Adv. Theor. Math. Phys. **11** (2007) 683–705

Toroidal orbifolds à la Vafa–Witten

Jimmy Dillies

Department of Mathematics Texas A&M College Station, TX 77843-3368
jimmy@math.tamu.edu

Abstract

We classify orbifolds obtained by taking the quotient of a 3-torus by Abelian extensions of $\mathbb{Z}/n \times \mathbb{Z}/n$ automorphisms, where each torus has a multiplicative $\mathbb{Z}/n$ action ($n \in \{3, 4, 6\}$). This 'completes' the classification of orbifolds of the above type initiated by Donagi and Faraggi [4] and Donagi and Wendland [5].

1 Introduction

In 1985, Dixon, Harvey, Vafa and Witten pioneered the study of string theory on orbifolds [3]. A recurrent model, the $\mathbb{Z}/2 \times \mathbb{Z}/2$ orbifold, which was introduced by Vafa and Witten [6] has been studied extensively. In particular, Donagi and Faraggi classified further quotients using symmetric shifts [4]. They deduce that the three generation vacua are not obtained in this manner. Seeking a better model, Donagi and Wendland [5] studied quotients of 3-tori by Abelian extensions of the $\mathbb{Z}/2 \times \mathbb{Z}/2$ automorphisms.

We classify here the orbifolds obtained by taking the quotient of 3-tori by Abelian extensions of $\mathbb{Z}/n \times \mathbb{Z}/n$ automorphisms, where each torus has a multiplicative $\mathbb{Z}/n$ action ($n \in \{3, 4, 6\}$).

e-print archive: http://lanl.arXiv.org/abs/arXiv:hep-th/0609714

2 Construction and results

2.1 Construction

Consider the elliptic curve E_3, quotient of the complex plane by the sublattice $\Lambda_n = \mathbb{Z} \oplus \omega_3\mathbb{Z}$, where $\omega_3 = \mathrm{e}^{i\pi/3}$. This curve is endowed with a multiplicative automorphism: if we write $[x]$ for the class in E_3 of $x \in \mathbb{C}$ and ζ_3 for a primitive third root of unity, the map is given explicitly by $[x] \mapsto [\zeta_3 x]$. This automorphism generates an action of $\mathbb{Z}/3\mathbb{Z}$ on E_3. Consequently, if we take the variety obtained by taking three copies of E_3, $X_3 = E_3 \times E_3 \times E_3$, it comes with an action of $(\mathbb{Z}/3\mathbb{Z})^3$. This action restricts to an action of $(\mathbb{Z}/3\mathbb{Z})^2$ as schematized in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z}/3\mathbb{Z})^2 & \hookrightarrow & (\mathbb{Z}/3\mathbb{Z})^3 & \xrightarrow{+} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\ & & & \searrow & \downarrow & & & & \\ & & & & \mathrm{Aut}(X_3) & & & & \end{array}$$

Define T_3 to be the set of points of X_3 which are fixed by $(\mathbb{Z}/3\mathbb{Z})^2$. It is a subgroup of $E_3[3] \times E_3[3] \times E_3[3]$ isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$ ($E_3[3]$ denotes the 3-torsion points of the elliptic curve). The group T_3 acts by translation on X_3 and its action commutes with the one of $(\mathbb{Z}/3\mathbb{Z})^2$ which was just introduced. We have therefore an action of the direct product $V_3 := (\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3$ which we will call the Vafa–Witten action of order 3.

In the following sections, we will deal with quotients of X_3 by subgroups of V_3 which project onto the multiplicative $(\mathbb{Z}/3\mathbb{Z})^2$.

The above construction can be mimiced with the elliptic curves $E_4 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ and $E_6 = E_3$, respectively, endowed with the automorphisms $[x] \mapsto [\zeta_6 x]$ and $[x] \mapsto [\zeta_4 x]$. As a result, we construct an action of $V_4 = ((\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^3)$ and $V_6 = ((\mathbb{Z}/6\mathbb{Z})^2 \times \mathrm{Id})$, respectively, on X_4 and X_6; these actions will be called the Vafa–Witten actions of order 4 and 6. As for X_3, we will be interested in quotients of $X_{n=4,6}$ by subgroups of $V_{n=4,6}$ which project onto the multiplicative part.

We can synthesize the situation in the following table.

n	ω_n	ζ_n	T_n
3	$\mathrm{e}^{i\pi/3}$	$\mathrm{e}^{2i\pi/3}$	$(\mathbb{Z}/3\mathbb{Z})^3 \subset E_3[3]^3$
4	i	i	$(\mathbb{Z}/2\mathbb{Z})^3 \subset E_4[2]^3$
6	$\mathrm{e}^{i\pi/3}$	$\mathrm{e}^{i\pi/3}$	$\{0\}$

2.2 Notation

The element $g : ([z_1], [z_2], [z_3]) \mapsto ([\zeta_i^{m_1} z_1 + a_1 t_n], [\zeta_i^{m_2} z_2 + a_2 t_n], [\zeta_i^{m_3} z_3 + a_3 t_n])$ will be denoted as $g = (m_1, m_2, m_3; a_1, a_2, a_3)$. The element t_n is a generator of $T_n|_{E_n}$. Notice that the m_i add up to zero in $\mathbb{Z}/n\mathbb{Z}$. We will call the m_i's the *twist* part and the a_i's the *shift* part.

Definition 2.1. The *rank* of G is its number of generators minus 2.

The definition is such that, when G is a direct product of the multiplicative group by some subgroup of T_n, the rank corresponds to the number of generators of the translation part.

2.3 Results

Definition 2.2. We call $\mathcal{H}_n$ the set of orbifolds obtained by taking the quotient of X_n by subgroups of the nth Vafa–Witten group, V_n, which surject onto the multiplicative component.

In the next sections we get the following classification.

Proposition 2.3. *The sets $\mathcal{H}_n$ contain only finitely many homeomorphism classes of orbifolds. The number of classes is 8 for $\mathcal{H}_3$ and $\mathcal{H}_4$ and is 1 for $\mathcal{H}_6$.*

The Hodge numbers of the spaces are given in the tables which follow. For each homeomorphism class of $\mathcal{H}_n$, we give a representative group G by listing its generators, as well as the Hodge numbers of X_n/G.

2.3.1 Case $n = 3$

For $n = 3$, the generators are (where applicable): $(1, 2, 0; a_1, a_2, a_3)$, $(2, 0, 1; b_1, b_2, b_3)$, $(0, 0, 0; c_1, c_2, c_3)$ and $(0, 0, 0; d_1, d_2, d_3)$.

Number	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(c_1, c_2, c_3)	(d_1, d_2, d_3)	(h_{11}, h_{12})
III.1	$(0, 0, 0)$	$(0, 0, 0)$			$(84, 0)$
III.2	$(0, 0, 0)$	$(0, 1, 0)$			$(24, 12)$
III.3	$(0, 0, 0)$	$(1, 1, 0)$			$(18, 6)$
III.4	$(0, 0, 1)$	$(1, 1, 0)$			$(12, 0)$
III.5	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 1, 1)$		$(40, 4)$
III.6	$(0, 0, 0)$	$(0, 0, 0)$	$(1, 1, 1)$		$(36, 0)$
III.7	$(0, 0, 0)$	$(0, 1, 0)$	$(1, 0, 1)$		$(16, 4)$
III.8	$(0, 0, 0)$	$(0, 0, 0)$	$(1, 1, 1)$		$(18, 6)$

All the above orbifolds are simply connected, except III.4 whose fundamental group is $\mathbb{Z}/3\mathbb{Z}$.

