

# Quotients of the conifold in compact Calabi–Yau threefolds and new topological transitions

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## Abstract

The moduli space of multiply connected Calabi–Yau threefolds is shown to contain codimension-one loci on which the corresponding variety develops a certain type of hyperquotient singularity. These have local descriptions as discrete quotients of the conifold, and are referred to here as hyperconifolds. In many (or possibly all) cases such a singularity can be resolved to yield a distinct compact Calabi–Yau manifold. These considerations therefore provide a method for embedding an interesting class of singularities in compact Calabi–Yau varieties, and for constructing new Calabi–Yau manifolds. It is unclear whether the transitions described can be realized in the string theory.

## 1 Introduction

Calabi–Yau threefolds have played a central role in string theory compactifications since the seminal work [1]. A vast number of these manifolds have

now been constructed, the best-known classes of which are the complete intersections in products of projective spaces (CICYs), and hypersurfaces in weighted  $\mathbb{P}^4$  or more general toric fourfolds [2–5].

In order to obtain semi-realistic heterotic compactifications, it is usually necessary to consider multiply connected Calabi–Yau manifolds, on which discrete Wilson lines can be used to help break unwanted gauge symmetries. Two of the best-known examples are the heterotic model on Yau’s “three generation” manifold [6–8], with fundamental group  $\mathbb{Z}_3$ , and the models constructed on a manifold with fundamental group  $\mathbb{Z}_3 \times \mathbb{Z}_3$  [9–12]. Recent progress has been made on calculating the spectra of many heterotic models on these types of spaces, opening up new realistic model-building possibilities [13, 14]. Several such multiply connected manifolds were known in the early days of string phenomenology [1, 15, 16], and recently a number of new examples with various fundamental groups have been constructed in [17]. Any such manifold necessarily arises as a quotient of a simply connected covering space by a freely acting discrete group.

Smooth Calabi–Yau manifolds are not the only ones relevant to string theory. The moduli spaces of families of smooth Calabi–Yau manifolds have boundaries corresponding to singular varieties, and these are moreover at a finite distance as measured by the moduli space metric [18]. Even more remarkable is that moduli spaces of topologically distinct families can meet along such singular loci, and in fact it has been speculated that the moduli space of all Calabi–Yau threefolds is connected in this way [19]. It was shown soon after that this was true for almost all known examples, and suggested that the associated string vacua may also be unified as a single physical moduli space [20, 21]. A series of beautiful papers in the 1990s established that type II string theories can indeed pass smoothly through singular geometries, realizing spacetime topology change via so-called “flops” and “conifold transitions” [22–24]. Conifold transitions can also be used as a tool for finding new Calabi–Yau manifolds, as in [17, 25].

The most generic singularities which occur in threefolds are ordinary double points, or nodes, which are usually referred to as conifold singularities in the physics literature. The purpose of this paper is to point out that for multiply connected threefolds, there are worse singularities which are just as generic, in that they also occur on codimension-one loci in moduli space. Specifically, if the moduli are chosen such that the (generically free) group action on the covering space actually has fixed points, these turn out always to be singular points, generically nodes. The singularities on the quotient are therefore quotients of the conifold, and, as we will see, have local descriptions as toric varieties. Standard techniques from toric geometry are therefore utilized throughout; the reader unfamiliar with these ideas can

consult one of several reviews in the physics literature [22, 26, 27] or the textbook [28].

Quotients of hypersurface singularities were dubbed “hyperquotients” by Reid. We will refer to the particular examples considered herein as “hyperconifolds,” or sometimes  $G$ -hyperconifolds to be explicit about the quotient group  $G$ . Although the toric formalism allows us to find local crepant resolutions (i.e., resolutions with trivial canonical bundle) of these singularities in each case, the important question is whether these preserve the Calabi–Yau conditions when embedded in the compact varieties of interest. In particular, the existence of a Kähler form is a global question. For most of the examples we can argue that a Calabi–Yau resolution does indeed exist for all varieties containing the singularities of interest, and furthermore we can calculate the Hodge numbers of such a resolution. This therefore gives a systematic way of constructing new Calabi–Yau manifolds from known multiply connected ones. By analogy with conifold transitions, the process by which we pass from the original smooth Calabi–Yau through the singular variety to its resolution will be dubbed a “hyperconifold transition”. Like a conifold transition, the Hodge numbers of the new manifold are different, but unlike flops or conifold transitions, the fundamental group also changes.

Quotients of the conifold have been considered previously in the physics literature, mostly in the context of D3-branes at singularities (e.g. [29–33]), although the most-studied group actions in this context have fixed-point sets of positive dimension. The most simple example in this paper, the  $\mathbb{Z}_2$ -hyperconifold, has however appeared in numerous papers (e.g. [34–37], and recently in the context of heterotic theories with flux [38]), while the  $\mathbb{Z}_3$  case appears in an appendix in [39]. To the best of my knowledge hyperconifold singularities have not before been explicitly embedded in compact varieties. This paper gives a general method to find compactifications of string/brane models based on these singularities.

The layout of the paper is as follows. In Section 2, the  $\mathbb{Z}_5$  quotient of the quintic is presented as an example of a compact Calabi–Yau threefold which develops a hyperconifold singularity. The toric description of such singularities is also introduced here. Section 3 contains the main mathematical result of the paper. It is demonstrated that if one starts with a family of threefolds generically admitting a free group action, then specialises to a sub-family for which the action instead develops a fixed point, then this point is necessarily a singularity (generically a node). The quotient variety therefore develops a hyperconifold singularity; the toric descriptions of these are given, and their topology described. In Section 4 Calabi–Yau resolutions are shown to exist for many of these singular varieties, and the Hodge numbers of these resolutions are calculated. In Section 5 a few initial observations are made

relating to the possibility of hyperconifold transitions being realised in string theory.

The notation used throughout the paper is as follows:

- $\tilde{X}$  is a generic member of a family of smooth Calabi–Yau threefolds which admit a free holomorphic action of the group  $G$ .
- $X$  is the (smooth, Calabi–Yau) quotient  $\tilde{X}/G$ .
- $\tilde{X}_0$  is a ( $G$ -invariant) deformation of  $\tilde{X}$  such that the action of  $G$  is no longer free.
- $X_0$  is the singular quotient space  $\tilde{X}_0/G$ . This can be thought of as living on the boundary of the moduli space of smooth manifolds  $X$ .
- $\hat{X}$  will denote a crepant resolution of  $X_0$ , with projection  $\pi : \hat{X} \rightarrow X_0$ . We will denote by  $E$  the exceptional set of this resolution.

