Adv. Theor. Math. Phys. **8** (2004) 955–985

On quantum symmetries of the non–ADE graph F_4

Robert Coquereaux and Esteban Isasi

Centre de Physique Théorique
Campus de Luminy, Case 907
F-13288 Marseille, France

coque@cpt.univ-mrs.fr
isasi@cpt.univ-mrs.fr

Abstract

We describe quantum symmetries associated with the F_4 Dynkin diagram. Our study stems from an analysis of the (Ocneanu) modular splitting equation applied to a partition function which is invariant under a particular congruence subgroup of the modular group.

1 Introduction

Modular invariant partition functions of the affine $SU(2)$ conformal field theory models have been classified long ago [**1**]; they follow an ADE classification. There are three infinite series called A_n, D_{2n}, D_{2n+1} and three

e-print archive:
http://lanl.arXiv.org/abs/hep-th/0409201
E. Isasi is partially supported by a fellowship of "Fundación Gran Mariscal de Ayacucho", Venezuela.

exceptional cases called E_6, E_7 and E_8. The terminology was justified from the fact that exponents of the corresponding Lie groups appear in the expression of the corresponding partition functions but, originally, this labelling using Dynkin diagrams was only a name since the diagram itself was not an ingredient in the construction. Later, and following in particular the work of [**17**], [**16**], the corresponding field theory models were built directly from the data associated with the diagrams themselves.

About ten years ago, the occurrence of ADE diagrams in the affine $SU(2)$ classification was understood in a rather different way. One observation (already present in reference [**17**]) is that the vector space spanned by the vertices of a diagram A_n possesses an associative and commutative algebra structure encoded by the diagram itself : this so-called "graph algebra" has a unit τ_0 (the first vertex), one generator τ_1 (the next vertex) and multiplication of a vertex τ_p by τ_1 is given by the sum of the neighbors of τ_p. So, $\tau_1\tau_p = \tau_{p-1} + \tau_{p+1}$ when $p < n-1$ and $\tau_1\tau_{n-1} = \tau_{n-2}$. The structure constants of this algebra (which can be understood as a quantum version of the algebra of $SU(2)$ irreps at roots of unity) are positive integers. In some cases (A_n, D_{2n}, E_6, E_8), the vector space spanned by the vertices of a chosen Dynkin diagram of type ADE also enjoys self-fusion, i.e., admits, like A_n itself, an associative algebra structure, with structure constants that are positive integers, together with a multiplication table "related" to the graph algebra of the corresponding A_n. Another important observation is that this vector space is always a module over the graph algebra of A_n where $n+1$ is the Coxeter number of the chosen diagram. For instance the vector spaces spanned by vertices of the diagrams E_6 and D_7 are modules over the graph algebra of A_{11} (their common Coxeter number is 12). The Ocneanu construction [**12**] associates with every ADE Dynkin diagram G a special kind of weak Hopf algebra (or quantum groupoid). This bialgebra $B(G)$ is finite dimensional and semi-simple for its two associative structures. Existence of a coproduct on the underlying vector space (and on its dual) allows one to take tensor producst of irreducible representations (or co-representations) and decompose them into irreducible components. One obtains in this way two – usually distinct – algebras of characters. The first, called the fusion algebra of G, and denoted $A(G)$, can be identified with the graph algebra of A_n, where $n+1$ is the Coxeter number of G (this number is defined in a purely combinatorial way, and does not require any reference to the theory of Lie algebras or Coxeter groups). The second algebra of characters is called the algebra of quantum symmetries and denoted $Oc(G)$; it is an associative – but not necessarily commutative – algebra with two (usually distinct) generators. It comes with a particular basis, and the multiplication of its basis elements by the two generators is encoded by a graph called the Ocneanu graph of G. The character algebra $Oc(G)$ is a bimodule over $A(G)$

with integer structure constants; this bimodule structure is encoded by a set of “toric matrices” and one of them can be identified with the modular invariant partition function for a physical system. The others, which are not modular invariant, can be physically interpreted, in the framework of Boundary Conformal Field Theory, as partition functions in the presence of defects, see [**18**]. Physical applications of this general formalism may be found in statistical mechanics [**15**], string theory or quantum gravity, but this is not the subject of the present paper.

Modular invariance can be investigated either directly, by associating vertices τ_p of the graph A_n with explicit functions χ_p defined on the upper - half plane (the characters of an affine Kac-Moody algebra), or more simply, by checking commutation relations between a particular toric matrix and the two generators S and T of the $SL(2,\mathbb{Z})$ group in a particular representation (Verlinde-Hurwitz, [**11**], [**21**]). The matrix S, as given by the Verlinde formula, is actually a non-commutative analog of the table of characters for a finite group and can be obtained directly from the multiplication table of the graph algebra of A_n.

The theory that was briefly summarized above can be generalized from $SU(2)$ to $SU(N)$ and, more generally, to any affine algebra associated with a chosen Lie group. The familiar simply laced ADE Dynkin diagrams are associated with the affine $SU(2)$ theory, but more general Coxeter-Dynkin systems (each system being a collection of diagrams together with their corresponding quantum groupoids) can be obtained [**13**]. For instance the Di Francesco - Zuber diagrams [**8**], [**9**] are associated with the $SU(3)$ system.

What notion comes first? The chosen diagram, member of some Coxeter-Dynkin system? Its corresponding quantum groupoid? Or the associated modular invariant? The starting point may be a matter of taste... However, from a practical point of view, it is probably better to start from the combinatorial data provided by a given modular invariant. Indeed, apart from $SU(2)$ and, to some extent, $SU(3)$, the Coxeter-Dynkin system itself is not a priori known, whereas existence of several algorithms, mostly due to T. Gannon, allows one to explore up to rather high levels the possibly new modular invariants associated with every choice of an affine Kac-Moody algebra. The primary data is then a given modular invariant – a sesquilinear form in the characters of some affine Kac-Moody algebra. From the resolution, over the positive integers, of a particular equation, called “equation of modular splitting” (more about it later), one can determine first the set of toric matrices, and then the algebra of quantum symmetries. The associated Dynkin diagram – an ADE diagram in the case of the $SU(2)$ system – or, more generally, the particular member of some higher Coxeter-Dynkin system becomes an *outcome* of the construction, not a starting point. The $SU(3)$ system of diagrams was already essentially determined by [**8**], [**9**], then recovered by A. Ocneanu using the modular splitting technique together with the known classification of $SU(3)$ modular invariants obtained by T.Gannon [**10**]. It was also the route followed by [**14**] in his classification of the $SU(4)$ system. In all these examples, starting from a modular invariant partition function, one obtains a diagram that is simply laced (i.e., an ADE diagram, for the $SU(2)$ system), or is a generalization of what could be defined as a "simply laced diagram", for the higher systems.

In the present paper we are only interested in the affine $SU(2)$ system and we want to start from a partition function which is not modular invariant but which is nevertheless invariant under some particular congruence subgroup. Our starting point is the so-called "F_4 partition function" which appears as a kind of $\mathbb{Z}_2$ orbifold of the E_6 modular invariant (its name comes from the fact that exponents of the Lie group F_4 appear in its expression). After the work of [**7**], this partition function was discussed in [**22**]. Our purpose is neither to propose any physical conformal field theory model that would lead to this expression nor to investigate its analytical properties but rather to analyze the equation of modular splitting corresponding to this particular choice of a modular non-invariant expression and see what algebraic structure it gives. Starting from this data, we shall determine, in turn, a set of toric matrices and an algebra of quantum symmetries (described by an Ocneanu graph). Actually we do not even suppose that we "know" what the diagram F_4 is : it will appear as a subgraph of the graph of quantum symmetries. We shall not try here to generalize our study to cover other non-ADE cases of the $SU(2)$ system, and shall not investigate, either, the non simply laced diagrams belonging to higher systems. Since we want to focus on the modular splitting technique itself, we shall not try to use or modify, in this non-simply laced case, the definitions of the product and coproduct that usually lead to a quantum groupoïd structure, but the fact that one can solve the equation of modular splitting, define toric matrices and determine an algebra of quantum symmetries together with two algebras of characters obeying the usual quadratic sum rules is by itself a non trivial result that suggests further developments.

The algorithms that we use in order to solve the relevant equations and determine the algebraic quantities of interest have been developed by the authors, but it may well be that more efficient techniques have been found by others[1] . If so, this information is not available. Although devoted to the study of a particular member of an unusual class (a non-ADE example), we

[1]The results presented long ago in [**12**] or [**14**], for instance, require the use of analogous techniques.

believe that the present paper may be of interest for the reader who wants to see how the general technique based on the equation of modular splitting works, since it does not seem to be documented in the literature. Apart from considerations of computing efficiency, it should not be too hard to adapt the following analysis to study other cases, simply laced or not, associated with any Coxeter-Dynkin system[2] .

Before ending this introduction, let us stress the fact that the present paper does not require any knowledge of the theory of Lie algebras or quantum groups (or quantum groupoids). All the algebras used in the following are associative algebras, not Lie. A sentence like "the algebra A_n" means actually "the associative graph algebra corresponding to the diagram A_n" and we usually identify a given Dynkin diagram with the vector space spanned by its vertices.

