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Closed String Tachyons, Non-Supersymmetric Orbifolds and Generalised McKay Correspondence

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Abstract

We study closed string tachyon condensation on general non-supersymmetric orbifolds of $\mathbb{C}^2$. Extending previous analyses on Abelian cases, we present the classification of quotients by discrete finite subgroups of $GL(2;\mathbb{C})$ as well as the generalised Hirzebruch-Jung continued fractions associated with the resolution data. Furthermore, we discuss the intimate connexions with certain generalised versions of the McKay Correspondence.

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1 Introduction

Understanding dynamic processes in string theory has recently played a pivotal rôle in the field. Ever since Sen's pioneering work on non-supersymmetric configurations of D-branes and their evolution via tachyon condensation (cf. e.g. [12]), a host of activities ensued. These tachyonic instabilities, arising from various scenarios in which supersymmetry is absent, shed light on diverse topics ranging from the K-theory charges lattices to time evolution in cosmology.

Sen's tachyon condensation, which sparked interests such as the K-theory analysis of D-branes and revival of cubic string field theory, focused on the open string sector. There, boundary conformal field theory techniques can be applied to track the evolution of the tachyon toward supersymmetric configurations. The situation in the closed string sector is less tractable. We seem to require a full off-shell formulation which is thus far not well understood. Indeed it is believed that the general tachyon condensation process changes the very structure of spacetime.

An initial step was taken by Adams, Polchinski and Silverstein (APS) [1] where, in analogy to the provision of defects by D-branes in the open sector, closed string tachyon condensation was considered on singularities in spacetime. By localising the tachyon on so-called non-supersymmetric orbifolds, more familiar techniques could be used. In the substringy regime, when the tachyon vacuum expectation value is small, the D-brane probe technology of [23] may be applied.

With this technology we are familiar: we let the D-brane sit not on flat space, but rather let the transverse dimensions to the brane be an orbifold singularity. Indeed when the orbifold is Gorenstein (i.e., we orbifold by a discrete finite subgroup of the special unitary group), we have a local Calabi-Yau singularity and some supersymmetry is preserved. If however we consider subgroups of the general linear group as was done in [1], supersymmetry is generically broken and there are tachyons in the tree-level spectrum. Therefore in its most prototypical form and in the notation of [5] we study the propagation of superstrings on manifolds that could be locally modeled as singular algebraic varieties of the orbifold type (cf. [31] for some review on this subject). In other words, the background geometry of concern is

$$\mathbb{R}^{d-1,1} \times \mathbb{R}^{10-d}/\Gamma \tag{1}$$

where Γ is an orbifold group, embedded in some Lie isometry group of (subspaces of) the $\mathbb{R}^{10-d}$ factor. The algebraic structure of Γ determines the physics of the transverse $\mathbb{R}^{d-1,1}$, the low-energy dynamics of which are of vital phenomenological importance [23, 25, 24, 19, 27, 20, 21, 30].

A beautiful insight of [1] is that, as is with the open string sector, the condensation of, i.e., acquisition of VEV's by, the tachyon leads to a systematic decay of the orbifold that geometrically corresponds to the partial resolution thereof. The instability is finally resolved when the decay ends in a supersymmetric configuration, viz., when the orbifold finally becomes Calabi-Yau.

Much work followed. These included renormalisation group (RG) analysis of the two-dimensional worldsheet field theory [4, 5, 15]. Indeed if we considered the space of closed string field theory to be the space of two-dimensional field theories then the RG flow in such spaces will govern the dynamics of strings. Therefore we can investigate the relevant two-dimensional field theories and deformations thereof corresponding to operators which induce tachyon condensation; evolution in physical time can then be identified with the RG flow on the worldsheet. In other words, tachyon condensation corresponds to the addition of a relevant operator to the worldsheet Lagrangian which describes the background perturbative string propagation. The end process is the IR fixed point of the worldsheet RG flow.

Linear sigma model analyses and mirror symmetry can be applied (to the Abelian cases) to track the flow, leading to non-trivial non-supersymmetric dualities [4]. Also, a certain g_{cl} conjecture was proposed in [5], in identifying the analogue of the open string boundary entropy which decreases along the flow. Lifts to M-theory [9, 8] and to F-theory [2] were also considered as well as addition of flux branes and Wilson lines [6]. Tests of the g_{cl} conjecture on AdS orbifolds [10], study of bulk condensation [13] as well as relations to non-commutative field theory on $\mathbb{C}^3$-orbifolds [16] were performed. On a more phenomenological note, type II and heterotic model-building [7] and chiral phase transitions [14] in this context were also addressed.

In [5, 15] careful analysis was performed on the chiral ring in the worldsheet conformal field theory and the tachyon condensation process was geometrised to certain Hirzebruch-Jung resolutions of the orbifold $\mathbb{C}^n/\Gamma \subset GL(n;\mathbb{C})$ for Abelian Γ.

Indeed work thus far has been focusing exclusively on Abelian groups where methods from toric geometry are happily applicable. A systematic study using the Inverse Toric Algorithm of [18] was performed by [11] in this context. An obvious direction beckons us: what about general groups? It is the purpose of this writing to investigate arbitrary quotients of $\mathbb{C}^2$ which do not preserve supersymmetry. We will see that there is a generalised Hirzebruch-Jung (minimal) resolution for orbifolds by discrete finite subgroups of $GL(2;\mathbb{C})$ whose classification we present. We shall also see how extra subtleties arise for non-Abelian quotients and how they are intimately related to a generalised version of the McKay Correspondence due to Ishii et al. and a conjectured correspondence of [19, 27].

The paper is organised as follows. In Section 2 we define our problem and briefly review how closed string tachyon condensation is related to non-supersymmetric orbifolds and how to geometrically interpret the decay of spacetime. In Section 3 we recast Brieskorn's classification of the quotients of $\mathbb{C}^2$ into a form readily accessible to computation and present a first non-Abelian example in the present context. Section 4 then discusses how to view these issues through generalised McKay Correspondences. Finally we conclude in Section 5.