## 2 A $\mathbb{Z}_5$ example

We will begin with a simple example to illustrate the idea. Consider a quintic hypersurface in  $\mathbb{P}^4$ , and denote such a variety by  $\tilde{X}$ . Take homogeneous coordinates  $x_i$  for the ambient space, with  $i \in \mathbb{Z}_5$ , so that such a hypersurface is given by  $f = 0$ , where

$$f = \sum A_{ijklm} x_i x_j x_k x_l x_m. \tag{2.1}$$

If we denote by  $g_5$  the generator of the cyclic group  $G \cong \mathbb{Z}_5$ , we can define an action of this group on the ambient  $\mathbb{P}^4$  as follows:

$$g_5 : x_i \rightarrow \zeta^i x_i, \quad \text{where } \zeta = e^{2\pi i/5},$$

$\tilde{X}$  will be invariant under this action if  $A_{ijklm}$  is zero except when  $i + j + k + l + m \equiv 0 \pmod{5}$ . It is easy to see that for a general such choice of these coefficients, the  $\mathbb{Z}_5$  action on  $\tilde{X}$  has no fixed points, so the quotient variety, denoted  $X$ , is smooth. For special choices of complex structure though, the hypersurface given by  $f = 0$  will contain fixed points, and it is this case which will interest us here.

Consider the fixed point<sup>1</sup>  $[1, 0, 0, 0, 0] \in \mathbb{P}^4$ , and take local affine coordinates  $y_a = x_a/x_0$ ,  $a = 1, 2, 3, 4$  around this point. Then the  $\mathbb{Z}_5$  action is

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<sup>1</sup>The analysis is the same for any of the five fixed points of the  $\mathbb{Z}_5$  action.

given by

$$y_a \rightarrow \zeta^a y_a$$

and an invariant polynomial must locally be of the form

$$f = \alpha_0 + y_1 y_4 - y_2 y_3 + \text{higher-order terms}$$

where  $\alpha_0 := A_{00000}$  is one of the constant coefficients in (2.1) and we have chosen the coefficients of the quadratic terms by rescaling the coordinates.<sup>2</sup>

If we make the special choice  $\alpha_0 = 0$  (which corresponds to a codimension one locus in the moduli space of invariant hypersurfaces), we obtain a variety  $\tilde{X}_0$  on which the action is no longer free.

But now we see what turns out to be a general feature of this sort of situation: when  $\alpha_0 = 0$  we actually have  $f = df = 0$  at the fixed point, meaning it is a node, or conifold, singularity on  $\tilde{X}_0$ . This means that on its quotient  $X_0$  we get a particular type of hyperquotient singularity. We will now study this singularity by the methods of toric geometry.

### 2.1 The conifold and $\mathbb{Z}_5$ -hyperconifold as toric varieties

We will take the conifold  $\mathcal{C}$  to be described in  $\mathbb{C}^4$  by the equation

$$y_1 y_4 - y_2 y_3 = 0. \tag{2.2}$$

This is a toric variety whose fan consists of a single cone, spanned by the vectors

$$\begin{aligned} v_1 &= (1, 0, 0), & v_2 &= (1, 1, 1), \\ v_3 &= (1, 1, 0), & v_4 &= (1, 0, 1). \end{aligned} \tag{2.3}$$

We can see that the four vertices lie on a hyperplane; this is equivalent to the statement that the conifold is a non-compact Calabi–Yau variety.

We can also give homogeneous coordinates for the conifold, following the prescription for toric varieties originally described by Cox:  $\mathcal{C} = (\mathbb{C}^4 \setminus \mathcal{S}) / \sim$ , where the excluded set is given by  $\mathcal{S} = \{z_1 = z_2 = 0, (z_3, z_4) \neq (0, 0)\} \cup$

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<sup>2</sup>The quadratic terms correspond to some quadratic form  $\eta$  on  $\mathbb{C}^4$ . Assuming that  $\eta$  is non-degenerate, it will always take the given form in appropriate coordinates. For general choices of coefficients in (2.1),  $\eta$  will indeed be non-degenerate.

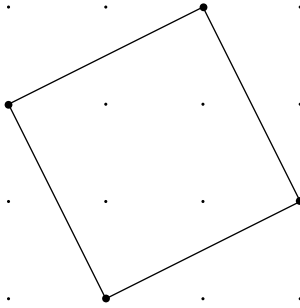


Figure 1: The toric diagram for the  $\mathbb{Z}_5$ -hyperconifold.

$\{z_3 = z_4 = 0, (z_1, z_2) \neq (0, 0)\}$ , and the equivalence relation is

$$(z_1, z_2, z_3, z_4) \sim (\lambda z_1, \lambda z_2, \lambda^{-1} z_3, \lambda^{-1} z_4) \text{ for } \lambda \in \mathbb{C}^*. \tag{2.4}$$

The explicit isomorphism between this representation and the hypersurface defined by (2.2) is given by

$$y_1 = z_1 z_3, \quad y_2 = z_1 z_4, \quad y_3 = z_2 z_3, \quad y_4 = z_2 z_4. \tag{2.5}$$

The  $\mathbb{Z}_5$ -hyperconifold singularity is obtained by imposing the equivalence relation  $(y_1, y_2, y_3, y_4) \sim (\zeta y_1, \zeta^2 y_2, \zeta^3 y_3, \zeta^4 y_4)$ , where  $\zeta = e^{2\pi i/5}$ . Using the above equations we can express this in terms of the  $z$  coordinates as

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^2 z_2, \zeta z_3, \zeta^2 z_4).$$

This equivalence relation must be imposed in addition to the earlier one.<sup>3</sup>

The resulting variety is again a toric Calabi–Yau variety; the intersection of its fan with the hyperplane on which the vertices lie is drawn in figure 1. The singularity could easily be resolved by sub-dividing the fan, but we will postpone a discussion of resolution of singularities until later. First, we want to prove that the example presented here is far from unique.

### 3 Local hyperconifold singularities in general

The above discussion can be generalized to many other families of multiply connected Calabi–Yau threefolds. To this end consider a CY threefold  $\tilde{X}$  which, for appropriately chosen complex structure, admits a free holomorphic action by some discrete group  $G$ . Then there exists a smooth quotient

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<sup>3</sup>Note that the power of  $\zeta$  multiplying  $z_1$  can always be chosen to be trivial by simultaneously applying a rescaling from (2.4).

$X = \tilde{X}/G$ , the deformations of which correspond to  $G$ -invariant deformations of  $\tilde{X}$ . For simplicity I will herein consider only the case in which  $G$  is cyclic,  $G \cong \mathbb{Z}_N$ . This is not a great restriction, since there seem to be very few free actions of non-Abelian groups on Calabi–Yau manifolds, and in any case, every non-Abelian group has Abelian subgroups, to which the following discussion applies.