2 Toric matrices from modular splitting

Invariance of the partition function

For the $SU(2)$ system, there are three modular invariant partition functions at Coxeter number $\kappa = 12$: they are respectively associated with the diagrams A_{11}, D_7 and E_6. The first case (also called "diagonal") is given by

$$Z_{A_{11}} = \sum_{n=0}^{10} |\chi_n|^2 .$$

Modular invariance of this expression can be explicitly checked by performing the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$; to do that, one can use explicit expressions for the eleven characters χ_p of affine $SU(2)$ at level 10, for instance in terms of theta functions. Another possibility, which is simpler, is to represent S and T by 11×11 matrices, namely $S_{ij} = \sqrt{\frac{2}{\pi}} \sin(\pi \frac{(i+1)(j+1)}{\kappa})$, $0 \leq i, j \leq \kappa - 2$, and $T_{ij} = \exp[2i\pi(\frac{(j+1)^2}{4\kappa} - \frac{1}{8})]\delta_{ij}$, and check that they commute with the modular matrix M associated with Z (the relation between the two being of course, $Z = \overline{\chi} M \chi$). Commutation is obvious since M is the diagonal unit matrix $\mathbb{1}_{11}$.

The E_6 partition function is given by

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2 .$$

[2] See the forthcoming paper [**20**] where an exceptional simply laced example of the $SU(4)$ system is studied.

Its modular invariance can be checked as above but the associated modular matrix is non-trivial. Z_{E_6} is a sum of three modulus squared "generalized characters" $\lambda_1 = \chi_0 + \chi_6$, $\lambda_2 = \chi_3 + \chi_7$, and $\lambda_3 = \chi_4 + \chi_{10}$. M has 12 non - zero entries equal to 1. This is related to the fact that the Ocneanu graph has 12 points, three of them being ambichiral. The corresponding algebra is commutative.

We now turn to F_4, which is our object of study, with partition function

$$Z_{F_4} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2 .$$

The fact that it is *not* modular invariant is obvious from the modular transformations of the generalized characters of E_6: Under $\tau \to -1/\tau$,

$$\begin{aligned}
\lambda_1 &\to (1/2)(\lambda_1 + \lambda_2) - (1/\sqrt{2})\lambda_2, \\
\lambda_2 &\to (1/\sqrt{2})(\lambda_3 - \lambda_1), \\
\lambda_3 &\to (1/2)(\lambda_1 + \lambda_2) + (1/\sqrt{2})\lambda_2.
\end{aligned}$$

Under $\tau \to \tau + 1$, $\lambda_1 \to e^{\frac{19i\pi}{24}}\lambda_1$, $\lambda_2 \to e^{\frac{10i\pi}{24}}\lambda_2$, $\lambda_3 \to e^{\frac{-5i\pi}{24}}\lambda_3$. However, these relations also show that Z_{F_4} is invariant under the transformations $\tau \longrightarrow \tau + 2$ and $\tau \longrightarrow \frac{\tau}{2\tau+1}$ that span a congruence subgroup[3] $\Gamma_0^{(2)}$ of $SL(2,\mathbb{Z})$ at level 2. It is actually easier to show this by checking explicitly that the modular matrix associated with Z_{F_4}, namely

$$M = \begin{pmatrix}
1 & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & 1
\end{pmatrix}$$

commutes with the generators T^2 and $ST^{-2}S$ of $\Gamma_0^{(2)}$. Notice that the exponents of the Lie group F_4 appear on the diagonal of M, hence the name chosen for this partition function. From the fact that Z_{F_4} is a sum of two squares, one expects two ambichiral points in the Ocneanu graph (indeed it will turn out to be so). M has 8 non - zero entries equal to 1, this could suggest that the corresponding Ocneanu graph has the same number of vertices... however, as we shall see, this is *not* so.

---

[3]We do not use the corresponding projective group.

Toric matrices (definition)

Let $m, n, \ldots$ (rather than τ_m, τ_n, or χ_m, χ_n) denote the vertices of A_{11} and $x, y, \ldots$ the vertices of the Ocneanu graph Oc to be found. As already mentioned, Oc should be a bi-module over A_{11} so that there exist a collection of 11×11 toric matrices $W_{x,y}$, with positive integer entries, such that

$$m\, x\, n = \sum_y \left(W_{x,y}\right)_{m,n} y.$$

These are "toric matrices with two twists" (x and y). If o is the number of vertices in Oc – it will be determined later – the number of such matrices is of course o^2 but many of them may coïncide. Since A_{11} and Oc play a dual role, it is useful to introduce the 11^2 matrices $V_{m,n}$ of size $o \times o$ by

$$(V_{m,n})_{x,y} = \left(W_{x,y}\right)_{m,n}.$$

The algebra of quantum symmetries has a unit, called $\underline{0}$, a particular vertex of Oc, so that we have also " matrices with one twist" $W_{0,y}$ and $W_{x,0}$. When the example under study corresponds to a simply laced situation (for instance the ADE cases of the usual $SU(2)$ system) and if the algebra Oc is commutative. one shows that $W_{x,y} = W_{y,x}$. However, we are now in a new situation and should keep our mind open. The resolution of the general modular splitting equation will, in any case, determine all these quantities.

Equations of modular splitting

The general equation of modular splitting expresses associativity of a bi-module structure and reads

$$(m\, n)\, x(\, p\, q) = (m\, (n\, x\, p)\, q).$$

The products $(m\, n)$ or $(\, p\, q)$ belong to the fusion algebra, i.e., for instance, to A_{11}, and involve its structure constants defined by $m\, n = \sum_p N_{m,n}^p\, p$. They are completely symmetric in their three indices. These positive integers are determined, recursively, by the – truncated – $SU(2)$ algebra of compositions of spins. For A_{11} one obtains 11 matrices N_m of size 11×11, determined by the equations $N_1\, N_m = N_{m-1} + N_{m+1}$, with $N_0 = \mathbb{1}_{11}$ and N_1, the adjacency matrix of the diagram A_{11}. One obtains in particular, $N_1\, N_{10} = N_9$ since N_{11} vanishes. Of course $N_{m,n}^p = \left(N_m\right)_{p\, n}$. Using toric matrices, the general modular splitting equation reads therefore

$$\sum_{m'', n''} \left(N_{m''}\right)_{m', m} \left(N_{n''}\right)_{n', n} \left(W_{x,y}\right)_{m'' n''} = \sum_z \left(W_{x,z}\right)_{mn} \left(W_{z,y}\right)_{m' n'}$$

This general equation, valid for any simply laced graph belonging to a Coxeter-Dynkin system, together with its proof and interpretation, in terms of associativity of a bi-module structure, was obtained in the thesis [**19**]; it generalizes the "modular splitting equation" described in [**14**]. The later can be obtained from the former by setting $x = y = 0$, so that its left hand side involves only known quantities, namely the matrix $M = W_{0,0}$ associated with the given graph (in the simply laced cases, it is *the* modular invariant), and the fusion coefficients. It reads

$$\sum_{m'',n''} (N_{m''})_{m',m} (N_{n''})_{n',n} M_{m''n''} = \sum_{z} (W_{z,0})_{m'n'} (W_{0,z})_{mn}$$

or, in terms of tensor products,

$$\sum_{mn} N_m \otimes N_n \; M_{m,n} = \sum_{z \in Oc} \tau \circ (W_{z,0} \otimes W_{0,z}).$$

Here τ denotes a tensorial flip (compare explicit indices in the two previous equations). The left and side – call it K – is therefore a known matrix of size 121×121 (in this case), with positive integer entries, and the right hand side involves a set of toric matrices (with one twist), to be determined. Each term of this right hand side should be a matrix of rank 1 with positive integer coefficients. Only one member of this family is a priori known, namely $M = W_{0,0}$ which is our initial data. The indexing set on the right hand side of the modular splitting equation defines the set of vertices of the Ocneanu graph. The problem at hand is analogous to those related with convex decompositions in abelian monoïds. In the simply laced cases (where $W_{x,0} = W_{0,x}$ for all x), each term appearing on the right hand side is a tensor square composed with the flip. In the non simply laced case, as we shall see, the situation is slightly more complicated. Notice that, in any case, calling "toric vectors" $w_{x,y}$ the line-vectors obtained by "flattening" the toric matrices $W_{x,y}$ (i.e., $(w_{x,y})_k = W_{x,y}[p,q]$ with $k = ((p-1) \times 11) + q$), one can write the modular splitting equation as follows: $K = \sum_z K_z$ where each matrix K_z – of size 121×121 with positive integral entries – should be of rank 1, and its k-th line is equal to

$$K_z[k] = (w_{z,0})_k \, w_{z,0}.$$

Resolution of the equations

The algorithm used to solve this set of equations (over the positive integers) is slightly different for the known simply-laced case and for the example that we study in this paper. Let us first summarize what we would do in the usual situation and call n the total number of vertices of the corresponding fusion graph.