Nomenclature

Unless otherwise stated, we shall adhere to the following notations throughout: $\Gamma = \langle a_i | f_j(a_i) \rangle$ is a discrete finite group of order $|\Gamma|$, number of conjugacy classes $\#\text{conj}(\Gamma)$ and generated by elements a_i subject to relations $f_j(a_i)$; The set of irreducible representations of Γ is denoted $\text{Irrep}(\Gamma)$, the non-trivial ones, $\text{Irrep}^0(\Gamma)$; The centre of a group G is denoted $Z(G)$, the derived subgroup of G is written as G', and $H \triangleleft G$ means that H is a normal subgroup of G; The primitive n-th root of unity is denoted as $\omega_n := e^{\frac{2\pi i}{n}}$; Finally (a, b) means the great common divisor between integers a and b.

2 Closed String Tachyon Condensation

We shall throughout this writing focus on a subclass of (1), viz., dimension two orbifolds of the form

$$\mathbb{C}^2/(\Gamma \subset GL(2;\mathbb{C})) \ . \tag{2}$$

The localisation by APS of the tachyon to such an orbifold does not affect the stability of the bulk; however the local structure of spacetime singularity does change as the tachyon condenses. APS showed that, as is with Sen's open string case, the denouement of such a decay process is actually the restoration of supersymmetry.

There are two regimes to this description. When the tachyon VEV is small, we can study the decay at the *sub-stringy* scale where one could apply the usual D-brane probe techniques of [23]. On the other hand, when the VEV becomes large, and α' corrections become important, we are in the *gravity* regime and a full worldsheet RG technology needs to be applied.

We will focus on the brane probe regime. As advertised above, our chief concern will be two-dimensional quotients. In other words, we take $d = 6$ in (1) and the orbifold directions transverse to the D-brane world-volume to be $x^{6,7,8,9}$. We complexify as $z_1 = x^6 + ix^7$ and $z_2 = x^8 + ix^9$ which constitute the coördinates of $\mathbb{C}^2$. In terms of our coördinates, there will be a general twist acting on $\mathbb{C}^2$ as $R = \exp(\frac{2\pi i}{p}(J_{67} + pJ_{89}))$ with J_{67} and J_{89} being rotations in the z_1 and z_2 planes respectively.

Now we need to consider the general case of (2), and the canonical example is the generalisation of type A, the cyclic subgroups. Such a quotient, in the notation of [1, 5], is $\mathbb{Z}_{n(p)}$ (also cf. [17]) defined as the cyclic group $\mathbb{Z}_n$, but with the following matrix action on (z_1, z_2)

$$\begin{array}{rcl} \mathbb{Z}_{n(p)} & = & \langle \begin{pmatrix} \omega_n & 0 \\ 0 & \omega_n^p \end{pmatrix} \rangle \qquad \omega_n := e^{\frac{2\pi i}{n}}; \\ n & \in & \mathbb{Z}^+, p \in [-(n-1), \ldots, 0, 1, 2, \ldots, n-1], (n,p) = 1 \ . \end{array} \tag{3}$$

Of course $\mathbb{Z}_{n(-1)}$ is none other than our familiar A-type Kleinian singularity (ALE space) since in this case the group action embeds into $SL(2;\mathbb{C})$ (see Section 3.2). Because of its Abelian nature, the quotient (3) is an affine toric variety [17] (the reader is encouraged to consult [11] for a nice treatment of the toric resolution of such singularities). The toric diagram is given in Figure 1; it consists of a single two-dimensional cone spanned by $v_0 = (0, 1)$ and $v_n = (n, -p)$.

Resolution of toric singularities proceeds by stellar division of the cone. In other words we insert all vectors between v_0 and v_n which are lattice vectors as shown (in blue) in Figure 1; the resolved space corresponds to a fan consisting of cones each of which is spanned by adjacent vectors v_j and v_{j+1}. Furthermore there is a relation

$$a_j v_j = v_{j-1} + v_{j+1} \qquad j \in [1, 2, \ldots n-1] \tag{4}$$

for non-negative integers a_j. Each interior vector v_j then corresponds to an exceptional $\mathbb{P}^1$ divisor E_j, such that the intersection number are $E_j \cdot E_{j+1} = 1$ and $E_j \cdot E_j = -a_j$. These integers a_j are obtained from the so-called **Hirzebruch-Jung** continued fraction for $\frac{n}{k}$ where $k \in [0, 1, \ldots, n-1]$ and $k \equiv p(\mod n)$:

$$\frac{n}{k} := a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots}} := [[a_1, a_2, \ldots, a_r]] \ . \tag{5}$$

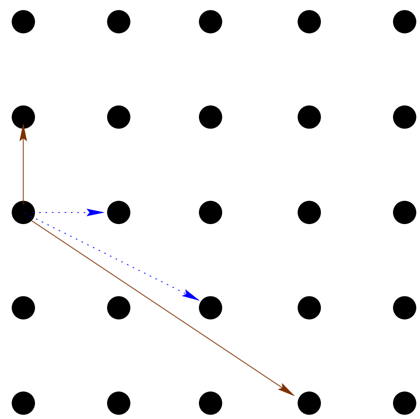

Figure 1: The toric diagram for $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ consists of single 2D cone spanned by $v_0 = (0, 1)$ and $v_n = (n, -p)$. Minimal resolution proceeds by inserting lattice vectors v_j in the interior of the cone.

Indeed as we are expanding a rational, the continued fraction terminates after finite steps, signifying the finite number of exceptional divisors. We point out that this resolution scheme is *minimal* in the sense that r, the number of exceptional divisors is smallest amongst all possible resolutions. In this case all the integers $a_j \geq 2$.

Physically, the exceptional divisors are in one-one correspondence with the generators of the chiral fusion ring of the $\mathcal{N} = 2$ worldsheet conformal field theory [29, 5]. The chiral ring structure is captured by the representation ring of the orbifold group (and hence the quivers which we are about to describe in Section 3), which in turn is encoded in the intersection numbers of the exceptional divisors [27, 33], i.e., with the interior vectors v_j.