As we have seen for the  $\mathbb{Z}_5$ -symmetric quintic, it may be that for special choices of the complex structure of  $X$  (generally on a codimension-one locus in its moduli space) the action of  $\mathbb{Z}_N$  on  $\tilde{X}$  will no longer be free. One might expect the resulting singularities on  $X$  to be simple orbifold singularities, locally of the form  $\mathbb{C}^3/\mathbb{Z}_N$ . In the case of the quintic though, we actually obtained a quotient of the conifold. We now demonstrate that this is a general phenomenon.

### 3.1 Analysis of fixed points

Let  $g_N$  denote the generator of  $\mathbb{Z}_N$ , suppose that  $\tilde{X}$  is locally determined by  $k$  equations  $f_1 = \dots = f_k = 0$  in  $\mathbb{C}^{k+3}$ , on which some  $\mathbb{Z}_N$  action is given, and let  $P_0 \in \mathbb{C}^{k+3}$  be a fixed point of this action. Then we can choose local coordinates  $x_1, \dots, x_{k+3}$  at  $P_0$  such that the action of  $g_N$  is given by  $x_i \rightarrow \zeta^{q_i} x_i$ , where  $\zeta = e^{2\pi i/N}$  and  $q_i \in \{0, \dots, N-1\}$ . Let  $I$  be the set of fixed points of this action, and order the coordinates such that  $I$  is given locally by  $x_{\dim I+1} = \dots = x_{k+3} = 0$ . This is equivalent to  $q_1 = \dots = q_{\dim I} = 0$  and  $q_i \neq 0$  for  $i > \dim I$ .

By taking linear combinations of the polynomials if necessary, we can assume that  $g_N \cdot f_a = \zeta^{Q_a} f_a$ . What we mean by this is that  $f_a(g_N \cdot P) = \zeta^{Q_a} f_a(P)$  for  $P \in \mathbb{C}^{k+3}$ . This immediately implies that if  $Q_a \neq 0$ , then we must have  $f_a|_I \equiv 0$ . But since by assumption  $\tilde{X}$  does not generically intersect  $I$ , at least  $\dim I + 1$  of the polynomials must be non-trivial when restricted to  $I$ , so that they have no common zeros. We conclude that at least  $\dim I + 1$  of the polynomials must be invariant under the group action.

Now suppose that we choose special polynomials such that the corresponding variety  $\tilde{X}_0$  intersects  $I$  at a point, and identify this point with  $P_0$  above:  $I \cap \tilde{X}_0 = \{P_0\}$ . The expansion of an invariant polynomial  $f_a$  (i.e.,  $Q_a = 0$ ) around  $P_0$  is then

$$f_a = \sum_{i=1}^{\dim I} C_{a,i} x_i + \mathcal{O}(x^2).$$

Now we can see what goes wrong. At  $P_0$  we have

$$\frac{\partial f_a}{\partial x_i} = \begin{cases} C_{a,i}, & i \leq \dim I, \\ 0, & i > \dim I, \end{cases}$$

so the matrix  $\partial f_a / \partial x_i$ , for  $f_a$  ranging over invariant polynomials, has maximal rank  $\dim I$ . But since, as argued above, there are at least  $\dim I + 1$  invariant polynomials, at the point  $P_0$  we get  $f_a = 0$  for all  $a$  and

$$df_1 \wedge \cdots \wedge df_{\dim I+1} = 0, \quad \text{and hence} \quad df_1 \wedge \cdots \wedge df_k = 0.$$

So the variety  $\tilde{X}_0$  is singular at this point, and in fact generically it will have a node, or conifold singularity. This means that the quotient variety  $X_0$  develops a worse local singularity: a quotient of the conifold by a  $\mathbb{Z}_N$  action fixing only the singular point. This is what we will now call a  $\mathbb{Z}_N$ -hyperconifold.

It should be noted that there is no reason for any other singularities to occur on  $X_0$ , and indeed it can be checked in specific cases that only one singular point develops.

### 3.2 The hyperconifolds torically

We now want to give explicit descriptions of the types of singularities whose existence in compact Calabi–Yau varieties we demonstrated above. There are known Calabi–Yau threefolds with fundamental group  $\mathbb{Z}_N$  for  $N = 2, 3, 4, 5, 6, 7, 8, 10, 12$ , and all cases except  $N = 7$  occur as quotients of CICYs [16, 17, 40, 41]. For these we can perform analyses similar to that presented earlier for the  $\mathbb{Z}_5$  quotient of the quintic, and obtain singular varieties containing isolated hyperconifold singularities. Each of these has a local toric description, which will be presented below. Since they are all toric Calabi–Yau varieties, the vectors generating the one-dimensional cones of their fans lie on a hyperplane; figures 2 and 3 are collections of diagrams showing the intersection of the fans with this hyperplane. It should be noted that from the diagrams it is obvious that each singularity admits at least one toric crepant resolution. However, we are only interested in those which give Calabi–Yau resolutions of the compact variety in which the singularity resides. Determining whether such a resolution exists requires more work, which we defer to Section 4.