2.3.2 Case $n = 4$

For $n = 4$, the generators are (where applicable): $(1, 3, 0; a_1, a_2, a_3)$, $(3, 0, 1; b_1, b_2, b_3)$, $(0, 0, 0; c_1, c_2, c_3)$ and $(0, 0, 0; d_1, d_2, d_3)$.

Number	(a, b, c)	(a', b', c')	(c_1, c_2, c_3)	(d_1, d_2, d_3)	(h_{11}, h_{12})
IV.1	$(0, 0, 0)$	$(0, 0, 0)$			$(90, 0)$
IV.2	$(0, 0, 0)$	$(0, 1, 0)$			$(54, 0)$
IV.3	$(0, 0, 0)$	$(1, 1, 0)$			$(42, 0)$
IV.4	$(0, 0, 1)$	$(1, 1, 0)$			$(30, 0)$
IV.5	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 1, 1)$		$(61, 1)$
IV.6	$(0, 0, 0)$	$(0, 0, 0)$	$(1, 1, 1)$		$(54, 0)$
IV.7	$(0, 0, 0)$	$(0, 1, 0)$	$(1, 0, 1)$		$(38, 0)$
IV.8	$(0, 0, 0)$	$(0, 0, 0)$	$(1, 1, 1)$		$(42, 0)$

All these orbifolds have trivial fundamental group.

2.3.3 Case $n = 6$

For $n = 6$, there is a unique case corresponding to the quotient by the Vafa–Witten group. The orbifold X_6/V_6 has Hodge numbers $(80, 0)$ and is simply connected.

3 Homeomorphism classes of $\mathcal{H}_n$

In this section, we will classify the elements of $\mathcal{H}_n$ up to homeomorphism. The classification will be made according to the rank of the group acting on X_n.

3.1 General lemmas

To identify groups, we will make recurrent use of the following lemma.

Lemma 3.1. *Let G be a subgroup of V_4 of V_4 (resp. V_3) which surjects onto the multiplicative part, then G has at least two generators. The first two generators can be taken of the form $g_1 = (1, 2, 0; *, *, *)$ and $g_2 = (2, 0, 1; *, *, *)$ (resp. $g_1 = (1, 3, 0; *, *, *)$ and $g_2 = (3, 0, 1; *, *, *)$). If there are more than two generators, then they can be taken of the form $g_{i>2} = (0, 0, 0; *, *, *)$.*

Proof. Since G must surject onto the multiplicative part, it clearly has at least two generators. These two generators g_1, g_2 can be chosen to be the lifts of the generators of the multiplicative group. For $n = 4$ (resp. $n = 3$), this group admits as generators $(1, 2, 0)$ and $(2, 0, 1)$ (resp. $(1, 3, 0)$ and $(3, 0, 1)$), so the nature of the first two generators is settled.

Let g_i be another generator of G. Since the twist part of g_1 and g_2 generates the multiplicative part, there exists a word w in the first two generators so that $w \cdot g_i = (0, 0, 0; *, *, *)$. We can now substitute g_i by $w \cdot g_i$. □

The most powerful tool to identify groups will be conjugation by torsion elements of E_n^3. For simplicity we will work on a single torus. Consider the transformation $z \mapsto \zeta_i^a z + \tau t_i$. We will conjugate it with the translation by λ, any given element of E_n:

$$(\zeta_i^a(z + \lambda) + \tau t_i) - \lambda = \zeta_i^a z + \underbrace{\tau t_i + (\zeta_i^a \lambda - \lambda)}_{\text{new shift}}.$$

If $a \neq 0$, we can choose a λ such that $\tau t_i + (\zeta_i^a \lambda - \lambda) = 0$, which implies that our transformation is conjugate to $\zeta_i^a z$.

3.2 Quotients of X_3

3.2.1 Rank 0

Although there are a priori $(3^3)^2 = 729$ possible choices of $g_1 = (1, 2, 0; a_1, a_2, a_3)$ and $g_2 = (2, 0, 1; b_1, b_2, b_3)$, by using conjugation and symmetry, we can, to begin with, restrict ourselves to at most four cases.

Lemma 3.2. *Without loss of generality, we can assume g_1 and g_2 to be of the form $(1, 2, 0; 0, 0, a_3)$ and $(2, 0, 1; b_1, b_2, 0)$ with $a_3, b_1, b_2 \in \{0, 1\}$.*

Proof. Let G be a group with generators $g_1 = (1, 2, 0; a_1, a_2, a_3)$ and $g_2 = (2, 0, 1; b_1, b_2, b_3)$. Conjugate both elements by $(0, 0, 0; \alpha_1, \alpha_2, \alpha_3)$. The element g_1 gets mapped to $(1, 2, 0; a_1 + (\zeta\alpha_1 - \alpha_1), a_2 + (\zeta^2\alpha_2 - \alpha_2), a_3)$ while g_2 gets mapped to $(2, 0, 1; b_1 + (\zeta^2\alpha_1 - \alpha_1), b_2, b_3 + (\zeta\alpha_3 - \alpha_3))$. We can take $\alpha_{1,\ldots,3}$ so that

$$a_1 + (\zeta\alpha_1 - \alpha_1) = a_2 + (\zeta^2\alpha_2 - \alpha_2) = b_3 + (\zeta\alpha_3 - \alpha_3) = 0.$$

Finally, the symmetry between t_3 and $2t_3$ allows us to take the remaining entries in $\{0, 1\}$. □

Lemma 3.3. *If we take generators as in the previous lemma, the entries b_1 and b_2 are symmetric.*

Proof. Note that $g_1 = (1,2,0;0,0,\delta_3)$ with $\delta_3 \in \{0,1\}$ and $g_2 = (2,0,1;b_2,b_3,0)$. The group spanned by g_1, g_2 is the same as the group spanned by g_1^2, g_1g_2, that is, $(2,1,0;0,0,2\delta_3)$ and $(0,2,1;b_1,b_2,\delta_3)$. We now rearrange the order of the tori: $(1\ 2\ 3) \rightsquigarrow (2\ 1\ 3)$ so that the new generators read $(1,2,0;0,0,\delta_3)$ and $(2,0,1;b_2,b_1,\delta_3)$. By conjugating by an appropriate element of the third torus, we get as second generator the required $(2,0,1;b_2,b_1,0)$. □

The above lemmas restrict the number of cases to 6:

- $g_1 = (1,2,0;0,0,1), \quad g_2 = (2,0,1;0,0,0)$;
- $g_1 = (1,2,0;0,0,1), \quad g_2 = (2,0,1;1,0,0)$;
- $g_1 = (1,2,0;0,0,1), \quad g_2 = (2,0,1;1,1,0)$;
- $g_1 = (1,2,0;0,0,0), \quad g_2 = (2,0,1;0,0,0)$;
- $g_1 = (1,2,0;0,0,0), \quad g_2 = (2,0,1;1,0,0)$;
- $g_1 = (1,2,0;0,0,0), \quad g_2 = (2,0,1;1,1,0)$.

We will show that among those six classes, two are redundant. Also, we will simplify the notation further: e.g., $\underline{a} = (1,1,1)$ will denote the element $g_1 = (1,2,0;1,1,1)$ while $\underline{b} = (1,1,0)$ will denote the element $g_2 = (2,0,1;1,1,0)$.