Now the matter content of the orbifold theory is simply the McKay quiver associated to Γ. We need to be careful that since we here are no longer protected by supersymmetry we need to construct (generically quite) different quivers for the fermions and the bosons (cf. [21]). A quintessential idea of [1] is that turning on marginal or tachyonic deformations in the twisted sectors of the orbifold theory induces partial resolutions of the initial non-SUSY singularity. The process can be applied consecutively, each stage being a $\mathbb{P}^1$-blowup. In other words, the *stepwise tachyonic condensation process corresponds precisely, in the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ example, to the Hirzebruch-Jung resolution outlined in (5).*

As we proceed in this *decay of spacetime* where the very topology of the singular spacetime changes, the symmetry of the theory subsequently changes. At each stage of the decay we should be careful to preserve the quantum symmetry of the theory while turning on VEV's of tachyons. Turning on VEV's to break symmetry is of course nothing but the Higgsing procedure. Indeed, in the case of the supersymmetric quiver theories, especially the toric ones, such Higgsing procedures can be systematically studied via the tuning of Fayet-Illiopoulos parametres and so-called Inverse Algorithm [18, 11].

The $\mathcal{N} = 2$ worldsheet CFT provides the link to study the decay process from a geometric point of view. We can parametrise the $\mathbb{C}^2$ by the chiral superfields corresponding to the coördinates. Up to normalisation, the fusion rules for the chiral ring is dictated precisely by the Hirzebruch-Jung continued fraction, i.e., by the self-intersection numbers of the exceptional divisors. Having translated the problem from chiral rings to geometric resolutions, the decay process becomes rather visual. In general $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ can decay to an orbifold of lower rank,

$$\mathbb{C}^2/\mathbb{Z}_{n(p)} \to \mathbb{C}^2/\mathbb{Z}_{n'(p')} \oplus \mathbb{C}^2/\mathbb{Z}_{n-n'\ (q)} \tag{6}$$

(in particular it can decay to one of its subgroups in which case n' divides n). If the end product is such that $p' = -1$ we have then reached our familiar A_n singularity and hence a supersymmetric orbifold. This is the crucial outcome: supersymmetry restoration via tachyon condensation in (localised) closed string sector. We remark that the case of $p = +1$ is also, though not immediately obvious, supersymmetric because $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ and $\mathbb{C}^2/\mathbb{Z}_{n(-p)}$ are isomorphic by conjugation of complex structure: $z_2 \leftrightarrow z_2^*$. The resolution data corresponding to (6) is of course captured by the respective continued fraction expansions, the general pattern is

$$\frac{n}{p} = [[a_1, a_2, \ldots, a_r]] \to [[a_1, a_2, \ldots, a_{n'-1}]] \oplus [[a_{n'+1}, \ldots a_r]] \ . \tag{7}$$

3 The Classification of the Discrete Finite Subgroups of $GL(2;\mathbb{C})$

Having refreshed the readers' minds and motivated their hearts on the interesting physics of tachyon condensation on the non-supersymmetric orbifold $\mathbb{C}^2/\mathbb{Z}_{n(p)}$, one task is immediate. Can we perform a similar analysis to all the non-SUSY orbifolds of $\mathbb{C}^2$? This seems to require a classification of the discrete finite subgroups of $GL(2;\mathbb{C})$. At first glance this may appear to be a rather intractable problem because we seem to have the liberty to quotient $\mathbb{C}^2$ by any finite group which affords a two-dimensional irreducible representation, the list of which is certainly overwhelming. In what follows however we shall see that it in fact suffices to consider what are known as **small groups** and our candidate pool reduces considerably. Inspired by the extensive (and still ongoing) programme of the study of resolutions of quotient singularities, the classification of $\mathbb{C}^2$ orbifolds took form in [39]. To a comprehensive presentation in [40, 41] and especially in [41] is the reader referred. Before we proceed however, some preliminary technicalities should be addressed [41]. To these we now briefly turn.

3.1 Small Groups and Quotient Singularities

As mentioned above, we can restrict orbifolds of $\mathbb{C}^n$ to a very small subclass. In general we call a group $\Gamma \subset GL(n;\mathbb{C})$ acting on $\mathbb{C}^n$ a **reflection group** if it is generated by elements g which fix a hyperplane in $\mathbb{C}^n$, i.e., the eigenvalues of g are 1 of multiplicity $n-1$ together with $\omega_{k\geq 2}$, some root of unity. The complement thereof, i.e., a group which does not contain any reflections is called a **small group**.

The following Theorems of Prill [42] and [43] therefore narrow down our search considerably and make the small groups the building blocks of our orbifolds.

THEOREM 3.1. *1. Every quotient singularity is isomorphic to a quotient by a small group $G \subset GL(n;\mathbb{C})$.*

2. If G_1 and G_2 are small groups in $GL(n;\mathbb{C})$ and $\mathbb{C}^n/G_1 \simeq \mathbb{C}^n/G_2$, then G_1 and G_2 are conjugate in $GL(n;\mathbb{C})$.

Therefore the classification of conjugacy classes of small groups of $GL(n;\mathbb{C})$ suffices the classification of all complex quotient singularities.

3.2 Discrete Finite Subgroups of $SL(2;\mathbb{C})$

Before moving on to the small groups, we first set the notation by briefly reminding the reader that the discrete finite subgroups of $SL(2;\mathbb{C})$ give rise to the so-called Kleinian surface singularities (cf. e.g. [19, 31] for some implications in string theory). These groups can be brought, by conjugation, to the subgroups of $SU(2)$ and fall under 3 types, namely A, D and E. Type A is an (reducible) infinite family which are abelian, in fact $A_n := \mathbb{Z}_{n+1}$, the cyclic group on $n+1$ elements. Thus

$$A_{n-1} := \langle \zeta_n := \begin{pmatrix} \omega_n & 0 \\ 0 & \omega_n^{-1} \end{pmatrix} \rangle \qquad |A_{n-1}| = n \ . \tag{8}$$

Type D is another (reducible imprimitive) infinite family, the so-called binary dihedral group, defined as

$$D_n := \langle \zeta_{2n}, \gamma := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle, \qquad |D_n| = 4n \ . \tag{9}$$