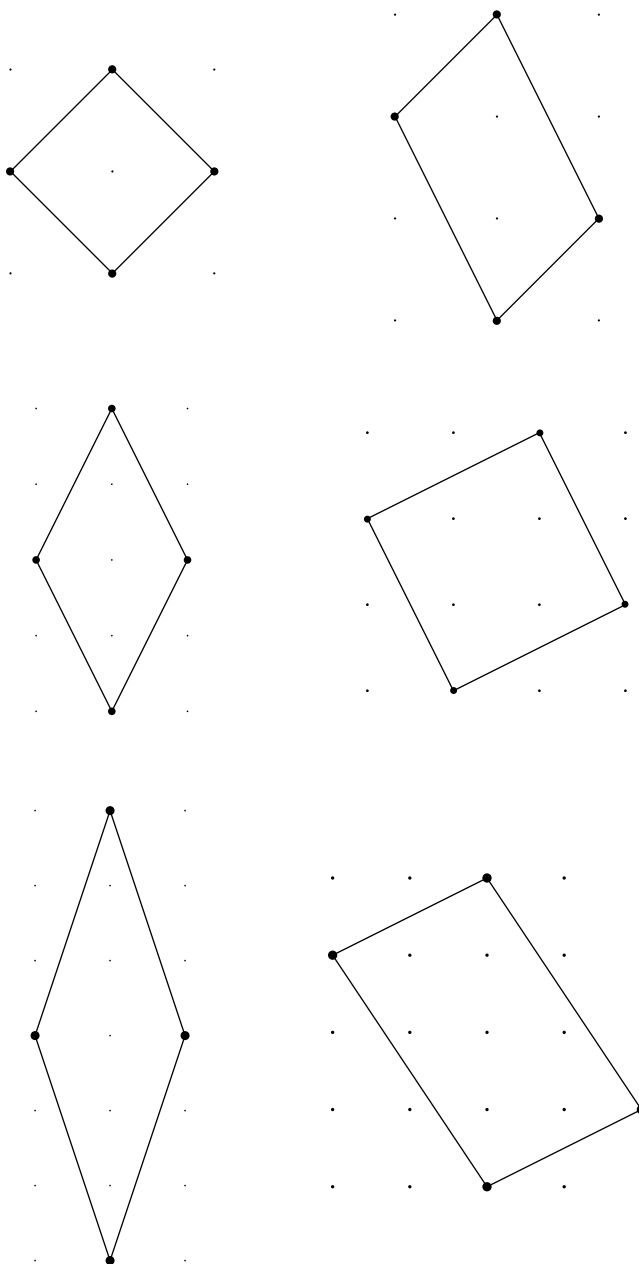


Figure 2: The toric diagrams for the  $\mathbb{Z}_N$ -hyperconifolds, where  $N = 2, 3, 4, 5, 6$  and  $8$ .

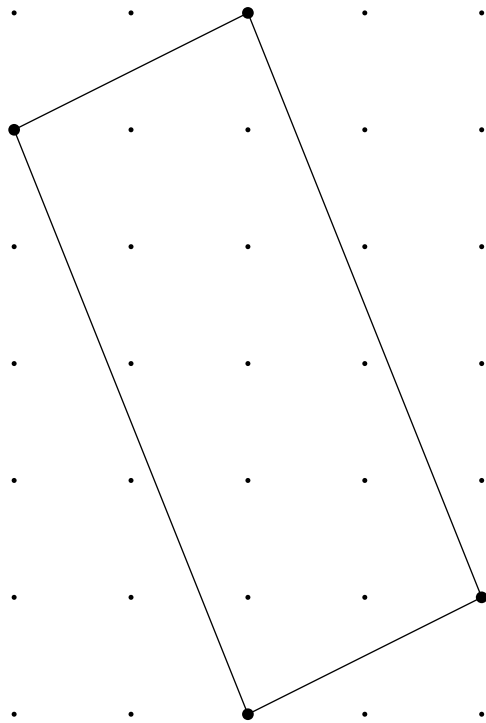
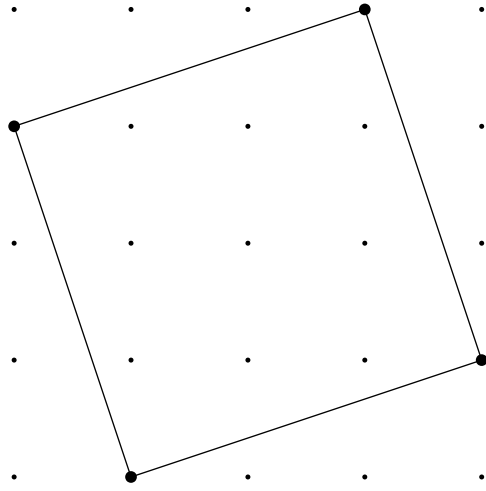


Figure 3: The toric diagrams for the  $\mathbb{Z}_{10}$  and  $\mathbb{Z}_{12}$ -hyperconifold singularities.

$\mathbb{Z}_2$  quotient

Note that, as discussed above, the only point on the conifold fixed by the group actions we are considering will be the singular point itself. As such, there is only a single possible action of  $\mathbb{Z}_2$ :

$$(y_1, y_2, y_3, y_4) \rightarrow (-y_1, -y_2, -y_3, -y_4).$$

In terms of the homogeneous coordinates this gives the additional equivalence relation

$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, -z_3, -z_4).$$

The resulting singularity is one which *has* appeared in the physics literature, as mentioned earlier. The difference here is that we have given a prescription for embedding this singularity in a compact Calabi–Yau variety, in such a way that it admits both a smooth deformation and, as we will see later, a resolution.

$\mathbb{Z}_3$  quotient

Similarly to the  $\mathbb{Z}_2$  case, there is only a single action of  $\mathbb{Z}_3$  on the conifold with an isolated fixed point:

$$(y_1, y_2, y_3, y_4) \rightarrow (\zeta y_1, \zeta y_2, \zeta^2 y_3, \zeta^2 y_4),$$

where  $\zeta = e^{2\pi i/3}$ . In terms of the homogeneous coordinates this leads to

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta z_2, \zeta z_3, \zeta z_4).$$

$\mathbb{Z}_4$  quotient

The group  $\mathbb{Z}_4$  has a  $\mathbb{Z}_2$  subgroup, which must also act non-trivially on each coordinate  $y_a$ , so again there is a unique action consistent with this:

$$(y_1, y_2, y_3, y_4) \rightarrow (i y_1, i y_2, -i y_3, -i y_4).$$

In terms of the homogeneous coordinates this implies

$$(z_1, z_2, z_3, z_4) \sim (z_1, -z_2, i z_3, i z_4).$$

$\mathbb{Z}_5$  quotient

This is the first case where there are two actions of the group on the conifold which fix only the origin. This is true for  $\mathbb{Z}_5$  and several of the

larger cyclic groups discussed below, but in each case only one of the actions actually occurs in known examples. For  $\mathbb{Z}_5$  it is

$$(y_1, y_2, y_3, y_4) \rightarrow (\zeta y_1, \zeta^2 y_2, \zeta^3 y_3, \zeta^4 y_4), \tag{3.1}$$

where  $\zeta = e^{2\pi i/5}$ . We have already seen this in our original example of the quintic. In terms of the homogeneous coordinates the new equivalence relation is

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^2 z_2, \zeta z_3, \zeta^2 z_4).$$

*$\mathbb{Z}_6$  quotient*

For  $\mathbb{Z}_6$  we can once again find the action by general arguments. If we require all elements of the group to act with only a single fixed point, there is only one possibility:

$$(y_1, y_2, y_3, y_4) \rightarrow (\zeta y_1, \zeta y_2, \zeta^5 y_3, \zeta^5 y_4),$$

where  $\zeta = e^{\pi i/3}$ . The identification on the homogeneous coordinates is therefore

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^4 z_2, \zeta z_3, \zeta z_4).$$

*$\mathbb{Z}_8$  quotient*

As in the  $\mathbb{Z}_5$  case, there are multiple actions of  $\mathbb{Z}_8$  on the conifold, which fix only the origin, but only one is realised in the present context. The only free  $\mathbb{Z}_8$  actions I know on compact Calabi–Yau threefolds are the one described in [16] and those related to it by conifold transitions [40, 42]. These can be deformed to obtain a local conifold singularity with the following quotient group action:

$$(y_1, y_2, y_3, y_4) \rightarrow (\zeta y_1, \zeta^3 y_2, \zeta^5 y_3, \zeta^7 y_4),$$

where  $\zeta = \exp \pi i/4$ . The equivalence relation on the homogeneous coordinates is therefore