Lemma 3.4. *The group generated by $\underline{a} = (0,0,1)$ and $\underline{b} = (0,0,0)$ is isomorphic to the group spanned by $\underline{a} = (0,0,0)$, $\underline{b} = (0,1,0)$.*

Proof. We can replace the generators g_1 and g_2 by their squares: $(2,1,0;0,0,2)$ and $(1,0,2;0,0,0)$. If we rearrange the terms in the order $(1\ 2\ 3) \rightsquigarrow (1\ 3\ 2)$ and we permute the generators, we get $(1,2,0;0,0,0)$ and $(2,0,1;0,2,0)$. This is what we want up to relabeling. □

Lemma 3.5. *The elements $\underline{a} = (0,0,1)$ and $\underline{b} = (1,0,0)$ and the elements $\underline{a} = (0,0,0)$, $\underline{b} = (1,1,0)$ generate isomorphic groups.*

Proof. The basis $g_2, g_1^2g_2^2$ is equivalent to g_1, g_2. It is made out of the vectors $(2,0,1;1,0,0)$ and $(0,1,2;2,0,2)$. We now rearrange the tori using the permutation $(1\ 2\ 3) \rightsquigarrow (3\ 1\ 2)$ to get the basis $(1,2,0;0,1,0)$, $(2,0,1;2,2,0)$. We now conjugate with an appropriate translation on the second tori to get $(1,2,0;0,0,0)$, $(2,0,1;2,2,0)$. □

As a conclusion, we have

Proposition 3.6. *There are four homeomorphism classes of quotients of* X_3 *by groups of rank* 0 *in* $\mathcal{H}_3$*. We have written a representative of each class in the following table.*

Number	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(h_{11}, h_{12})	π_1
III.1	$(0,0,0)$	$(0,0,0)$	$(84,0)$	1
III.2	$(0,0,0)$	$(0,1,0)$	$(24,12)$	1
III.3	$(0,0,0)$	$(1,1,0)$	$(18,6)$	1
III.4	$(0,0,1)$	$(1,1,0)$	$(12,0)$	$\mathbb{Z}/3$

Proof. We have seen through the previous lemmas that there are at most four types of isomorphism of groups. By computing the Hodge diamond of the associated Calabi–Yau 3-fold (see Section 3), we deduce that they yield four different varieties. □

3.2.2 Rank 1

We label the third generator $g_3 = (0,0,0; c_1, c_2, c_3)$ or $\underline{c}$. We will extend the list of rank 0 groups using the following rules.

Lemma 3.7 (Reduction principle) [4]. *Let* G *be a group of which one of the generators is of the form* $(0,0,0; x_1, x_2, x_3)$ *and exactly one of the* $x_i \neq 0$*. The quotient* X_n/G *is then homeomorphic to* $X_n/\bar{G}$*, where* $\bar{G}$ *is the quotient of* G *by the subgroup spanned by that generator.*

The idea is that if there is an element which consists in a translation on a unique curve, we can first take the quotient by this element and have another 3-torus on which the rest of the group acts.

As a corollary, we can restrict ourselves to g_3's where at least two of the c_k's are not zero. Furthermore, we have the following simplifications.

Lemma 3.8. *If* $c_k \neq 0$ *and* $a_k = b_k = 0$*, we can assume* $c_k = 1$*.*

Proof. It follows from the symmetry between t_3 and $2t_3$. □

Lemma 3.9. *We can assume that the shift part of* $\underline{c}$ *is not a non-zero multiple of the shift part of* $\underline{a}$ *or* $\underline{b}$*.*

Proof. Without loss of generality, assume that $\underline{c}$ is a multiple of $\underline{a}$, then we can substitute g_1 by $g_1 g_3^k$ (so the shift part is $(0,0,0)$) to get a new first generator without translation. In other words, we have reduced the group to a previous case. □

We will now try to discern the groups.

1. $\underline{a} = (0,0,0)$, $\underline{b} = (0,0,0)$. We can choose $\underline{c}$ to be either $(1,1,0)$ or $(1,1,1)$. All other cases resume to these two using the previous points and S_3 symmetry.
2. $\underline{a} = (0,0,0)$, $\underline{b} = (0,1,0)$. We can choose, using the previous points, a $\underline{c}$ of the form $(\delta_1, c_2, \delta_3)$ where the $\delta_i \in \{0,1\}$. However, we can also assume that $c_2 \in \{0,1\}$.

   Proof. The generator g_3 is equivalent to g_3^2, so since we can relabel the first and third translations without loss of generality, we see that we can assume c_2 to be in $\{0,1\}$. □

   It now seems that we have four possible choices for $\underline{c}$, namely $(1,1,0)$, $(0,1,1)$, $(1,0,1)$ and $(1,1,1)$. We will show that the first two are actually redundant.

   Proof. Consider the group obtained by adjoining the element $(0,0,0;0,1,0)$ to the group generated by $\underline{a} = (0,0,0)$, $\underline{b} = (0,1,0)$ and $\underline{c} = (1,1,0)$. It is easy to see that up to S_3 symmetry, this group is equivalent to the action of the group obtained by joining $(0,0,0;0,1,0)$ to the group generated by $\underline{a} = (0,0,0)$ $\underline{b} = (0,0,0)$ and $\underline{c} = (0,1,1)$. By the reduction principle, all these groups generate the same space up to homeomorphism, so the first case reduces to an ancient case.
   For the second case, notice that the group generated by g_1, g_2, g_3 is the same as the group spanned by $g_1, g_2g_3^2, g_3$. The element $g_2g_3^2 = (2,0,1;0,0,2)$. If we conjugate these new elements by an adequate translation on the third torus, the first and last generator are unchanged while $g_2g_3^2 \rightsquigarrow (2,0,1;0,0,0)$. □

   The computation of the Hodge numbers will assure us that the two remaining cases are independent.
3. $\underline{a} = (0,0,0)$, $\underline{b} = (1,1,0)$.
   We can choose $\underline{c}$ to be of the form (c_1, c_2, δ_3), where $\delta_3 \in \{0,1\}$ (same argument as previously). Now, we could replace g_3 by its square and, since we can change the last component of $\underline{c}$ freely, it means that we could replace (c_1, c_2, δ_3) by (c_1^2, c_2^2, δ_3). We will do this to have a minimal number of entries equal to 2 in (c_1, c_2). Using the above rules, the possible $\underline{c}$'s are $(1,2,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,1)$ or $(1,2,1)$. Actually, none of these cases is new:
   - $(1,2,0)$. We have $(g_1, g_2, g_3) = (g_1, g_2g_3^2, g_3)$. The element $g_2g_3^2 = (2,0,1;0,2,0)$. After conjugation, and up to transforming the third generator g_3, we can let $g_2g_3^2 \rightsquigarrow (2,0,1;0,0,0)$.