Finally type E, the irreducible primitives, consists of 3 exceptional members $E_{6,7,8}$, respectively the binary tetrahedral, octahedral and icosahedral groups, generated as

$$\begin{array}{rclr} E_6 & := & \langle S, T \rangle, & |E_6| = 24; \\ E_7 & := & \langle S, U \rangle, & |E_7| = 48; \\ E_8 & := & \langle S, T, V \rangle, & |E_8| = 120. \end{array} \tag{10}$$

where

$$\begin{array}{ll} S := \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}, & T := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ U := \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, & V := \begin{pmatrix} \frac{i}{2} & \frac{(1-\sqrt{5})-i(1+\sqrt{5})}{4} \\ \frac{-(1-\sqrt{5})-i(1+\sqrt{5})}{4} & -\frac{i}{2} \end{pmatrix}. \end{array} \tag{11}$$

3.3 Surface Quotient Singularities and Discrete Finite Subgroups of $GL(2;\mathbb{C})$

Now let us return to the $GL(2;\mathbb{C})$ case, the members of which are generated from the above. Subsection 3.1 has shewn us that all relevant quotients of the type (2) can be obtained from the *small subgroups* of $GL(2;\mathbb{C})$. We here recast the classification of [39] (cf. also [41]) into a notation convenient for physical applications. The interested reader may consult the Appendix on an outline of how these groups arise. The small discrete finite subgroups $\Gamma \subset GL(2;\mathbb{C})$ and hence all orbifolds of $\mathbb{C}^2$ fall under 5 types.

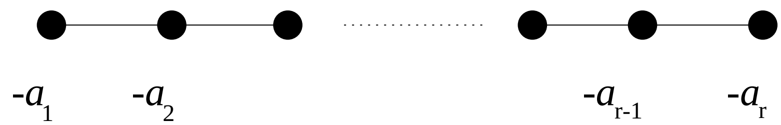

Figure 2: The (minimal) resolution diagram for the type $A_{n,p}$ quotient, i.e., $\mathbb{C}^2/\mathbb{Z}_{n(p)}$. Each node corresponds to an exceptional $\mathbb{P}^1$-divisor, each line corresponds to a single intersection between two $\mathbb{P}^1$'s. Each divisor is of self-intersection number $-a_j$ which is obtained from the continued fraction expansion $\frac{n}{p} = [[a_1, a_2, \ldots, a_r]]$.

3.3.1 Type $A_{n,p}$

This is $\mathbb{Z}_{n(p)}$, the generalisation of the A_n singularity presented in Section 2 and studied in [1, 5]. The minimal resolution thereof is dictated by (5). We can encode the resolution into a McKay-like quiver [35] where each node corresponds to an exceptional divisor. The adjacency matrix is given by the intersection of distinct exceptional divisors. Thus, the resolution (5) is drawn as in Figure 2. In the figure, adjacent divisors intersect once while each divisor is of self-intersection number $-a_j$, obtained from the continued fraction expansion of $\frac{n}{p}$.

3.3.2 Type $A_m D_n$

These are composed of the A and D groups; defining

$$\tilde{\zeta}_q := \left(\begin{smallmatrix} \omega_q & 0 \\ 0 & \omega_q \end{smallmatrix} \right) \subset Z(GL(2;\mathbb{C})) \ , \tag{12}$$

and using the notation of (8), (9) and (11), there are two subtypes:

$$A_m D_n^{(I)} := \langle \tilde{\zeta}_{2m}, \zeta_{2n}, \gamma \rangle \tag{13}$$

such that $m = (b-1)n - q$ is odd, as well as

$$A_m D_n^{(II)} := \langle \tilde{\zeta}_{4m}, \zeta_{2n}, \gamma \rangle \tag{14}$$

such that $m = (b-1)n - q$ is even.

The parametres b and q determine the minimal resolution in the spirit of (5). Let us digress a moment to settle notation. The adjacency matrix and self-intersection numbers of the exceptional $\mathbb{P}^1$ divisors in the minimal resolution of the general quotient of $\mathbb{C}^2$ can be associated with the septuple

$$\mathcal{R} := (b; \quad 2, 1; \quad n_2, q_2; \quad n_3, q_3) \tag{15}$$

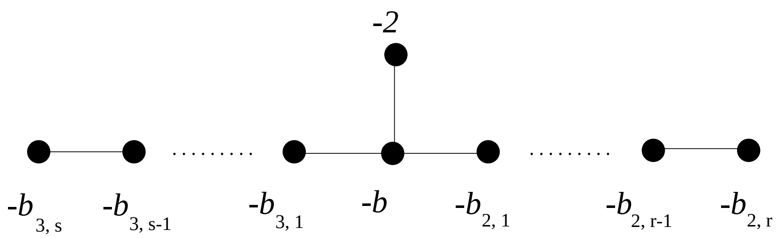

Figure 3: The (minimal) resolution diagram for the general $\mathbb{C}^2/\Gamma \subset GL(2;\mathbb{C})$ quotient, to each of which is associated a determining septuple data $\mathcal{R} := (b;\quad 2,1;\quad n_2,q_2;\quad n_3,q_3)$. Every node corresponds to an exceptional $\mathbb{P}^1$-divisor and every line corresponds to a single intersection between two $\mathbb{P}^1$'s. Each divisor is of self-intersection number $-b$, -2, $-b_{2,j=1,\ldots r}$ or $-b_{3,j=1,\ldots s}$, the last two of which are obtained from the continued fraction of $\frac{n_i}{q_i} := [[b_{i,1}, b_{i,2}, \ldots b_{i,r(s)}]]$.

or equivalently, with the McKay-type quiver drawn in Figure 3. Again, the nodes correspond to the exceptional $\mathbb{P}^1$-divisors and each line signifies the one-time intersection between two divisors. The self-intersection numbers of each $\mathbb{P}^1$ are also marked.