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^4 z_2, \zeta z_3, \zeta^3 z_4).$$

*$\mathbb{Z}_{10}$  quotient*

Several free actions of  $\mathbb{Z}_{10}$  on Calabi–Yau manifolds were described in [17]. If we allow one of these to develop a fixed point, the resulting action on the

conifold is

$$(y_1, y_2, y_3, y_4) \rightarrow (\zeta y_1, \zeta^3 y_2, \zeta^7 y_3, \zeta^9 y_4),$$

where  $\zeta = \exp \pi i/5$ . The corresponding equivalence relation on the homogeneous coordinates is

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^6 z_2, \zeta z_3, \zeta^3 z_4).$$

*$\mathbb{Z}_{12}$  quotient*

The largest cyclic group known to act freely on a Calabi–Yau manifold is  $\mathbb{Z}_{12}$ , and this was discovered only recently [41]. The resulting action on the conifold is

$$(y_1, y_2, y_3, y_4) \rightarrow (\zeta y_1, \zeta^5 y_2, \zeta^7 y_3, \zeta^{11} y_4),$$

where  $\zeta = \exp \pi i/6$ . The corresponding equivalence relation on the homogeneous coordinates is

$$(z_1, z_2, z_3, z_4) \sim (z_1, \zeta^6 z_2, \zeta z_3, \zeta^5 z_4).$$

**3.3 Topology of the singularities**

Topologically, the conifold is a cone over  $S^3 \times S^2$ . It would be nice to relate the group actions described herein to this topology. Evslin and Kuperstein [43] have provided a convenient parametrisation of the base of the conifold for just this sort of purpose, which I will use here:

Parametrize the conifold as the set of degenerate  $2 \times 2$  complex matrices

$$W = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \quad \det W = 0, \tag{3.2}$$

and identify the base with the subset satisfying  $\text{Tr}(W^\dagger W) = 1$ . Now identify  $S^3$  with the underlying topological space of the group  $SU(2)$ , and  $S^2$  with the space of unit two-vectors modulo phases. Then we map the point  $(X, v) \in S^3 \times S^2$  to

$$W = X v v^\dagger. \tag{3.3}$$

This is shown to be a homeomorphism in [43]. The actions of  $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$  and  $\mathbb{Z}_6$ , described in Section 3.2, are all realized in this description by

$$W \rightarrow \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} W, \tag{3.4}$$

where  $\zeta = e^{2\pi i/N}$ ,  $N = 2, 3, 4$  and  $6$ , respectively. We see from (3.3) that this can be considered as an action purely on the  $S^3$  factor of the base, the quotient by which is the lens space  $L(N, 1)$ . Topologically then, the singularity is locally a cone over  $L(N, 1) \times S^2$ . In fact these same spaces were considered in [44].

The more complicated cases of  $\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{10}$  and  $\mathbb{Z}_{12}$  quotients do not have such a straightforward topological description, but could be analysed along the lines of [43].

## 4 Global resolutions

In the preceding section we have described the local structure of the hyperconifolds using toric geometry; now we want to address the question of their resolution. Certainly if we consider each case as a non-compact variety, they can all easily be resolved using toric methods. The interesting question is whether or not the compact varieties containing these singularities can be resolved to yield new Calabi–Yau manifolds.

### 4.1 Blowing up a node

It will be useful to first consider blowing-up an ordinary node, and only then turn to its quotients. Again we take the conifold  $\mathcal{C}$  to be given in  $\mathbb{C}^4$  by (2.2):

$$y_1 y_4 - y_2 y_3 = 0.$$

The singularity lies at the origin, and we can resolve it by blowing up this point. To do so we introduce a  $\mathbb{P}^3$  with homogeneous coordinates  $(t_1, t_2, t_3, t_4)$ , and consider the equations  $y_i t_j - y_j t_i = 0$  in  $\mathbb{C}^4 \times \mathbb{P}^3$ . This has the effect of setting  $(t_1, t_2, t_3, t_4) \propto (y_1, y_2, y_3, y_4)$  when at least one  $y_i$  is non-zero, but leaving the  $t$ 's completely undetermined at the origin. In this way, we “blow up” a single point to an entire copy of  $\mathbb{P}^3$ , and have a natural projection map  $\pi$ , which blows it down again. The blow-up of the conifold

is then defined to be the closure of the pre-image of its smooth points:

$$\widehat{\mathcal{C}} = \overline{\pi^{-1}(\mathcal{C} \setminus \{\mathbf{0}\})}.$$

Therefore,  $\widehat{\mathcal{C}}$  is isomorphic to  $\mathcal{C}$  away from the node, but the node itself is replaced by the surface in  $\mathbb{P}^3$  given by

$$t_1 t_4 - t_2 t_3 = 0,$$

which is in fact just  $\mathbb{P}^1 \times \mathbb{P}^1$ . This is called the *exceptional divisor* of the blow-up, and we will denote it by  $E$ . Another important piece of information is the normal bundle  $\mathcal{N}_{E|\widehat{\mathcal{C}}}$  to  $E$  inside  $\widehat{\mathcal{C}}$ . If  $\mathcal{O}(n, m)$  denotes the line bundle, which restricts to the  $n^{\text{th}}$  (resp.  $m^{\text{th}}$ ) power of the hyperplane bundle on the first (resp. second)  $\mathbb{P}^1$ , then in this case the normal bundle is  $\mathcal{O}(-1, -1)$ . This can be verified by taking an affine cover and writing down transition functions, but the toric formalism, to which we turn shortly, will let us see this much more easily. In any case, with this information we can demonstrate that  $\widehat{\mathcal{C}}$  is *not* Calabi–Yau. To see why, recall the adjunction formula for the canonical bundle of the hypersurface  $E$  in terms of that of  $\widehat{\mathcal{C}}$

$$\omega_E = \omega_{\widehat{\mathcal{C}}}|_E \otimes \mathcal{N}_{E|\widehat{\mathcal{C}}}. \tag{4.1}$$

Therefore, if  $\omega_{\widehat{\mathcal{C}}}$  were trivial, we would have  $\omega_E \cong \mathcal{N}_{E|\widehat{\mathcal{C}}} \cong \mathcal{O}(-1, -1)$ , but it is a well-known fact that  $\omega_E = \mathcal{O}(-2, -2)$ , so we conclude that  $\widehat{\mathcal{C}}$  has a non-trivial canonical bundle. This is why the blow-up of a node does not generally feature in discussions of Calabi–Yau manifolds. We will see soon why it becomes relevant once we want to consider quotients.

