} + \frac{2k^2 H(x,y)}{(1-\nu)^2(x-y)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right) - \frac{H(y,x)(dt-\Omega)^2}{H(x,y)}, \qquad (A.4)$$

where Ω is as defined at the beginning of Section 1.

This provides an orthonormal frame, and is in particular convenient for studying the signature of the metric.

There are other interesting identities fulfilled by the rational functions involved in the definition of the PS metric. To write them we need to add to the rational functions the explicit dependencies on the variables λ, ν in addition to x and y. If $\mathcal{Q}(x,y,\lambda,\nu)$ denotes any of the rational functions $H(x,y,\lambda,\nu)$, $F(x,y,\lambda,\nu)$, and $J(x,y,\lambda,\nu)$, then an explicit computation shows the property

$$\mathcal{Q}(x,y,\lambda,\nu) = -x^2y^2\nu^3\mathcal{Q}\left(\frac{1}{x},\frac{1}{y},\frac{\lambda}{\nu},\frac{1}{\nu}\right). \qquad (A.5)$$

In addition, we also have the relations

$$M(x,y,\lambda,\nu) = M\left(\frac{1}{x},\frac{1}{y},\frac{\lambda}{\nu},\frac{1}{\nu}\right), \quad P(x,y,\lambda,\nu) = P\left(\frac{1}{x},\frac{1}{y},\frac{\lambda}{\nu},\frac{1}{\nu}\right),$$
$$G(x,\lambda,\nu) = -x^4 G\left(\frac{1}{x},\frac{\lambda}{\nu},\frac{1}{\nu}\right), \qquad (A.6)$$

which again are shown by expanding explicitly all the functions involved. A straightforward consequence of these properties is contained in the following result:

Proposition A.1. *The PS family of metrics is invariant under the transformation*

$$x \mapsto \frac{1}{x}, \quad y \mapsto \frac{1}{y}, \quad \lambda \mapsto \frac{\lambda}{\nu}, \quad \nu \mapsto \frac{1}{\nu}, \quad k \mapsto k\nu. \qquad (A.7)$$

A direct consequence of Proposition A.1 is that all the properties of the PS metric which have been proven in this paper under the assumption that $(x,y,\lambda,\nu) \in \Omega_0$ hold *mutatis mutandi* when (x,y,λ,ν) belong to the set $\tilde{\Omega}_0$ defined by

$$\tilde{\Omega}_0 := \{(x,y,\nu,\lambda) \in \mathbb{R}^4 \ ; \ |x| \geq 1,\ -1 \leq y \leq y_{\rm h},\ 1 < \nu,\ 2\sqrt{\nu} \leq \lambda < 1+\nu\},$$

which is the image of Ω_0 under the map (A.7). Since $\nu > 1$ then we have $|y_{\rm h}| < 1$ (see Proposition A.2) and therefore for points in $\tilde{\Omega}_0$ we always have $|y| < 1$. We can now localize the asymptotically flat ends, horizons, and so on by just applying the map (A.7) to the corresponding regions in Ω_0 which were discussed in the paper. In this way we find that the point $(-1,-1,\lambda,\nu) \in \tilde{\Omega}_0$ corresponds to an asymptotically flat end and $(x, y_{\rm c}, \lambda, \nu) \in \tilde{\Omega}_0$ is the event horizon associated to the d.o.c. of this asymptotically flat end (note that $y_{\rm c}$ is mapped to $y_{\rm h}$, $y_{\rm h}$ is mapped to $y_{\rm c}$ under (A.7) and their absolute value is less than unity when $\nu > 1$).

The determinant of the metric reads

$$\det(g_{\mu\nu}) = -\frac{4k^4 H(x,y)\left(J(x,y)^2 + F(x,y)F(y,x)\right)}{(\nu-1)^4 G(x)G(y)H(y,x)(x-y)^4}.$$