The simply laced case

- The first line of K – a line vector with n^2 components – is just the "flattened" invariant matrix M.
- One does not assume that $W_{x,y}$ is equal to $W_{y,x}$ but takes nevertheless $W_x \doteq W_{x,0} = W_{0,x}$.
- The rank $r(K)$ of K can be calculated. This tells us that the dimension of the vector space spanned by the n^2 lines of K can be expanded on a set of $r(K)$ basis vectors $w_x = w_{x,0}$ (the toric vectors). Once a toric vector w is found, the corresponding toric matrix is obtained by partitioning its entries into n lines of length n.
- The problem is to determine whether a given line of K is a toric vector or if it is a positive integral linear combination of such vectors, and in that later case, one has to find the number of such terms in the sum. We choose the (non canonical) scalar product for which the basis of toric vectors is orthonormal; for every line of K we write $K[p] = \sum_x a(x)\, w_x$ and the norm square of the vector $K[p]$ is therefore $\sum a(x)^2$. The fundamental observation is that this number (call it $\ell[p]$) is equal to the diagonal matrix element $K[p,p]$.
- Since K is known, we first consider the list of values of p for which $K[p,p] = 1$. Since several line vectors $K[p]$ and $K[p']$ may coïncide for distinct values of p, p', we actually build a restricted list. For p in this list, every line vector $K[p]$ is then automatically a toric vector.
- We next consider the list (actually a restricted list, as above) of values of p for which $K[p,p] = 2$. The corresponding line vectors $K[p]$ should be the sum of two toric vectors, and there are three cases. Either $K[p]$ is the sum of two already determined toric vectors, or it is the sum of an already determined toric vector and a new one, else it is equal to twice a new toric vector. For every value of p belonging to the new restricted list, it is enough to calculate the set of differences $K[p] - w_x$ were w_x runs in the set of the already determined toric vectors, and impose that all the components of such differences should be positive integers.
- The next step is to consider the set of values of p for which $K[p,p] = 3$, etc. and generalize the previous discussion in an straightforward way. The process stops, ultimately, since the rank of the system is finite. At the end, we obtain a set of $r(K)$ toric vectors which are either lines of K of linear combinations of lines of K.
- The integer $r(K)$ may be strictly smaller than the number o of vertices of the Ocneanu graph. This happens when distinct quantum symmetries x are associated with the same toric matrix W_x. This is for example the case of the graph D_4 where the rank is 5 but where $o = 8$. A method to determine such multiplicities is to plug the results for w_x

(actually for the matrices W_x) back into the modular splitting equation. If there are no multiplicities, this equation is readily checked. If it does not hold, one has to introduce appropriate multiplicities in the right hand side (introduce unknown coefficients and solve). In the case of E_6, the rank is 12, the final list of toric vectors is obtained from lines $1, 2, 3, 10, 11, 12, 13, 14, 21, 22, 23, 4-10$ of K, the last one being equal to a difference of two lines[4] , and the modular splitting equation holds "on the nose", so $o = 12$ also. In the case of D_4, the rank is 5, the list of toric vectors is $K[1], K[2], K[6], K[3]/2, K[7]/3$ but the modular splitting equation holds only by introducing multiplicities 2 and 3 for the last two[5] , the number of quantum symmetries o is therefore 8.

The case of F_4 (non simply laced)

- As usual, the first line $K[1]$ of K – a line vector with 121 components – is just the "flattened" invariant matrix M.
- The rank of K is 20. We present the results according to the decomposition number $\ell[p]$ relative to the line vector $K[p]$ of K, with p running from 1 to 121. The twenty toric vectors can be taken as follows.

$$
\begin{aligned}
\ell[p] = 1 \quad & w[p] = K[p] \quad \text{with} \quad p = 1, 2, 3, 10, 11, 12, 13, 14, 21, 22, 23, 24 \\
\ell[p] = 2 \quad & w[4] = K[4] - w[10] = K[4] - K[10] \\
& w[15] = K[15] - w[21] = K[15] - K[21] \\
& w[25] = K[25]/2, \quad w[34] = K[34] - w[22] = K[34] - K[22] \\
& w[35] = K[35] - w[21] = K[35] - K[21] \\
\ell[p] = 3 \quad & w[26] = (K[26] - w[24])/2 = (K[26] - K[24])/2 \\
& w[36] = (K[36] - w[14])/2 = (K[36] - K[14])/2 \\
\ell[p] = 5 \quad & w[37] = (K[37] - w[13] - w[15] - w[35])/2 \\
& = K[37] - K[13] - K[15] + 2K[21] - K[35])/2.
\end{aligned}
$$

- We re - label the toric vectors in such a way that the index x of w_x runs from 0 to 19.

$w_0 = w[1], w_1 = w[2], w_2 = w[3], w_3 = w[4], w_4 = w[10],$
$w_5 = w[11], w_6 = w[12], w_7 = w[13], w_8 = w[14], w_9 = w[15],$
$w_{10} = w[21], w_{11} = w[22], w_{12} = w[23], w_{13} = w[24], w_{14} = w[25],$
$w_{15} = w[26], w_{16} = w[34], w_{17} = w[35], w_{18} = w[36], w_{19} = w[37].$

[4] Using the notations of [**2**] or [**3**], these toric vectors correspond respectively to $W_{00}, W_{01}, W_{02}, W_{05}, W_{40}, W_{10}, W_{11}, W_{21}, W_{51}, W_{50}, W_{20}, W_{30}$.

[5] Using the notations of [**12**] or [**3**], they correspond respectively to $W_0, W_{1\epsilon}, W_1, W_2 = W_2', W_\epsilon = W_{2\epsilon} = W_{2\epsilon}'$.

> There is a one to one correspondence between toric matrices $W_{x,0}$ and the previously determined toric vectors w_x which have 121 components: just partition them into 11 lines of 11 elements. At a later stage we shall write the generators O_x of the algebra Oc in terms of tensor products, for this reason, the following distinct notations appear in this paper: $W_x = W_{x,0} = W_{\underline{ab'}}$ whenever $O_x = a \dot{\otimes} b$. We hope that no confusion should arise between the notations $W_{\underline{ab'}}$ and $W_{x,y}$.

Explicitly the twenty toric matrices W_x are as follows.

$$W_0 = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix} \quad W_5 = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$$W_7 = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&2&\cdot&1&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} \quad W_{10} = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&1&\cdot&2&\cdot&1&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$$W_{14} = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} \quad W_{19} = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$$W_1 = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix} \quad W_2 = \begin{pmatrix} \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot \end{pmatrix}$$

$$W_3 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} \quad W_4 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$W_8 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad W_9 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$W_{15} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

We have moreover the following identifications

$$W_6 = (W_1)^T; \quad W_{12} = (W_2)^T; \quad W_{16} = (W_3)^T; \quad W_{11} = (W_4)^T$$
$$W_{13} = (W_8)^T \quad W_{17} = (W_9)^T \quad W_{18} = (W_{15})^T$$

Notice that $W_0, W_5, W_7, W_{10}, W_{14}, W_{19}$ are symmetric (self-dual). Among them, two will be called "ambichiral", namely W_0 and W_5 for the reason that they correspond to the ambichiral generators of the algebra Oc (intersection of the two chiral sub-algebras).

As expected from the presence of the 1/2 coefficients in the list giving the vectors $w[p]$, the modular splitting equation does not hold if we impose $W_{x,0} = W_{0,x}$ and sum only on the corresponding 20 terms on the right hand side. One possibility is two introduce a multiplicity two for entries $w_{14} = w[25], w_{15} = w[26], w_{18} = w[36], w_{19} = w[37]$; this indeed works, in the sense that the modular splitting equation is then satisfied. With such a choice, the number of quantum symmetries would be $o = 24$, rather than 20. However, the algebra of quantum symmetries later determined by this choice incorporates several arbitrary coefficients that cannot be fixed by requirements of positivity and integrality alone. Since we are in "Terra Incognita" (namely quantum symmetries of non simply laced diagrams), we prefer to explore another possibility, which also allows us to solve the modular splitting equation and leads to a nice algebra of quantum symmetries (and, as we shall see, to the emergence of the diagram F_4). Our choice is to keep only the previously determined 20 terms, no more, but without imposing equality of $W_{x,0}$ and $W_{0,x}$. This choice is natural in view of the fact that these matrices actually "count" a number of essential paths between the origin 0 and x on the Ocneanu graph itself, and the fact that in the present case, the graph is not symmetric (all edges are not bi-oriented). In general, the solution to the modular splitting equation, for a given invariant matrix M is not necessarily unique, although some later considerations may impose extra conditions that, ultimately, lead to rejection of one or another solution. At the moment, we investigate one solution (which is both minimal in terms of number of quantum symmetries and natural from the path interpretation point of view) and explore the consequences.

In order for the modular splitting equation to be satisfied, we therefore take $W_{0,x} = 2W_{x,0}$ when $x = 14, 15, 18, 19$ and $W_{0,x} = W_{x,0}$ for the others. As we shall see, this corresponds to the fact that the F_4 diagram contains an oriented edge.

3 Quantum symmetries and Ocneanu graph

Determination of the two chiral generators O_1 and O'_1

Call K_0 the rectangular matrix (121×20) obtained by decomposing each line vectors of K on the (flattened) toric matrices. For instance the fourth line $K[4]$ of K is equal to $w[4] + w[5]$, so its components are

$$(0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

Call L_0 the rectangular matrix obtained by transposing the matrix (20×121) obtained by flattening each component of the column vector (twenty lines) containing the toric matrices.

When $W_{x,0} = W_{0,x}$ it is easy to see that $K_0 = L_0$ but this is not so in the present case.

If E_0 denotes the essential matrix (also called "intertwiner") associated with the origin of an ADE diagram (for the $SU(2)$ system or higher generalizations), and if G_1 denotes the corresponding adjacency matrix, it is so that $E_1 \doteq N_1 \,.\, E_0 = E_0 \,.\, G_1$ where N_1 is the generator of the fusion algebra (adjacency matrix of the appropriate A_n diagram). E_1 coïncides with the essential matrix associated with the next vertex (after the origin) and describes essential paths emanating from it.

We have the following analogy: K_0 (or L_0) play the same role as E_0, but now G_1 should be replaced by one of the two generators of the algebra of quantum symmetries, and N_1 should be replaced by $N_0 \otimes N_1$ (so we replace the fusion algebra by its tensor square). In other words, we determine the generator O_1 by solving the intertwining equation

$$N_0 \otimes N_1 \,.\, K_0 = K_0 \,.\, O_1$$

The other chiral generator O'_1 is determined by solving the same equation, but replacing $N_0 \otimes N_1$ by $N_1 \otimes N_0$. At this level it is interesting to recall that, in this analogy between the vector space of a diagram and its algebra of quantum symmetries, the fusion algebra should be replaced by its tensor square, and the role of fused adjacency matrices (the F_{ab} matrices of references [**3**] that represent the action of A_n on a given diagram) is played by the toric matrices themselves.