As in complete analogy with (5) as appeared in [5], we have continued fractions

$$\frac{n_i}{q_i} := b_{i,1} - \frac{1}{b_{i,2} - \frac{1}{b_{i,3} - \ldots}} \ , \qquad i = 2,3 \ , \tag{16}$$

such that the intersection of different blowups are given by the graph in Figure 3 and the self-intersection numbers are $-b_{i,j}$ such that when $i = 2$, j indexes up to say, r and when $i = 3$, j goes up to s. Therefore, in all the minimal resolution requires $2 + r + s$ exceptional divisors.

The curious reader may note that $(2, n_2, n_3)$ always forms a *Platonic triple*, i.e., $\frac{1}{2} + \frac{1}{n_2} + \frac{1}{n_3} > 1$ and $2 \le n_2 \le n_3$. Moreover, these $A_m D_n$ groups are two dimensional analogues of what was called ZD-groups in the brane box constructions of [22].

Bearing this notation in mind, $A_m D_n^{(I)}$ has the resolution data $(b;\quad 2,1;\quad 2,1;\quad n,q)$ and so too does $A_m D_n^{(II)}$. Henceforth, we shall present this resolution data $\mathcal{R}$ in addition to the group structure. Therefore by subtype of groups we are being a little more refined than merely referring to the group, but also to its resolution information.

3.3.3 Type $A_m E_6$

There are three subtypes of this category, composed of the cyclic with the binary tetrahedral:

$$\begin{array}{rclllll} A_m E_6^{(I)} & := & \langle \tilde{\zeta}_{2m}, S, T \rangle & \mathcal{R} = (b; & 2,1; & 3,2; & 3,2), & m = 6(b-2)+1; \\ A_m E_6^{(II)} & := & \langle \tilde{\zeta}_{2m}, S, T \rangle & \mathcal{R} = (b; & 2,1; & 3,1; & 3,1), & m = 6(b-2)+5; \\ A_m E_6^{(III)} & := & \langle \tilde{\zeta}_{6m}, S, T \rangle & \mathcal{R} = (b; & 2,1; & 3,1; & 3,2), & m = 6(b-2)+3 \ . \end{array} \quad (17)$$

3.3.4 Type $A_m E_7$

Next we compose with the octahedral group. There are 4 subtypes. These all have the same structure

$$A_m E_7 := \langle \tilde{\zeta}_{2m}, S, U \rangle, \quad (18)$$

but for different values of m, the resolution data differ:

$$\begin{array}{rcllll} \mathcal{R}^{(I)} & = & (b; \quad 2,1; & 3,2; & 4,3), & m = 12(b-2)+1 \\ \mathcal{R}^{(II)} & = & (b; \quad 2,1; & 3,1; & 4,3), & m = 12(b-2)+5 \\ \mathcal{R}^{(III)} & = & (b; \quad 2,1; & 3,2; & 4,1), & m = 12(b-2)+7 \\ \mathcal{R}^{(IV)} & = & (b; \quad 2,1; & 3,1; & 4,1), & m = 12(b-2)+11 \ . \end{array}$$

3.3.5 Type $A_m E_8$

Finally we compose with the icosahedral group. There are 8 subtypes: the group is:

$$A_m E_8 := \langle \tilde{\zeta}_{2m}, S, T, V \rangle; \quad (19)$$

again for different values of m, the subtypes have distinct resolution data.

$$\begin{array}{rcllll} \mathcal{R}^{(I)} & = & (b; \quad 2,1; & 3,2; & 5,4), & m = 30(b-2)+1 \\ \mathcal{R}^{(II)} & = & (b; \quad 2,1; & 3,2; & 5,3), & m = 30(b-2)+7 \\ \mathcal{R}^{(III)} & = & (b; \quad 2,1; & 3,1; & 5,4), & m = 30(b-2)+11 \\ \mathcal{R}^{(IV)} & = & (b; \quad 2,1; & 3,2; & 5,2), & m = 30(b-2)+13 \\ \mathcal{R}^{(V)} & = & (b; \quad 2,1; & 3,1; & 5,3), & m = 30(b-2)+17 \\ \mathcal{R}^{(VI)} & = & (b; \quad 2,1; & 3,2; & 5,1), & m = 30(b-2)+19 \\ \mathcal{R}^{(VII)} & = & (b; \quad 2,1; & 3,1; & 5,2), & m = 30(b-2)+23 \\ \mathcal{R}^{(VIII)} & = & (b; \quad 2,1; & 3,1; & 5,1), & m = 30(b-2)+29 \ . \end{array}$$

We thus conclude the presentation of the quotient singularities of $\mathbb{C}^2$, i.e., conjugacy classes of the discrete finite small subgroups of $GL(2;\mathbb{C})$, together with their quivers for the minimal resolution in the sense of Figure 3, each of which has a generalised Hirzebruch-Jung continued fraction associated therewith. We see that all these groups are very simple in that they afford the direct product structure or a simple quotient thereof. Indeed for types $A_n E_{7,8}$ for example, these are direct products whenever n is odd. The remarkable fact is that every orbifold of $\mathbb{C}^2$ is isomorphic to a member of this simple list above.

3.4 A non-Abelian Example

Armed with this list, let us discuss tachyon condensation associated therewith. Thus far all the examples addressed in the literature have been focusing on Abelian non-SUSY orbifolds to which toric resolutions in the manner of (5) have been applied. From the classification above, we see that the continued-fraction scheme of resolutions is not limited to the Abelian case of type $A_{n,p}$ but persists, via a resolution graph, to all subgroups.

The mathematics of the situation is therefore clear. Let us take the example of the group $G = A_5 E_6^{(II)}$. This is the group

$$G := \langle \tilde{\zeta}_{10}, S, T \rangle \ , \tag{20}$$

which is none other than $E_6 \times \mathbb{Z}_5$ (where we have chosen $m = 5$ and hence $b = 2$). The resolution data is:

$$\mathcal{R} = (2; \quad 2,1; \quad 3,1; \quad 3,1) \ . \tag{21}$$

The resolution graph is drawn in part (a) of Figure 4, we see that the minimal resolution has only 4 exceptional divisors, of self intersections $-2, -2, -3, -3$ (the continued fractions in this case are rather trivial).