Using the identity (A.3) one obtains

$$\det(g_{\mu\nu}) = -\frac{16k^8 H(x,y)^2}{(\nu-1)^6(x-y)^8}. \tag{A.8}$$

The restriction of the metric to the hyperplanes $\mathrm{Span}\{\partial_t, \partial_\varphi, \partial_\psi\}$ is

$$\begin{aligned} ds^2 = &-\frac{2d\psi dt M(x,y)H(y,x)}{H(x,y)} - \frac{2d\varphi dt P(x,y)H(y,x)}{H(x,y)} \\ &- \frac{dt^2 H(y,x)}{H(x,y)} - d\psi^2\left(\frac{H(y,x)M(x,y)^2}{H(x,y)} + \frac{F(x,y)}{H(y,x)}\right) \\ &- 2d\psi d\varphi\left(\frac{J(x,y)}{H(y,x)} + \frac{H(y,x)M(x,y)P(x,y)}{H(x,y)}\right) \\ &- d\varphi^2\left(\frac{H(y,x)P(x,y)^2}{H(x,y)} - \frac{F(y,x)}{H(y,x)}\right). \end{aligned} \tag{A.9}$$

The determinant of the restricted metric is given by

$$\det(g_{ij}) = \frac{J(x,y)^2 + F(x,y)F(y,x)}{H(x,y)H(y,x)}, \tag{A.10}$$

which, after replacing all the functions simplifies to

$$\det(g_{ij}) = \frac{4k^4 G(x)G(y)}{(\nu-1)^2(x-y)^4}. \tag{A.11}$$

Clearly $\det(g_{ij}) < 0$ if $y_{\mathrm{h}} < y < -1$ or $1 < y$, where $\{y = y_{\mathrm{h}}\}$ is a Killing horizon with respect to the following Killing vector

$$\xi = \frac{\partial}{\partial t} + \frac{\sqrt{(\nu+1)^2-\lambda^2}}{2k(\lambda+\nu+1)}\frac{\partial}{\partial\psi} + \frac{\left(\lambda^2+(\nu-1)^2\right)\sqrt{\nu}\sqrt{(\nu+1)^2-\lambda^2}}{k\lambda(\lambda+\nu+1)\left(\lambda(\nu+1)-\sqrt{\lambda^2-4\nu}(\nu-1)\right)}\frac{\partial}{\partial\varphi},$$

and $\{y = y_{\mathrm{c}}\}$ is a Killing horizon with respect to the Killing vector

$$\tilde{\xi} = \frac{\partial}{\partial t} + \frac{\sqrt{(\nu+1)^2-\lambda^2}}{2k(\lambda+\nu+1)}\frac{\partial}{\partial\psi} + \frac{\left(\lambda^2+(\nu-1)^2\right)\sqrt{\nu}\sqrt{(\nu+1)^2-\lambda^2}}{k\lambda(\lambda+\nu+1)\left(\lambda(\nu+1)+\sqrt{\lambda^2-4\nu}(\nu-1)\right)}\frac{\partial}{\partial\varphi}.$$

From the expression for the restriction of the metric to $\{t, \varphi, \psi\}$ one easily gets the expression from the restriction of the metric to $\{\varphi, \psi\}$. We denote this restricted metric by g_{AB} and its determinant is

$$\begin{aligned}\det g_{AB} = {} & \frac{1}{H(x,y)(-1+\lambda-\nu)H(y,x)^2} \\ & \times\Bigg[F(y,x)\Bigg\{F(x,y)H(x,y)(1-\lambda+\nu)+4k^2\lambda^2(1+y)^2 \\ & (1+\lambda+\nu)\Big(1+\lambda-\nu+2\nu x-\nu xy\big(2+x(-1+\lambda+\nu)\big)\Big)^2\Bigg\} \\ & +(1-\lambda+\nu)\Bigg\{H(x,y)J(x,y)^2+4k^2\lambda^2y\sqrt{\nu}(-1+x^2) \\ & (1+\lambda+\nu)\Bigg(y\sqrt{\nu}F(x,y)(-1+x^2)(-1+\lambda-\nu)+2J(x,y) \\ & (1+y)\Big(-1-\lambda+\nu-2\nu x+\nu xy\big(2+x(-1+\lambda+\nu)\big)\Big)\Bigg)\Bigg\}\Bigg],\end{aligned} \tag{A.12}$$

which can be rewritten in the form

$$\det(g_{AB}) = \frac{4k^4(-1+x^2)(1+y)}{(-1+\lambda-\nu)(-1+\nu)^2(x-y)^4 H(x,y)}\Theta(x,y,\lambda,\nu). \qquad \text{(A.13)}$$

Clearly $\Theta(x,y,\lambda,\nu)$ is a rational function of x, y, λ and $\sqrt{\nu}$. However, a Mathematica calculation shows that $\Theta(x,y,\lambda,\nu)$ is a polynomial in x, y, λ and ν.

