Adv. Theor. Math. Phys. **12** (2008) 853–888

Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes

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Abstract

The present work extends our short communication L. Andersson, M. Mars and W. Simon, *Local existence of dynamical and trapping horizons*, Phys. Rev. Lett. **95** (2005), 111102. For smooth marginally outer trapped surfaces (MOTS) in a smooth spacetime, we define stability with respect to variations along arbitrary vectors v normal to the MOTS. After giving some introductory material about linear non-self-adjoint elliptic operators, we introduce the stability operator L_v and we characterize stable MOTS in terms of sign conditions on the principal eigenvalue of L_v. The main result shows that given a strictly stable MOTS $\mathcal{S}_0 \subset \Sigma_0$ in a spacetime with a reference foliation Σ_t, there is an open marginally outer trapped tube (MOTT), adapted to the reference foliation, which contains $\mathcal{S}_0$. We give conditions under which the MOTT can be completed. Finally, we show that under standard energy conditions on the spacetime, the MOTT must be either locally achronal, spacelike or null.

1 Introduction

The singularity theorems of Hawking and Penrose [2] assert the presence of incomplete geodesics in the time evolution of Cauchy data with physically reasonable matter containing a trapped surface. Studying the structure of these singularities and understanding whether generically they entail a loss of predictability power of the theory have become central issues in classical general relativity. More specifically, the weak cosmic censorship conjecture asserts, roughly speaking, that in the asymptotically flat case the singularity will be generically hidden from infinity by an event horizon and that a black hole will form. Since this conjecture aims precisely at showing that a black hole forms, any sensible approach to its proof should not make strong a priori assumptions on the global structure of the spacetime. It is therefore necessary to replace to concept of black hole, which requires full knowledge of the future evolution of a spacetime, with a quasi-local concept that captures its main features and that can be used as a tool to show the existence of a black hole.

This approach has been successfully applied in spherically symmetric spacetimes with matter fields, where the existence of a complete future null infinity and an event horizon can in fact be inferred from the presence of at least one trapped surface in the data [3], plus some extra assumptions. An important tool in the analysis is to study the sequence of the marginally trapped surfaces bounding the region with trapped surfaces within each slice of a spherically symmetric foliation.

Quasi-local versions of black holes are important not only in the context of cosmic censorship, but also they are relevant in any physical situation involving black holes where no global knowledge of the spacetime is available. An outstanding example is the dynamics and evolution of black holes. In the strong field regime, these evolutions are so complex that they can only be approached with the aid of numerical methods. In most cases, numerical computations can only evolve the spacetime a finite amount of time, which makes the global definition of black hole of little practical use. A quasilocal definition becomes necessary even to define what is understood by a black hole in this context. More importantly, such a definition is crucial in order to be able to track the location of the black holes and to extract relevant physical information from their evolution. Over the last few years, marginally outer trapped surfaces (MOTS) and the hypersurfaces in spacetime which they sweep out during a time evolution, have become standard as quasilocal replacements of black holes and have been studied extensively, using both numerical methods (see, e.g. [4–6]) as well as analytically, with either mathematical [7] or more physical scope [8]. See [9] for a review of some of these issues. However, many open problems still remain. Only in spherical symmetry a rather complete picture has been obtained thanks to [3, 10, 11]. In general even finding examples is not easy, see however [12–14] for interesting non-spherical examples.

A MOTS is a spacelike surface of codimension two, such that the null expansion θ with respect to the outgoing null normal vanishes. The notions of "outer" and "outgoing" are simply defined by the choice of a null section in the two-dimensional normal bundle of the surface. We call a 3-surface foliated by MOTS a marginally outer trapped tube (MOTT) [15, 16]; an alternative terminology adopted in [1] is "trapping horizon", c.f. [17].

The wide range of applicability of MOTS motivated us to study their propagation in spacetime from an analytical point of view and in a general context, i.e., assuming neither symmetries nor the presence of any trapped regions a priori. In the context of the initial value problem in general relativity, namely in a smooth spacetime foliated by smooth hypersurfaces Σ_t, it is natural to ask the following: Given a MOTS $\mathcal{S}_0$ on some initial leaf Σ_0, does it "propagate" to the adjacent leaves Σ_t of the foliation? In other words, is there a MOTT starting at $\mathcal{S}_0$ whose marginally outer trapped leaves lie in the time slices Σ_t?

It turns out that the key property of a MOTS $\mathcal{S}_0$ relevant for this question is its "stability" with respect to the initial leaf Σ_0. This concept has interesting applications even when considered purely inside a hypersurface Σ_0, in particular for the topology of $\mathcal{S}_0$, and also for the property of being a "barrier" for weakly outer trapped and weakly outer untrapped surfaces (defined by $\theta \leq 0$ and $\theta \geq 0$, respectively). We shall discuss these two issues, which both originate in the work of Hawking [18, 19], before turning to the question of propagation of $\mathcal{S}_0$ off Σ_0. Hawking's analysis was extended by Newman [20] who calculated the general variation δ_v of the expansion θ with respect to any transversal direction v. A central issue in these papers was to show that stable MOTS have spherical topology in the generic case. The classification of the "rigidity case," in which the torus is allowed, was investigated first for minimal surfaces [21] and subsequently also for generic MOTS and in higher dimensions [22–24]; see [25] for a review.

A key tool, both for the topological issues as well as for the present purposes, is the linear elliptic stability operator L_v defined by $\delta_{\psi v}\theta = L_v\psi$ for MOTS (introduced in [1], in deformation form already present in [20, 22]) where v is a suitably scaled vector. L_v is not self-adjoint in general, except in special cases as for example when the MOTS lies in a time symmetric slice. Nevertheless, linear elliptic operators always have a real principal eigenvalue, and the corresponding principal eigenfunction can be chosen to be positive [26–29]. While we define stability and strict stability of MOTS in terms of sign conditions for preferred variations $\delta_v \theta$, we can show that this definition is equivalent to requiring that the principal eigenvalue of L_v is non-negative or positive, respectively. In [1] we also showed that strictly stable MOTS are barriers for all weakly outer trapped and weakly outer untrapped surfaces in a neighbourhood. In addition to the properties of the stability operator, this result uses a representation of MOTS as graphs over some reference 2-surfaces. In terms of this representation, the condition of vanishing expansion θ characterizing a MOTS becomes a quasilinear elliptic equation for the graph function, for which a maximum principle holds. A similar application of the maximum principle for the functional θ is contained in the uniqueness results of Ashtekar and Galloway [7] where null hypersurfaces through a given MOTS and their intersection with a given spacelike MOTT were considered.

A barrier property of trapped and marginally trapped surfaces, which complements the one discussed above has been considered by Kriele and Hayward [30]. They showed that the boundary of the trapped region, i.e., the set of all points in a spacelike hypersurface contained in a bounding, trapped surface is a MOTS, under the assumption that it is *piecewise smooth*. By [1], this MOTS is necessarily stable. Andersson and Metzger [31] recently showed that the boundary of the trapped region is a smooth, embedded, stable MOTS, without any additional smoothness assumption.

Turning now to our main problem, namely the propagation of the MOTS $\mathcal{S}_0$ off Σ_0 to adjacent slices, we can prove this for some open time interval if the MOTS are strictly stable (c.f. Theorem 9.1). As a tool we first extend the graph representation of $\mathcal{S}_0 \subset \Sigma_0$ to 2-surfaces $\mathcal{S}_t \subset \Sigma_t$. The linearization of the expansion operator θ is precisely the stability operator, and strict stability guarantees that this operator is invertible. As $\mathcal{S}_0$ is a solution of the equation $\theta = 0$ on Σ_0, we could apply in our earlier paper [1] the implicit function theorem to get solutions of $\theta = 0$ in a neighbourhood. Here we cut this procedure short by using standard results on perturbations of differential operators [32, 33] whose linearizations are elliptic and invertible. Naturally, these results also make use of the implicit function theorem. As an easy corollary to Theorem 9.1, we find that the MOTT constructed in this interval is *nowhere* tangent to the Σ_t. Much more subtle results are Theorem 9.2 and Corollary 9.2. Under some genericity condition they show in particular that, if the $\mathcal{S}_t$ converge smoothly to a limiting MOTS $\mathcal{S}_\tau \subset \Sigma_\tau$ whose principal eigenvalue λ_τ vanishes, the resulting MOTT is *everywhere* tangent to Σ_τ.

We wish to stress that the existence result contained in [1] is local in time. The attempt to formulate a global result in [1] required the *assumptions* (implicit in the definitions of that paper) that the MOTS remain compact, embedded, smooth and strictly stable during the evolution. In the present work we allow immersed rather than embedded MOTS in our existence theorems for MOTT, but we have to deal with the other potential pathologies. To do so we apply a recent result by Andersson and Metzger [34] which shows that, in four-dimensional spacetimes, a sequence of stable and smooth MOTS which lie in a compact set such that the area of the sequence stays bounded, converges to a smooth and stable MOTS. This leads to Proposition 9.1. Moreover, if the dominant energy condition holds, it is easy to show that the area stays in fact bounded provided the MOTS remain strictly stable in the limit. This gives Theorem 9.3 as a sharper version of our existence result.

Two final comments are in order. In general the stability of a MOTS depends on the foliation Σ_t, and so does the location of the MOTT, in particular in the strictly stable case. This raises the question whether there is an "optimal" choice for the foliation. Given a MOTS $\mathcal{S}$, the optimal non-timelike direction of Σ to make $\mathcal{S}$ "as stable as possible" is the one along the null direction complementary to the one defining θ, c.f. Proposition 5.2. However other possible criteria for "optimality" of a foliation have been considered, in particular one can require the area of the resulting MOTT to grow as fast as possible [35].

The heuristic picture of the MOTT in the asymptotically flat case has it inside the event horizon and reaching timelike infinity. In the spherical symmetric case with matter, criteria have been given which ensure (or exclude) this behaviour [3, 10, 11]. A substantially more involved task is to extend this analysis to the general case. A first step along these lines has been recently performed by Korzyński [13] who analysed the evolution of MOTS in a simple axially symmetric example.

This paper is organized as follows. In Section 2 we explain the most important items of our notation. In Section 3 we discuss the variation of the expansion, and introduce the stability operator. The somewhat technical computation of the variation, which simplifies the derivation by Newman, is given in Appendix A.

We proceed with some technical material on linear elliptic operators with first-order term, cf. Section 4. Here and in Appendix B we give an exposition of, in particular, the Krein–Rutman theorem [36] on the principal eigenvalue and eigenfunction of linear elliptic operators, and the maximum principle for operators with non-negative principal eigenvalue [37]. We continue in

Section 5 by discussing in detail stability definitions for MOTS, in particular the relation between the variational definitions and the sign condition on the principal eigenvalue, and we give a result on the dependence of stability on the direction. Section 6 contains the graph representation of a MOTS. In Section 7 we describe the barrier properties of MOTS which satisfy suitable stability conditions, along the lines sketched above, slightly extending our earlier paper [1]. In Section 8, we show, roughly speaking, that strictly stable MOTS inherit the symmetries of the ambient geometry, and that the same is true for the principal eigenfunction. In the final Section 9 we prove existence for MOTT in the three Theorems 9.1, 9.2 and 9.3 already sketched above. Our final Theorem 9.4 is a slight extension of a result of [1] and shows that under standard energy conditions, suitable (non-)degeneracy conditions for the initial MOTS $\mathcal{S}_0$ and for spacelike or null reference foliations, the MOTT through $\mathcal{S}_0$ is either spacelike or null everywhere on $\mathcal{S}_0$.

2 Some basic definitions

A spacetime $(\mathcal{M}, g)$ is an n-dimensional oriented and time-oriented Hausdorff manifold endowed with a smooth metric of Lorentzian signature $+2$. Some results below require $n = 4$. $\mathcal{S}$ will denote an orientable, closed (i.e., compact without boundary) codimension 2, immersed submanifold of $\mathcal{M}$ with positive definite first fundamental form h. An object with all these properties is simply called "surface" throughout this paper. The area of $\mathcal{S}$ will be denoted by $|\mathcal{S}|$. Spacetime tensors will carry Greek indices and tensors in $\mathcal{S}$ will carry capital Latin indices. Our conventions for the second fundamental form (-vector) and the mean curvature (-vector) are $K(X,Y) \equiv -(\nabla_X Y)^\perp$ and $H = \mathrm{tr}_h K$. Here X, Y are tangent vectors to $\mathcal{S}$, $\perp$ denotes the component normal to $\mathcal{S}$ and ∇ is the Levi-Civita connection of g. $\mathcal{S}$ will always be assumed to be smooth unless otherwise stated.