As a contrast we present in part (b) of Figure 4, the true McKay quiver for G. This quiver is of course the standard one of [23, 24] (cf. [31] or section 2 of [19] for a review) and dictates the matter content coming form the projection of the parent theory on the orbifold. In particular we choose the spacetime orbifold action on the fermions (resp. bosons) as embedded in the $SU(4)$ R-symmetry of the parent $\mathcal{N} = 4$ theory on the D-brane probe; this is a complex four-dimensional representation R^4 of Γ (resp. the adjoint six dimensional). As we do not have supersymmetry here, the 4 and the 6 need not be related. For illustrative purposes we choose $R^4 = R^1 \oplus R^1 \oplus R^2$ where

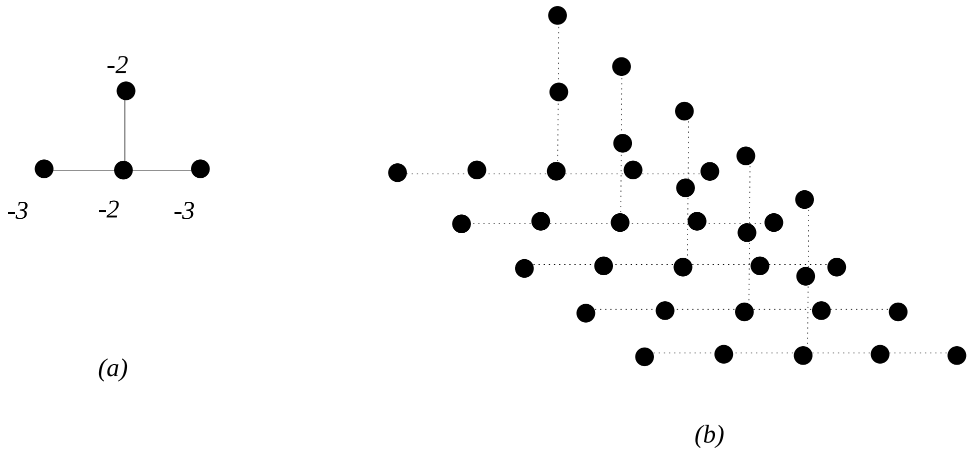

Figure 4: (a) The (minimal) resolution graph for $\mathbb{C}^2/A_5E_6^{(II)}$. There are 4 exceptional divisors. (b) The McKay quiver for $\mathbb{C}^2/A_5E_6^{(II)}$, with the fundamental defining 2-dimensional irrep chosen. As group is simply $E_6\times\mathbb{Z}_5$, we have 5 copies of the familiar E_6 quiver interlaced. The dotted lines stand for how these are linked together, the precise details of which are not important here. What we wish to emphasize is that (a) differs markedly from (b) and is only a small subgraph thereof.

R^2 is the fundamental defining irrep of Γ and R^1 is the trivial 1-dimensional irrep. Then

$$R^2 \otimes R^i = \bigoplus_j a_{ij} R^j \tag{22}$$

where i, j indexes over the irreps of Γ. The finite graph for which a_{ij} is the adjacency matrix is the McKay Quiver; this gives the (fermion) matter content a_{ij} which counts the bi-fundamentals (The trivial R^1's give self-linking arrows to each node and are adjoint fields). It is this quiver which we draw in part (b) of Figure 4. Indeed, because we have a direct product, the quiver is nothing but 5 copies of the familiar E_6 quiver appropriately interlaced.

Whereas the mathematics is clear, the physics on the other hand becomes very involved. Naïvely, we could perform a similar routine as in (7) to study the partial resolution of G to one of its subgroups, say $\mathbb{Z}_{5(-1)}$, and we would have the rather curious process

$$[[3,3,2,2]] \to [[1,1,1,1,1]]$$

where we end up with a longer sequence than we had stated with. The reason for this failure has to do with the weakness of the McKay Correspondence in this situation and will be addressed in Section 4. Furthermore it is not clear at all to what operators these blowup modes should correspond.

Indeed, the careful analysis of [5, 15] required the one-one correspondence between the basis for the chiral ring of the worldsheet CFT and the exceptional divisors of the resolution so that the acquisition of tachyon VEV's and deformations to the CFT may be completely re-phrased in terms of the geometric resolution of the orbifold. This is true for Abelian orbifolds and thus far only true in this case (cf. e.g. [27, 33]). For non-Abelian quotients subtleties arise [19, 27]. The McKay quivers, in comparison with the fusion graphs of the corresponding orbifold CFT, needs artificial truncation, i.e., only subsectors of the generators of the chiral ring are in correspondence with some of the blowup modes of the orbifold. The identification of which operators to which divisor would be a very interesting problem to which our list above provides half of the story.

4 Tachyons, Orbifolds and McKay Correspondence

We mentioned above that the subtleties involved in the analysis of the general non-supersymmetric orbifold are related to the Mckay Correspondence; in this section let us see this in some detail.

The traditional McKay Correspondence [35] dictates that for the discrete finite subgroups of $SL(2;\mathbb{C})$ (q.v. Section 3.2), the McKay quiver drawn for the defining 2-dimensional representation as in (22) is precisely the Dynkin diagram for the (simply laced) affine Lie groups $\widehat{ADE}$, each corresponding to the finite group of the same name. The subsequent work on the geometrisation of this correspondence (cf. [36] for an excellent account) hinges on the fact that for $\Gamma \subset SL(2;\mathbb{C})$, there is a one-one correspondence between the conjugacy classes of Γ and the exceptional divisors in the (minimal, crepant) resolution $\widehat{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$:

$$\text{conj}(\Gamma) \sim \widehat{\mathbb{C}^2/\Gamma}, \qquad \Gamma \subset SL(2;\mathbb{C}) \ . \tag{23}$$

In particular, (23) implies that the number of $\mathbb{P}^1$ blowups is equal to the number of conjugacy classes of Γ, which in turn, by an elementary theorem on the representation of finite groups, is equal to the number of irreducible representations of Γ. Therefore, the number of nodes in the McKay quiver each corresponds to a $\mathbb{P}^1$ and the intersection matrix amongst the $\mathbb{P}^1$'s is the adjacency matrix of the quiver. Hence we can re-phrase (23) as

$$\text{irrep}^0(\Gamma) \sim H^*(\widehat{\mathbb{C}^2/\Gamma}, \mathbb{Z}) \tag{24}$$

where the cohomology picks up the exceptional divisors which are -2 curves and $\text{irrep}^0(\Gamma)$ means the non-trivial irreducible representations of Γ. The nodes of the quiver are usually labeled by the dimension of the corresponding irrep, which coincides with the Coxeter number of the associated Dynkin diagram. In summary, the McKay quiver, the minimal resolution graph for the exceptional $\mathbb{P}^1$'s and the affine ADE Dynkin diagrams are identical for $\Gamma \subset SL(2;\mathbb{C})$. Everything therefore fits nicely for Calabi-Yau orbifolds in dimension two [27].