- $(0,1,1)$. We have $(g_1,g_2,g_3)=(g_1^2,g_1g_2g_3^2,g_3)$, where $g_1^2=(2,1,0;0,0,0)$ and $g_1g_2g_3^2=(0,2,1;1,0,2)$. Conjugating by some appropriate translation on the third torus, the second generator becomes $g_1g_2g_3^2 \rightsquigarrow (0,2,1;1,0,0)$. We now reorder the tori $(1,2,3)\rightsquigarrow(2,1,3)$ and we get a previous case.
- $(1,0,1)$. We have $(g_1,g_2,g_3)=(g_1,g_2g_3^2,g_3)$. The element $g_2g_3^2=(2,0,1;0,1,2)$. After conjugating with an element of translation on the third torus, the second generator can be taken to be $(2,0,1;0,1,0)$.
- $(1,1,1)$. We have $(g_1,g_2,g_3)=(g_1,g_2g_3^2,g_3)$. The element $g_2g_3^2=(2,0,1;0,0,1)$. Again, we can conjugate by an element of translation on the third torus to have our second generator $(2,0,1;0,0,0)$.
- $(1,2,1)$. We have $(g_1,g_2,g_3)=(g_1,g_2g_3^2,g_3)$. The element $g_2g_3^2=(2,0,1;0,2,2)$. We can conjugate by an element of translation on the third torus to have our second generator $\rightsquigarrow(2,0,1;0,2,0)$. Up to renaming, we again reduced to a previous case.

So we conclude that there are no interesting extensions in this case.

4. $\underline{a}=(0,0,1)$, $\underline{b}=(1,1,0)$.
We can choose $\underline{c}$ to be of the form (c_1,c_2,c_3). Let us first undercover some symmetry: we have $(g_1,g_2,g_3)=(g_1^2,\ g_1g_2,g_3)$, and if we permute the order of the tori $(1,2,3)\rightsquigarrow(2,1,3)$, we get the generators $(1,2,0;0,0,2)$, $(2,0,1;1,1,1)$ and $(0,0,0;c_2,c_1,c_3)$. Now we can conjugate by a translation on the third torus to let the second generator become $(2,0,1;1,1,0)$ and leave the other two unchanged. Now we can relabel the new a_3 into a_1, and we must therefore substitute the new c_3 by its square. Finally, we get the generators $(1,2,0;0,0,1)$, $(2,0,1;1,1,0)$ and $(0,0,0;c_2,c_1,c_3^2)$.
So we can let c_3 be 0 or 1 up to a permutation of the c_1,c_2.
Using the above symmetry, we have the following possibilities which we show to be reducible to previous cases:

- $(1,2,0)$. We have $(g_1,g_2,g_3)=(g_1,g_2g_3,g_3)$, where $g_2g_3=(2,0,1;2,0,0)$. Using the symmetry of the first translation entries of the second generator (up to variation of g_3), we have reduced to a previous case.
- $(1,0,1)$. We have $(g_1,g_2,g_3)=(g_1g_3^2,g_2,g_3)$, where $g_1g_3^2=(1,2,0;2,0,0)$. Up to changing g_2, we can conjugate by a translation element on the first torus to get the first generator to become $(1,2,0;0,0,0)$. So we reduced to a previous case.
- $(0,1,1)$. We have $(g_1,g_2,g_3)=(g_1g_3^2,g_2,g_3)$, where $g_1g_3^2=(1,2,0;0,2,0)$. We can conjugate by a translation element on the second torus to get the first generator to become $(1,2,0;0,0,0)$. So we reduced to a previous case.

- $(1,1,1)$. We have $(g_1,g_2,g_3)=(g_1,g_2g_3^2,g_3)$, where $g_2g_3^2=(2,0,1;0,0,2)$. We can conjugate by a translation element on the third torus to get the second generator to become $(2,0,1;0,0,0)$.
- $(2,1,1)$. We have $(g_1,g_2,g_3)=(g_1g_3^2,g_2,g_3)$, where $g_1g_3^2=(1,2,0;1,2,0)$. Up to changing g_2, we can conjugate by a translation element on the first and second torus to get the first generator to become $(1,2,0;0,0,0)$. So we reduced to a previous case.
- $(1,2,1)$. We have $(g_1,g_2,g_3)=(g_1g_3^2,g_2,g_3)$, where $g_1g_3^2=(1,2,0;1,2,0)$. Up to changing g_2, we can conjugate by a translation element on the first and second torus to get the first generator to become $(1,2,0;0,0,0)$. So we reduced to a previous case.

From the above discussion, we conclude:

Proposition 3.10. *There are four homeomorphism classes in $\mathcal{H}_3$ coming from groups of rank* 1*. We have written a representative of each class in the following table.*

Number	(a_1,a_2,a_3)	(b_1,b_2,b_3)	(c_1,c_2,c_3)	(h_{11},h_{12})	π_1
III.5	$(0,0,0)$	$(0,0,0)$	$(0,1,1)$	$(40,4)$	1
III.6			$(1,1,1)$	$(36,0)$	1
III.7	$(0,0,0)$	$(0,1,0)$	$(1,0,1)$	$(16,4)$	1
III.8			$(1,1,1)$	$(18,6)$	1

3.2.3 Rank 2

Since we do not consider rank 2 groups which we can reduce to a lower rank, we have restrictions on the third and fourth generator: by the reduction principle, they cannot generate an element with only one non-zero entry in the shift part.

Consider the third and fourth generators as elements of $\mathbb{F}_3^3$ (they just have shift parts).

Lemma 3.11. *There are exactly 4 linear planes in $\mathbb{F}_3^3$ which do not intersect the coordinate axes outside the origin.*

Proof. Let H be a plane which does not intersect the coordinate axes outside the origin. Since it contains the origin, it intersects the three coordinate planes in a line. For each coordinate plane P_i, the only two possible lines are the diagonal Δ_i and, the only other line which is not a coordinate axis, l_i. The choice of any two lines. not in the same coordinate plane, out of $\{l_1,l_2,\Delta_1,\Delta_2\}$ gives an adequate plane. In particular, there are four of them. If we include Δ_3 and l_3 in the picture, we look at the coplanarity of the lines, which is readily checked and can be visualized in figure 1. Each plane is represented by either an edge of the triangle or the inscribed circle. □

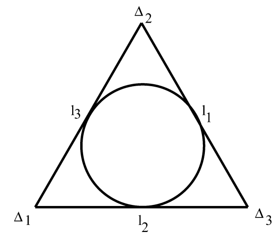

Figure 1: Planes in $\mathbb{F}_3^3$ not crossing the coordinate axes outside the origin.

Remarks.

- The three planes represented by the edges of the triangle in figure 1 are clearly S_3 symmetric. Therefore, it is enough to consider the adjunction of any one of those three to the first case of rank 0 subgroups.
- Since all rank 1 cases associated to $\underline{a} = (0, 0, 0)$, $\underline{b} = (1, 1, 0)$ and $\underline{a} = (0, 0, 1)$, $\underline{b} = (1, 1, 0)$ were reduced to previous cases, all rank 2 extensions will also be reducible, so we just need to deal with $\underline{a} = (0, 0, 0)$, $\underline{b} = (0, 0, 0)$ and $\underline{a} = (0, 0, 0)$, $\underline{b} = (0, 1, 0)$.

We will now work out the rank 2 cases.

- Using S_3 symmetry, for $\underline{a} = \underline{b} = (0, 0, 0)$, we have as possible extension $\underline{c} = (0, 1, 1)$, $\underline{d} = (1, 1, 0)$ and $\underline{c} = (0, 1, 2)$, $\underline{d} = (1, 2, 0)$. Now, we can replace $\underline{d}$ by its square and since c_3 and d_1 are freely chosen, we see that this case is equivalent to the previous one.
- All four possible extensions of $\underline{a} = (0, 0, 0)$, $\underline{b} = (0, 1, 0)$ contain either the element $(0, 2, 2)$ or $(0, 2, 1)$. Using this element, it is easy to see that we can reduce g_2 to $(2, 0, 1; 0, 0, 0)$: multiply g_2 by this element and conjugate by an appropriate translation on the third coordinate. We show hereby that there are no further rank 2 cases.