The normal bundle of $\mathcal{S}$ is a Lorentzian vector bundle which admits a null basis $\{l^\alpha, k^\alpha\}$ which we always take future directed, smooth and normalized so that $l^\alpha k_\alpha = -2$. This basis is defined up to a boost $l^\alpha \to \kappa l^\alpha$, $k^\alpha \to \kappa^{-1} k^\alpha$, $\kappa > 0$. The *null expansions* of $\mathcal{S}$ are $\theta_l = H^\alpha l_\alpha$ and $\theta_k = H^\alpha k_\alpha$ and the mean curvature in this null basis reads $H^\alpha = -\frac{1}{2}\left(\theta_k l^\alpha + \theta_l k^\alpha\right)$.

Definition 2.1. A surface $\mathcal{S}$ is a MOTS if H^α is proportional to one of the elements of the null basis of its normal bundle.

This condition is more restrictive than just demanding H^α to be null because it excludes the possibility that H^α points along l^α in some open set and along k^α on its complement (c.f. [38]). The null vector to which H^α is proportional is called l^α, and the direction to which it points is called the *outer* direction (in the case $H^\alpha \equiv 0$, both l^α and k^α are outer directions). In other words, the term outer does not refer to a direction singled out *a priori*, but to the fact that we only have information about, or we are only interested in, *one of* the expansions. Equivalently, $\mathcal{S}$ is a MOTS iff $\theta_l = 0$. If furthermore H^α is either future or past directed everywhere $\mathcal{S}$ is called *marginally trapped*. Hence a marginally trapped surface satisfies $\theta_l = 0$ and either $\theta_k \leq 0$ or $\theta_k \geq 0$ everywhere. Next, $\mathcal{S}$ is called *weakly outer trapped* iff at least one of the expansions in non-positive, say $\theta_l \leq 0$. *Weakly outer untrapped* surfaces satisfy the reverse inequality. Finally, in order for $\mathcal{S}$ to be a future (past) *trapped surface* we require that the strict inequalities $\theta_l < 0$ and $\theta_k < 0$, ($\theta_l > 0$, $\theta_k > 0$) hold. Since we will only deal with θ_l from now on, we simplify the notation and refer to it simply as θ.

A MOTT $\mathcal{G}$ is a smooth collection of MOTS. More precisely, we state the following

Definition 2.2. Let $I \subset \mathbb{R}$ be an interval. A hypersurface $\mathcal{G}$, possibly with boundary, is a MOTT if there is a smooth immersion $\Phi : \mathcal{S} \times I \to M$, such that $\mathcal{G} = \Phi(\mathcal{S} \times I)$ and

(i) for fixed $s \in I$, $\Phi(\mathcal{S}, s)$ is a MOTS with respect to a smooth field of null normals l^α on $\mathcal{G}$ and
(ii) $\Phi_\star(\partial_s)$ is nowhere zero.

Suppose $(\mathcal{M}, g)$ contains a foliation by hypersurfaces $\{\Sigma_t\}_{t\in J}$. A MOTT $\mathcal{G}$ is said to be adapted to the reference foliation $\{\Sigma_t\}$ if for each $s \in I$, it holds that $\mathcal{S}_{\sigma(s)} = \Phi(\mathcal{S}, s)$ is a MOTS in $\Sigma_{\sigma(s)}$, where $\sigma : I \to J$ is a smooth, strictly monotone function.

Remark. Note that if I contains at least one of its boundary points, then the MOTT $\mathcal{G}$ is a hypersurface with boundary.

Condition (ii) is required in order to allow self-intersections of the MOTS but not in the direction of propagation. For embedded MOTS, the MOTT is also embedded and its definition is equivalent to Hayward's "trapping horizons" [18]. The terms "dynamical horizon" and "isolated horizon" [15, 16] are particular cases in which the causal structure is restricted *a priori*.

3 Varying the expansion

A fundamental ingredient in our existence theorem is the first-order variation of the vanishing null expansion θ of a MOTS. This variation was given in full generality by Newman in [20] for arbitrary immersed surfaces (not necessarily MOTS). We give here a simplified derivation. We first have to introduce some notation.

Let ∇_α and $G_{\alpha\beta}$ denote the covariant derivative and Einstein tensor of $(\mathcal{M}, g)$, respectively. Let $(\mathcal{S}, h)$ be a spacelike codimension-two surface immersed in $\mathcal{M}$ (not necessarily closed), e^α_A any basis of the tangent space of $\mathcal{S}$, D_A the covariant derivative on $(\mathcal{S}, h)$ and $R_\mathcal{S}$ its curvature scalar. Let us fix a null basis $\{l^\alpha, k^\alpha\}$ in the normal bundle of $\mathcal{S}$ and define a one-form on $\mathcal{S}$ as

$$s_A = -\frac{1}{2} k_\alpha \nabla_{e_A} l^\alpha.$$

We shall calculate how θ changes when the surface $\mathcal{S}$ is varied arbitrarily. This variation is defined by an arbitrary spacetime vector q^α defined along $\mathcal{S}$. More specifically, let $0 \in I \subset \mathbb{R}$ be an open interval and $\Phi : \mathcal{S} \times I \to \mathcal{M}$ be a differentiable map such that for fixed σ, $\Phi(\cdot, \sigma)$ is an immersion and for fixed p, $x^\alpha_p(\sigma) \equiv \Phi^\alpha(p, \sigma)$ is a curve starting at $p \in \mathcal{S}$ with tangent vector $q^\alpha(p)$. Define the family of 2-surfaces $\mathcal{S}_\sigma \equiv \Phi(\mathcal{S}, \sigma)$. Let l^α_σ be a nowhere zero null vector on the normal bundle of $\mathcal{S}_\sigma$ which is differentiable with respect to σ, and let θ_σ be the null expansion of each surface $\mathcal{S}_\sigma$. The variation of θ_σ, defined as $\delta_q \theta \equiv \partial_\sigma \theta_\sigma|_{\sigma=0}$, depends only on q^α and on the null vector field l^α and its first variation (if $\mathcal{S}$ is a MOTS this last dependence also drops out), but not on the details of the map Φ. For a MOTS, the variation is moreover linear in the sense that

$$\delta_{aq_1+q_2}\theta = a\delta_{q_1}\theta + \delta_{q_2}\theta \tag{3.1}$$

for any *constant* a, while in general $\delta_{\psi q}\theta \neq \psi \delta_q \theta$ for functions ψ.

It should be noted that in the context of trapping and dynamical horizons, derivatives of θ have been employed frequently (for instance in the definition of outer trapping horizon by Hayward [17] or in the uniqueness results by Ashtekar and Galloway [7]). These are *not* the variations we are considering in this paper. The former are derivatives of a scalar function defined in a neighbourhood of the horizon by extending θ off the horizon, using the Raychaudhuri equation. In this case, the derivative is obviously linear with respect to multiplication by scalar functions, unlike the geometric variation employed here.

The variation vector q^α can be decomposed into a tangential part $q^{\parallel\alpha}$ and an orthogonal part $q^{\perp\alpha}$ with respect to $\mathcal{S}$, i.e., $q^\alpha = q^{\perp\alpha} + q^{\parallel\alpha}$. The normal component can in turn be decomposed in terms of the null basis as $q^{\perp\alpha} = bl^\alpha - \frac{u}{2}k^\alpha$ where b, u are functions on $\mathcal{S}$. The following result, in essence due to Newman [20] and proven in the Appendix A, gives the explicit expression for the variation of θ along q^α.

Lemma 3.1. *Let $\mathcal{S}$, l^α, θ and $q^\alpha = q^{\parallel\alpha} + bl^\alpha - \frac{u}{2}k^\alpha$ be as before. Then, the variation of θ along q is*

$$\delta_q\theta = a\theta + q^{\parallel}(\theta) - b\left(K^\mu_{AB}K^{\nu\,AB}l_\mu l_\nu + G_{\mu\nu}l^\mu l^\nu\right) - \Delta_{\mathcal{S}}u + 2s^A D_A u \\ + \frac{u}{2}\left(R_{\mathcal{S}} - H^2 - G_{\mu\nu}l^\mu k^\nu - 2s^A s_A + 2D_A s^A\right),$$

where $\Delta_{\mathcal{S}} = D_A D^A$ is the Laplacian on $(\mathcal{S}, h)$ and $a = -\frac{1}{2}k_\alpha \partial_\sigma l^\alpha_\sigma|_{\sigma=0}$.

The decomposition of $q^{\perp\alpha}$ in the null basis $\{l^\alpha, k^\alpha\}$ is natural for the surface $\mathcal{S}$ as a codimension-two submanifold in spacetime, and Lemma 3.1 gives the general variation with respect to arbitrary vectors on $\mathcal{S}$. However, we will later refer the variations of $\mathcal{S}$ to some foliation of $\mathcal{M}$ by hypersurfaces Σ_t with $\mathcal{S} \subset \Sigma_0$ (c.f. Section 7), and for this we will employ a natural alternative decomposition of $q^{\perp\alpha}$ adapted to the foliation. We consider an arbitrary normal vector field v^α to $\mathcal{S}$ which is, at each point, linearly independent of l^α, and we impose the normalization $v^\alpha l_\alpha = 1$ which does not restrict the causal character of v^α anywhere on $\mathcal{S}$. v^α is uniquely defined by a scalar function V on $\mathcal{S}$ according to

$$v^\alpha = -\frac{1}{2}k^\alpha + Vl^\alpha. \tag{3.2}$$

We use $\{v^\alpha, l^\alpha\}$ as a basis in the normal space. Inverting (3.2) we get $k^\alpha = 2(Vl^\alpha - v^\alpha)$. We next define a vector $u^\alpha = \frac{1}{2}k^\alpha + Vl^\alpha$, which is orthogonal to v^α and satisfies $u^\alpha u_\alpha = -v^\alpha v_\alpha = -2V$. The quantities

$$W = K^\mu_{AB}K^{\nu\,AB}l_\mu l_\nu + G_{\mu\nu}l^\mu l^\nu, \tag{3.3}$$

$$Y = VK^\mu_{AB}K^{\nu\,AB}l_\mu l_\nu + G_{\mu\nu}l^\mu u^\nu \tag{3.4}$$

will appear frequently below. Clearly W is non-negative provided the null energy condition holds and Y is non-negative if u^α is causal and the dominant energy condition holds.

The following definition introduces an object which plays a key role in this paper.

Definition 3.1. The *stability operator* is defined by

$$L_v\psi = -\Delta_{\mathcal{S}}\psi + 2s^A D_A\psi + \left(\frac{1}{2}R_{\mathcal{S}} - Y - s^A s_A + D_A s^A\right)\psi. \tag{3.5}$$

The following lemma is a trivial specialization of Lemma 3.1.

Lemma 3.2. *Let $\mathcal{S}$ be a MOTS, i.e., $\theta = 0$. The variation of the expansion θ on $\mathcal{S}$ with respect to the null vector ψl^α, and with respect to any vector*

ψv^α orthogonal to $\mathcal{S}$ with $l^\alpha v_\alpha = 1$, respectively, are given by

$$\delta_{\psi l}\theta = -\psi W, \tag{3.6}$$

$$\delta_{\psi v}\theta = L_v \psi. \tag{3.7}$$

We note that (3.6) is the Raychaudhuri equation. For arbitrary vectors w^α orthogonal to $\mathcal{S}$ and linearly independent of l^α, not necessarily normalized to satisfy $w^\alpha l_\alpha = 1$, we can define another elliptic operator $L_w \psi = \delta_{\psi w}\theta$. In terms of the normalized vector $v \equiv F^{-1} w$, where $F = w^\alpha l_\alpha$ we obviously have

$$L_w \psi = L_{Fv}\psi = L_v(F\psi). \tag{3.8}$$

Hence, L_v and L_{Fv} contain essentially the same information. To see the dependence of L_v on the vector v, i.e. on the function V we calculate, from linearity (3.1), and (3.6),

$$L_v = \frac{1}{2} L_{-k} - VW. \tag{3.9}$$

Due to the presence of the first-order term in (3.5), L_v is not self-adjoint in general. However, self-adjoint extensions still exist in special cases. For example, if s_A is a gradient, i.e., $s_A = D_A \zeta$ for some ζ, L_v is self-adjoint with respect to a suitable measure depending on ζ, c.f. Section 4.