In the present situation (non simply laced), we could hesitate between L_0 and K_0, but the choice actually does not matter : it turns out that this arbitrariness corresponds to the arbitrariness in the association between the asymmetric F_4 graph and a particular adjacency matrix or its transpose.

In general, after having solved the modular splitting equation (and there is no necessarily uniqueness of the solution), we have to solve a generalized intertwining identity (the one just given) in order to find the two chiral generators. Notice that the solution could be non unique, even after imposing integrality and positivity, but in the present case the solution is unique and we list[6] below the two matrices O_1 and O'_1, of dimension 20×20, which solve these equations.

$$O_1 = \begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & 1 & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 2 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 2 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & .
\end{pmatrix}$$

[6]The reader already recognizes, from the structure of O_1, two subdiagrams of type E_6 and two others of type F_4.

$$O'_1 = \begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}$$

Now that we have obtained the two algebraic generators, the algebra Oc that they span is determined as well (take linear combinations of powers and products). However we shall exhibit a particular basis of linear generators. To find them, one possibility is to first determine the set of matrices $V_{m,n}$ that describe, in a dual way, the full set of toric matrices $W_{x,y}$ with two twists.

Determination of the toric matrices with two twists

The $V_{m,n}$ are obtained by solving the general intertwining equation (notice that $O_1 = V_{0,1}$ and $O'_1 = V_{1,0}$).

$$N_m \otimes N_n \,.\, K_0 = K_0 \,.\, V_{m,n}.$$

The solution is unique. Of course, we shall not list these 11^2 square matrices of dimension 12×12 and we shall not give, either, the list of toric matrices with two twists, but remember that they are determined by the relation

$$(W_{x,y})_{m,n} = (V_{m,n})_{x,y}.$$

One can then check that the generalized equation of modular splitting (the one that involves the $W_{x,y}$ rather than the $W_{x,0}$) is satisfied.

Determination of the linear generators O_x of Oc

The structure constants O_{xyz} are defined by the equations

$$W_{y,x} = \sum_z O_{xyz}\, W_z$$

where $W_z = W_{z,0}$. Notice that, in the present case, $O_{xyz} = O_{zyx}$ but $O_{xyz} \neq O_{yxz}$ in general. Matrices O_x are defined by their coefficients as follows : $(O_x)_{yz} = O_{xyz}$. We have

$$O_x \, O_y = \sum_z O_{xzy} \, O_z$$

and any two generators O_x and O_y commute, because of the symmetry properties of the structure constants.

One could be tempted to consider the (non-commutative) family of matrices Z_y defined by the equation $(Z_y)_{xz} = O_{xyz}$ but one can see that this family is not multiplicatively closed; moreover Z_0 does not even coïncide with the identity matrix, since it has diagonal coefficients equal to 2 in positions $x = 14, 15, 18, 19$.

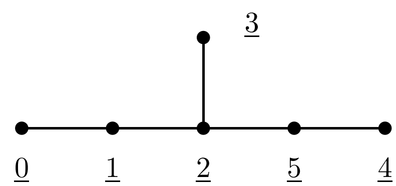
When the Ocneanu diagram possesses geometric symmetries, for instance in the case of the D diagrams, it may be that the general solution involve parameters that should be fixed by imposing positivity and integrality, and that one solution is only determined up to a discrete transformation reflecting the classical symmetries (this amounts to re-label the vertices x). In the present case, however, everything is perfectly determined and we obtain the twenty generators of the Ocneanu diagram – They are 20×20 matrices. Rather than giving this list explicitly (it would be typographically heavy !), we shall express them in terms of the already known and explicitly given chiral generators O_1 and O_1'. Call O_0 the unit matrix $\mathbb{1}_{20}$.

O_0	$O_6 = O_1'$	$O_{12} = O_6.O_1' - O_0$	$O_{16} = O_{12}.O_1' - O_6 - O_{11}$
O_1	$O_7 = O_1 \cdot O_1'$	$\downarrow$	$\downarrow$
$O_2 = O_1^2 - O_0$	$O_8 = O_2 \cdot O_1'$	$O_{13} = O_{12} \cdot O_1$	$O_{17} = O_{16} \cdot O_1$
$O_5 = O_1^4 - 4O_1^2 + 2O_0$	$O_9 = O_3 \cdot O_1'$	$O_{14} = O_{13} \cdot O_1 - O_{12}$	$O_{18} = O_{17} \cdot O_1 - O_{16}$
$O_4 = O_1 \cdot O_5$	$O_{10} = O_4 \cdot O_1'$	$O_{15} = O_{14} \cdot O_1 - 2O_{13}$	$O_{19} = O_{18} \cdot O_1 - 2O_{17}$
$O_3 = O_2.O_1 - O_4 - O_1$	$O_{11} = O_5 \cdot O_1'$	$O_{14} = O_{15} \cdot O_1$(check)	$O_{18} = O_{19} \cdot O_1$(check)

The full multiplication table (that we don't display because it is 20×20) defines a 20 dimensional algebra Oc with linear generators O_x, with $x \in \{0, 1, 2, 3, \ldots 18, 19\}$. It is generated, as an algebra, by the two matrices associated with vertices 1 and 6 (called *chiral generators*). We call it "algebra of quantum symmetries of F_4". Multiplication of any single linear generator O_x by the two chiral ones is encoded by a graph: the Ocneanu graph of F_4. It will be described later.

The Ocneanu algebra as a the tensor square of a graph algebra

For all simply laced diagrams belonging to the $SU(2)$ system or to an higher system, the algebra of quantum symmetries turns out to be related, in one way or another, to the tensor square of some graph algebra. For instance $Oc(E_6)$ is the tensor square of the graph algebra of E_6 taken above the graph subalgebra generated by the ambichiral vertices $0, 4, 3$. We remind the reader that E_6 admits self - fusion, with graph algebra given by the following table.



$*$	$\underline{0}$	$\underline{3}$	$\underline{4}$	$\underline{1}$	$\underline{2}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{3}$	$\underline{4}$	$\underline{1}$	$\underline{2}$	$\underline{5}$
$\underline{3}$	$\underline{3}$	$\underline{0}+\underline{4}$	$\underline{3}$	$\underline{2}$	$\underline{1}+\underline{5}$	$\underline{2}$
$\underline{4}$	$\underline{4}$	$\underline{3}$	$\underline{0}$	$\underline{5}$	$\underline{2}$	$\underline{1}$
$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{5}$	$\underline{0}+\underline{2}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{2}+\underline{4}$
$\underline{2}$	$\underline{2}$	$\underline{1}+\underline{5}$	$\underline{2}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{0}+\underline{2}+\underline{2}+\underline{4}$	$\underline{1}+\underline{3}+\underline{5}$
$\underline{5}$	$\underline{5}$	$\underline{2}$	$\underline{1}$	$\underline{2}+\underline{4}$	$\underline{1}+\underline{3}+\underline{5}$	$\underline{0}+\underline{2}$

It is natural to try to realize $Oc(F_4)$, that we obtained by solving the modular splitting equation, directly in terms of some analogous algebraic construction. ¿From the fact that F_4 is an orbifold of E_6, it is easy to make an educated guess, and, by calculating the corresponding multiplication table, check that it is indeed correct. We claim that $Oc(F_4) = E_6 \dot{\otimes} E_6$ where $\dot{\otimes}$ denotes the tensor product taken, not above the complex numbers but above the subalgebra J generated by vertices $\underline{0}$ and $\underline{4}$ of E_6. In other words, we identify $a * u \otimes b$ and $a \otimes u * b$ as soon as $u \in J$.

The twenty generators of $Oc(F_4)$ are realized as follows.

$$\underline{0} = \underline{0} \dot{\otimes} 0 = \underline{4} \dot{\otimes} \underline{4} = \underline{0}$$

$$1 = \underline{1} \dot{\otimes} \underline{0} = \underline{5} \dot{\otimes} \underline{4} = \underline{1} \qquad 6 = \underline{0} \dot{\otimes} \underline{1} = \underline{4} \dot{\otimes} \underline{5} = \underline{1'}$$

$$2 = \underline{2} \dot{\otimes} \underline{0} = \underline{2} \dot{\otimes} \underline{4} = \underline{2} \qquad 12 = \underline{0} \dot{\otimes} \underline{2} = \underline{4} \dot{\otimes} \underline{2} = \underline{2'}$$

$$3 = \underline{3} \dot{\otimes} \underline{0} = \underline{3} \dot{\otimes} \underline{4} = \underline{3} \qquad 16 = \underline{0} \dot{\otimes} \underline{3} = \underline{4} \dot{\otimes} \underline{3} = \underline{3'}$$

$$4 = \underline{5} \dot{\otimes} \underline{0} = \underline{1} \dot{\otimes} \underline{4} = \underline{5} \qquad 11 = \underline{0} \dot{\otimes} \underline{5} = \underline{4} \dot{\otimes} \underline{1} = \underline{5'}$$

$$5 = \underline{4} \dot{\otimes} \underline{0} = \underline{0} \dot{\otimes} \underline{4} = \underline{4}$$

$$7 = \underline{1} \dot{\otimes} \underline{1} = \underline{5} \dot{\otimes} \underline{5} = \underline{11'}$$