The final important point is that the blowing-up procedure automatically gives us another projective variety (i.e. a Kähler manifold if it is smooth), since the blow-up is embedded in the product of the original space with a projective space.

## 4.2 The toric picture and the $\mathbb{Z}_2$ -hyperconifold

We can also blow up the node on the conifold using toric geometry. Recall that the fan for  $\mathcal{C}$  consists of a single cone spanned by the four vectors given in (2.3), plus its faces. To this set of vectors we want to add  $v_5 = v_1 + v_2 = v_3 + v_4$ , and sub-divide the fan accordingly. The result is shown in figure 4. We now have five homogeneous coordinates and two independent

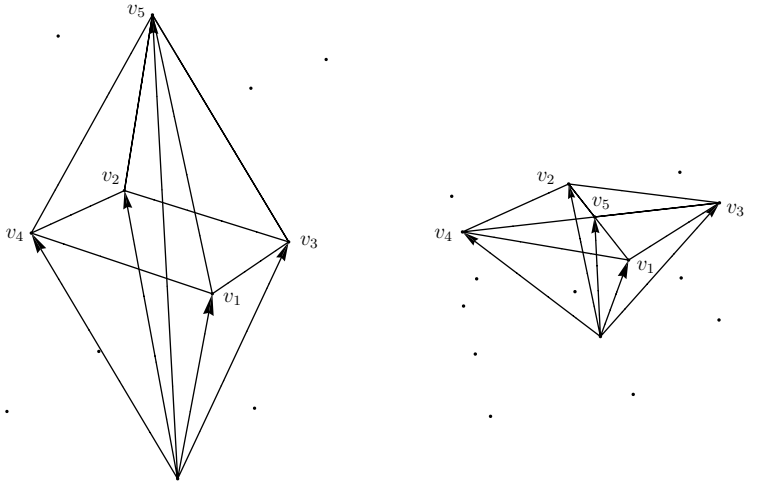


Figure 4: The fans for the blow-up of the conifold at its singular point and its  $\mathbb{Z}_2$  quotient.

rescalings (we won't explicitly describe the set to be removed before taking the quotient — this can be read from the fan).

$$(z_1, z_2, z_3, z_4, z_5) \sim (\lambda z_1, \lambda z_2, \mu z_3, \mu z_4, \lambda^{-1}\mu^{-1}z_5) \quad \lambda, \mu \in \mathbb{C}^*. \quad (4.2)$$

From this data we can easily see that  $(z_1, z_2)$  parametrize a  $\mathbb{P}^1$ , as do  $(z_3, z_4)$ , and  $z_5$  is a coordinate on the fibre of  $\mathcal{O}(-1, -1)$ . When  $z_5 \neq 0$ , we can choose  $\mu = \lambda^{-1}z_5$ , and we obtain the isomorphism to  $\mathcal{C} \setminus \{0\}$ . The remaining points are on the toric divisor given by  $z_5 = 0$ , and it is clear that this is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . So this toric variety is indeed the blow-up of  $\mathcal{C}$  at the origin. The toric formalism makes it clear that the resolution is not crepant, since the new vector does not lie on the same hyperplane as the others.

We now turn our attention to the  $\mathbb{Z}_2$ -hyperconifold, which was described in Section 3.2, and which we will denote by  $\mathcal{C}_2$ . In this case, the blow-up of the singular point is obtained by adding a vector that lies on the same hyperplane as the first four, meaning that the resulting resolution  $\widehat{\mathcal{C}}_2$  is also Calabi–Yau. Its fan is shown in figure 5. The relations are now  $2v_5 = v_1 + v_2 = v_3 + v_4$ , and the five vectors can be taken to be

$$\begin{aligned} v_1 &= (1, -1, 0), & v_2 &= (1, 1, 0), \\ v_3 &= (1, 0, -1), & v_4 &= (1, 0, 1), \\ v_5 &= (1, 0, 0). \end{aligned}$$



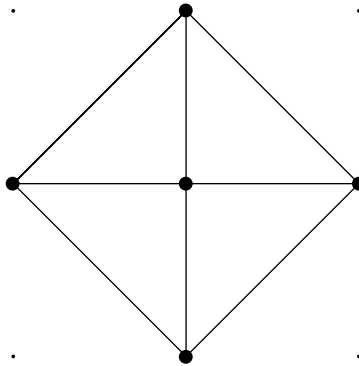


Figure 5: The toric diagram for the resolution of the  $\mathbb{Z}_2$ -hyperconifold.

The resulting equivalence relations on the homogeneous coordinates are

$$(w_1, w_2, w_3, w_4, w_5) \sim (\lambda w_1, \lambda w_2, \mu w_3, \mu w_4, \lambda^{-2} \mu^{-2} w_5) \quad \lambda, \mu \in \mathbb{C}^*. \quad (4.3)$$

This is very similar to (4.2), but in this case  $w_5$  is seen to be a coordinate on  $\mathcal{O}(-2, -2)$  rather than  $\mathcal{O}(-1, -1)$ . As such, the adjunction formula (4.1) says that the canonical bundle of  $\widehat{\mathcal{C}}_2$  restricts to be trivial on the exceptional divisor, consistent with  $\widehat{\mathcal{C}}_2$  being Calabi–Yau.

There is another nice way to think about the resolution of  $\mathcal{C}_2$ . We begin by noticing that the blown-up conifold  $\widehat{\mathcal{C}}$  is actually a ramified double cover of  $\widehat{\mathcal{C}}_2$ , with the explicit map being given by

$$w_i = z_i, \text{ for } i = 1, 2, 3, 4, \quad w_5 = (z_5)^2. \quad (4.4)$$

This deserves some elaboration. It is clear from (4.4) that the map is two-to-one everywhere except along the exceptional divisor given by  $z_5 = 0$ . In fact, it can be thought of as an identification  $z_5 \sim -z_5$  on the fibres of  $\mathcal{O}(-1, -1)$  over  $\mathbb{P}^1 \times \mathbb{P}^1$ . Since the fixed point set of the involution  $z_5 \rightarrow -z_5$  is of complex codimension one, taking the quotient actually does not introduce any singularity (which is clear here since  $\widehat{\mathcal{C}}_2$  is manifestly smooth). So, we can think of the resolution of  $\mathcal{C}_2$  in two ways: either we blow up the singular point of  $\mathcal{C}_2$ , or we blow up the node on the covering space, and then take the  $\mathbb{Z}_2$  quotient.