One can give an alternative form for $\det(g_{AB})$ if we use the parameterization of the angular components of the metric tensor given in (A.13):

$$\begin{aligned}\det(g_{AB}) = {}& \frac{-4k^4\lambda^2\nu(-1+x^2)^2(1+y)^2\Theta_{\phi\psi}(x,y)^2}{(-1+\nu)^4 H(x,y)^2(x-y)^2} \\ &- \frac{4k^4(1+y)(-1+x^2)\Theta_{\phi\phi}(x,y)\Theta_{\psi\psi}(x,y)}{(-1+\nu)^4(1-\lambda+\nu)H(x,y)^2(x-y)^4}. \end{aligned} \qquad \text{(A.14)}$$

Comparing (A.13) and (A.14) we deduce the relation

$$\begin{aligned}&(\nu-1)^2 H(x,y)\Theta(x,y,\lambda,\nu) \\ &\quad = \lambda^2\nu(x-y)^2(1+y)(1-x^2)(1+\nu-\lambda)\Theta_{\phi\psi}(x,y)^2 \\ &\qquad - \Theta_{\phi\phi}(x,y)\Theta_{\psi\psi}(x,y). \end{aligned} \qquad \text{(A.15)}$$

The non-zero components of $g^{\mu\nu}$ read

$$\begin{aligned} g^{tt} &= \big(-M(x,y)(F(y,x)M(x,y) + 2J(x,y)P(x,y))H(y,x)^2 \\ &\quad + H(x,y)J(x,y)^2 + F(x,y)\left(F(y,x)H(x,y) - H(y,x)^2P(x,y)^2\right)\big) \\ &\quad \times \left(H(y,x)\left(J(x,y)^2 + F(x,y)F(y,x)\right)\right)^{-1}, \\ g^{t\psi} &= \frac{H(y,x)(F(y,x)M(x,y) + J(x,y)P(x,y))}{J(x,y)^2 + F(x,y)F(y,x)}, \\ g^{t\varphi} &= -\frac{H(y,x)(F(x,y)P(x,y) - J(x,y)M(x,y))}{J(x,y)^2 + F(x,y)F(y,x)}, \\ g^{xx} &= \frac{(\nu-1)^2 G(x)(x-y)^2}{2k^2 H(x,y)}, \\ g^{yy} &= -\frac{(\nu-1)^2 G(y)(x-y)^2}{2k^2 H(x,y)}, \\ g^{\psi\psi} &= -\frac{F(y,x)H(y,x)}{J(x,y)^2 + F(x,y)F(y,x)}, \end{aligned}$$

$$g^{\psi\varphi} = -\frac{H(y,x)J(x,y)}{J(x,y)^2 + F(x,y)F(y,x)},$$
$$g^{\varphi\varphi} = \frac{F(x,y)H(y,x)}{J(x,y)^2 + F(x,y)F(y,x)}.$$

Alternative forms for some of the above can be obtained using the identity (A.3)

$$\begin{aligned} g^{tt} &= (\nu-1)^2 \bigg(-\frac{4k^4 G(x)G(y)H(x,y)^2}{16H(y,x)(x-y)^4} - F(y,x)M(x,y)^2 \\ &\quad + F(x,y)P(x,y)^2 - 2J(x,y)M(x,y)P(x,y)\big)(x-y)^4 \bigg) \\ &\quad \times \big(4k^4 G(x)G(y)H(x,y)\big)^{-1}, \\ g^{t\psi} &= \frac{(\nu-1)^2(F(y,x)M(x,y) + J(x,y)P(x,y))(x-y)^4}{4k^4 G(x)G(y)H(x,y)}, \\ g^{t\varphi} &= -\frac{(\nu-1)^2(F(x,y)P(x,y) - J(x,y)M(x,y))(x-y)^4}{4k^4 G(x)G(y)H(x,y)}, \\ g^{\psi\psi} &= -\frac{(\nu-1)^2 F(y,x)(x-y)^4}{4k^4 G(x)G(y)H(x,y)}, \\ g^{\psi\varphi} &= -\frac{(\nu-1)^2 J(x,y)(x-y)^4}{4k^4 G(x)G(y)H(x,y)}, \\ g^{\varphi\varphi} &= \frac{(\nu-1)^2 F(x,y)(x-y)^4}{4k^4 G(x)G(y)H(x,y)}. \end{aligned}$$