$$10 = \underline{5} \dot{\otimes} \underline{1} = \underline{1} \dot{\otimes} \underline{5} = \underline{15'}$$

$$8 = \underline{2} \dot{\otimes} \underline{1} = \underline{2} \dot{\otimes} \underline{5} = \underline{21'} \qquad 13 = \underline{1} \dot{\otimes} \underline{2} = \underline{5} \dot{\otimes} \underline{2} = \underline{12'}$$

$$14 = \underline{2} \dot{\otimes} \underline{2} = \underline{22'}$$

$$9 = \underline{3} \dot{\otimes} \underline{1} = \underline{3} \dot{\otimes} \underline{5} = \underline{31'} \qquad 17 = \underline{1} \dot{\otimes} \underline{3} = \underline{5} \dot{\otimes} \underline{3} = \underline{13'}$$

$$15 = \underline{3} \dot{\otimes} \underline{2} = \underline{32'} \qquad 18 = \underline{2} \dot{\otimes} \underline{3} = \underline{23'}$$

$$19 = \underline{3} \dot{\otimes} \underline{3} = \underline{33'}$$

Labels on the left correspond to the original notation $\{0, 1, 2 \ldots 19\}$ that we have been using before, while underlined labels on the right refer to tensor products of E_6 vertices (as defined[7] by the above E_6 diagram, like in [**12**], [**2**] or [**19**]). Because of this labeling convention, notice that $4 = \overline{5}$ and $5 = \overline{4}$. Using the above realization, one recovers the multiplication table of quantum symmetries. For instance, $\underline{21'} \times \underline{23'} = (\underline{22'})_2 + (\underline{2'})_2$. Indeed,

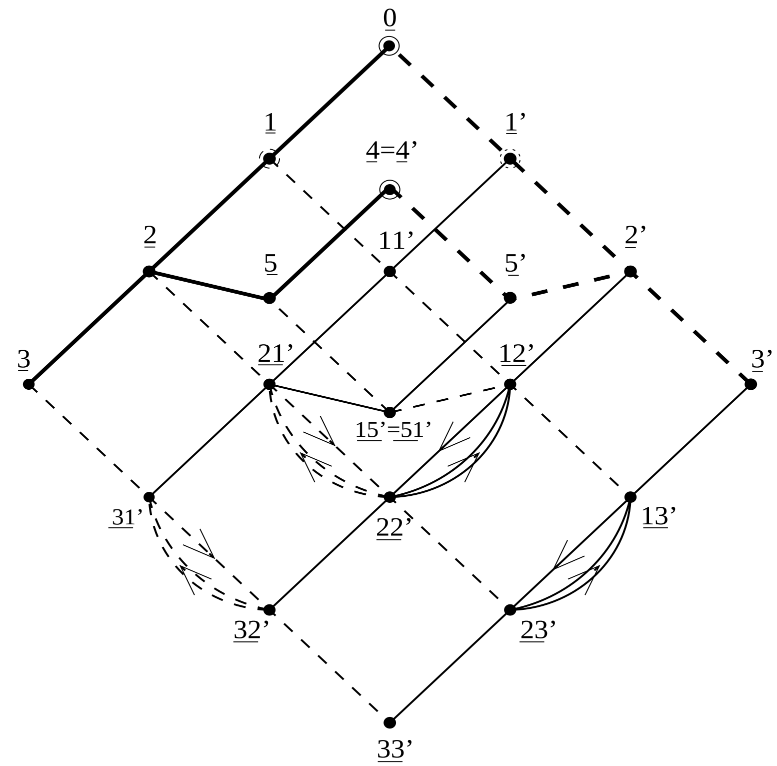
$$\begin{aligned} 8 \times 18 &= (\underline{2} \dot{\otimes} \underline{1}) \times (\underline{2} \dot{\otimes} \underline{3}) = (\underline{2} * \underline{2}) \dot{\otimes} (\underline{1} * \underline{3}) = (\underline{0} + \underline{2} + \underline{2} + \underline{5}) \dot{\otimes} (\underline{2}) \\ &= (\underline{0} \otimes \underline{2})_2 + (\underline{2} \otimes \underline{2})_2 = 12 + 12 + 14 + 14 \end{aligned}$$

We therefore recover the matrix product equality $O_8 \times O_{18} = 2O_{12} + 2O_{14}$.

The Ocneanu graph

Using $E_6 \dot{\otimes} E_6$ notation for the vertices, the F_4 Ocneanu graph is given as follows:

[7] Warning: There are several conventions in the literature.



It is the Cayley graph of multiplication of the linear generators of Oc by the two generators O_1 and O_1', called the *chiral* generators. It is the union of two distinct graphs called left and right graphs, involving the same set of vertices. They are drawn in two different colors (or solid and dashed lines). One can obtain one graph from the other by performing a symmetry with respect to the vertical axis. The six vertices that belong to this axis of symmetry are the *self-dual points*.

The subalgebra generated by the unit and the left chiral generator $\underline{1}$ is the left chiral subalgebra. It is spanned by vertices $\{\underline{0}, \underline{1}, \underline{2}, \underline{5}, \underline{4}, \underline{3}\}$ and corresponds to the connected component of the identity of the left chiral graph. The subalgebra generated by the unit and the right chiral generator $\underline{1}'$ is the right chiral subalgebra. It is spanned by vertices $\{\underline{0}, \underline{1}', \underline{2}', \underline{5}', \underline{4}', \underline{3}'\}$ and corresponds to the connected component of the identity of the right chiral graph. The intersection of these two subalgebras is spanned by the set of *ambichiral points*, namely the two-element set $\{\underline{0}, \underline{4}\}$. These points are self-dual.

Because of these symmetry considerations, we discuss only the left graph. It is itself given by the union of four disjoint connected graphs, two of type

E_6, that we call respectively $E = E_6[1]$ (the left chiral subalgebra) and $e = E_6[2]$ (span of $\underline{1'}, \underline{11'}, \underline{21'}, \underline{51'}, \underline{5'}, \underline{31'}$), and two of type F_4, that we call $F = F_4[1]$ (span of $\underline{32'}, \underline{22'}, \underline{12'}, \underline{2'}$) and $f = F_4[2]$ (span of $\underline{33'}, \underline{23'}, \underline{13'}, \underline{3'}$). The description of the right graph is similar. Notice that F_4 Dynkin diagrams emerge from our resolution of the equations of modular splitting.

The first E_6 (left) subgraph called E describes the subalgebra generated by O_1. The other E_6 (left) subgraph called e is not a subalgebra of Oc but a module over E. The two subgraphs of type F_4 called F and f are also modules over E, but their properties are very different. Writing the full multiplication table would be too long, but the interested reader can easily do it, either using 20×20 matrices, or, more simply, using the multiplication table of the graph algebra of E_6 together with the realization of generators of $Oc(F_4)$ in terms of tensor products (see previous section). We have the following relations between the different subspaces:

$\times$	E	e	F	f
E	E	e	F	f
e	e	$E+F$	$e+f$	F
F	F	$e+f$	$E+F$	e
f	f	F	e	E

4 Actions, coactions and sum rules

As written in the introduction, the Ocneanu quantum groupoid associated with a simply laced diagram G (with r vertices) belonging to the $SU(2)$ system, or to an higher system, possesses two – usually distinct – algebras of characters. The first, called the fusion algebra of G, and denoted $A(G)$, can be identified with the graph algebra of the graph A_n (for a proper choice of n). The second algebra of characters, called the algebra of quantum symmetries, is denoted $Oc(G)$. The fusion algebra $A(G) = A_n$ acts on the vector space spanned by the vertices of the graph G, and this action is explicitly described by the so-called "fused adjacency matrices" F_p. These matrices have therefore the same commutation relations as the fusion matrices N_p (the generators of A_n) but their size is smaller since it is $r \times r$, where r is the number of vertices of G. In the same way, but at the dual level (coaction), the algebra of quantum symmetries $Oc(G)$ acts on G and this is described by the so-called "quantum symmetry fused matrices" Σ_x of G. These matrices have therefore the same commutation relations as the quantum symmetry matrices O_x (the generators of $Oc(G)$) but their size is smaller since it is again $r \times r$. See [**2**], [**3**] for explicit expressions of these matrices in various ADE cases. In the simply laced situation, there is a general theory [**12**] (see also [**6**], [**18**]) that tells us how to build first, a product law – composition – defined as composition of graded endomorphisms of essential paths ("horizontal paths") on the given graph, then a co-product law – associated, via the choice of a scalar product, to convolution – by using the composition of endomorphisms of the so - called "vertical paths". However, to our knowledge, for a non simply-laced diagram like the one we study here, the general theory is not known. Our purpose, in this section, is therefore very modest, in the sense that we shall only mimic what we would have done in the simply laced situation, and describe what we find. This is admittedly rather naive, since when counting dimensions, for instance, we take the oriented double line of the F_4 diagram (between vertices a_2 and a_1) as a pair of two essential paths of length one, and this is maybe not what should be done.