For higher dimensions, the correspondence weakens substantially; though still shown to hold for Abelian quotients in dimension three (cf. [37]), the generic group has been so far intractable. In terms of the physics, the nice work of [3] for example, has discussed how in the open sector, the $D\bar{D}$ system exhibits tachyon condensation on three-dimensional Calabi-Yau (i.e., $SL(3;\mathbb{C})$) orbifolds and how boundary CFT modes can be interpreted via the McKay correspondence from the derived category point of view. Here we are dealing with another generalisation, viz., non-Calabi-Yau orbifolds in dimension two and works by Ishii, Riemenschneider et al. shed some light.

4.1 A Mismatch

First of all we can immediately see that there is a mismatch between the conjugacy classes (irreps) and the exceptional divisors for $\Gamma \subset GL(2;\mathbb{C})$. The number of nodes in Figure 2 is r which is the number of terms in the continued fraction expansion of $\frac{n}{p}$; however, the number of conjugacy classes of this Abelian group is n. As another example, in Figure 4, (a) and (b) differ significantly: the number of exceptional divisors is 4 while there are 35 conjugacy classes (irreps). Indeed one can see that this is the general pattern: instead of having (23), we now have

$$\#\text{Irrep}(\Gamma) > \#\text{Exceptional Div}\left(\widehat{\mathbb{C}^2/\Gamma}\right), \qquad \Gamma \subset GL(2;\mathbb{C}) \ . \tag{25}$$

This discrepancy of course shows up in the physics. The twisted sector operators in the orbifold CFT are counted by the conjugacy classes of the group (we can see this when summing up the torus partition function). When we are dealing with D-brane probes, the equivariant K-theory of the orbifold gives the representation ring of Γ and so there are still enough distinct D-branes at the orbifold point. However in the resolution of the singularity, (25) means that there are not enough cycles for the distinct D-branes to wrap. In a very nicely detailed gauged linear sigma model analysis, [15] has shown that in the toric case (i.e., $\mathbb{Z}_{n(p)}$), while $r+1$ charges come from the

Hirzebruch-Jung resolution (Higgs branch), an extra set of $n-r-1$ D-branes lives on the Coulomb branch whereby conserving the total D-brane charges.

We remark that this mismatch phenomenon is quite generic and stands largely unresolved for arbitrary quotients. It was pointed out in [27] that the ADE meta-pattern, ubiquitous in various fields, seems to enjoy the specialty of Calabi-Yaus in dimension two. Indeed as was shown in [19], attempts to relate the chiral fusion ring and twisted sector operators associated with blowups (i.e., the full geometrisation of the orbifold conformal field theory), even in the Calabi-Yau case for dimension 3 met similar mismatches as (25). Graphical truncation of the quivers are needed and only subsets of the chiral operators can be placed in correspondence with either the representation ring of the group, or with the exceptional divisors in the resolution.

4.2 Ishii's McKay Correspondence for $GL(2;\mathbb{C})$

For the general two-dimensional quotients we should not be discouraged by the mismatch mentioned above, and there is a partial remedy. For this we need to first turn to the work by Ito and Nakamura. It is known that the symmetric orbifold, $(\mathbb{C}^2)^n/S_n$ where S_n is the symmetric group on n elements, has a smooth resolution, the so-called Hilbert Scheme of points on $\mathbb{C}^2$, denoted as $\mathrm{Hilb}^n(\mathbb{C}^2)$. It was realised in [37] that if one took $n=|\Gamma|$ and $\Gamma \subset SL(2;\mathbb{C})$, and considered the Γ-equivariant version of the said resolution, one in fact has

$$\widehat{\mathbb{C}^2/\Gamma} = \mathrm{Hilb}^\Gamma(\mathbb{C}^2) := (\mathrm{Hilb}^n(\mathbb{C}^2))_\Gamma \to \left((\mathbb{C}^2)^n/S_n\right)_\Gamma \simeq \mathbb{C}^2/\Gamma \ . \qquad (26)$$

This construction distinguished the Hilbert scheme as the (crepant) minimal resolution and provided an explicit mapping of the (non-trivial) irreducible representations with the exceptional divisors.

Into the Hilbert scheme resolution we shall not delve too far. The key point for our purposes is the following theorem due to A. Ishii:

THEOREM 4.2. *For small subgroups $\Gamma \subset GL(2;\mathbb{C})$, the Ito-Nakamura construction for the McKay Correspondence still holds if instead of considering $Irrep^0(\Gamma)$, we consider a subset, the so-called* **special representations**.

The definition of these representations is rather technical and we briefly touch upon it. Let $\mu : \mathbb{C}^2 \to \mathbb{C}^2/\Gamma$ be the canonical projection and $\pi : \widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$ be the resolution for $\Gamma \subset GL(2;\mathbb{C})$. Moreover let V_{ρ^*} be the representation vector space (module) associated to the dual ρ^* of the irrep

ρ and $\mathcal{O}_{\mathbb{C}^2}$ the sheaf of holomorphic functions on $\mathbb{C}^2$. We can then define (a so-called reflexive sheaf) $M_\rho := \mu_* \left(\mathcal{O}_{\mathbb{C}^2} \otimes V_{\rho^*}\right)^\Gamma$. A representation ρ is *special* if

$$H^1(\widetilde{\mathbb{C}^2/\Gamma}; (\pi^* M_\rho)^*) = 0 \ . \tag{27}$$

The upshot is that we can check (27) against all the (non-trivial) irreducible representations - easily read off from the character tables - of the groups in the classification of Section 3; the only ones which satisfy the condition will be in one-one correspondence with the exceptional divisors in the minimal resolution, i.e., with the nodes in the resolution quivers (Figure 3) and with the terms in the generalised Hirzebruch-Jung continued fraction (16). In the Abelian case of type $A_{n,p}$, these special representations are associated to the $r+1$ "obvious" D-brane charges sitting at the Higgs branch as discussed in [15].