The $x - y$ part of the PS metric,

$$ds^2_{x,y} = \frac{2k^2 H(x,y)}{(\nu-1)^2(x-y)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right),$$

can be written in a form which is conformally flat [17] by introducing new coordinates ρ, z defined as

$$\rho^2 = -\frac{4k^4 G(x)G(y)}{(-1+\nu)^2\,(x-y)^4}, \qquad z = \frac{k^2(1-xy)(\lambda(x+y) + 2xy\nu + 2)}{(1-\nu)(x-y)^2}.$$

The line element becomes

$$ds^2_{\rho,z} = \Lambda(x,y)(d\rho^2 + dz^2),$$

where

$$\Lambda(x,y) = \frac{2(y-x)H(x,y)}{k^2(xy\lambda+(\nu+1)(x+y)+\lambda)} \times \Big(\big(\lambda^2 \left(x^2 \left(-\left(y^2-1\right)\right)\right. \\ \left. + 4xy + y^2 - 1\big) + 4(\nu+1)\left(x^2y^2\nu+1\right) \\ + 4\lambda(x+y)\left(xy\nu+1\right)\right)\Big)^{-1}.$$

Finally, we mention a property of one of the polynomial factors in $G(x)$:

Proposition A.2. *Let $p(\xi) \equiv \nu\xi^2 + \lambda\xi + 1$ and assume that $2\sqrt{\nu} \leq |\lambda| < 1+\nu$. Then the real roots $\xi_- \leq \xi_+$ of the polynomial $p(\xi)$ fulfill the inequalities $|\xi_\pm| > 1$ if $0 < \nu < 1$ and $|\xi_\pm| < 1$ if $1 < \nu$.*

Proof. The roots of $p(\xi)$ are given by

$$\xi_\pm = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\nu}}{2\nu},$$

and therefore, these are real if and only if $2\sqrt{\nu} \leq |\lambda|$, as assumed. The roots $\xi_\pm$ can be regarded as functions of λ and ν and these functions are continuous on the set $\mathcal{Z} := \{(\lambda,\nu) : \nu > 0,\ 2\sqrt{\nu} \leq |\lambda| < 1+\nu\}$. Now the equations $\xi_\pm = 1$, $\xi_\pm = -1$ admit as respective solutions the values $\nu = -1-\lambda$ and $\nu = \lambda - 1$, and therefore no point lying in $\mathcal{Z}$ has the property that $|\xi_\pm| = 1$. The conclusion is then that for any $(\lambda,\nu) \in \mathcal{Z}$ we have that either $|\xi_\pm| > 1$ or $|\xi_\pm| < 1$ and given the continuity of $\xi_\pm$ on $\mathcal{Z}$ as a function of λ, ν only one of these alternatives will hold on each connected component of $\mathcal{Z}$. These connected components are

$$\begin{aligned} \mathcal{Z}_1 &:= \{(\lambda,\nu) : 0 < \nu < 1,\ 0 < \lambda,\ 2\sqrt{\nu} \leq |\lambda| < 1+\nu\}, \\ \mathcal{Z}_2 &:= \{(\lambda,\nu) : 0 < \nu < 1,\ 0 > \lambda,\ 2\sqrt{\nu} \leq |\lambda| < 1+\nu\}, \\ \mathcal{Z}_3 &:= \{(\lambda,\nu) : 1 < \nu,\ 0 < \lambda,\ 2\sqrt{\nu} \leq |\lambda| < 1+\nu\}, \\ \mathcal{Z}_4 &:= \{(\lambda,\nu) : 1 < \nu,\ \lambda < 0,\ 2\sqrt{\nu} \leq |\lambda| < 1+\nu\}. \end{aligned}$$