A priori, there is thus only one rank 2 case.

(a_1, a_2, a_3)	(b_1, b_2, b_3)	(c_1, c_2, c_3)	(d_1, d_2, d_3)
$(0, 0, 0)$	$(0, 0, 0)$	$(0, 1, 1)$	$(1, 1, 0)$

However, it can be reduced to the III.1 case: by applying the reduction principle of Donagi and Faraggi, this case is equivalent to the Vafa–Witten group case (adjoin any translation) which in turn is equivalent to the III.1 case.

3.2.4 Rank 3

The only group is the Vafa–Witten group. It reduces to the III.1 case.

3.3 Quotients of X_4

Since there are only two fixed points per torus, the structure of the translation locus is simpler. A translation element will be of the form $(\delta_1, \delta_2, \delta_3)$ with $\delta_i \in \{0, 1\}$.

3.3.1 Rank 0

Using the same argument as for $\mathbb{Z}/3\mathbb{Z}$ (being careful to replace square by inverse), we quickly get the following list.

Proposition 3.12. *There are four homeomorphism classes in $\mathcal{H}_3$ obtained by taking the quotients by rank 0 groups. We have written a representative of each class in the following table:*

Number	(a, b, c)	(a', b', c')	(h_{11}, h_{12})
IV.1	$(0, 0, 0)$	$(0, 0, 0)$	$(90, 0)$
IV.2	$(0, 0, 0)$	$(0, 1, 0)$	$(54, 0)$
IV.3	$(0, 0, 0)$	$(1, 1, 0)$	$(42, 0)$
IV.4	$(0, 0, 1)$	$(1, 1, 0)$	$(30, 0)$

3.3.2 Rank 1

Again, we will use the same arguments as for $n = 3$.

Lemma 3.13. *If $\underline{a} = (0, 0, 0)$, $\underline{b} = (0, 0, 0)$, then we can assume $\underline{c}$ to be of the form:*

- $(0, 1, 1)$;
- $(1, 1, 1)$.

Proof. Using S_3 symmetry, all other $\underline{c}$'s with two non-zero entries are equivalent to the one listed here. Also, the reduction principle excludes all $\underline{c}$'s with a single non-zero entry. □

Lemma 3.14. *If $\underline{a} = (0, 0, 0)$, $\underline{b} = (0, 1, 0)$, then we can assume that $\underline{c}$ is one of the following:*

- $(1, 0, 1)$;
- $(1, 1, 1)$.

Proof. The case where $\underline{c}$ is $(0, 1, 1)$ is reducible: replace g_2 by $g_2 g_3 = (3, 0, 1; 0, 0, 1)$ and conjugate by a translation element on the third torus to get a second generator which is translation free.

The case where $\underline{c}$ is $(1,1,0)$ is reducible: adjoin the element $(0,0,0;0,1,0)$ and use the same argument as for the order 3 case. □

Lemma 3.15. *If* $\underline{a}=(0,0,0)$, $\underline{b}=(1,1,0)$, *then there are no new cases.*

Proof. If $\underline{c}$ is

- $(0,1,1)$. This case is reducible: we replace g_1 by g_1^{-1} and g_2 by $g_2g_3=(0,3,1;1,0,1)$. Secondly, we conjugate by a translation element on the third torus to let the second generator become $(0,3,1;1,0,0)$. Finally, we arrange the torus in the order $(1,2,3)\rightsquigarrow(2,1,3)$. We reduced to an extension of the previous type of rank 0.
- $(1,0,1)$. This case is reducible: replace g_2 by $g_2g_3=(3,0,1;0,1,1)$ and conjugate by a translation element on the third torus to get a second generator of the form $(3,0,1;0,1,0)$. We reduced to an extension of the previous type of rank 0.
- $(1,1,1)$. This case is reducible: replace g_2 by $g_2g_3=(3,0,1;0,0,1)$ and conjugate by a translation element on the third torus to get a second generator which is translation free. □

Lemma 3.16. *If* $\underline{a}=(0,0,1)$, $\underline{b}=(1,1,0)$, *then there are no new cases.*

Proof. If $\underline{c}$ is

- $(0,1,1)$. We have $(g_1,g_2,g_3)=(g_1g_3,g_2,g_3)$, where $g_1g_3=(1,3,0;0,1,0)$. We can conjugate by a translation element on the second torus to get the first generator to become $(1,3,0;0,0,0)$. So we reduced to a previous case.
- $(1,0,1)$. We have $(g_1,g_2,g_3)=(g_1g_3,g_2,g_3)$, where $g_1g_3=(1,3,0;1,0,0)$. Up to changing g_2, we can conjugate by a translation element on the first torus to get the first generator to become $(1,3,0;0,0,0)$. So we reduced to a previous case.
- $(1,1,1)$. We have $(g_1,g_2,g_3)=(g_1g_3,g_2,g_3)$, where $g_1g_3=(1,3,0;1,1,0)$. Up to changing g_2, we can conjugate by a translation element on the first and second torus to get the first generator to become $(1,3,0;0,0,0)$. So we reduced to a previous case. □

Using those lemmas, we conclude

Proposition 3.17. *There are four homeomorphism classes generated by groups of rank* 1 *in* $\mathcal{H}_3$. *We have written a representative of each class in* *the following table:*

Number	(a_1, a_2, a_3)	(b_1, b_2, b_3)	(c_1, c_2, c_3)	(h_{11}, h_{12})
IV.5	$(0,0,0)$	$(0,0,0)$	$(0,1,1)$	$(61,1)$
IV.6			$(1,1,1)$	$(54,0)$
IV.7	$(0,0,0)$	$(0,1,0)$	$(1,0,1)$	$(38,0)$
IV.8			$(1,1,1)$	$(42,0)$

3.3.3 Rank 2

Since we do not want cases which reduce to lower ranks, we need the third and fourth generators to span a subgroup which does not contain elements where only one of the entries is non-zero. There is actually a unique possibility. In geometric language:

Lemma 3.18. *There is a unique two-dimensional vector subspaces of* $\mathbb{F}_2^3$ *which does not intersect the coordinate axes outside the origin.*

Proof. Let H be a plane verifying the above conditions. Since H passes through the origin, it must intersect each coordinate plane in at least a line. Since H does not intersect the coordinate axes, the intersection lines must be the first diagonals. Therefore $\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\} \subset H$ and since a plane over $\mathbb{F}_2$ contains four elements, we are done. □

Since the group of translations is isomorphic to $\mathbb{F}_3^3$, we get the direct result that

Corollary 3.19. *There is at most one rank 2 group associated to each group of rank 0 — which we classified earlier.*

Corollary 3.20. *There is no non-reducible rank 2 case associated to the last three types of rank 0.*

Proof. In the last two types of rank 0, one generator has a shift with two non-zero entries. Since this shift belongs to the plane H, the case can be reduced to a previous one.

In the second case of rank 0, $\underline{b} = (0,1,0)$, so replacing g_2 with its composition with $(0,0,0;0,1,1)$ we get $(3,0,1;0,0,1)$. We can now conjugate with an adequate translation on the third torus to get the second generator in the form $(3, 0, 1; 0, 0, 0)$, i.e., we are back to the first type of rank 0. □

So we conclude that apparently there is only one type of rank 2 subgroup, namely:

$$\frac{(a_1, a_2, a_3) \quad (b_1, b_2, b_3) \quad (c_1, c_2, c_3) \quad (d_1, d_2, d_3)}{(0, 0, 0) \qquad (0, 0, 0) \qquad (0, 1, 1) \qquad (1, 1, 0)}.$$

However, as in the case of X_3, it is reducible to IV.1.