Since L_v is linear and elliptic, it has discrete eigenvalues and the corresponding eigenfunctions are regular. However, in general, the eigenvalues and eigenfunctions are complex. Nevertheless, the principal eigenvalue, i.e., the eigenvalue with smallest real part, and its corresponding eigenfunction behave in the same manner as for self-adjoint operators. In particular, they can be used to give a very efficient reformulation of the maximum principle. In the next section we collect some material on linear elliptic operators which will be the key tools in the subsequent discussion of stability.

4 Properties of linear elliptic operators

The results of this section hold for connected compact smooth manifolds $\mathcal{S}$ without boundary, and for arbitrary smooth, linear, second-order, elliptic operators on $\mathcal{S}$, which can be written as

$$L = -\Delta_h + 2t^A D_A + c, \tag{4.1}$$

where $\Delta_h \psi = D_A(h^{AB} D_B \psi)$, h^{AB} is positive definite and smooth, D_A is the corresponding Levi-Civita covariant derivative and t^A and c are smooth.

Lemma 4.1. *Let L be an elliptic operator of the form (4.1) on a compact manifold. Then, the following holds.*

(i) *There is a real eigenvalue λ, called the principal eigenvalue, such that for any other eigenvalue μ the inequality $\mathrm{Re}(\mu) \geq \lambda$ holds. The corresponding eigenfunction ϕ, $L\phi = \lambda\phi$ is unique up to a multiplicative constant and can be chosen to be real and everywhere positive.*
(ii) *The adjoint $L^\dagger$ (with respect to the L^2 inner product) has the same principal eigenvalue λ as L.*

Applications of Lemma 4.1 to the stability operator (3.5) in a spacetime will be described in the next section. Note that our terminology follows Evans [29]; in particular, regarding the sign of L it is opposite to other references [26, 37] cited below. This entails that when comparing with these references one always has to interchange "sup" and "inf" when acting on expressions containing L.

The existence of the principal eigenvalue and eigenfunction was first proven by Donsker and Varadhan [26] using parabolic theory and the Krein–Rutman theorem [36]. For uniqueness of the principal eigenfunction, c.f. Berestycki et al. [37]. All these papers actually deal with the Dirichlet problem for bounded domains in $\mathbb{R}^n$. However, the proof is easily adapted to the case of compact connected manifolds without boundary (it is in fact simpler). For completeness and since this result is not widely known, we provide in Appendix B a sketch of the proof in the closed manifold case. The sketch follows the argument in Smoller [28] and Evans [29].

For self-adjoint operators $L_0 = -\Delta_h + c$, the principal eigenvalue λ_0 is given by the Rayleigh–Ritz formula

$$\lambda_0 = \inf_u \langle u, L_0 u\rangle = \inf_u \int_{\mathcal{S}} \left(D_A u D^A u + cu^2\right) \eta_{\mathcal{S}}, \tag{4.2}$$

where $\eta_{\mathcal{S}}$ is the surface element on $(\mathcal{S}, h)$ and the infimum is taken over smooth functions u on $\mathcal{S}$ with $||u||_{L^2} = 1$.

For non-self-adjoint operators such a characterization is no longer true. However, Donsker and Varadhan [26] have given alternative variational representations of the eigenvalue, namely

$$\lambda = \inf_{\mu_{\mathcal{S}}} \sup_{\psi} \int_{\mathcal{S}} \psi^{-1} L\psi \mu_{\mathcal{S}}, \tag{4.3}$$

$$\lambda = \sup_{\psi} \inf_{x\in\mathcal{S}} \psi^{-1}(x) L\psi(x). \tag{4.4}$$

Here the infimum in (4.3) is taken over all probability measures $\mu_{\mathcal{S}}$ on $\mathcal{S}$, while the suprema are over all smooth, positive functions ψ on $\mathcal{S}$.

To get (4.4) from (4.3) we first note that the "inf" and "sup" in (4.3) can be interchanged (which is non-trivial but follows from a min–max theorem of Sion [39], c.f. [26]). This done, the infimum of the integral with respect to all probability measures is achieved by a Dirac delta measure concentrated at the location where the integrand takes its infimum.

In order to approach a characterization closer to a Rayleigh–Ritz expression, we note that, since probability measures can be approximated by smooth positive functions, we can assume $\mu_{\mathcal{S}} = u^2 \eta_{\mathcal{S}}$ with smooth $u > 0$ and $||u||_{L^2} = 1$. Following Donsker and Varadhan, ψ can be decomposed as $\psi = ue^{\omega}$. Direct substitution in (4.3) and rearrangement gives

$$\lambda = \inf_u \sup_\omega \int_{\mathcal{S}} \left(D_A u D^A u + \left(c + t_A t^A - D_A t^A\right) u^2 - \left(D_A \omega + t_A\right)^2 u^2 \right) \eta_{\mathcal{S}}.$$

To reformulate this expression we use the Hodge decomposition $t_A = D_A \zeta + z_A$, where ζ is a function and z_A is divergence free. This decomposition is unique except for a constant additive term in ζ. The supremum over ω only affects the last term, and it only depends on z_A and not on ζ. Thus, we need to determine $\inf_\omega \int_{\mathcal{S}} \left(D_A \omega + z_A\right)^2 u^2 \eta_{\mathcal{S}}$, for each u. A standard argument shows that the infimum is achieved and is given by the solution of the linear elliptic equation

$$-D_A D^A \omega[u] - 2u^{-2} D_A \omega[u] D^A u = 2u^{-2} z^A D_A u, \tag{4.5}$$

where we write $\omega[u]$ to emphasize that the solution depends on u. A trivial Fredholm argument shows that this equation has a unique solution satisfying

$$\int_{\mathcal{S}} u^2 \omega[u] \eta_{\mathcal{S}} = 0. \tag{4.6}$$

It therefore follows that the Donsker–Varadhan characterization of the principal eigenvalue can be rewritten as

$$\lambda = \inf_u \int_{\mathcal{S}} \left(D_A u D^A u + Q u^2 - \left(D_A \omega[u] + z_A\right)^2 u^2 \right) \eta_{\mathcal{S}}, \tag{4.7}$$

where $Q = c + t_A t^A - D_A t^A$ and the infimum is taken over smooth functions of unit L^2 norm. The symmetrized operator $L_s = -\Delta_h + Q$ is self-adjoint and has a principal eigenvalue λ_s given by the Rayleigh–Ritz formula, as in (4.2). Since the last term in (4.7) is non-positive, the inequality

$$\lambda_s \geq \lambda \tag{4.8}$$

follows immediately. This inequality has recently been demonstrated by Galloway and Schoen [23] using direct estimates. It is interesting that (4.8) is a trivial consequence of the Rayleigh–Ritz type formula (4.7) for the principal eigenvalue.

As a second application of the characterization (4.7), we note that the last term can be rewritten as

$$- \left(D_A\omega[u] + z_A\right)^2 u^2 = \left(D_A\omega[u]D^A\omega[v] - z_A z^A\right) u^2 \\ - 2D_A \left(\omega[u]u^2 \left(z^A + D^A\omega[u]\right)\right)$$

after using the equation (4.5) and the fact that z^A is divergence-free. Integration on $\mathcal{S}$ gives the alternative representation for λ

$$\lambda = \inf_u \int_{\mathcal{S}} \left(D_A u D^A u + \left(Q - z_A z^A\right) u^2 + u^2 D_A\omega[u]D^A\omega[u]\right) \eta_{\mathcal{S}}. \tag{4.9}$$

Since the last term is now non-negative, dropping it decreases the integrand and therefore also the infimum. Thus, defining the alternative symmetrized operator $L_z \equiv L_s - z_A z^A$ and denoting by λ_z the corresponding principal eigenvalue, it follows that

$$\lambda \geq \lambda_z.$$

Therefore the principal eigenvalue of a non-self-adjoint operator is always sandwiched between the principal eigenvalues of two canonical symmetrized elliptic operators, and we also note that $\inf_{\mathcal{S}}(z_A z^A) \leq \lambda_s - \lambda_z \leq \sup_{\mathcal{S}}(z_A z^A)$. Obviously, when L is self-adjoint (w.r.t. to the L^2 norm), i.e., $t_A \equiv 0$, the two symmetrized operators L_s and L_z coincide with it. More generally, when t_A is a gradient (so that L is self-adjoint with a suitable measure) the characterization of λ given by (4.7) coincides with the Rayleigh–Ritz expression because whenever $z_A = 0$, the unique solution to (4.5) and (4.6) is just $\omega[u] = 0$ for all u.

An important tool in the analysis of the properties of the stability operator will be the maximum principle. The textbook formulations normally require that the coefficient of the zero-order term of the elliptic operator is non-negative, at least if a source term is present (see for example [40, Section 3]). We give here a reformulation, used in several places below, which instead requires non-negativity or positivity of the principal eigenvalue.

Lemma 4.2. *Let L be a linear elliptic operator of the form (4.1) on a compact manifold. Let λ and $\phi > 0$ be the principal eigenvalue and eigenfunction of L, respectively, and let ψ be a smooth solution of $L\psi = f$ for some smooth function $f \geq 0$. Then the following holds.*

(i) *If $\lambda = 0$, then $f \equiv 0$ and $\psi = C\,\phi$ for some constant C.*
(ii) *If $\lambda > 0$ and $f \not\equiv 0$, then $\psi > 0$.*
(iii) *If $\lambda > 0$ and $f \equiv 0$, then $\psi \equiv 0$.*

Remark. Clearly analogous results hold for $f \leq 0$.

Proof. For a positive principal eigenfunction ϕ of L, we define χ by $\psi = \chi\phi$ and an operator $\Gamma[\phi]$ by the first equation in

$$\Gamma[\phi]\chi = -D_A\left(\phi^2 h^{AB} D_A\chi\right) + 2\phi^2 t^A D_A\chi + \lambda\phi^2\chi = \phi f, \tag{4.10}$$

while the second equality follows by computation. The strong maximum principle [40, Theorem 3.5], applied to (4.10) gives the desired results. $\square$

We end this section with a result which is essentially a straightforward application of linear algebra.

Lemma 4.3. *Let L be a second-order elliptic operator on $\mathcal{S}$ of the form (4.1). Let $\lambda, \phi, \phi^\dagger$ be the real principal eigenvalue, and the corresponding real eigenfunctions of L and its adjoint $L^\dagger$. Let $\mathbb{P}$ be the projection operator defined by*

$$\mathbb{P}f = \phi\frac{\langle\phi^\dagger, f\rangle}{\langle\phi^\dagger, \phi\rangle}$$

and let $\mathbb{Q} = \mathbb{I} - \mathbb{P}$. Then

(i) *$L = \lambda\mathbb{P} + A$, where A has spectrum $\sigma(A)$ such that for some constant $c_0 > 0$,*

$$\min_{\mu\in\sigma(A),\mu\neq 0} |\mu - \lambda| > c_0 \tag{4.11}$$

(ii) *For any $s \geq 2$, $p > 1$, there is a constant C such that the inequality*

$$||\mathbb{Q}u||_{W^{s,p}} \leq C||(A - \lambda\mathbb{Q})u||_{W^{s-2,p}} \tag{4.12}$$

holds, where $W^{s,p}$ is the usual Sobolev space of functions with s derivatives in $L^p(\mathcal{S})$.

Remark. We refer to the constant c_0 as the spectral gap.

Proof. We shall consider complex eigenvalues and eigenvectors, and use the sesquilinear L^2 pairing

$$\langle u, v\rangle = \int_{\mathcal{S}} \bar{u}v.$$

It is straightforward to check that $L\mathbb{P} = \mathbb{P}L = \lambda\mathbb{P}$, and if we define the operator A by $A = L\mathbb{Q}$, so that

$$L = \lambda\mathbb{P} + A,$$

then

$$A\mathbb{P} = \mathbb{P}A = 0, \quad A\mathbb{Q} = \mathbb{Q}A = A, \quad A^\dagger\mathbb{P}^\dagger = 0, \quad A^\dagger\mathbb{Q}^\dagger = A^\dagger, \tag{4.13}$$

where we used that $\mathbb{P}^\dagger$ is the projector onto $\phi^\dagger$ i.e. $\mathbb{P}^\dagger f = \phi^\dagger\langle\phi, f\rangle/\langle\phi^\dagger, \phi\rangle$. It follows that the range of A lies in the space orthogonal to $\phi^\dagger$.