Again, by continuity, it is enough to check the values of $\xi_\pm$ at particular points of $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, $\mathcal{Z}_4$ to draw the desired conclusions. Our explicit choices are as follows

$$\left(\lambda = 1, \nu = \frac{1}{9}\right) \in \mathcal{Z}_1 \Rightarrow \left|\xi_\pm\left(1, \frac{1}{9}\right)\right| > 1,$$
$$\left(\lambda = \frac{16}{5}, \nu = \frac{9}{4}\right) \in \mathcal{Z}_3 \Rightarrow \left|\xi_\pm\left(\frac{16}{5}, \frac{9}{4}\right)\right| < 1,$$

$$\left(\lambda = -\frac{16}{5}, \nu = \frac{9}{4}\right) \in \mathcal{Z}_4 \Rightarrow \left|\xi_\pm\left(-\frac{16}{5}, \frac{9}{4}\right)\right| < 1,$$
$$\left(\lambda = -1, \nu = \frac{1}{9}\right) \in \mathcal{Z}_2 \Rightarrow \left|\xi_\pm\left(-1, \frac{1}{9}\right)\right| > 1.$$

Therefore $|\xi_\pm| > 1$ on $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $|\xi_\pm| < 1$ on $\mathcal{Z}_3$, $\mathcal{Z}_4$, which finishes the proof. □

Appendix B Emparan–Reall limit of the PS metric

In this section we verify that the Emparan–Reall solutions are a special case of the PS metrics. We take the Emparan–Reall metric in the form given in [11],

$$\begin{aligned} ds^2 &= \frac{R^2 F(x)}{(x-y)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} + \frac{G(x)}{F(x)}d\varphi^2 - \frac{G(y)}{F(y)}d\psi^2\right) \\ &\quad - \frac{F(y)}{F(x)}\left(dt - \frac{CR(1+y)}{F(y)}d\psi\right)^2, \end{aligned} \tag{B.1}$$

where

$$F(z) = 1 + \lambda z, \quad G(z) = (1-z^2)(1+\nu z), \quad C = \sqrt{\frac{\lambda(1+\lambda)(\lambda-\nu)}{1-\lambda}}.$$

The parameters are assumed in that paper to range over

$$0 < \nu \le \lambda < 1.$$

However, the requirement that there are no struts imposes the supplementary relation

$$\lambda = \frac{2\nu}{1+\nu^2}.$$

The coordinates of (B.1) are not the ones of the original paper of Emparan and Reall [10]. If we denote by $\{\hat{t}, \hat{x}, \hat{y}, \hat{\psi}, \hat{\varphi}\}$ the original coordinates of Emparan & Reall, then we have the relation

$$t = \hat{t}, \;\; x = \frac{\hat{\lambda} - \hat{x}}{-1 + \hat{\lambda}\hat{x}}, \;\; y = \frac{\hat{\lambda} - \hat{y}}{-1 + \hat{\lambda}\hat{y}}, \;\; \varphi = \frac{1 - \hat{\lambda}\hat{\nu}}{\sqrt{1-\hat{\lambda}^2}}\hat{\varphi}, \;\; \psi = \frac{1 - \hat{\lambda}\hat{\nu}}{\sqrt{1-\hat{\lambda}^2}}\hat{\psi}, \tag{B.2}$$

where

$$\hat{\nu} = \frac{\nu - \lambda}{\lambda\nu - 1}, \quad \hat{\lambda} = \lambda, \quad \nu = \frac{\hat{\nu} - \hat{\lambda}}{\hat{\lambda}\hat{\nu} - 1}.$$

The transformation (B.2) brings the metric (B.1) into the form (as in [10] with hats on all the coordinates and functions and R replaced by $\hat{A}$)

$$\begin{aligned} ds^2 = &-\frac{\hat{F}(\hat{x})}{\hat{F}(\hat{y})}(d\hat{t} + \hat{A}\sqrt{\hat{\lambda}\hat{\nu}}(1+\hat{y})d\hat{\psi})^2 \\ &+ \frac{\hat{A}^2}{(\hat{x}-\hat{y})^2}\left(\hat{F}(\hat{y})^2\left(\frac{d\hat{x}^2}{\hat{G}(\hat{x})} + \frac{\hat{G}(\hat{x})}{\hat{F}(\hat{x})}d\hat{\varphi}^2\right)\right. \\ &- \left.\hat{F}(\hat{x})\left(\frac{\hat{F}(\hat{y})}{\hat{G}(\hat{y})}d\hat{y}^2 + \hat{G}(\hat{y})d\hat{\psi}^2\right)\right), \end{aligned} \tag{B.3}$$

where

$$\hat{F}(z) = 1 - \hat{\lambda}z, \quad \hat{G}(z) = (1-z^2)(1-\hat{\nu}z), \quad \hat{A} = -R\sqrt{\frac{(1-\hat{\lambda}\hat{\nu})}{1-\hat{\lambda}^2}}.$$