Fused matrices F_p relative to the fusion generators N_p of A_{11}

We first consider the action of A_{11} implemented by matrices F_p, to be found. The simplest determination stems from the fact that A_{11} is a truncated version of the algebra of characters of $SU(2)$, so that the F_p's are obtained from the usual recurrence formula (composition of spins) $F_pF_1 = F_{p-1} + F_{p+1}$, and the seed: $F_0 = \mathbb{1}_4$ and F_1 is equal to the adjacency matrix G_1 of the graph F_4. This recurrence relation has a period (2×12) and one can check that $F_{10}F_1 = F_9$ since F_{11} vanishes. For $12 \leq p \leq 23$, $F_{p+12} = -F_p$ but we are only interested here in the positive part. Notice that F_1 is not symmetric, since the graph F_4 itself, with vertices a_0, a_1, a_2, a_3, is not[8] (two oriented edges between vertices a_2 and a_1 but only one oriented edge between a_1 and a_2). We obtain the following 11 matrices and check that $(F_pF_q)\,a_i = F_p\,(F_q\,a_i)$, as it should.

$$F_0 \quad \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \qquad F_1 \quad \begin{pmatrix} . & 1 & . & . \\ 1 & . & 2 & . \\ . & 1 & . & 1 \\ . & . & 1 & . \end{pmatrix} \qquad F_2 \quad \begin{pmatrix} . & . & 2 & . \\ . & 2 & . & 2 \\ 1 & . & 2 & . \\ . & 1 & . & . \end{pmatrix}$$

$$F_3 \quad \begin{pmatrix} . & 1 & . & 2 \\ 1 & . & 4 & . \\ . & 2 & . & 1 \\ 1 & . & 1 & . \end{pmatrix} \qquad F_4 \quad \begin{pmatrix} 1 & . & 2 & . \\ . & 3 & . & 2 \\ 1 & . & 3 & . \\ . & 1 & . & 1 \end{pmatrix} \qquad F_5 \quad \begin{pmatrix} . & 2 & . & . \\ 2 & . & 4 & . \\ . & 2 & . & 2 \\ . & . & 2 & . \end{pmatrix}$$

and $F_p = F_{10-p}$ for $p = 6, \ldots, 10$. From the fused adjacency matrices F_p we can obtain four essential matrices E_a that are rectangular 4×11

[8] The reader may check that $C = 2\,\mathbf{1} - G_1$ is the usual Cartan matrix of F_4

matrices defined by $(E_a)_{b,n} = (F_n)_{a,b}$. The F's and the E's determine the induction/restriction rules between the graphs A_{11} and F_4.

Fused matrices Σ_x relative to the quantum symmetry generators O_x of Oc

We now turn to the determination of matrices Σ_x. We shall present two methods. The most direct uses the fact that these matrices provide a 4×4 realization of Oc. It is enough to define $\Sigma_0 = \mathbb{1}_4$, to set $\Sigma_1 = \Sigma_6$ equal to the adjacency matrix of the F_4 diagram and use the same relations that determine all matrices O_x from O_0, O_1 and O_6 (equivalently, solve the system of equations $\Sigma_x \Sigma_y = O_{xzy}\, \Sigma_z$ with given structure constants O_{xyz}).

Another method uses our realization of $Oc(F_4)$ as a fibered tensor product $E_6 \dot{\otimes} E_6$. Remember that the (left, for instance) graph of quantum symmetries is a union of four graphs E, e, F, f, two of type E_6, two of type F_4, so that any single connected component describes a module action over the subalgebra associated with the first subgraph (E). In this way we obtain four sets of matrices: s_u^E (of dimension 6×6), s_u^e (of dimension 6×6), s_u^F (of dimension 4×4), and s_u^f (of dimension 4×4). In all cases, u runs from 0 to 5, so that these are six-elements sets. The elements of the first set s^E coïncide with the already known generators of the graph algebra of E_6. What we have to use in this section is the second module[9] of type F_4 (called f) and the matrices s^f that express the action $E \times f \subset f$. Remember also that we have $f \times f \subset E$. These matrices are as follows:

$$
\overset{s_0^f}{\begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}}
\quad
\overset{s_1^f}{\begin{pmatrix} . & 1 & . & . \\ 1 & . & 2 & . \\ . & 1 & . & 1 \\ . & . & 1 & . \end{pmatrix}}
\quad
\overset{s_3^f}{\begin{pmatrix} . & . & 2 & . \\ . & 2 & . & 2 \\ 1 & . & 2 & . \\ . & 1 & . & . \end{pmatrix}}
\quad
\overset{s_3^f}{\begin{pmatrix} . & . & . & 2 \\ . & . & 2 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}}
$$

and $s_4^f = s_1^f$, $s_5^f = s_0^f$. From the expressions giving the linear generators of Oc as tensor products of E_6 vertices, we obtain $\Sigma_x = s_a . s_b$, whenever $x = a \dot{\otimes} b$.

[9]In this respect the situation is similar to the analysis of E_7, which appears as a subgraph of its own algebra of quantum symmetries and as a module over the graph algebra of D_{10}, itself a subalgebra of $Oc(E_7)$.

The two methods give the same result and we obtain the following 20 matrices:

$$\Sigma_0 \quad \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \qquad \Sigma_1 \quad \begin{pmatrix} . & 1 & . & . \\ 1 & . & 2 & . \\ . & 1 & . & 1 \\ . & . & 1 & . \end{pmatrix} \qquad \Sigma_2 \quad \begin{pmatrix} . & . & 2 & . \\ . & 2 & . & 2 \\ 1 & . & 2 & . \\ . & 1 & . & . \end{pmatrix}$$

$$\Sigma_3 \quad \begin{pmatrix} . & . & . & 2 \\ . & . & 2 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix} \qquad \Sigma_7 \quad \begin{pmatrix} 1 & . & 2 & . \\ . & 3 & . & 2 \\ 1 & . & 3 & . \\ . & 1 & . & 1 \end{pmatrix} \qquad \Sigma_8 \quad \begin{pmatrix} . & 2 & . & 2 \\ 2 & . & 6 & . \\ . & 3 & . & 2 \\ 1 & . & 2 & . \end{pmatrix}$$

and $\Sigma_5 = \Sigma_{19}/2 = \Sigma_0$, $\Sigma_4 = \Sigma_6 = \Sigma_{11} = \Sigma_{15}/2 = \Sigma_{18}/2 = \Sigma_1$, $\Sigma_9 = \Sigma_{12} = \Sigma_{17} = \Sigma_2$, $\Sigma_{16} = \Sigma_3$, $\Sigma_{10} = \Sigma_{14}/2 = \Sigma_7$, $\Sigma_{13} = \Sigma_8$. Notice that $\Sigma_{14}, \Sigma_{15}, \Sigma_{18}, \Sigma_{19}$, as matrices over positive integers, can be divided by 2.

As another check of the correctness of the previous calculation, there exists a relation that holds between matrices F_p and matrices Σ_x. It could actually be used to determine the former from the later, although this method would be more complicated than the one we followed. This relation reads: $F_n = \sum_y (W_{\underline{0},y})_{n,0}\,\Sigma_y$ and stems from the compatibility between actions of A_{11} and Oc on the diagram F_4 and the fact that the unit $\underline{0}$ of Oc indeed acts trivially.

Sum rules

- Quadratic sum rule. Being both semi-simple and co-semi-simple, the following quadratic sum rule holds for the Ocneanu quantum groupoid associated with a simply laced diagram: $\sum_p d_p^2 = \sum_x d_x^2$, where $d_p = \sum_{a,b} (F_p)_{a,b}$ gives the dimensions of the simple blocks for the first algebra structure, and $d_x = \sum_{a,b} (\Sigma_x)_{a,b}$ gives the dimensions of the simple blocks for the second algebra structure (actually the algebra structure defined on the dual). If we take $G = E_6$ for instance, we get

$$d_p = \{6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6\}$$

$$d_x = \{6, 8, 6, 10, 14, 10, 10, 14, 10, 20, 28, 20\}$$

and we check that $\sum_p d_p^2 = \sum_x d_x^2 = 2512$.

In the case of the non simply laced diagram $G = F_4$, analogous calculations for d_p and d_x lead to the following values:

$$d_p = \{4, 7, 10, 13, 14, 14, 14, 13, 10, 7, 4\}$$

$$d_x = \{4, 7, 10, 6, 7, 4, 7, 14, 20, 10, 14, 7, 10, 20, 28, 14, 6, 10, 14, 8\}.$$

Notice that $\sum_p d_p^2 = 1256$, which is half the E_6 result; this could be expected since F_4 is a $\mathbb{Z}_2$ orbifold of E_6. However, this value is not equal to $\sum_x d_x^2$. This could also be expected if we remember that vertices $14, 15, 18, 19$ play a special role (like the "long roots" in the theory of Lie algebras): these are the values for which $W_{0,x} = 2\,W_{x,0}$ (no factor 2 for the others) and for which the Σ_x matrices can be divided by 2. For this reason, we introduce another set of matrices, setting $\widetilde{\Sigma}_x = \Sigma_x/2$ for those four values, and equality otherwise. We also introduce the corresponding dimensions $\widetilde{d}_x$, which are equal to d_x except for the four special vertices where the values are divided by 2. We find $\sum_x d_x \widetilde{d}_x = 2512$ and notice that this value is twice the value of the sum $\sum_p d_p^2$. It would be natural to introduce a quadratic form in the vector space Oc, diagonal and taking the value 1 on the basis generators O_x, except in positions $15, 16, 19, 20$ where the coefficients would be equal to 2 . The conclusion is that the usual quadratic sum rule almost works, in the sense that it is somehow twisted by the appearance of factors 2 which should be understood from the fact that basis generators O_x of quantum symmetries have two different lengths (corresponding to short and long roots in the theory of Lie algebras). These results are not compatible with the existence of a quantum groupoïd structure (in the usual sense) since the candidates for algebras of characters associated with the two multiplicative structure – that would be respectively described by the semi-simple algebras Oc (20 blocks) or the direct sum of two copies of the graph algebra A_{11}(twice 11 blocks) – do not have the same dimension. A more general type of algebraic structure seems to be needed.