Note that the procedure described above is completely local (we blew up a point), and therefore can be performed inside any compact Calabi–Yau variety  $X_0$  in which the singularity  $\mathcal{C}_2$  occurs, to yield a new compact Calabi–Yau manifold. This should be compared to the small resolution

of the conifold, which involves blowing up a sub-variety which extends “to infinity” in  $\mathcal{C}$  (in fact the variety given by  $y_1 = y_2 = 0$ ), so that the existence of the Calabi–Yau resolution depends on the global structure.

### 4.3 The $\mathbb{Z}_{2M}$ -hyperconifolds

Having demonstrated the existence of crepant projective resolutions (i.e., Calabi–Yau resolutions) for Calabi–Yau varieties containing the singularity  $\mathcal{C}_2$ , we can easily do the same for the quotients of the conifold by all cyclic groups of even order. This is achieved by breaking the process down into several steps.<sup>4</sup>

The unique  $\mathbb{Z}_2$  subgroup of  $\mathbb{Z}_{2M}$  fixes exactly the singular point, and we can blow this up by adding the ray through the point  $v_5 = \frac{1}{4}(v_1 + v_2 + v_3 + v_4)$  and sub-dividing the fan accordingly. Alternatively, we can think of this as first taking the quotient by  $\mathbb{Z}_2 \subset \mathbb{Z}_{2M}$ , blowing up the resulting  $\mathcal{C}_2$  singularity, and then taking the quotient by the induced action of  $\mathbb{Z}_{2M}/\mathbb{Z}_2 \cong \mathbb{Z}_M$ . Either way, we obtain a variety with only  $\mathbb{Z}_M$  orbifold singularities. There are then two cases:

- If  $M$  is odd, it turns out that there is a unique way to further sub-divide the fan to obtain a smooth variety. In [45], it is shown that for a projective threefold with only orbifold singularities, one obtains a global projective crepant resolution by choosing an appropriate crepant resolution on each affine patch. If there is a unique choice for each, we therefore automatically obtain the projective resolution, so we are done.
- If  $M$  is even, then  $\mathbb{Z}_M$  contains a unique  $\mathbb{Z}_2$  subgroup, and the fixed point set of this subgroup is a pair of disjoint curves, which are toric orbits (this follows from inspecting the diagrams case-by-case). These are given by two-cones, spanned by  $v_i, v_j$ , and in each case the vector  $\frac{1}{2}(v_i + v_j)$  is integral, so can be added to the fan to blow up the fixed curve (in fact this is just the well-known resolution of the  $A_1$  surface singularity, fibred over the curves). We iterate this process until we are left with  $\mathbb{Z}_{M'}$  orbifold singularities for  $M'$  odd, and the fan then has a unique smooth sub-division.

Note that I am not claiming that the resolutions obtained are the unique Kähler ones. Several of the hyperconifolds admit multiple resolutions differing by flops, and it is possible that more than one of these corresponds to a projective resolution.

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<sup>4</sup>The following argument is partly due to Balázs Szendrői.

The preceding prescription is easy to understand in particular cases, as we will now illustrate with the complicated  $\mathbb{Z}_{12}$ -hyperconifold. We begin with the fan in figure 3, and blow up the singular point, which adds a ray through the geometric centre of the top-dimensional cone, and divides it into four (see figure 6). The result is a fan for a variety containing a chain of

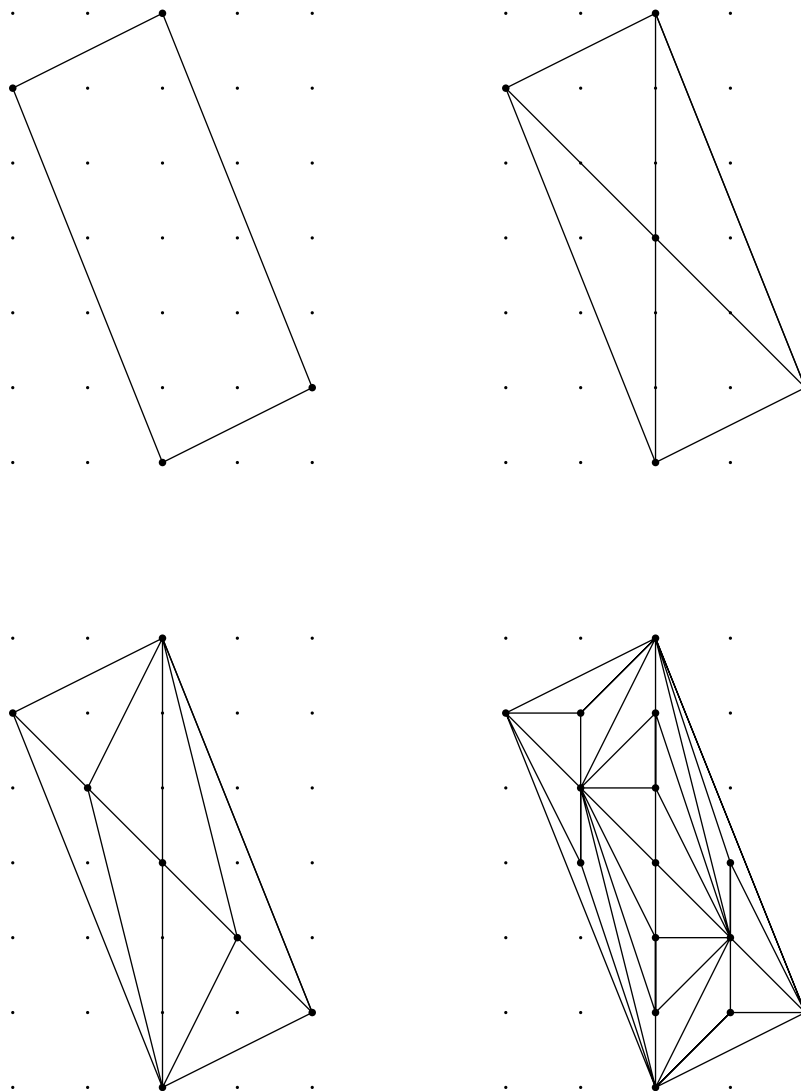


Figure 6: The three steps involved in resolving the  $\mathbb{Z}_{12}$ -hyperconifold singularity. First, we blow up the singular point, then the two curves fixed by  $\mathbb{Z}_2 \subset \mathbb{Z}_6$ , and finally we perform the unique maximal sub-division of the resulting fan.