Let ψ be an eigenvector for L corresponding to a non-principal eigenvalue $\mu \neq \lambda$. We find that

$$\mu\langle\phi^\dagger,\psi\rangle = \langle\phi^\dagger, L\psi\rangle = \langle L^\dagger\phi^\dagger,\psi\rangle = \lambda\langle\phi^\dagger,\psi\rangle.$$

Since $\mu \neq \lambda$ by assumption, we conclude that $\langle\phi^\dagger,\psi\rangle = 0$. Combining the previous facts, it follows easily that the spectrum $\sigma(A)$ of A satisfies

$$\sigma(A) = (\sigma(L) \setminus \{\lambda\}) \cup \{0\}.$$

Since λ is the principal eigenvalue of L, so that $\lambda \leq \mathrm{Re}(\mu)$ for any non-principal eigenvalue μ, (4.11) follows from the fact that L has discrete spectrum.

To prove point (ii), we first note that by construction, $A - \lambda\mathbb{Q}$ is a Fredholm operator which maps the space orthogonal to $\phi^\dagger$ into itself. By point (i), the spectrum of $A - \lambda\mathbb{Q}$ is bounded from below by $c_0 > 0$. Thus, by the Fredholm alternative $A - \lambda\mathbb{Q} : \langle\phi^\dagger\rangle^\perp \to \langle\phi^\dagger\rangle^\perp$ is invertible on Sobolev spaces. $\square$

5 Stability of MOTS

The concept of stability of a MOTS with respect to a given slice Σ is a central issue of our paper and crucial for the application to existence of MOTTs. We first briefly comment on stability of minimal surfaces embedded in Riemannian manifolds (Σ,γ), disregarding any embeddings in spacetimes. Letting m^i be the unit normal to $\mathcal{S}$, the stability operator $L_m(\zeta) \equiv \delta_{\psi m} p$ where p is the mean curvature of $\mathcal{S}$ reads (e.g., [42]) $L_m\zeta = -\Delta_{\mathcal{S}}\zeta - (R_{ij}m^i m^j + K_{ij}K^{ij})\zeta$, where and R_{ij} and K_{ij} are the Ricci tensor of (Σ,γ) and second fundamental form of $\mathcal{S}$, respectively. This expression also follows from (3.5) by taking $\mathcal{S}$ immersed in a time symmetric spacelike hypersurface (Σ,γ) and choosing $v = m$. As L_m is self-adjoint its principal eigenvalue λ can be represented as the Rayleigh–Ritz formula (4.2). In terms of the latter, the standard formula for the variation of the area A of a minimal surface along a vector $v^i = \psi m^i$ gives

$$\delta_v^2 A = \delta_v^2 \int_{\mathcal{S}} \eta_{\mathcal{S}} = \int_{\mathcal{S}} \psi\delta_v p \eta_{\mathcal{S}} = \int_{\mathcal{S}} \psi L_m \psi \eta_{\mathcal{S}} \geq \lambda \int_{\mathcal{S}} \psi^2 \eta_{\mathcal{S}}. \tag{5.1}$$

The minimal surface is called *stable* if $\lambda \geq 0$, and (5.1) together with (4.2) shows that this is equivalent to $\delta_v^2 A \geq 0$, for all smooth variations δ_v.

We wish to characterize stable MOTS embedded in arbitrary hypersurfaces in spacetime in a similar way as stable minimal surfaces in Riemannian manifolds. In the general case we lose the connection between stability and the second variation of the area (except if the variation is directed along l^α). Using the discussion in the previous section, we can now define stability either in terms of a sign condition on suitable first variations of θ at the MOTS or, in view of (3.5) by a sign condition on the principal eigenvalue of the stability operator. In either case, we wish to establish a relation between both concepts. Here we choose the variational formulation as definition, from which we show the properties of the principal eigenvalue.

We could obtain a variational definition by replacing (4.2) by one of (4.3), (4.4) (4.7) or (4.9), but the resulting definition does not seem very illuminating. Therefore, instead of defining stability in terms of sign requirements on *integrals over* $\mathcal{S}$ for *all* positive variations of θ, we now require the existence of *at least one* variation along which θ has a sign *everywhere* on $\mathcal{S}$. This definition also enables us to introduce an important refinement, namely to distinguish whether this preferred variation of θ is just non-negative everywhere, or even positive somewhere. (We slightly expand our earlier presentation [1].) We recall that v^α is linearly independent of l^α everywhere on $\mathcal{S}$.

Definition 5.1. A MOTS $\mathcal{S}$ is called *stably outermost* with respect to the direction v iff there exists a function $\psi \geq 0$, $\psi \not\equiv 0$, on $\mathcal{S}$ such that $\delta_{\psi v}\theta \geq 0$. $\mathcal{S}$ is called *strictly stably outermost* with respect to the direction v if, moreover, $\delta_{\psi v}\theta \neq 0$ somewhere on $\mathcal{S}$.

We will omit the phrase "with respect to the direction v" when this is clear from the context. We now establish the connection between stability and the sign of the principal eigenvalue.

Proposition 5.1. *Let $\mathcal{S} \subset \Sigma$ be a MOTS and let λ be the principal eigenvalue of the corresponding operator L_v. Then $\mathcal{S}$ is stably outermost iff $\lambda \geq 0$ and strictly stably outermost iff $\lambda > 0$.*

Proof. If $\lambda \geq 0 (> 0)$, choose ψ in the definition of (strictly) stably outermost as a positive eigenfunction ϕ corresponding to λ. Then $\delta_{\phi v}\theta = L_v\phi = \lambda\phi \geq 0 (> 0)$. For the converse, we note that from Lemma 4.1 the adjoint $L_v^\dagger$ (with respect to the standard L^2 inner product $\langle , \rangle$ on $\mathcal{S}$) has the same principal eigenvalue as L_v, and a positive principal eigenfunction $\phi^\dagger$. Thus, for ψ as in the definition of (strictly) stably outermost,

$$\lambda\langle\phi^\dagger, \psi\rangle = \langle L_v^\dagger\phi^\dagger, \psi\rangle = \langle\phi^\dagger, L_v\psi\rangle \geq 0 (> 0),$$

Since $\langle\phi^\dagger, \psi\rangle > 0$, the proposition follows. □

The stability properties discussed above always refer to a MOTS $\mathcal{S}$ with respect to a fixed direction v^α normal to $\mathcal{S}$ or a fixed hypersurface to which

v^α is tangent. This raises the question as to how the stability properties depend on this direction. Using definition (3.2) we can change v^α by adjusting the function V, which can take any value, and study the dependence of the principal eigenvalue λ on this function. This yields the following

Proposition 5.2. *Let $\mathcal{S}$ be a MOTS and let λ_v, $\lambda_{v'}$ be the principal eigenvalues of the stability operators in the directions v^α and v'^α defined by (3.2), with some functions V and V', respectively. Let ϕ' and $\phi^\dagger$ be principal eigenfunctions of $L_{v'}$ and $L_v^\dagger$ respectively. Then*

$$(\lambda_v - \lambda_{v'})\langle \phi^\dagger,\ \phi' \rangle = \langle \phi^\dagger, (V' - V)W\phi' \rangle. \tag{5.2}$$

Proof. From (3.9) it follows that,

$$L_v = L_{v'} + (V' - V)W. \tag{5.3}$$

Hence, $0 = \langle \phi^\dagger, (L_v - L_{v'} - (V' - V)W)\,\phi' \rangle$ and the proposition follows using point (ii) of Lemma 4.1. □

Applying some trivial estimates to (5.2) we find

$$\lambda_{v'} + \inf_{\mathcal{S}}[(V' - V)W] \le \lambda_v \le \lambda_{v'} + \sup_{\mathcal{S}}[(V' - V)W]. \tag{5.4}$$

The inequality (5.4) implies in particular that, in spacetimes satisfying the null energy condition, a MOTS which is stable or strictly stable with respect to v^α will have the same property with respect to all directions "tilted away" from the null direction l^α defining the MOTS, and it puts a limit to the allowed amount of tilting of v^α "towards" l^α which preserves stability or strict stability. In particular, if a MOTS $\mathcal{S}$ is (strictly) stable with respect to some spacelike direction, then it is also (strictly) stable with respect to the null direction $-k^\alpha$ complementary to l^α in the normal basis $\{l^\alpha, k^\alpha\}$.

6 The graph representation of MOTS

We now assume that $(\mathcal{M}, g)$ is foliated by hypersurfaces Σ_t defined as the level sets of a smooth function t. Assume also that one of the elements of the foliation, say Σ_0, contains a MOTS $\mathcal{S}_0$. We further assume that Σ_0 is transverse to the null normal vector l^α on $\mathcal{S}$, i.e., $T_pM = T_p\Sigma_0 \oplus (l^\alpha|_p)$ for all $p \in \mathcal{S}_0$. However, for most results we do not assume any specific causal character for the foliation Σ_t and we allow leaves which are spacelike, timelike, null or which change their causal character. The transversality above allows us to fix l^α so that $l^\mu \partial_\mu(t) = 1$.

The main goal of this paper is to examine under which conditions there is a MOTT $\mathcal{G}$ such that $\mathcal{S}_t = \mathcal{G} \cap \Sigma_t$ is a MOTS for all t. For this purpose it is useful to define an abstract copy $\widehat{\mathcal{S}}$ of $\mathcal{S}_0$ detached from spacetime. We are looking for a 1-parameter family of smooth immersions $\chi_t : \widehat{\mathcal{S}} \longrightarrow \Sigma_t$ for t in some open interval $I \ni 0$ such that $\mathcal{S}_t \equiv \chi_t(\widehat{\mathcal{S}})$ is a MOTS and the map

$$\begin{aligned} \Psi : I \times \widehat{\mathcal{S}} &\longrightarrow \mathcal{M} \\ (t,p) &\longrightarrow i_{\Sigma_t} \circ \chi_t(p) \end{aligned}$$

is smooth, where $i_{\Sigma_t} : \Sigma_t \hookrightarrow \mathcal{M}$ is the natural inclusion. In other words, the collection of MOTS should depend smoothly on t. Using arbitrary local coordinates y^i on Σ_t, x^A in $\widehat{\mathcal{S}}$ and writing $y^\mu = \{t, y^i\}$, the immersion Ψ takes the local form

$$\Psi^\mu(t, x^A) = (t, \chi^i_t(x^A)). \tag{6.1}$$

For the null expansion we obtain

$$\theta = l^t_\mu h_t^{AB} K^\mu_{AB} = h_t^{AB} l^t_\mu \left(\Gamma^{(\mathcal{S})\,C}_{t\,AB} \frac{\partial \Psi^\mu}{\partial x^C} - \frac{\partial^2 \Psi^\mu}{\partial x^A \partial x^B} - \Gamma^\mu_{\nu\rho}\big|_{x=\Psi^\alpha} \frac{\partial \Psi^\nu}{\partial x^A} \frac{\partial \Psi^\rho}{\partial x^B} \right), \tag{6.2}$$

where $h_{t\,AB}$ is the induced metric on $\mathcal{S}_t$, $\Gamma^{(\mathcal{S})\,C}_{t\,AB}$ are the corresponding Christoffel symbols and $\Gamma^\mu_{\nu\rho}$ are the spacetime Christoffel symbols. The vector $l^t_\alpha(x^A)$ satisfies $l^{t=0}_\alpha = l_\alpha$ and solves

$$l^t_\mu l^{t\,\mu} = 0, \quad l^t_\mu \frac{\partial \Psi^\mu}{\partial x^A} = 0, \quad l^{t\,\mu}\partial_\mu t = 1. \tag{6.3}$$

under the condition that l^t_α depends continuously (and hence smoothly) on t. The condition we want to solve is $\theta = 0$ which is a scalar condition on the immersion χ_t. Since the problem is diffeomorphism invariant, we need to choose coordinates in order to convert the geometric problem into a PDE problem. The idea is to construct a suitable coordinate system adapted to the initial MOTS $\mathcal{S}_0$ and to restrict the class of allowed MOTS to be suitable graphs in this coordinate system. Since $\mathcal{S}_0$ may have self-intersections, the coordinate system and the graphs we will use are only local in the sense that for any point $p \in \widehat{\mathcal{S}}$, there is a spacetime neighbourhood $\mathcal{V}_p$ of its image $\Phi(p)$ such that the spacetime metric takes a special form in suitable coordinates adapted to the surface (strictly speaking, to the connected component of $\mathcal{S}_0 \cap \mathcal{V}_p$ containing $\Phi(p)$). In the following lemma we introduce this adapted coordinate system. In order to avoid cumbersome notation, we will assume $\mathcal{S}_0$ to be embedded for this lemma. Since the applications of this result below will always be local on $\mathcal{S}_0$ in the sense just described (and all immersions are locally embeddings) this suffices for our purposes.

The following lemma does not require $\mathcal{S}_0$ to be a MOTS.