- Linear sum rule. It is an observational fact (not yet understood) that the following linear sum rule also holds[10] : $\sum_p d_p = \sum_x d_x$, for most ADE cases; and when it does not, one also knows how to "correct" the rule by introducing natural prefactors. In the case of E_6, for instance, this sum equals 156. For the graph F_4 however, $\sum_p d_p = 110$ whereas $\sum_x d_x = 220$. This is also compatible with the previous discussion.
- Quantum sum rule. For ADE diagrams G with n vertices σ_i the quantum mass $m(G)$ is defined by:

$$m(G) = \sum_{a=0}^{n-1} \left(q\,\dim(\sigma_i)\right)^2$$

where the quantum dimensions $q\dim$ of the vertex σ_i is given by the i-component of the normalized Perron-Frobenius vector, associated with the highest eigenvalue (here $\beta = \frac{1+\sqrt{3}}{\sqrt{2}}$). To get these quantities for the vertices of Oc, we assign β to both chiral generators, impose that $q\dim$ is an algebra morphism and use recurrence formulae for O_x.

[10] This was noticed in [**18**]

For ADE cases, the following property can be verified (see [**19**]): if we denote the fusion algebra of the graph G by $A(G)$ (a graph algebra of type A_n) and the algebra of quantum symmetries by $Oc(G)$, one finds that $m(A(G)) = m(Oc(G))$. Moreover, for a graph with self-fusion, and if it is so that $Oc(G)$ is isomorphic, as an algebra, with $G \otimes_J G$, then $m(Oc(G)) = (m(G) \times m(G))/m(J)$. For instance in the E_6 case

$$m(E_6) = 4(3+\sqrt{3}) \quad \text{and}$$

$$m(Oc(E_6)) = \tfrac{m(E_6)\times m(E_6)}{m(J)} = 24(2+\sqrt{3}) = m(A_{11}).$$

However, for the non simply laced diagram F_4, the $q\dim$ are as follows ($x = 0, \ldots 19$)

$$\begin{aligned} q\dim\{E,e,F,f\} = \quad & \Big\{\Big(1, \tfrac{1+\sqrt{3}}{\sqrt{2}}, 1+\sqrt{3}, \sqrt{2}, \tfrac{1+\sqrt{3}}{\sqrt{2}}, 1\Big), \\ & \Big(\tfrac{1+\sqrt{3}}{\sqrt{2}}, 2+\sqrt{3}, \sqrt{2}(2+\sqrt{3}), 1+\sqrt{3}, 2+\sqrt{3}, \tfrac{1+\sqrt{3}}{\sqrt{2}}\Big), \\ & \big(1+\sqrt{3}, \sqrt{2}(2+\sqrt{3}), 2(2+\sqrt{3}), \sqrt{2}(1+\sqrt{3})\big), \\ & \big(\sqrt{2}, 1+\sqrt{3}, \sqrt{2}(1+\sqrt{3}), 2\big)\Big\} \end{aligned}$$

Like for the quadratic sum rule we introduce quantum dimensions $\widetilde{q\dim}(x)$ equal to $q\dim(x)$ except for the four vertices $14, 15, 18, 19$ where the values are divided by 2. One finds[11] :

$$\begin{aligned} m(Oc(F_4)) \quad &= \textstyle\sum_{a=0}^{n-1} \Big(q\dim(x)\widetilde{q\dim}(x)\Big)^2 \\ &= m(E) + m(e) + m(F) + m(f) = 48(2+\sqrt{3}) = 2m(A_{11}) \end{aligned}$$

with $m(E) = m(f) = 4(3+\sqrt{3})$ and $m(e) = m(F) = 4(9+5\sqrt{3})$.

- Quadratic modular double sum rule. The modular splitting relation implies the following. Call $d_p^N = \sum_{q,r}(N_p)_{q,r}$, $d_x^{W'} = \sum_{y,z}(W_{x,0})_{y,z}$ and $d_x^{W''} = \sum_{y,z}(W_{0,x})_{y,z}$, then (take traces):

$$\sum_{p,q} d_p^N d_q^N M_{p,q} = \sum_x d_x^{W'} d_x^{W''}.$$

Once the W_x are determined, one should verify that this sum rule holds. In the simply laced case E_6, for instance, one easily checks this identity, with $d_x^{W'} = d_x^{W''}$ for all x given by

$$d^W = \{20, 28, 20, 20, 28, 20, 12, 16, 12, 34, 48, 34\}$$

(sum of squares is 8328) and

$$d^N = \{11, 20, 27, 32, 35, 36, 35, 32, 27, 20, 11\}$$

[11] Using the graph algebra of F_4 defined in the following section, one finds rather $m(F_4) = m(E_6)/2$.

for A_{11} so that the M-norm square defined by the E_6 modular matrix M is also 8328. In the case of F_4 we have the same dimension vector d^N with its M-norm square (equal to 4232) now defined by the M matrix of F_4, but $d_x^{W\prime} \neq d^{W\prime\prime}$ and we have to take

$$d^{W\prime} = \{8, 12, 16, 8, 12, 8, 12, 18, 24, 12, 18, 12, 16, 24, 16, 8, 8, 12, 8, 4\}$$

together with

$$d^{W\prime\prime} = \{8, 12, 16, 8, 12, 8, 12, 18, 24, 12, 18, 12, 16, 24, 32, 16, 8, 12, 16, 8\}$$

– notice the factor 2 for entries $15, 16, 19, 20$ – so that the right hand side of this sum rule is also 4232, as it should.

5 Miscellaneous comments

Comparison with a direct method using self-fusion on F_4

Remember that the diagram F_4 emerged from our analysis of the modular splitting equation and that we started from a given partition function. Now, we would like to reverse the machine and start from the diagram F_4 itself. We imitate techniques initiated in [**2**] and developed in [**3**], [**19**].

- From the diagram, we find its adjacency matrix and call it G_1. From its eigenvalues $2cos(r\,\pi/12)$, we find the exponents $r = 1, 5, 7, 11$. In particular the highest eigenvalue is $\beta = \frac{1+\sqrt{3}}{\sqrt{2}} = 2\cos(\frac{\pi}{\kappa})$ with $\kappa = 12$ gives the value of the Coxeter number – not the dual Coxeter number, which is different (and equals 9) since the diagram is not simply laced. The quantum dimensions of the vertices are given by the normalized Perron-Frobenius vector associated with β, and we obtain the q-numbers $[1], [2], [2], [1]$, for $q = e^{\frac{i\pi}{\kappa}}$, so $q^{2\times 12} = 1$.
- The fused adjacency matrices F_p are obtained by checking that the vector space of the diagram F_4 is indeed an A_{11} module and by imposing the usual $SU(2)$ recurrence relation for the F_p's, together with the seed $F_0 = \mathbb{1}_4$ and $F_1 = G_1$. Vertices of the diagram F_4 are labelled as follows:

a) Oriented graph F_4 b) Coxeter-Dynkin F_4 diagram

- One is tempted to analyze the possibility of defining a graph algebra structure for the diagram F_4. This is indeed possible. The multiplication table given below is determined by imposing associativity, once the multiplications by 0 (unity) and 1 (adjacency matrix) have been defined.

$\cdot$	a_0	a_1	a_2	a_3
a_0	a_0	a_1	a_2	a_3
a_1	a_1	$a_0 + a_2$	$2a_1 + a_3$	a_2
a_2	a_2	$2a_1 + a_3$	$2a_2 + 2a_0$	$2a_1$
a_3	a_3	a_2	$2a_1$	$2a_0$

  The graph matrices G_a, $a = 0, 1, 2, 3$, obtained from this table, coïncide with the first four matrices Σ_x, but the reader will notice immediately that this table cannot be obtained, by restriction, from the multiplication table of Oc. Indeed, there are only two candidates (F or f). The second one is ruled out by the fact that $f \times f \subset E$. The first one is not a subalgebra either since $F \times F \subset E + F$, and even if we artificially project the result (right hand side) to F, the obtained table will differ from the one just given. The conclusion is that there is no hope to use the above graph algebra structure on F_4 to recover the modular matrix M that we used as the starting point of the whole analysis carried out in this paper. Let us nevertheless proceed.
- Potential candidates for the ambichiral vertices can be obtained by imposing that the eigenvalues of the T modular operator (they are well defined for the vertices of A_{11}) are also well defined under the induction rules (see [**4**] for details and examples). This constraint selects the set $\{a_0, a_3\}$ so that a natural guess for the corresponding algebra of quantum symmetries would be $F_4 \dot{\otimes} F_4$ where the algebra structure of each factor was described in the previous paragraph and where the tensor product is taken above the subalgebra generated by $\{a_0, a_3\}$. The new modular matrix $M^{new} = (W_{0,0})^{\text{new}}$ is then given by $E_0^{\text{red}}(E_0^{\text{red}})^T$ where the reduced essential matrix E_0^{red} is obtained from E_0 by keeping only the first and last column and replacing the two others by zeros. It is equal to

$$M^{\text{new}} = \begin{pmatrix} 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . & . & 1 \end{pmatrix}$$

  As expected, it differs from the matrix $M = W_{\underline{0}} = W_{0,0}$. However it is interesting to notice that both are related by by a conjugacy: $M^{\text{new}}/2 = S^{-1}\, M\, S$, where S stands for one of the two generators of the modular group in this representation. We find that the bilinear form obtained from W_{00} is invariant under the action of the congruence subgroup $S^{-1}\Gamma_0^{(2)}S = \text{gen}\left\{S^{-1}T^2S,\ S^{-1}(ST^{-2}S)S\right\}$ conjugated with $\Gamma_0^{(2)}$. This can be directly verified by calculating the commutators

$$\left[S^{-1}T_{11}^2S,\ W_{00}\right] = 0, \qquad \left[S^{-1}(ST^{-2}S)_{11}S,\ W_{00}\right] = 0.$$

Notice that M^{new} (which has 20 non-zero entries) is equal to the sum of the three matrices $W_{\underline{0}}$, $W_{\underline{4}}$ and $W_{\underline{33'}}$ associated with three self-dual points of the graph $Oc(F_4)$.