3.3.4 Rank 3

The only case is the whole group V_4. It is reducible to the III.1 case.

3.4 Quotients of V_6

There is a unique case as there are no translations commuting with the multiplicative $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ action. The Hodge numbers of the quotient are $(80, 0)$.

4 The Hodge structure

To compute the orbifold cohomology of the spaces that we have classified in the previous section, we will use the orbifold cohomology introduced in physics (see e.g., [3]) and formalized by Chen and Ruan [2].

Definition 4.1. The orbifold cohomology of the quotient of the manifold X_n by the finite group G, which is also the cohomology of a crepant resolution of this quotient, is

$$H^{i,j}(X_n/G) = \bigoplus_{\substack{[g] \in G \\ U \in X_n^g}} H^{i-\kappa(g), j-\kappa(g)}(U)^{C(g)}.$$

where we sum over the conjugacy classes of G and the components of the fixed locus of each element. $C(g)$ denotes the centralizer of g in G. The set X_n^g is the fixed locus of any element in the conjugacy class of g, i.e., $\{x \in X_n \text{ such that } g \cdot x = x\}$; U denotes one of its components. Moreover, if the action of g sends $[z_i]_{i=1,\ldots,3}$ to $[\mathrm{e}^{2\pi i\theta_i} z_i]_{i=1,\ldots,3}$ with $0 \leq \theta_i < 1$, then we define the shift function[1] by $\kappa(g) = \sum_{i=1}^{3} \theta_i$.

[1]Also known as *fermionic shift*, *degree shifting number* or *age*.

In our case, the definition of the Vafa–Witten groups forces the function κ to take its values in $\{0,1,2\}$. Also, since we are dealing with Abelian groups and given that the local action is essentially unique, the formula simplifies to

$$H^{i,j}\left(\frac{X_n}{G}\right) = \bigoplus_{g \in G} H^{i-\kappa(g),j-\kappa(g)}(X_n^g)^G.$$

Note that the action of an element of G on the cohomology depends uniquely on the multiplicative part: the action of ζ_i is $\zeta_i \cdot dz_i = \zeta_i dz_i$ and $\zeta_i \cdot d\bar{z}_i = \bar{\zeta}_i d\bar{z}_i$ while the action of T_i is trivial.

Example 4.2. Take $g = (m_1, m_2, m_3; a_1, a_2, a_3)$ and $\Omega = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3$. We have

$$\Omega \overset{g}{\longmapsto} \zeta_i^{(m_1+m_2)-(m_1+m_3)}\Omega = \zeta_i^{(m_2-m_3)}\Omega.$$

In order to compute the Hodge structure of the X_n/G, we first need to classify the possible fixed loci of an element of $g \in G$ on X_n.

Lemma 4.3. *Consider the element $g = (m_1, m_2, m_3; a_1, a_2, a_3) \in G$. Its fixed locus X_n^g can be of four different topological types.*

Proof. There are four exclusive forms of g which correspond to the four different fixed loci:

1. The identity element $(0,0,0;0,0,0)$ has as fixed locus the whole variety X_n.
2. If for a certain index $k \in \{1,2,3\}$ we have that $m_k = 0$, and $a_k \neq 0$, then the fixed locus of g is the empty set. Indeed, g acts by translation on each elliptic curve, and these translations do not fix any point.
3. The fixed locus of g where all m_k's are different from 0 is a collection of points.
4. The fixed locus of g, where for exactly one of the k's $m_k = a_k = 0$ (and excluding case 2), is made out of elliptic curves (the number depends on n).

Note that if two of the m_k's are zero, then the third one also has to be zero and therefore all cases have been exhausted in the above list. □

We will now compute the G invariant cohomology from these fixed loci.

Lemma 4.4. *For all $n \in \{3, 4, 6\}$, the G-invariant part of the cohomology of X_n is*[2]

$$H^*(X_n)^G = \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}.$$

Proof. The Hodge diamond of the 3-torus is

$$\begin{matrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{matrix}.$$

The invariance of $H^{0,0}$ and $H^{3,0}$ is straightforward.

For the other components, note that the action of G on $H^{1,0}$ and $H^{0,1}$, and thus on the whole of $H(X_n)$, is diagonal with respect to the standard basis. Therefore, it will be enough to check the behavior of the basis elements to find the G-invariant part. Let dz_k be a generator of $H^{1,0}$, and it is not fixed by the element $(1, 1, n-2; *, *, *)$. Therefore, we have ${h^{1,0}}^G = 0$.

Similarly, $dz_k \wedge dz_l$, a generator of $H^{2,0}$, is not fixed by the element $(1, 1, n-2; *, *, *)$ and thus ${h^{2,0}}^G = 0$.

Only the generators of $H^{1,1}$ of the form $dz_i \wedge d\bar{z}_i$ are invariant, the others are killed by the element $(1, n-1, 0; *, *, *)$. Therefore, the dimension of the G-invariant part of $H^{1,1}$ is 3.

Consider now $H^{2,1}$; by symmetry, we can restrict ourselves to the generators $dz_1 \wedge dz_2 \wedge d\bar{z}_3$ and $dz_1 \wedge dz_2 \wedge \; d\bar{z}_1$. The former is not fixed by $(1, 0, n-1; *, *, *)$, while the latter is not fixed by $(1, 1, n-2; *, *, *)$. We conclude that ${h^{2,1}}^G = 0$. Finally, the diamond is completed using $dz_k \leftrightarrow d\bar{z}_k$ and Hodge symmetry. □

Lemma 4.5. *Assume that after identification via G-action, $g \in G$ has as fixed locus a collection of n points. The contribution to the cohomology of g and g^{-1}, $H^*(X^g)^G \oplus H^*(X^{g^{-1}})^G$ is equivalent to the contribution of n projective lines: $H^*(n\mathbb{P}^1)$.*

Proof. The elements g and g^{-1} have the same fixed locus, whose cohomology is exclusively $H^{0,0}$. Given that the fixed locus of g is made of points, we

[2] We have tilted the Hodge diamonds 45° to the left to facilitate typesetting.

know that $\kappa(g)$ is non-zero and that none of the θ_i is 0. We claim that $\{\kappa(g), \kappa(g^{-1})\}$ is exactly $\{1, 2\}$. Indeed, if we denote by θ'_i the linearized action of g^{-1} on the ith component, then we have the relation $\theta'_i = 1 - \theta_i$. Therefore, $\kappa(g^{-1}) = 3 - \kappa(g)$. □

Lemma 4.6. *The G-invariant part of the cohomology of a fixed elliptic curve will be either the cohomology of the projective line* $\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$ *or the one of an elliptic curve* $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$. *In either case κ will be 1.*

Proof. Since $H^{0,0}$ is obviously invariant, only $H^{1,0}$ which is of dimension 1 might not preserved, in which case the invariant cohomology corresponds to that of a $\mathbb{P}^1$.
Since we do not deal with the fixed locus of the identity, κ is not 0. Moreover, we know that the action is trivial on one component. Therefore, we have that κ, which is the sum $\theta_1 + \theta_2 + \theta_3$ where one $\theta_i = 0$ and the two others of norm less than one, can only be one. □

4.1 Notation

We are now ready to compute the Hodge numbers of each orbifold.
Each group will be represented, as previously, by the shift part of its generators. We will list the group elements g which have a non-empty fixed locus X^g and their contribution, i.e., for each group, we will have a collection $\{(g, (h^{1,1}, h^{1,2}))\}$.