Lemma 6.1. *Let $(\mathcal{M}, g)$, Σ_t and t be as before. There exists a spacetime neighbourhood $\mathcal{V}$ of a smooth, embedded closed spacelike 2-surface $\mathcal{S}_0 \subset \Sigma_0 \subset$*

$\mathcal{M}$, local coordinates $\{t, r, x^A\}$ on $\mathcal{V}$ adapted to the foliation $\{\Sigma_t\}$ and functions Z, φ, η^A and h_{AB} such that the metric can be written as

$$g_{\mu\nu}dx^\mu dx^\nu = 2e^Z dtdr + \varphi dr^2 + h_{AB}\left(dx^A + \eta^A dr\right)\left(dx^B + \eta^B dr\right), \quad (6.4)$$

where $\mathcal{S}_0 \cap \mathcal{V} = \{t = 0, r = 0\}$, $Z(t = 0, r = 0, x^A) = 0$ and h_{AB} is a positive definite $(n-2)$-matrix. The function φ may have any sign (even a changing one) reflecting the fact that the hypersurfaces Σ_t, which coincide with the $t = const$ surfaces, are of any causal character.

Proof. Let x^A be a local coordinate system on $\mathcal{S}_0$ and let v^α be a smooth, nowhere zero vector field on $\mathcal{S}_0$ orthogonal to this surface, tangent to Σ_0 and satisfying $v^\alpha l_\alpha = 1$. We extend v^α to a smooth vector field (still denoted by v) in a neighbourhood of $\mathcal{S}_0$ within Σ_0 and define coordinates (r, x^A) on this neighbourhood by solving the ODE (again with slight abuse of notation)

$$\begin{aligned} v(r) &= 1, & r|_{\mathcal{S}_0} &= 0, \\ v(x^A) &= 0, & x^A|_{\mathcal{S}_0} &= x^A \end{aligned}$$

In some tubular neighbourhood of $\mathcal{S}_0$ within Σ_0, this solution defines a smooth coordinate system in which we have $v^\alpha \partial_\alpha = \partial_r$. Furthermore, for small enough r_0 the sets $\mathcal{S}_{r_0} = \{r = r_0\}$ define spacelike, closed, codimension-two surfaces embedded in Σ_0 and diffeomorphic to $\mathcal{S}_0$. After restricting the range of r if necessary, we can also extend l^α to a nowhere vanishing null vector field on Σ_0 orthogonal to each level surface $\{\mathcal{S}_r\}$ and satisfying $l^\alpha \partial_\alpha t = 1$.

We finally consider the null geodesics with tangent vector l^α whose "length" is fixed by $l^\mu \partial_\mu t = 1$ everywhere in a suitably small neighbourhood of $\mathcal{S}_0$ in $\mathcal{M}$. On this neighbourhood $\mathcal{V}$ define functions $\{r, x^A\}$ by solving $l^\alpha(r) = 0$, $l^\alpha(x^A) = 0$ and so that $r|_{\Sigma_0}$ and $x^A|_{\Sigma_0}$ coincide with the functions with the same name defined above. The functions $\{t, r, x^A\}$ define a coordinate system on $\mathcal{V}$. Since $l^\alpha \partial_\alpha = \partial_t$ everywhere we immediately have $g_{tt} = 0$ for the metric. On $\Sigma_0 = \{t = 0\}$ we have $(\partial_t, \partial_{x^A}) = 0$, where $(\, , \,)$ denotes scalar product with g. The geodesic equation $l^\alpha \nabla_\alpha l^\beta = b l^\beta$ becomes $\partial_t g_{\mu t} = b g_{\mu t}$ in this coordinate system (b need not be zero as the null vector l^α has already been chosen). Hence $g_{tx^A} = 0$ and $g_{tr} > 0$ on this neighbourhood (because $g_{tr} = 1$ on $\mathcal{S}_0$). $\square$

In terms of the coordinates (6.4), we can consider local graphs $\mathcal{S}_t$ given by $r = f(x^A)$ on the surface $\{r = 0\} \subset \Sigma_t$. Restricting the allowed immersions (6.1) to those local graphs, (6.2) becomes an operator on f, which we call $\theta_t[f]$. We formulate the ellipticity property of this operator in the subsequent lemma. We use the shorthands $f_A = \partial_{x^A} f$ and $\rho = [\varphi f^A f_A + \left(1 + \eta^A f_A\right)^2]_{r=f}$, where φ and η are the metric coefficients in (6.4) and capital Latin indices are moved with h_{AB} and its inverse h^{AB}.

Lemma 6.2. *Consider codimension-two immersions defined locally by a smooth function $f(x^A)$ according to*

$$\begin{aligned} \chi_t[f] : \widehat{\mathcal{S}} &\longrightarrow \mathcal{S}_t \subset \mathcal{M} \\ x^A &\longrightarrow \left(t = t, r = f(x^A), x^A = x^A\right). \end{aligned}$$

If $f^C f_C < C$ for a suitable positive constant C, the operator $\theta_t[f]$ is quasilinear and uniformly elliptic.

Proof. We give explicitly some steps of this straightforward calculation, which requires some care if Σ_t is not spacelike. We define the quantities

$$\gamma = \varphi^{-1} e^{2Z} \left(\rho^{-(1/2)}(1 + \eta^A f_A) - 1\right)\Big|_{r=f}, \quad \nu = \rho^{-(1/2)} e^Z|_{r=f}, \qquad (6.5)$$

which are smooth even at points where φ vanishes (i.e., where Σ_t becomes null). The one-form l^t_α satisfying (6.3) is

$$l^t_\alpha \, dx^\alpha = \gamma dt + \nu \left(dr - f_A dx^A\right).$$

For the metric induced on the $\mathcal{S}_t$ and its inverse we have

$$\begin{aligned} h_{t\,AB} &= \left(h_{AB} + \varphi f_A f_B + f_A \eta_B + f_B \eta_A + \eta^C \eta_C f_A f_B\right)\big|_{r=f}, \\ h_t^{AB} &= h^{AB} + \rho^{-1} \\ &\quad \times \left[f^D f_D \eta^A \eta^B - \varphi f^A f^B - \left(1 + \eta^D f_D\right)\left(\eta^A f^B + f^A \eta^B\right)\right]\big|_{r=f}. \end{aligned}$$

Note that h_t^{AB} is positive definite wherever $\rho > 0$. The operator $\theta_t[f]$ can now be written explicitly

$$\theta_t[f] = -\nu h_t^{AB} f_{AB} - \left(l^t_\mu(\Gamma^\mu_{rr} h_t^{AB}) f_A f_B + 2 l^t_\mu \Gamma^\mu_{rA} h^{AB} f_B + l^t_\mu \Gamma^\mu_{AB} h^{AB}\right)\Big|_{r=f}, \qquad (6.6)$$

where $f_{AB} = \partial_{x^A}\partial_{x^B} f$. Choosing C small enough it follows that $\rho > \epsilon$ for a positive ϵ. The assertion of the lemma is verified easily by estimating the eigenvalues of νh_t^{AB}. □

Remark. We note that for quasilinear elliptic equations such as $\theta_t[f] = 0$ there hold regularity results. In particular, if we required $C^{2,\alpha}$ for the function f instead of smooth as in the definitions above, the latter differentiability would in fact follow (see, e.g., Section 8.3 of Evans [29]).

We also remark that the stability operator (3.5) coincides with the linearization of the quasilinear operator (6.6) and can be obtained from the latter by making f infinitesimal. However, the expression (6.6) requires a choice of coordinates. On the other hand (3.5) was derived in a covariant manner and therefore holds independently of the coordinates on $\mathcal{S}$ or on the spacetime.

7 Barrier properties of stable MOTS

Let $\mathcal{G}$ be a MOTT, i.e., a 3-surface foliated by MOTS. Then there exists a positive variation along $\mathcal{G}$ such that $\delta_{\psi v}\theta = 0$. According to the definition in Section 5 the MOTS foliating a MOTT are stable with respect to the MOTT, but they are not strictly stable as $\delta_{\psi v}\theta = L_v\psi = 0$ implies that the principal eigenvalue vanishes. In general situations, since the variation of θ is essentially a derivative of the θs of adjacent surfaces, we expect that embedded strictly stable MOTS are "barriers", i.e., local boundaries separating regions containing weakly outer trapped and weakly outer untrapped surfaces. The difficulty in showing this lies again in the fact that strictly stable MOTS had to be defined in terms of a sign condition on a *single* variation, while they should be barriers for *all* weakly outer trapped surfaces in a neighbourhood.

Below we give a result on this issue, which in fact requires the full machinery developed in the preceding sections, namely the properties of the principal eigenvalue and eigenfunction of the stability operator as well as the maximum principle applied to the quasilinear elliptic equation (6.6) representing the MOTS as a graph. The maximum principles for non-linear operators normally require *uniform ellipticity* (c.f., e.g., [40, Section 10]) which is *not* the case for operators of prescribed mean curvature such as (6.6) when $f^C f_C$ is not small (c.f. Lemma 6.2). For this reason we prove the following theorem "from scratch." To state the result, we recall from [1] the definition of "locally outermost." Note that in the definition and in the theorem below we require that the MOTS is embedded rather than immersed as in the previous sections.

Definition 7.1. Let $\mathcal{S}$ be a MOTS embedded in a hypersurface Σ. $\mathcal{S}$ is called *locally outermost* in Σ, iff there exists a two-sided neighbourhood of $\mathcal{S}$ in Σ such that its exterior part does not contain any weakly outer trapped surface.

Theorem 7.1. (i) *An embedded, strictly stably outermost surface $\mathcal{S}$ is locally outermost. Moreover, $\mathcal{S}$ has a two-sided neighbourhood $\mathcal{U}$ such that no weakly outer trapped surface contained in $\mathcal{U}$ enters the exterior of $\mathcal{S}$ and no weakly outer untrapped surface contained in $\mathcal{U}$ enters the interior of $\mathcal{S}$.*

(ii) *A locally outermost surface $\mathcal{S}$ is stably outermost.*

Proof. The first statement of (i) is in fact contained in the second one. To show the latter, let ϕ be the positive principal eigenfunction of L_v. Since $L_v\phi > 0$ by assumption, flowing $\mathcal{S}$ in Σ along any extension of ϕv^α within Σ produces a family $\mathcal{S}_\sigma$, $\sigma \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. By choosing ϵ small enough, the $\mathcal{S}_\sigma$ have $\theta|_{\mathcal{S}_\sigma} < 0$ for $\sigma \in (-\epsilon, 0)$ and $\theta|_{\mathcal{S}_\sigma} > 0$ for $\sigma \in (0, \epsilon)$. We can now take $\mathcal{U}$ to be the neighbourhood of $\mathcal{S}$ given by $\mathcal{U} = \cup_{\sigma \in (-\epsilon,\epsilon)} \mathcal{S}_\sigma$. We first express the expansion of the level sets of σ in terms of connection coefficients. Setting $f = \sigma = const. > 0$ in (6.6), only the last term survives and we have

$$0 < \theta_t[\sigma] = - \left. l^t_\mu \Gamma^\mu_{AB} h^{AB} \right|_{f=\sigma} \tag{7.1}$$

and analogously for $\sigma < 0$.

Now let $\mathcal{B}$ be a weakly outer trapped surface (i.e., $\theta|_{\mathcal{B}} \leq 0$) contained in $\mathcal{U}$ which enters the exterior part of $\mathcal{U}$. Let p be the point where $\sigma|_{\mathcal{B}}$ achieves a maximum value. In a small neighbourhood of p, $\mathcal{B}$ is a graph given by a function f which achieves its maximum at p. From (6.6),

$$\begin{aligned} 0 \geq \theta|_{\mathcal{B}} = \theta_t[f] = &-\nu h_t^{AB} f_{AB} \\ &- \left. \left(l^t_\mu (\Gamma^\mu_{rr} h_t^{AB}) f_A f_B + 2 l^t_\mu \Gamma^\mu_{rA} h^{AB} f_B + l^t_\mu \Gamma^\mu_{AB} h^{AB} \right) \right|_{r=f} . \end{aligned} \tag{7.2}$$

At p, $f_A|_p = 0$ and $h^{AB} f_{AB}|_p \leq 0$. Since $\sigma = const$ and $\mathcal{B}$ have a common tangent plane at p, the last term in (7.2) coincides with (7.1), which yields the contradiction (we also use $\nu|_p = e^Z|_p$)

$$\theta_t[f]_p = -e^Z h^{AB} f_{AB}|_p - \left. l^t_\mu \Gamma^\mu_{AB} h^{AB} \right|_p > 0. \tag{7.3}$$

Hence $\mathcal{B}$ cannot enter the exterior part of $\mathcal{U}$, and analogously weakly outer untrapped surfaces cannot enter the interior.