So, we can indeed define self-fusion on the diagram F_4 but this associative algebra structure does not seem to be simply related with the so-called F_4 modular matrix. Still another possibility would be to work with a symmetrized form of the F_4 diagram, i.e., with an "adjacency matrix" that incorporates non-integer matrix elements (q-numbers equal to $\sqrt{2}$). This possibility is actually quite interesting but will not be discussed here. It does not seem to allow one to recover the F_4 modular matrix M, either.

A relative equation of modular splitting

From the fact that the diagram F_4 is, geometrically, a Z_2 orbifold of E_6 (identify pair of vertices $(0,4)$, $(1,5)$ of the later), we are tempted to consider an action of the graph algebra of E_6 (this graph has self-fusion) on the vector space of F_4. Call $G_u^{E_6}$ the six 6×6 generators of the E_6 graph algebra and $F_u^{E_6}$ the six 4×4 matrices implementing the action of E_6 on F_4. The multiplication table of E_6 was given before. Its graph matrices obey the usual relations:

$$G_0^{E_6} = \mathbb{1}_6 \qquad G_1^{E_6}$$

$$G_2^{E_6} = (G_1^{E_6})^2 - G_0^{E_6} \qquad G_3^{E_6} = -G_1^{E_6}\cdot(G_4^{E_6} - (G_1^{E_6})^2 + 2G_0^{E_6})G_1^{E_6}$$

$$G_4^{E_6} = (G_1^{E_6})^4 - 4(G_1^{E_6})^2\cdot G_1^{E_6} + 2G_0^{E_6} \qquad G_5^{E_6} = G_1^{E_6}G_4^{E_6}.$$

To obtain the fused matrices $F_u^{E_6}$ relative to this action, we set $F_0^{E_6} = \mathbb{1}_4$, $F_1^{E_6} = G_1$ (the adjacency matrix of F_4) and impose that the $F_u^{E_6}$ should obey the same algebra relations as the $G_u^{E_6}$.

Exactly as we had an action of A_{11} on F_4, implemented by matrices F_p, we have a (relative) action of E_6 on F_4, implemented by matrices $F_u^{E_6}$. For this reason we are led to consider a "relative" theory of modular splitting (and a corresponding equation) with A_{11} replaced by E_6. In particular the graph matrices N_p of A_{11} – the usual fusion matrices – are replaced by the generators $G_u^{E_6}$ of the graph algebra of E_6. With $u, v \in E_6$. we define "relative" toric matrices by

$$u\, x\, v = \sum_y (W_{x,y}^{E_6})_{u,v}\, y.$$

The relative equation of modular splitting reads (τ is a tensorial flip):

$$\sum_{u,v} G_u^{E_6} \otimes G_v^{E_6}\, M_{u,v}^{rel} = \sum_{x \in Oc} \tau \circ (W_{x,0}^{E_6} \otimes W_{0,x}^{E_6}).$$

M^{rel} describes the same F_4 partition function as before, but in terms of generalized characters[12] :

$$\chi_0 + \chi_6, \quad \chi_1 + \chi_5 + \chi_7, \quad \chi_2 + \chi_4 + \chi_6 + \chi_8,$$
$$\chi_3 + \chi_5 + \chi_9, \quad \chi_4 + \chi_{10}, \quad \chi_3 + \chi_7.$$

If P denotes the matrix of this linear transformation (it is the first essential matrix, i.e., the "intertwiner" of the E_6 theory), we have $M = PM^{\mathrm{rel}}P^T$. The E_6 invariant, in terms of these generalized characters, with the above ordering, is diagonal and reads $\mathrm{diag}\,(1,0,0,0,1,1)$ whereas the F_4 modular matrix is $M^{\mathrm{rel}} = \mathrm{diag}\,(1,0,0,0,1,0)$. The equation of modular splitting is then solved exactly as we did in a previous section, with the technical advantage that the size of the matrices that we have to manipulate is much smaller (36×36 rather than 121×121). Same comment for most objects of the theory: the analogue of K_0 is 36×20 (rather than 121×20) and the relative toric matrices are 6×6 (rather than 11×11). The twenty generators O_x are the same (their size is 20×20) and the graph of quantum symmetries is determined as before. It is easy to translate the relative E_6 theory to the "usual" one (that uses the action of the A_{11} graph algebra) by using the – rectangular – essential matrix P. It is technically easier to work with this relative theory, but the drawback is that the E_6 case should be analyzed first. This is why we did not follow this method in our presentation.

Acknowledgments

This work was certainly influenced by conversations with A. Ocneanu, O. Ogievetsky and G. Schieber. We want to thank them here.

[12]In our formalism, they are obtained from essential matrices of the E_6 diagram as described, for instance, in [**5**].

References

[1] A.Cappelli, C.Itzykson and J.B. Zuber, *The ADE classification of minimal and $A_1^{(1)}$ conformal invariant theories*, Comm. Math. Phys. **13** (1987), 1.

[2] R. Coquereaux, *Notes on the quantum tetrahedron*, Moscow Math. J. **2**(**1**) (Jan.-March 2002), 1–40, hep-th/0011006.

[3] R. Coquereaux and G. Schieber, *Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries*, J. of Geom. and Phys. **781** (2002), 1–43, hep-th/0107001.

[4] R. Coquereaux and G. Schieber, *Determination of quantum symmetries for higher ADE systems from the modular T matrix*, J. Math. Phys. **44** (2003), 3809–3837, hep-th/0203242.

[5] R. Coquereaux and M. Huerta, *Torus structure on graphs and twisted partition functions for minimal and affine models*, J. of Geom. and Phys. **48**(**4**) (2003), 580–634, hep-th/0301215.

[6] R. Coquereaux and R. Trinchero, *On Quantum symmetries of ADE graphs*, Advances in Theor. and Math. Phys. **8**(**1**) (2004), hep-th/0401140.

[7] B. Dubrovin, *Differential geometry of the space of orbits of a Coxeter group*, hep-th/9303152, SISSA-29/93/FM.

[8] P. Di Francesco and J.-B. Zuber, in 'Recent Developments in Conformal Field Theories', Trieste Conference 1989 (S. Randjbar-Daemi, E Sezgin and J.-B. Zuber eds.), World Scientific 1990;
P. Di Francesco, Int. J. Mod. Phys. **A7** (1992), 407–500.

[9] F. Di Francesco and J.-B. Zuber, *$SU(N)$ Lattice integrable models associated with graphs*, Nucl. Phys **B338** (1990), 602–646.

[10] T. Gannon, *The Classification of affine $su(3)$ modular invariants*, Comm. Math. Phys. **161** (1994), 233–263.

[11] A. Hurwitz, *Über endliche Gruppen, welche in der Theory der elliptischen Transzendenten außtraten*, Math. Annalen **27** (1886), 183–233.

[12] A. Ocneanu, *Paths on Coxeter diagrams: from Platonic solids and singularities to minimal models and subfactors*, notes taken by S. Goto, Fields Institute Monographs (Rajarama Bhat et al eds, AMS, 1999); Same title: talks given at the Centre de Physique Théorique, Luminy, Marseille, 1995.

[13] A. Ocneanu, *Higher Coxeter systems*, Talk given at MSRI, http://www.msri.org/publications/ln/msri/2000/subfactors/ocneanu.

[14] A. Ocneanu, *The Classification of subgroups of quantum $SU(N)$*, Lectures at Bariloche Summer School, Argentina, Jan.2000, AMS Contemporary Mathematics, **294** (R. Coquereaux, A. García and R. Trinchero, eds.).

[15] C.H. Otto Chui, C. Mercat, W. Orrick and P.A. Pearce, *Integrable Lattice Realizations of Conformal Twisted Boundary Conditions*, Phys.Lett. **B517** (2001), 429–435, hep-th/0106182.

[16] V. Pasquier, *Operator contents of the ADE lattice models*, J. Phys. **A20** (1987), 5707.

[17] V. Pasquier, *Two-dimensional critical systems labeled by Dynkin diagrams*, Nucl.Phys. **B285** (1987), 162.

[18] V.B. Petkova and J.-B. Zuber, *The many faces of Ocneanu cells*, Nucl. Phys. **B603** (2001), 449, hep-th/0101151.

[19] G. Schieber, *L'algèbre des symétries quantiques d'Ocneanu et la classification des systèmes conformes à 2D*, Ph.D. thesis (available in French and in Portuguese), UP (Marseille) and UFRJ (Rio de Janeiro), Sept. 2003.

[20] G. Schieber, *Bimodule structure and quantum symmetries: an exceptional affine $su(4)$ example*, in preparation.

[21] E. Verlinde, *Fusion rules and modular transformations in $2-D$ Conformal Field Theory*, Nucl. Phys. **B300** (1988), 360–376.

[22] J.-B. Zuber, *On Dubrovin topological field theories*, Mod. Phys. Lett. **A9** (1994), 749–760, hep-th/9312209.