To lighten the notation,

1. We will count twice the contribution of non-trivial elements which are not involutions but have a union of curves as fixed locus. In compensation, we will not be writing their inverse.
2. if a non-involutive element fixes a union of points, then we will count its contribution together with the one of its inverse. We will not write down its inverse.

4.2 Order 3

We have eight cases to compute:

- $(0,0,0)(0,0,0) \rightsquigarrow (h_{11}, h_{12}) = (84, 0)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(18,0)),$

$((2,0,1;0,0,0),(18,0)),\ ((0,1,2;0,0,0),(18,0)),$
$((1,1,1;0,0,0),(27,0))\};$

- $(0,0,0)(0,1,0) \rightsquigarrow (24,12)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,6)),$
  $((0,2,1;0,1,0),(6,6)),\ ((1,1,1;0,1,0),(9,0))\};$
- $(0,0,0)(1,1,0) \rightsquigarrow (18,6)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,6)),$
  $((1,1,1;1,1,0),(9,0))\};$
- $(0,0,1)(1,1,0) \rightsquigarrow (12,0)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,1,1;1,1,2),(9,0))\};$
- $(0,0,0)(0,0,0)(0,1,1) \rightsquigarrow (40,4)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,0)),\ ((2,0,1;0,0,0),(6,0)),$
  $((0,1,2;0,0,0),(6,0)),\ ((0,1,2;0,1,1),(2,2)),\ ((0,1,2;0,2,2),(2,2)),$
  $((1,1,1;0,0,0),(9,0)),\ ((1,1,1;0,1,1),(3,0)),\ ((1,1,1;0,2,2),(3,0))\};$
- $(0,0,0)(0,0,0)(1,1,1) \rightsquigarrow (36,0)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(6,0)),$
  $((2,0,1;0,0,0),(6,0)),\ ((0,1,2;0,0,0),(6,0)),\ ((1,1,1;0,0,0),(9,0)),$
  $((1,1,1;1,1,1),(3,0)),\ ((1,1,1;2,2,2),(3,0))\};$
- $(0,0,0,)(0,1,0)(1,0,1) \rightsquigarrow (16,4)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(2,2)),$
  $((0,2,1;0,1,0),(2,2)),\ ((1,1,1;0,1,0),(3,0)),\ ((1,1,1;1,1,1),(3,0)),$
  $((1,1,1;2,1,2),(3,0))\};$
- $(0,0,0)(0,1,0)(1,1,1) \rightsquigarrow (18,6)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,2,0;0,0,0),(2,2)),$
  $((2,0,1;2,0,2),(2,2)),\ ((0,2,1;0,1,0),(2,2)),\ ((1,1,1;0,1,0),(3,0)),$
  $((1,1,1;1,2,1),(3,0)),\ ((1,1,1;2,0,2),(3,0))\}.$

4.3 Order 4

For this family, we also have eight cases to compute:

- $(0,0,0)(0,0,0) \rightsquigarrow (h_{11},h_{12}) = (90,0)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(8,0)),$
  $((3,0,1;0,0,0),(8,0)),\ ((0,3,1;0,0,0),(8,0)),\ ((1,1,2;0,0,0),(12,0)),$
  $((1,2,1;0,0,0),(12,0)),\ ((2,1,1;0,0,0),(12,0)),$
  $((2,2,0;0,0,0),(9,0)),\ ((2,0,2;0,0,0),(9,0)),\ ((0,2,2;0,0,0),(9,0))\};$
- $(0,0,0)(1,1,0) \rightsquigarrow (42,0)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(4,0)),$
  $((1,1,2;0,0,0),(8,0)),\ ((1,2,1;1,1,0),(4,0)),\ ((2,1,1;1,1,0),(4,0)),$
  $((2,2,0;0,0,0),(5,0)),\ ((2,0,2;0,0,0),(7,0)),\ ((0,2,2;0,0,0),(7,0))\};$
- $(0,0,0)(0,1,0) \rightsquigarrow (54,0)$
  $\{((0,0,0;0,0,0),(3,0)),\ ((1,3,0;0,0,0),(4,0)),$

$((0,1,3;0,1,0),(4,0))$, $((1,1,2;0,0,0),(8,0))$, $((1,2,1;0,1,0),(4,0))$, $((2,1,1;0,1,0,[3],0),(8,0))$, $((2,2,0;0,0,0),(7,0))$, $((2,0,2;0,0,0),(9,0))$, $((0,2,2;0,0,0),(7,0))\}$;

- $(0,0,1)(1,1,0) \rightsquigarrow (30,0)$
  $\{((0,0,0;0,0,0),(3,0))$, $((1,1,2;0,0,0),(4,0))$, $((1,2,1;0,1,0),(4,0))$, $((2,1,1;0,1,0),(4,0))$, $((2,2,0;0,0,0),(5,0))$, $((2,0,2;0,0,0),(5,0))$, $((0,2,2;0,0,0),(5,0))\}$;
- $(0,0,0)(0,0,0)(0,1,1) \rightsquigarrow (61,1)$
  $\{,((0,0,0;0,0,0),(3,0))$, $((1,3,0;0,0,0),(4,0))$, $((3,0,1;0,0,0),(4,0))$, $((0,3,1;0,0,0),(4,0))$, $((1,1,2;0,0,0),(6,0))$, $((1,2,1;0,0,0),(6,0))$, $((2,1,1;0,0,0),(6,0))$, $((2,2,0;0,0,0),(6,0))$, $((2,0,2;0,0,0),(6,0))$, $((0,2,2;0,0,0),(5,0))$, $((0,3,1;0,1,1),(2,0))$, $((1,1,2;0,1,1),(2,0))$, $((1,2,1;0,1,1),(2,0))$, $((2,1,1;0,1,1),(4,0))$, $((0,2,2;0,1,1),(1,1))\}$;
- $(0,0,0)(0,0,0)(1,1,1) \rightsquigarrow (54,0)$
  $\{((0,0,0;0,0,0),(3,0))$, $((1,3,0;0,0,0),(4,0))$, $((3,0,1;0,0,0),(4,0))$, $((0,3,1;0,0,0),(4,0))$, $((1,1,2;0,0,0),(6,0))$, $((1,2,1;0,0,0),(6,0))$, $((2,1,1;0,0,0),(6,0))$, $((2,2,0;0,0,0),(5,0))$, $((2,0,2;0,0,0),(5,0))$, $((0,2,2;0,0,0),(5,0))$, $((1,1,2;1,1,1),(2,0))$, $((1,2,1;1,1,1),(2,0))$, $((2,1,1;1,1,1),(2,0))\}$;
- $(0,0,0)(0,1,0)(1,0,1) \rightsquigarrow (37,0)$
  $\{((0,0,0;0,0,0),(3,0))$, $((1,3,0;0,0,0),(2,0))$, $((0,1,3;0,1,0),(2,0))$, $((1,1,2;0,0,0),(4,0))$, $((1,1,2;1,0,1),(2,0))$, $((1,2,1;0,1,0),(2,0))$, $((1,2,1;1,1,1),(2,0))$, $((2,1,1;0,1,0),(4,0))$, $((2,1,1;1,1,1),(2,0))$, $((2,2,0;0,0,0),(4,0))$, $((2,0,2;0,0,0),(5,0))$, $((2,0,2;1,0,1),(1,0))$, $((0,2,2;0,0,0),(4,0))\}$;
- $(0,0,0)(0,1,0)(1,1,1) \rightsquigarrow (42,0)$
  $\{((0,0,0;0,0,0),(3,0))$, $((1,3,0;0,0,0),(2,0))$, $((1,3,0;1,1,0),(4,0))$, $((3,0,1;1,0,0),(4,0))$, $((0,1,3;0,1,0),(4,0))$, $((1,1,2;0,0,0),(4,0))$, $((1,1,2;1,1,0),(6,0))$, $((1,2,1;0,1,0),(2,0))$, $((1,2,1;1,0,0),(6,0))$, $((2,1,1;0,1,0),(6,0))$, $((2,1,1;1,0,0),(2,0))$, $((2,2,0;0,0,0),(5,0))$, $((2,2,0;1,1,0),(1,1))$, $((2,0,2;0,0,0),(6,0))$, $((0,2,2;0,0,0),(6,0))\}$.