To show (ii), assume $\mathcal{S}$ is locally outermost but not stably outermost. From Proposition 5.1, the principal eigenvalue λ is then negative. Arguing as above one constructs a foliation outside $\mathcal{S}$ with leaves which are outer trapped near $\mathcal{S}$, contradicting the assumption. □

Returning to the example of a MOTT mentioned above, it follows that a MOTS within a MOTT is not only not strictly stable with respect to the direction tangent to the MOTT, as we already knew, but also moreover not locally outermost.

The barrier arguments suggest that a region of a slice Σ_t bounded by an outer trapped surface $\mathcal{S}_1$ and outer untrapped surface $\mathcal{S}_2$ contains at least one MOTS. This has been proven recently by Andersson and Metzger [31], based on an argument proposed by Schoen [41].

8 Symmetries

As examples one normally considers spacetimes with symmetries, i.e., invariant under the action of some symmetry group. If the symmetry is spacelike, it is natural to consider spacelike foliations with the same symmetry. In order to locate the MOTS contained in the leaves and to compute the principal eigenfunctions and eigenvalues, it is useful to know whether the MOTS and the eigenfunctions inherit the symmetries from the ambient geometry. In this section we address these questions, starting with the MOTS.

Theorem 8.1. *Let Ξ be a local isometry of $(\mathcal{M}, g)$ (i.e., $\mathcal{L}_\xi g_{\mu\nu} = 0$ for the Lie-derivative $\mathcal{L}_\xi$ w.r. to the corresponding Killing field ξ^α), and let $\mathcal{S}$ be a MOTS which is stable with respect to a normal direction v^α such that the normal component $\xi^{\perp\alpha}$ of ξ^α satisfies $\xi^{\perp\alpha} = \psi v^\alpha$ for some function ψ. Then either Ξ leaves the MOTS invariant (i.e., ξ^α is tangential to the MOTS), or $\Xi(\mathcal{S})$ is a MOTT.*

Proof. Clearly ξ^α leaves the MOTS invariant iff $\xi^{\perp\alpha}$ and hence ψ are identically zero. Assume this is not the case. From (3.7), $0 = \delta_\xi\theta = \delta_{\psi v}\theta = L_v\psi$ shows that ψ is an eigenfunction of L_v with eigenvalue zero. Since the MOTS is stable with respect to v^α, ψ is the unique (up to a constant) principal eigenfunction of L_v and has therefore a constant sign, which after reversing ξ^α if necessary can be taken to be positive. This implies that Ξ in fact generates a MOTT as stated. □

Note that this proof shows in particular that if $\mathcal{S}$ is strictly stable then $\mathcal{S}$ must remain invariant under the isometry.

If $\mathcal{S}$ lies in some hypersurface Σ invariant under the isometry, then Theorem 8.1 implies that either Ξ leaves $\mathcal{S}$ invariant, or $\Xi(\mathcal{S}) \subset \Sigma$. Clearly, in the second case $\mathcal{S}$ is not strictly stable. Moreover, it is not locally outermost in the sense of Definition 7.1.

We also remark that the range of validity of Theorem 8.1 can be extended with the help of Proposition 5.2. In this theorem it suffices to require that the MOTS is stable with respect to any direction normal to $\mathcal{S}$ which lies "between l^μ and $\xi^{\perp\mu}$" (which is in fact a conical segment).

The following theorem on the symmetry of the principal eigenfunction ("ground state") is well known for self-adjoint operators, with numerous physical applications. We have formulated it here for the general linear elliptic operator (4.1), and therefore it holds in particular for the stability operator (3.5).

Theorem 8.2. *If L is invariant under a 1-parameter group of isometries generated by η^α, the principal eigenfunction ϕ is invariant as well.*

Proof. The invariance of L under the isometry implies that this operator commutes with the action of the corresponding Lie derivative $\mathcal{L}_\eta$. Hence, if ϕ is an eigenfunction with principal eigenvalue λ, it follows that

$$L\mathcal{L}_\eta\phi = \mathcal{L}_\eta L\phi = \lambda\mathcal{L}_\eta\phi, \tag{8.1}$$

which means that $\mathcal{L}_\eta\phi$ is a principal eigenfunction as well. But from Lemma 4.1(i), the latter is unique up to a factor, i.e., $\mathcal{L}_\eta\phi = \alpha\phi$ for some real number α. This expression integrates to zero on $\mathcal{S}$ because $\mathcal{L}_\eta\phi = D_A(\phi\eta^A)$. Since ϕ has constant sign, it follows that $\alpha = 0$. This proves the assertion. $\square$

As an example we consider axially symmetric data on Σ with Killing vector $\eta^\alpha = \partial/\partial\varphi$, which has in addition a (t, φ) symmetry, i.e., invariance under simultaneous sign reversal of φ and t. This is the case, in particular, for data on a $t = const.$ slice of the Kerr metric in Boyer–Lindquist coordinates. It turns out that the vector s_A introduced in Section 3 is proportional to the axial Killing vector and divergence free, i.e., $s_A = z_A$ in the notation of Section 4. By Theorem 8.2 the principal eigenfunction is axially symmetric and therefore satisfies a second-order ODE. Finding the principal eigenvalue and eigenfunction is therefore equivalent to solving a one-dimensional Sturm–Liouville problem. Moreover, the first-order term in the stability operator vanishes when acting on this eigenfunction. However, this does not imply that the stability operator is self-adjoint in this situation or that the principal eigenvalue coincides with any of the symmetrized eigenvalues λ_s or λ_z.

9 Existence and properties of MOTTs

The main objective of this paper is to show that MOTS "propagate" from a given slice to adjacent leaves of a given foliation to form a MOTT. Setting $\theta = 0$ in (6.2) determines the location of a MOTT in terms of immersions. These may be viewed as defining a family of quasilinear elliptic equations which do not contain any derivatives along the presumptive "evolution" direction. Since we have assumed that $\mathcal{S}_0$ is marginally outer trapped, the elliptic equation is satisfied for $t = 0$ and we can adopt a perturbational approach. In Section 6 we have derived the "graph" representation (6.6) in a special coordinate system, which will be used in the existence proof. The proof makes use of a general existence result [32] which shows the existence of solutions near a known solution of a 1-parameter family of non-linear differential equations (or systems) of arbitrary order and arbitrary type whose linearization is elliptic. Since we will apply this result to a 1-parameter family of quasilinear elliptic equations for which regularity results hold, we will formulate the result here for smooth functions only. Equations of general type require some care regarding differentiability.

Lemma 9.1 ([32, 33]). *Let*

$$F(x, u, \partial u, \partial^2 u; t) = 0 \tag{9.1}$$

be a 1-parameter family of quasilinear, second-order differential equations, where F is a smooth function of all arguments. Assume that u_0 is a smooth solution of (9.1) *for $t = t_0$, and that the linearized equation around u_0, $Lv = f$ is elliptic and has a unique solution for any smooth f. Then* (9.1) *has a unique smooth solution for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.*

This result is an easy application of the implicit function theorem; alternatively, the latter theorem can also be applied directly to (6.6), as we suggested in [1]. The result is, in view of the above discussion, that a *smooth* horizon exists in some *open* neighbourhood of the given slice. More precisely, we have the following theorem and a corollary.

Theorem 9.1. *Let $(\mathcal{M}, g_{\alpha\beta})$ be a spacetime foliated by smooth hypersurfaces Σ_t, $t \in [0, T]$ and assume that Σ_0 contains a smooth, immersed, strictly stable MOTS $\mathcal{S}_0$. Then for some $\tau \in (0, T]$ there is a smooth adapted MOTT*

$$\mathcal{G}_{[0,\tau)} = \Phi(\mathcal{S}_0 \times [0, \tau))$$

such that for each $t \in [0, \tau)$, $\mathcal{S}_t = \Phi(\mathcal{S}_0, t)$ is a smooth, immersed, strictly stable MOTS with $\mathcal{S}_t \subset \Sigma_t$.

Corollary 9.1. *The MOTT $\mathcal{G}_{[0,\tau)}$ constructed in Theorem* 9.1 *is nowhere tangent to the leaves Σ_t of the foliation.*

Proof. Let $\{t, r, x^A\}$ be the local coordinate system introduced in Lemma 6.1 and restrict the allowed MOTT to be local graphs on $\mathcal{S}_0$ as described in Section 6. Thus, we are looking for functions $f(t, x^A)$ which satisfy $\theta_t[f] = 0$ for all t near $t = 0$ and satisfying $f(0, x^A) = 0$. In order to apply Lemma 9.1 we only need to check that the linearization of $\theta_t[f]$ is elliptic and invertible. The immersion Φ defining the MOTT is given in this coordinate system by $(t, r = f(t, x^A), x^A)$. Recall that $l^\alpha \partial_\alpha = \partial_t$ and that on $\mathcal{S}_0$ the vector $v^\alpha \partial_\alpha = \partial_r$ satisfies $l^\alpha v_\alpha = 1$. Linearizing the operator around $(t = 0, f = 0)$ corresponds to fixing $t = 0$ and making f infinitesimal, or more precisely to evaluating $L(f') \equiv \partial_\epsilon \theta_{t=0}[f_\epsilon]|_{\epsilon=0}$ where f_ϵ depends smoothly on ϵ and satisfies $f_\epsilon|_{\epsilon=0} = 0$, $\partial_\epsilon f_\epsilon|_{\epsilon=0} = f'$. It follows that L corresponds to performing a geometric variation of θ along the vector $f'\partial_r$. The general variation formula in Section 3 gives $L(f') = \delta_{f'v} = L_v(f')$. Since by assumption, $\mathcal{S}_0$ is strictly stably outermost with respect to v, it follows that L_v and therefore L, is invertible. Existence of $f(t, x^A)$ for small t solving $\theta_t[f] = 0$ follows readily from Lemma 9.1. Moreover, the "evolution" vector to the MOTT, $q^\alpha \equiv \Phi_\star(\partial_t) = \partial_t + f'\partial_r$ is by construction nowhere tangent to Σ_t, which proves the corollary. $\square$

As a complement to this corollary we now discuss the "tangency" property of the MOTT when the principal eigenvalue λ_τ vanishes, assuming smooth convergence of the $\mathcal{S}_t$ to a stable MOTS $\mathcal{S}_\tau$.

Theorem 9.2. *Let $(\mathcal{M}, g_{\alpha\beta})$ be a spacetime containing a smooth reference foliation $\{\Sigma_t\}_{t\in[0,T]}$. Assume that Σ_0 contains a smooth, immersed, strictly stable MOTS $\mathcal{S}_0$. Assume furthermore that the adapted MOTT $\mathcal{G}_{[0,\tau)}$ through $\mathcal{S}_0$ constructed in Theorem 9.1 is such that as $t \to \tau$, the surfaces $\mathcal{S}_t$ converge to a smooth, compact, stable MOTS $\mathcal{S}_\tau$. Let λ_t be the principal eigenvalue of the stability operator of $\mathcal{S}_t$ and $\phi_t^\dagger$ the principal eigenfunction of its adjoint. Assume that $\lambda_t^{-1}\langle\phi_t^\dagger, W|_{\mathcal{S}_t}\rangle$ has a limit (finite or infinite) as $t \to \tau$.*

Then the closure $\mathcal{G}_{[0,\tau]} = \mathcal{G}_{[0,\tau)} \cup \mathcal{S}_\tau$ is an adapted MOTT. If in addition $\lambda_\tau = 0$ and $\langle\phi_\tau^\dagger, W|_{\mathcal{S}_\tau}\rangle \neq 0$, then $\mathcal{G}_{[0,\tau]}$ is tangent to Σ_τ everywhere on $\mathcal{S}_\tau$.

Corollary 9.2. *If the null energy condition holds on $\mathcal{S}_\tau$, $\lambda_\tau = 0$ and $W|_{\mathcal{S}_\tau} \neq 0$ somewhere, then $\mathcal{G}_{[0,\tau]}$ is tangent to Σ_τ everywhere on $\mathcal{S}_\tau$.*

Proof. If $\lambda_\tau > 0$, then by uniqueness the MOTT through $\mathcal{S}_\tau$ constructed using Theorem 9.1 agrees with $\mathcal{G}_{[0,\tau)}$ for $t \in [0, \tau)$, and hence the closure $\mathcal{G}_{[0,\tau]}$ is an adapted MOTT. It remains to consider the case where $\lambda_\tau = 0$.