To check the limit as $\nu \to 0$ of PS metrics, we start by rewriting the PS solution in the form

$$\begin{aligned} ds^2 \;=\; &\frac{Q\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right)H(x,y)}{(x-y)^2} - 2\frac{d\varphi d\psi J(x,y)}{H(y,x)} - \frac{H(y,x)(dt+\Omega)^2}{H(x,y)} \\ &- \frac{d\psi^2 F(x,y)}{H(y,x)} + \frac{d\varphi^2 F(y,x)}{H(y,x)}. \end{aligned} \tag{B.4}$$

Here $Q > 0$ is a constant which can be eliminated by a rescaling of the metric together with an appropriate rescaling of the coordinates t, ψ, φ. If we set $Q = 2k^2/(1-\nu)^2$ in (B.4), we recover (1.1). One checks that the metric (B.4) reduces to the Emparan–Reall solution (B.1) if we set $\nu = 0$ and $Q = 2k^2/(1+\lambda^2)$, and if

$$\nu_{\rm e} = \lambda, \quad R_{\rm e} = -\sqrt{2}k$$

where the Emparan–Reall independent parameters are denoted by $\nu_{\rm e}$ and $R_{\rm e}$.

References

[1] M. T. Anderson, *On stationary vacuum solutions to the Einstein equations*, Ann. H. Poincaré **1** (2000), 977–994; arXiv:gr-qc/0001091.

[2] B. Carter, *Global structure of the Kerr family of gravitational fields*, Phys. Rev. **174** (1968), 1559–1571.

[3] —, *Killing horizons and orthogonally transitive groups in space-time*, J. Math. Phys. **10** (1969), 70–81.

[4] P. T. Chruściel and J. Cortier, *Maximal analytic extensions of the Emparan-Reall black ring*, J. Differential Geometry, 85 (2010), 425–459.

[5] P. T. Chruściel and J. Lopes Costa, *On uniqueness of stationary black holes*, Astérisque **321** (2008), 195–265; arXiv:0806.0016v2 [gr-qc].

[6] P. T. Chruściel and G. Galloway, *Uniqueness of static black-holes without analyticity*, (2010); arXiv:1004.0513 [gr-qc].

[7] P. T. Chruściel and S. Szybka, *Stable causality of the Pomeransky-Senkov black holes*, (2010), arXiv:1010.0213 [hep-th].

[8] M. Durkee, *Geodesics and symmetries of doubly spinning black rings*, Class. Quantum Grav. **26** (2009), 085016, 33, MR MR2524560

[9] H. Elvang and M. J. Rodriguez, *Bicycling black rings*, J. High Energy Phys. **04** (2008), 045, 30, MR MR2425258 (2009f:83043)

[10] R. Emparan and H. S. Reall, *A rotating black ring in five dimensions*, Phys. Rev. Lett. **88** (2002), 101101, arXiv:hep-th/0110260.

[11] —, *Black rings*, Class. Quantum Grav. **23** (2006), R169–R197, arXiv:hep-th/0608012.

[12] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973, Cambridge Monographs on Mathematical Physics, No. 1, MR MR0424186 (54 #12154)

[13] S. Hollands and S. Yazadjiev, *A uniqueness theorem for stationary Kaluza-Klein black holes*, (2008), arXiv:0812.3036 [gr-qc].

[14] —, *Uniqueness theorem for 5-dimensional black holes with two axial Killing fields*, Commun. Math. Phys. **283** (2008), 749–768, arXiv:0707.2775 [gr-qc] MR MR2434746

[15] H. J. Kunduri, J. Lucietti, and H. S. Reall, *Near-horizon symmetries of extremal black holes*, Class. Quantum Grav. **24** (2007), 4169–4190, arXiv:0705.4214 [hep-th].

[16] J. M. Martín-García, *xAct: Efficient tensor computer algebra*, http://metric.iem.csic.es/Martin-Garcia/xAct.

[17] A. A. Pomeransky and R. A. Senkov, *Black ring with two angular momenta*, (2006), hep-th/0612005.

[18] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, in 'Topics in Calculus of Variations (Montecatini Terme, 1987), Lecture Notes in Math. **1365**, Springer, Berlin, 1989, pp. 120–154, MR MR994021 (90g:58023)