4.4 Quotients of V_6

For $n = 6$, we have a single orbifold. In order to further shorten the notation, we will use S_3 symmetry in $(\mathbb{Z}/6\mathbb{Z})^3$: we will write one element per S_3 orbit (the size of the orbit is is written between square brackets).

$\{((0,0,0;0,0,0),(3,0))[1]$, $((1,5,0;0,0,0),(6,0))[6]$, $((1,4,1;0,0,0),(6,0))[3]$, $((1,3,2;0,0,0),(24,0))[6]$,

$((2,4,0;0,0,0),(24,0))[6]$, $((2,2,2;0,0,0),(5,0))[1]$,
$((3,3,0;0,0,0),(12,0))[3]\}$
$(h_{11},h_{12})=(80,0)$.

5 The fundamental group

We will compute π_1 of our orbifolds using the fact that they are the quotients of a simply connected space ($\mathbb{C}^3$).

Consider $E_n \times E_n \times E_n/G$ as the quotient of $\mathbb{C}^3$ by $\tilde{G}$, extension of G by the lattice group Λ_n. Let $F=\{g \in \tilde{G} : \exists x \in \mathbb{C}^3 \mid g \cdot x = x\}$ and $N(F)$ the group generated by F.

Theorem 5.1. *The fundamental group of $E_n \times E_n \times E_n/G$ is $\tilde{G}/N(F)$.*

A proof can be found in [1] or a more heuristic argument is given in [3].

5.1 Orders 4 and 6

Proposition 5.2. *Let G be a subgroup of the Vafa–Witten Group with $n=4$ or 6 which surjects onto the multiplicative part. The closure $N(F)$ of F is the whole of $\tilde{G}$.*

Proof. We will generalize slightly the notation used up to now. We will write $g=(a_1,a_2,a_3;\tau_1 t_i,\tau_2 t_i,\tau_3 t_i)$, where before we would have omitted the t_i. This permits us to write the elements of the lattice group Λ in the form $(0,0,0;\alpha_1+\beta_1\omega,\alpha_2+\beta_2\omega,\alpha_3+\beta_3\omega)$. Let g be an element of $\tilde{G}-F$, and it is of the form $(a_1,a_2,a_3;v_1,v_2,v_3)$, where, for at least one k, $v_k \neq 0$ and $a_k=0$.
The key is that $\tilde{G}$ always contains an element of the form $h=(b_1,b_2,b_3;*,*,*)$ such that $a_k+b_k \neq 0$ and $b_k \neq 0$ for all $k \in \{1,2,3\}$.

5.1.1 Case 1

Only one of the $a_k=0$. We can assume without loss of generality that $k=1$. Pick $\epsilon,\delta \in \mathbb{Z}/\mathrm{i}$ such that $\epsilon,\delta \neq 0$, $\delta \neq -\epsilon$, $\delta \neq a_2$ and $\delta \neq a_2-\epsilon$. For n large enough (that is, $n>3$), we see that there exists $n^2-5n+6>0$ such possible pairs. Indeed, the previous conditions take away (in the right order) n, $n-1$, $n-1$, $n-2$ and $n-2$ possibilities from the n^2 possible pairs of $\mathbb{Z}/n \times \mathbb{Z}/n$. (see figure 2) Now, we just pick $h=(\epsilon,\delta-a_2,a_2-\delta+\epsilon;*,*,*)$.

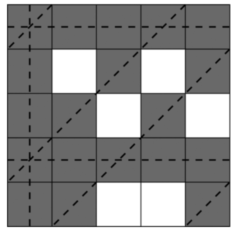

Figure 2: "Good pairs" of indices.

5.1.2 Case 2

All $a_k = 0$. We can pick h to be $(1, 1, n-2; *, *, *)$. Since $n > 3$, $n - 2 \neq 0$. In both cases, h and $h^{-1} \cdot g$ belong to F, so from the trivial equality $g = h \cdot (h^{-1} \cdot g)$, we deduce that $g \in N(F)$. □

Corollary 5.3. *Let G be a subgroup of the Vafa–Witten Group which surjects onto the multiplicative part with $n = 4$ or 6, $\pi_1(X(G), x)$ is trivial.*

5.2 Order 3

Proposition 5.4. *All orbifolds obtained when n is 3 are simply connected, except the quotient by $III.4$ whose fundamental group has order 3.*

Proof. Let $j = (0, 0, 0; v_1, v_2, v_3)$ be a translation element in $\tilde{G}$. We can decompose it as

$$j = (jg_2^{-2}g_1^{-1})(g_1g_2^2).$$

Both $g_1g_2^2 = (2, 2, 2; *, *, *)$ and $jg_2^{-2}g_1^{-1} = (1, 1, 1; *, *, *)$ belong to F and hence j belongs to F as well. The group of translations, T, in $\tilde{G}$ is a normal subgroup (as is the case in any Euclidean group), and so $G/N(F) \cong (G/T)/(N(F)/T)$. Since G/T is generated by the classes of g_1 and g_2 and $g_1g_2^2 \in F$, the group $G/N(F)$ is a quotient of $\mathbb{Z}/3\mathbb{Z}$.
For all groups III.x, x different from 4, g_1 is an element of F, so the fundamental group is trivial. For III.4, it is easy to see that only powers of $g_1g_2^2$ lie in F and so the fundamental group is cyclic of order 3. □

Acknowledgments

This work is part of the author's thesis at the University of Pennsylvania, which has been a benevolent host. It was partially supported through a fellowship of the BAEF (Belgian American Educational Foundation).

I would like to thank Professor Ron Donagi for suggesting this problem and for many fruitful conversations.

References

[1] R. Brown and P.J. Higgins, *The fundamental groupoid of the quotient of a Hausdorff space by a discontinuous action of a discrete group is the orbit groupoid of the induced action*, University of Wales, Bangor, Maths preprint 02.25; `arXiv:math.AT/0212271`.

[2] W. Chen and Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. 248(1) (2004), 1–31.

[3] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Strings on orbifolds (I/II)*, Nucl. Phys. **B261/B274**, (1985/1986), 261/285.

[4] R. Donagi, and A. Faraggi, *On the number of chiral generations in $Z_2 \times Z_2$ orbifolds*, Nucl. Phys. B **695**, (2004), 187–205; `arXiv:hep-th/0403272`.

[5] R. Donagi and K. Wendland, in preparation.

[6] C. Vafa and E. Witten, *On orbifolds with discrete torsion*, J. Geom. Phys. **15**, (1995), 189–214; `arXiv:hep-th/9409188`.