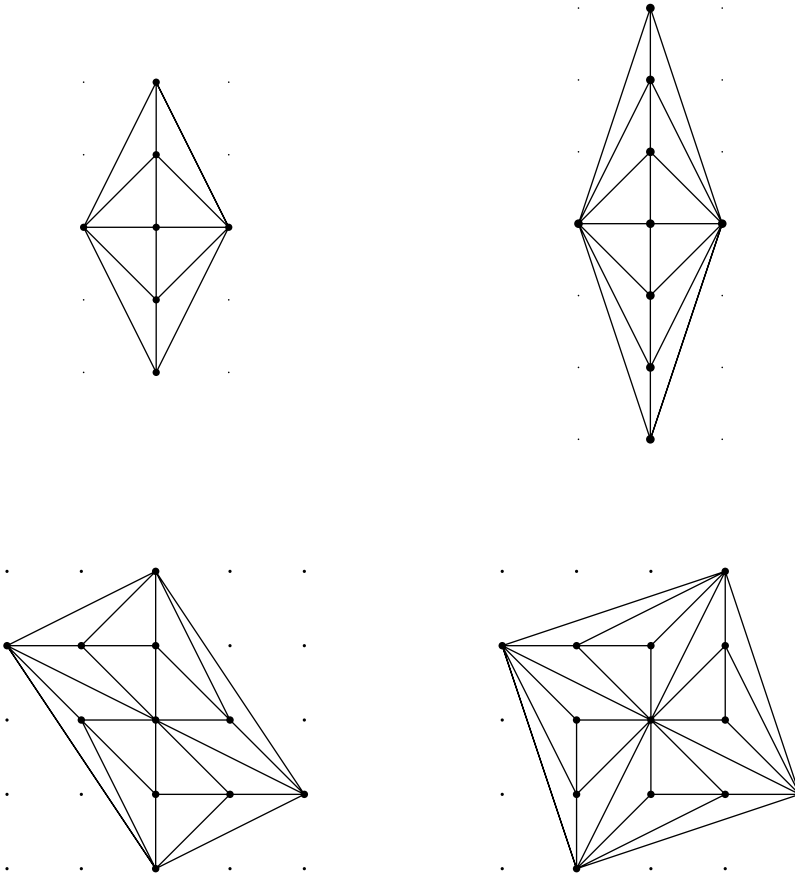


Figure 7: The fans for the Kähler crepant resolutions of the  $\mathbb{Z}_{2M}$  hyperconifold singularities, where  $M = 2, 3, 4$  and  $5$ .

four genus zero curves meeting in points. Two of these are curves of  $\mathbb{C}_2/\mathbb{Z}_2$  orbifold singularities, and the other two of  $\mathbb{C}_2/\mathbb{Z}_3$  singularities. The four points of intersection are locally  $\mathbb{C}_3/\mathbb{Z}_6$  orbifold singularities. We can blow up the (disjoint)  $\mathbb{Z}_2$  curves by bisecting the corresponding two-cones and sub-dividing the fan accordingly. This leaves us with eight top-dimensional simplicial cones, each of which has a unique crepant sub-division, giving us the final smooth, crepant, Kähler resolution of the singularity.

We can perform the same analysis for each  $\mathbb{Z}_{2M}$ -hyperconifold, obtaining the fans in figure 7. The reader may find it amusing to follow the steps in each case, and verify the resulting fans. At present I can provide no argument that varieties containing the  $\mathbb{Z}_3$ - and  $\mathbb{Z}_5$ -hyperconifolds also admit Kähler crepant resolutions, but one is naturally drawn to conjecture that this is the case. The following comments would then apply to these cases too.

It is easy to obtain certain topological data about these resolutions. From the toric diagrams we see that in each case the exceptional set  $E$  is simply connected, which is the case for any toric variety whose fan contains a top-dimensional cone. Therefore, the resolution of  $X_0$  is simply connected, even though the smooth Calabi–Yau  $X$ , of which  $X_0$  is a deformation, had fundamental group  $\mathbb{Z}_N$ . This contrasts with the case of a conifold transition, where the fundamental group does not change.

We can also simply read off the diagram that the exceptional set of the resolution of the  $\mathbb{Z}_N$ -hyperconifold has Euler characteristic  $\chi(E) = 2N$ , since  $\chi$  is just the number of top-dimensional cones in the fan. We can therefore calculate  $\chi(\widehat{X})$  quite easily. Topologically,  $\widehat{X}_0$  is obtained from  $\widetilde{X}$  by shrinking an  $S^3$  to a point  $P_0$ , so  $\chi(\widehat{X}_0) = \chi(\widetilde{X}) + 1$ . We delete  $P_0$ , quotient by  $\mathbb{Z}_N$ , then glue in  $E$ , so

$$\chi(\widehat{X}) = \chi(\widetilde{X})/N + \chi(E) = \chi(X) + 2N. \tag{4.5}$$

Finally, the resolution of the  $\mathbb{Z}_N$ -hyperconifold introduces  $N - 1$  new divisor classes, so we can actually calculate all the Betti numbers of  $\widehat{X}$  in terms of those of  $X$ :

$$b_1(\widehat{X}) = b_5(\widehat{X}) = 0, \quad b_2(\widehat{X}) = b_4(\widehat{X}) = b_2(X) + N - 1, \quad b_3(\widehat{X}) = b_3(X) - 2. \tag{4.6}$$

## 5 Hyperconifold transitions in string theory?

A natural question to ask is whether the “hyperconifold transitions” described in this paper can be realized in string theory, as their cousins flops and conifold transitions can. At this stage I will merely make some suggestive observations.<sup>5</sup>

Consider type IIB string theory on a Calabi–Yau manifold  $X$  with fundamental group  $\mathbb{Z}_N$ , and vary the complex structure moduli until we approach a singular variety  $X_0$ . We have seen (at least in the simplest cases) that topologically this looks like shrinking a three-cycle  $L(N, 1)$  to a point. Therefore, just as in the conifold case, there will be D3-brane states becoming massless [23]. This manifests in the low-energy theory as a hypermultiplet charged under a  $U(1)$  gauge group coming from the R–R sector, and although it becomes massless there is still a D-term potential preventing its scalars from developing a VEV. However, these D-brane states are not

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<sup>5</sup>The following is superseded by the follow-up paper [46].

necessarily the only ones becoming massless at the hyperconifold point — there are  $N - 1$  twisted sectors coming from strings wrapping non-trivial loops on  $L(N, 1)$ , and these strings attain zero length at the singular point. It is therefore conceivable that these twisted sectors give rise to a new branch of the low-energy moduli space, and that moving onto this branch corresponds to resolving the singularity of the internal space. Since there are  $N - 1$  new divisors/Kähler parameters on the resolution, there must be  $N - 1$  new flat directions.

The conjecture then is that in the low-energy field theory at the hyperconifold point, there is a new  $(N - 1)$ -dimensional branch of moduli space coming from the twisted sectors in the string theory. The new flat directions correspond to Kähler parameters on the resolution of the singular variety, and giving them VEVs resolves the singularity. It would be interesting to try to verify this picture.

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