By construction, $\mathcal{S}_t = \Sigma_t \cap \mathcal{G}_{[0,\tau)}$ is a MOTS for $t \in [0, \tau)$. Let v^α be the normal to $\mathcal{S}_t$ tangent to Σ_t (we drop the subindex t for simplicity). Consider the "evolution" vector q^α of the MOTS within $\mathcal{G}_{[0,\tau)}$, given by $q^\alpha\partial_\alpha = \Psi_\star(\partial_t) = \partial_t + (\partial_t f)\partial_r$. Recalling that $l^\alpha\partial_\alpha = \partial_t$ it follows that the normal component of q^α can be decomposed as

$$q^{\perp\alpha} = l^\alpha + uv^\alpha \tag{9.2}$$

for some function u. The variation of θ along q^α must vanish (this is precisely the condition that q^α is tangent to the MOTT). Hence, u must satisfy

$$\delta_q\theta = \delta_{l+uv}\theta = -W + L_v u = 0, \tag{9.3}$$

c.f. Lemma 3.2.

Consider the equation $L_v u = W$. Referring to Lemma 4.3, we decompose $u = \mathbb{P}u + \mathbb{Q}u$ and write this as $u = z\phi + w$ for some real z. We have $L_v u = \lambda z \phi + Aw$, where $A = L_v \mathbb{Q}$ as in Lemma 4.3. It follows that

$$z = \lambda^{-1} \frac{\langle \phi^\dagger, W \rangle}{\langle \phi^\dagger, \phi \rangle} \tag{9.4}$$

and

$$Aw = \mathbb{Q}W.$$

Since for $t \in [0, \tau]$, L_v is uniformly elliptic and its coefficients are uniformly smooth [43, Theorem 3.1, p. 208], applies to show that the spectral gap is lower semi-continuous in t, i.e., $\liminf c_0(t) \geq c_0(\tau)$. At τ, the spectral gap is positive. Thus, it follows that there exists $\epsilon > 0$ and a constant $c_0 > 0$ such that the spectral gap estimate (4.11) holds for $t \in (\tau - \epsilon, \tau]$. Recall that by construction $\mathbb{Q}w = w$. Since by assumption $\lambda > 0$ for $t < \tau$, we may apply (4.12) with $\lambda = 0$, to conclude that $\mathbb{Q}w$, and hence also w, is uniformly bounded in Sobolev spaces.

Suppose now that $\langle \phi_\tau^\dagger, W|_{\mathcal{S}_\tau} \rangle \neq 0$ and $\lambda_\tau = 0$. Then it follows that $z \to \infty$, as $t \to \tau$, and hence that u diverges. We see further that in this case,

$$\lim_{t \to \tau} z^{-1} u = \phi_\tau > 0,$$

where ϕ_τ is the eigenfunction of λ_τ. Define a normalized evolution vector $\hat{q}^\alpha = z^{-1} q^\alpha$. Then

$$\hat{q}^\alpha \partial_\alpha = z^{-1} \partial_t + z^{-1} u \partial_r.$$

It follows from the above that as $t \nearrow \tau$, the vectors $\hat{q}^\alpha$ converge smoothly to a limit, which is proportional to ∂_r. By construction $\hat{q}^\alpha$ is a smooth vectorfield tangent to the MOTT $\mathcal{G}_{[0,\tau)}$, and we have now shown that $\hat{q}^\alpha$ extends smoothly to the closure $\mathcal{G}_{[0,\tau]} = \mathcal{G}_{[0,\tau)} \cup \mathcal{S}_\tau$. In order to demonstrate that the closure is indeed a MOTT, it remains to renormalize the time parameter.

Thus, let $\phi_s : \mathcal{S}_0 \to \mathcal{G}_{[0,\tau]}$ be defined by the flow of the vector field $\hat{q}^\alpha$, and let $I = [0, s_*]$ denote the parameter interval required for this flow to sweep out $\mathcal{G}_{[0,\tau]}$. This construction defines a smooth monotone map $\sigma : I \to [0, \tau]$. We can now extend the immersion defining the MOTT $\mathcal{G}_{[0,\tau)}$ to an immersion $\Phi : \mathcal{S}_0 \times I \to \mathcal{M}$ such that for $s \in I$, we have $\mathcal{S}_{\sigma(s)} = \Phi(\mathcal{S}_0, s) \subset \Sigma_{\sigma(s)}$ for $s \in I$. It is clear from the construction that this defines a MOTT $\mathcal{G}_{[0,\tau]} = \mathcal{G}_{[0,\tau)} \cup \mathcal{S}_\tau$.

It remains to consider the case when $\langle \phi_\tau^\dagger, W|_{\mathcal{S}_\tau} \rangle = 0$. In this case, by applying the same argument as above, we find that u may or may not diverge as $t \to \tau$, depending on the detailed behaviour of λ and W as $t \to \tau$. In particular, we see from (9.4) that if $\lambda^{-1} \langle \phi^\dagger, W \rangle$ diverges as $t \to \tau$, then also z diverges, and we are essentially in the situation considered above. On the other hand, if $\lambda^{-1}\langle\phi^\dagger, W\rangle$ tends to a bounded limit as $t \to \tau$, then we are in a situation which is analogous to the case with $\lambda_\tau > 0$. In either case, the normalized evolution vector $\hat{q}^\alpha$ converges as $t \to \tau$, and an argument along the lines above shows that $\mathcal{G}_{[0,\tau]} = \mathcal{G}_{[0,\tau)} \cup \mathcal{S}_\tau$ is an adapted MOTT. □

In [1] we claimed that the evolution of the MOTS can be continued "as long as the MOTS remain strictly stably outermost." We emphasize, however, that in [1] the MOTS were taken to be *smooth and embedded by definition.* Moreover, we have tacitly assumed that the MOTT does not "run off to infinity." Hence, our control of the evolution may end not only when λ goes to zero but also when the MOTS develop self-intersections or when we lose compactness or smoothness. In the present setup, we do not worry about MOTS developing self-intersections as we have allowed them from the outset. However, to show in particular that existence continues up to and *including* Σ_τ (which was assumed in Theorems 9.1 and 9.2) we have to exclude the other pathologies.

To avoid that the MOTT "runs off to infinity" we simply require that it is contained in a compact subset of $\mathcal{M}$. Note that it may still happen that the area of the MOTS $\mathcal{S}_t$ grows without bound as we approach $\mathcal{S}_\tau$. In view of the curvature estimates of [34], this can happen in the four-dimensional case only if the MOTS "folds up" sufficiently. Assuming a uniform area bound, we have the following result, which follows from the work in [34].

Proposition 9.1. *Let $(\mathcal{M}, g_{\alpha\beta})$ be a spacetime of dimension 4, with a spacelike reference foliation $\{\Sigma_t\}_{t\in[0,T]}$. Assume that Σ_0 contains a smooth, immersed, strictly stable MOTS $\mathcal{S}_0$. Let $\mathcal{G}_{[0,\tau)}$ be the adapted MOTT through $\mathcal{S}_0$ constructed in Theorem* 9.1*, and let $\mathcal{S}_t = \mathcal{G}_{[0,\tau)} \cap \Sigma_t$ be the leaf of $\mathcal{G}_{[0,\tau)}$ in Σ_t. Assume that for $t \in [0,\tau)$, the leaves $\mathcal{S}_t$ have uniformly bounded area and are contained in a compact subset of $\mathcal{M}$. Then the $\mathcal{S}_t$ converge as $t \to \tau$ to a smooth compact surface $\mathcal{S}_\tau$ which is a smooth, immersed, stable MOTS.*

If the dominant energy condition is satisfied, the area of the limit set of the MOTT in fact stays bounded as long as the MOTS stay strictly stable, as a consequence of the following lemma. (This computation is known in the context of the topological results [20].) Recall that v^α is everywhere linearly independent of l^α.

Lemma 9.2. *In a four-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ in which the dominant energy condition holds, let $\mathcal{S}$ be a MOTS which is strictly stable with respect to a spacelike or null direction v, with principal eigenvalue λ and area $|\mathcal{S}|$. Then $\mathcal{S}$ is topologically a sphere and $\lambda|\mathcal{S}| \leq 4\pi$. Moreover, if $\lambda|\mathcal{S}| = 4\pi$ then $\mathcal{S}$ has constant curvature, i.e., $R_{AB} = \lambda h_{AB}$.*

Proof. We take ϕ to be the principal eigenfunction of L_v, and we call $y_A = -\phi^{-1}D_A\phi + s^A$ and $y^2 = y_A y^A$. Taking ψ as the eigenfunction ϕ in (3.5), we obtain

$$\lambda = D_A y^A - |y|^2 + \frac{1}{2}R_{\mathcal{S}} - Y, \tag{9.5}$$

where Y has been defined in (3.4). Integrating (9.5) and using the Gauss-Bonnet theorem gives

$$\lambda|\mathcal{S}| = 2\pi\chi - \int_{\mathcal{S}} (y^2 + Y), \tag{9.6}$$

where χ is the Euler number. The first assertion of the lemma now follows since $\lambda > 0$ and $Y \geq 0$. If $\lambda|\mathcal{S}| = 4\pi$, (9.6) implies $y_A \equiv 0$ and $Y \equiv 0$. Putting this back into (9.5), we have $R_{\mathcal{S}} = 2\lambda$ which implies the statement of the lemma as $R_{AB} = \frac{1}{2}Rh_{AB}$ in 2 dimensions. □

We remark that the same method shows that in the case $\lambda = 0$, the MOTS can be a torus, which necessarily must be flat ($R_{AB} = 0$).

With the help of Lemma 9.2, we can now sharpen Proposition 9.1 to formulate an existence result as follows.

Theorem 9.3. *Let $(\mathcal{M}, g_{\alpha\beta})$ be a spacetime of dimension 4, in which the dominant energy condition holds and which is foliated by smooth spacelike hypersurfaces Σ_t, $t \in [0, T]$. Assume that Σ_0 contains a smooth, immersed, strictly stable MOTS $\mathcal{S}_0$. Assume further that the MOTT $\mathcal{G}_{[0,\tau)}$ through $\mathcal{S}_0$ constructed in Theorem* 9.1 *is contained in a compact subset of $\mathcal{M}$ and that either*

(i) $\liminf_{t\to\tau} \lambda_t > 0$, *or*
(ii) $\lim_{t\to\tau} \lambda_t = 0$ *and* $\limsup_{t\to\tau} |\mathcal{S}_t| < \infty$.

Then, there is a smooth, compact, strictly stable MOTS $\mathcal{S}_\tau$ in Σ_τ such that $\mathcal{G}_{[0,\tau]} = \mathcal{G}_{[0,\tau)} \cup \mathcal{S}_\tau$ is an adapted MOTT, which is the closure of $\mathcal{G}_{[0,\tau)}$.

We remark that, as the surface $\mathcal{S}_\tau \subset \Sigma_\tau$ itself satisfies the requirements of the theorem, the evolution in fact continues in $[0, T]$ "as long as the MOTS $\mathcal{S}_t \subset \Sigma_t$ stay strictly stable".

We now reproduce and slightly extend the result [1] on the causal structure of MOTT foliated by strictly stably outermost MOTS, which must be either null *everywhere* or spacelike *everywhere*.

Theorem 9.4. *Let $(\mathcal{M}, g_{\alpha\beta})$ be a spacetime in which the null energy condition holds, which is foliated by smooth, locally achronal (i.e., spacelike or* *null at each point) hypersurfaces Σ_t nowhere tangent to l^μ, and that Σ_0 contains a strictly stable MOTS $\mathcal{S}_0$. Then, the following holds for the MOTT $\mathcal{G}$ through $\mathcal{S}_0$ and adapted to the foliation Σ_t.*

(i) *The MOTT $\mathcal{G}$ is achronal in a neighbourhood of $\mathcal{S}_0$.*
(ii) *If W does not vanish identically, $\mathcal{G}$ is spacelike everywhere near $\mathcal{S}_0$.*
(iii) *If W vanishes identically on $\mathcal{S}_0$, $\mathcal{G}$ is null everywhere on $\mathcal{S}_0$.*

Proof. Recall equation (9.3) for the normal variation vector $q^{\perp\alpha} = l^\alpha + uv^\alpha$. Applying points 4.2 and 4.2 of the maximum principle, Lemma 4.2, proves the results. □

We finally comment on a possible alternative construction for MOTTs. Consider a weakly outer trapped surface $\mathcal{S}$ on some initial slice and take the null cone emanating from it. By the Raychaudhuri equation (3.6) if $W > 0$, the null cone cuts each subsequent slice on an outer trapped surface which gives a trapped "barrier" $\mathcal{S}_1$. If we also assume an outer untrapped "barrier" $\mathcal{S}_2$ outside $\mathcal{S}_1$, which in particular always exists near spacelike or null infinity in asymptotically flat spacetimes, there is a MOTS in the region bounded by $\mathcal{S}_1$ and $\mathcal{S}_2$ by the results of [31, 41], c.f. the discussion in Section 7. In this way, one can show the existence of an outermost MOTS on subsequent slices. In general however, this collection of MOTS need not be a MOTT because it may jump, as it always tracks the outermost MOTS on each hypersurface Σ_t.

Acknowledgments

We thank Jan Metzger for several helpful discussions. We also wish to thank Alberto Carrasco for comments on the manuscript and Helmuth Urbantke for improving Theorem 8.2. We gratefully acknowledge the hospitality and support received from the Isaac Newton Institute, Cambridge, where part of the work was performed. M.M. and W.S. were supported by the projects FIS2006-05319 of the Spanish Ministerio de Educación y Tecnología and SA010CO of the Junta de Castilla y León. M.M. also acknowledges financial support from the Junta de Andalucía under project P06-FQM-01951. L.A. was partly supported by the NSF under contract no. DMS 0407732, with the University of Miami.

Appendix A Proof of Lemma 3.1

Due to the linearity of the variation and the fact that $q^{\|}$ generates a diffeomorphism of $\mathcal{S}$, which implies $\delta_{q^{\|}}\theta = q^{\|}(\theta)$, we can assume $q = q^{\perp}$

without loss of generality. Since the variation is a local calculation we may assume that all $\mathcal{S}_\sigma$ are embedded. Let k_σ be a null, future directed normal vector to $\mathcal{S}_\sigma$ satisfying $k^\alpha_\sigma l^\beta_\sigma g_{\alpha\beta} = -2$. We wish to extend l^α_σ and k^α_σ, for each value of σ to a 1-parameter family of vector fields defined on a neighbourhood of $\mathcal{S}$. Let Ω_σ be the null hypersurface generated by k^α_σ by affinely parametrized geodesics. Extend first l^α_σ to Ω_σ by parallel transport along k^α_σ. Then extend l^α_σ away from Ω_σ by affinely parametrized null geodesics and finally extend k^α_σ by parallel transport along l^α_σ. Notice that, in general, the two planes orthogonal to $\{l^\alpha_\sigma, k^\alpha_\sigma\}$ at each point do not define 2-surfaces. On $\mathcal{S}_\sigma$ however they obviously do. Being l^α_σ a geodesic field, the null expansion θ_σ can be rewritten as $\theta_\sigma(p) = (\nabla_\alpha l^\alpha_\sigma)|_{\Phi(p,\sigma)}$ for any $p \in \mathcal{S}$, where $\Phi(p,\sigma)$ is the variation of Section 3. Defining $U^\alpha = \partial_\sigma l^\alpha_\sigma|_{\sigma=0}$ (with partial derivative taken at a fixed spacetime point), we obtain directly from the definition of the variation

$$\begin{aligned}\delta_q\,\theta &= \nabla_\alpha U^\alpha + q^\beta \nabla_\beta \nabla_\alpha l^\alpha|_{\mathcal{S}}\\ &= \nabla_\alpha U^\alpha + q^\beta \nabla_\alpha \nabla_\beta l^\alpha - bG_{\alpha\beta} l^\alpha l^\beta + \frac{u}{2} R_{\alpha\beta} k^\alpha l^\beta|_{\mathcal{S}}, \qquad \text{(A.1)}\end{aligned}$$

where $l^\alpha = l^\alpha_{\sigma=0}$ and $k^\alpha = k^\alpha_{\sigma=0}$ and the Ricci identity has been used in the second equality. Let us next determine the divergence of U^α. From that fact that l^α_σ is null for all σ, it follows $U^\alpha l_\alpha = 0$ and hence $U^\alpha = al^\alpha + U^A e^\alpha_A$ for a as in the statement of the lemma and for suitable functions U^A. Here e^α_A is a basis of the orthogonal subspace to l and k at each point. Using the projector $h_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}(l_\alpha k_\beta + k_\alpha l_\beta)$ it is easily found that the divergence of any vector of the form $F^A e^\alpha_A$ is

$$\nabla_\alpha \left(F^A e^\alpha_A\right)|_{\mathcal{S}} = D_A F^A|_{\mathcal{S}}, \qquad \text{(A.2)}$$

where we used the fact that $l^\alpha \nabla_\alpha k^\beta = 0$ and $k^\alpha \nabla_\alpha l^\beta|_{\mathcal{S}} = 0$. We need to determine U^A on $\mathcal{S}$: in local coordinates $y^\alpha(x^A,\sigma)$ for the map Φ, orthogonality of l^α_σ to $\mathcal{S}_\sigma$ means

$$g_{\alpha\beta}(y^\mu(\sigma,x^B))\frac{\partial y^\alpha}{\partial x^A} l^\beta_\sigma(y^\mu(\sigma,x^B)) = 0.$$

Its σ-partial derivative at $\sigma = 0$ gives $(\nabla_{e_A} q, l) + U_A|_{\mathcal{S}} = 0$, i.e., $U_A|_{\mathcal{S}} = -D_A u + us_A$. From $l^\alpha_\sigma \nabla_\alpha l^\beta_\sigma = 0$ it follows $U^\alpha \nabla_\alpha l^\beta + l^\alpha \nabla_\alpha U^\beta = 0$, which after multiplication with k_β and the fact k^α is parallel along l^α gives $l^\alpha \partial_\alpha a = -U^A s_A$. Thus, using (A.2)

$$\nabla_\alpha U^\alpha|_{\mathcal{S}} = -\,\Delta_{\mathcal{S}} u + 2s^A D_A u + u\left(D_A s^A - s^A s_A\right) + a\theta\,\big|_{\mathcal{S}}. \qquad \text{(A.3)}$$

We next consider the second term in (A.1), i.e., $bl^\beta \nabla_\alpha \nabla_\beta l^\alpha - \frac{u}{2} k^\beta \nabla_\alpha \nabla_\beta l^\alpha$. An integration by parts and using that l^α is geodesic implies

$$l^\beta \nabla_\alpha \nabla_\beta l^\alpha|_{\mathcal{S}} = -\nabla_\alpha l_\beta \nabla^\beta l^\alpha|_{\mathcal{S}} = -K^\mu_{AB} K^{\nu\,AB} l_\mu l_\nu|_{\mathcal{S}}.$$

Decomposing $k^\beta \nabla_\beta l^\alpha = Q l^\alpha + r^A e^\alpha_A$ and using the fact that $Q|_{\mathcal{S}} = r^A|_{\mathcal{S}} = 0$, another integration by parts gives $k^\beta \nabla_\alpha \nabla_\beta l^\alpha|_{\mathcal{S}} = l^\mu \partial_\mu Q - K^\mu_{AB} K^{\nu\,AB} l_\mu k_\nu|_{\mathcal{S}}$. In order to determine $l^\alpha \partial_\alpha Q$, we first note that $-2Q = k^\alpha k^\beta \nabla_\alpha l_\beta$. Taking covariant derivative along l^α and using the Ricci identity we find $2 l^\mu \partial_\mu Q|_{\mathcal{S}} = l^\alpha k^\mu l^\beta k^\nu R_{\alpha\mu\beta\nu}|_{\mathcal{S}}$. Collecting terms we get

$$q^\alpha \nabla_\alpha \nabla_\beta l^\alpha|_{\mathcal{S}} = -b K^\mu_{AB} K^{\nu\,AB} l_\mu l_\nu + \frac{u}{2}\left(K^\mu_{AB} K^{\nu\,AB} l_\mu k_\nu - \frac{1}{2} l^\alpha k^\mu l^\beta k^\nu R_{\alpha\mu\beta\nu} \right)\Big|_{\mathcal{S}}. \tag{A.4}$$

For the last term in (A.1), the definition of the projector $h^{\alpha\beta}$ gives $R + 2R_{\alpha\beta} l^\alpha k^\beta = h^{\alpha\beta} h^{\mu\nu} R_{\alpha\mu\beta\nu} + \frac{1}{2} l^\alpha k^\mu l^\beta k^\nu R_{\alpha\mu\beta\nu}$. Making use of the Gauss identity $h^{\alpha\beta} h^{\mu\nu} R_{\alpha\mu\beta\nu}|_{\mathcal{S}} = R_{\mathcal{S}} - H^2 + K_{\mu\,AB} K^{\mu\,AB}$ we get

$$R_{\alpha\beta} l^\alpha k^\beta|_{\mathcal{S}} = -G_{\alpha\beta} l^\alpha k^\beta + R_{\mathcal{S}} - H^2 - K^\mu_{AB} K^{\nu\,AB} l_\mu k_\nu + \frac{1}{2} l^\alpha k^\mu l^\beta k^\nu R_{\alpha\mu\beta\nu}\Big|_{\mathcal{S}}. \tag{A.5}$$

Inserting (A.3), (A.4) and (A.5) into (A.1) completes the proof. □

Appendix B Proof of Lemma 4.1 (sketch)

The Krein–Rutman theorem states that on a Banach space B, a compact linear operator E that maps any non-zero element of a closed cone K (i.e., a topologically closed subset of B closed under addition and multiplication by non-negative scalars) into its topological interior necessarily has a unique eigenvector u in the interior of K of unit norm. Moreover, the corresponding eigenvalue α is real and positive and any other element β of the spectrum of E (complex in general) satisfies $\alpha \geq |\beta|$ where $|\cdot|$ denotes the complex norm. A proof of this theorem can be found in [8.1].

In the case of the elliptic operator L (4.1), let δ be a constant satisfying $\delta > \sup_{\mathcal{S}} -c$ and define the operator $L' = L + \delta$. The zero-order term is therefore positive everywhere and the PDE $L'f = g$ admits a unique solution in $C^{2,\alpha}(\mathcal{S})$ for any $g \in C^{0,\alpha}(\mathcal{S}) \equiv B$. Let $Q : B \to B$ be defined by $Q(g) = f$ and let K be the set of non-negative functions (which is obviously a cone). The maximum principle implies that if $g \in K$ and non-identically zero, then $f = Q(g)$ is strictly positive everywhere. Thus, all the conditions of the Krein–Rutman theorem are fulfilled and there exists a unique non-negative function ϕ of unit $C^{0,\alpha}(\mathcal{S})$ norm satisfying $Q(\phi) = \alpha\phi$. Since α is positive and ϕ is in the image of Q it follows that ϕ is strictly positive and in $C^{2,\alpha}(\mathcal{S})$. Elliptic regularity implies that ϕ is in fact smooth. It follows that $L\phi = (\alpha^{-1} - \delta)\phi$, so we have a positive eigenfunction (unique up to rescaling) and a real eigenvalue $\lambda = \alpha^{-1} - \delta$.

It only remains to show that any other eigenvalue of L has larger or equal real part. This does not follow directly from the Krein–Rutman theorem. However, we use the following argument, which we adapt from Evans [29]. Let ψ be a (possibly complex) eigenfunction of L with eigenvalue μ. Define $u = \phi^{-1}\psi$. A direct computation gives

$$-D_A D^A u + 2t'^A D_A u + (\lambda - \mu)u = 0, \tag{B.1}$$

where $t'^A = t^A - D^A\phi$. Using also the complex conjugate of (B.1) a short calculation gives

$$\left(-D_A D^A + 2t'^A D_A\right)|u|^2 = 2\left(\mathrm{Re}(\mu) - \lambda\right)|u|^2 - D_A u D^A \overline{u} \leq 2\left(\mathrm{Re}(\mu) - \lambda\right)|u|^2.$$

Thus, if $\mathrm{Re}(\mu) < \lambda$ the right-hand side is non-positive and the maximum principle would imply $|u|^2 = 0$. Thus, $\mathrm{Re}(\mu) \geq \lambda$ as claimed.

Finally, the result on the adjoint is a trivial consequence of the positivity of the principal eigenfunctions, c.f. [29]. Explicitly, if $\phi^\dagger$ and $\lambda^\dagger$ are the principal eigenfunction and eigenvalue of $L^\dagger$, it follows

$$0 = \langle L^\dagger\phi^\dagger, \phi\rangle - \langle\phi^\dagger, L\phi\rangle = \left(\lambda^\dagger - \lambda\right)\langle\phi^\dagger, \phi\rangle$$

and $\lambda^\dagger = \lambda$ as positive functions cannot be L^2 orthogonal.

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