5 Discussions and Prospects

In this paper we have generalised the problem of closed string tachyon condensation of [1] from the Abelian cases thus far addressed in the literature to non-Abelian groups. In particular we have considered arbitrary quotients of $\mathbb{C}^2$. To do so we have presented the classification of all relevant discrete finite subgroups $\Gamma \subset GL(2;\mathbb{C})$, namely the so-called small groups, the orbifolds thereby exhausts all isomorphism classes of non-supersymmetric orbifolds in dimension 2.

We have initiated the study of identifying tachyonic operators with blowups in the resolution of the orbifolds by D-brane probes and pointed out generalisations of the Hirzebruch-Jung continued fractions used in the toric analysis of the Abelian cases [5, 11]. A first non-Abelian example, namely type $A_5E_6^{(II)}$, has been studied in detail.

This geometrisation of the spacetime decay process is unlike supersymmetric orbifolds of the ALE spaces (local K3's) where the full McKay Correspondence conveniently gives one-one correspondences amongst the chiral ring generators, the irreducible representations of Γ as well as the exceptional divisors in the (minimal, crepant) resolution of the orbifold. Here we generically have far more conjugacy classes (and thus D-brane types) than geometric cycles and we turn to Ishii's generalised McKay Correspondence for $GL(2;\mathbb{C})$.

The work of [15] showed how to identify the "missing cycles" by going to the Coulomb branch to seek more D-brane charges from the K-theory. The general problem of matching the chiral ring, conjugacy class and blowups for arbitrary orbifolds [19, 27] still remains tantalising. We have provided the geometric data for non-supersymmetric non-Abelian quotients of $\mathbb{C}^2$, and it would be interesting to extend the analysis of [15] in finding the missing branes. Such a search would be guided by (27). The "obvious branes charges" associated to the fractional branes that wrap cycles which by Ishii's correspondence are in 1-1 relation with the special representations.

In this light if we were to use a world-sheet CFT analysis to find the extra charges we could find a strong version of generalised McKay [15]. This seems to be a *general philosophy:* even though there is a bijection between the representation ring of the orbifold group Γ and the exceptional divisors of the geometric orbifold $\mathbb{C}^n$ known only for the classical case of $\mathbb{C}^2/\{\Gamma \subset SU(2)\}$, string theory knows more. By the very virtue that the geometric resolution of the orbifold occurs only for the Higgs branch of the world-volume theory, the other phases of the moduli space should provide additional information. Could this help establish bijections between $\text{Rep}(\Gamma)$ and $H^*(\mathbb{C}^n/\Gamma)$ and in particular explain the graph truncations of [28, 27]?

Mathematically[2] this raises another fascinating point. In dimensions great than or equal to 2, the aforementioned mismatch between the blowups and conjugacy classes of the orbifold for the generic orbifold is only part of the problem: in general there is not even a notion of minimal resolution. If we could generalise the linear sigma model technique in studying the world-sheet CFT and identifying twisted operators with the blowups, could we physically distinguish a resolution of the orbifold (in a sense analogous to the Hilbert scheme resolution of Ito-Nakamura being a distinguished one)?

Moreover, it is well known that the ALE spaces do not admit discrete torsion in the sense that for $\Gamma \subset SU(2)$, the Schur multiplier $M(G) := H^2(\Gamma; U(1)) = 0$ (cf.[34]). However, now due to the direct product structure, we can have non-trivial projective representations of the D-brane probe. In general we know that (see e.g.[34])

$$M(G_1 \times G_2) \simeq M(G_1) \times M(G_2) \times \text{Hom}_{\mathbb{Z}}(G_1/G_1', G_2/G_2') \ , \qquad (28)$$

which for those members which afford the direct product structure, e.g. the cases of the for $A_n E_{7,8}$ (n odd), simplify to $\text{Hom}_{\mathbb{Z}}(G_1/G_1', G_2/G_2')$ because each $G_{i=1,2}$ is an $SL(2;\mathbb{C})$ group and has trivial Schur multiplier. Also we know that $A_n/A_n' \simeq A_n$ because A_n is Abelian. Moreover, we know that $D_n/D_n' \simeq \mathbb{Z}_4$, $E_6/E_6' \simeq \mathbb{Z}_3$, $E_7/E_7' \simeq \mathbb{Z}_2$, and $E_8/E_8' \simeq \mathbb{I}$. Finally,

[2]We thank Prof. Y. Ruan for discussions on this.

recalling that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \simeq \mathbb{Z}_{(m,n)}$, we can easily determine $M(G)$ for these product groups. Indeed we see that only the orbifolds of type AD and $AE_{6,7}$ admit discrete torsion. For example the group $A_9E_6^{(III)}$ has a $\mathbb{Z}_3$ discrete torsion. Therefore there should 3 disconnected pieces of the quiver diagram if one turns on the NS-NS B-field, one for each value of the torsion.

It would be worthwhile to investigate these torsion examples, the moduli space of the gauge theory as well as possible notions of non-commutativity and local mirror symmetry in such situations of dimension two. These and many other intriguing issues that stem from these non-supersymmetric orbifolds we leave to future investigation.

6 Appendix: From $SL(2;\mathbb{C})$ to $GL(2;\mathbb{C})$

We can obtain all the isoclasses of the discrete finite subgroups of $GL(2;\mathbb{C})$ from the ADE groups of $SL(2;\mathbb{C})$. We first note that there is a surjective homomorphism

$$\begin{aligned} \psi \quad : \quad & Z \times SL(2;\mathbb{C}) \to GL(2;\mathbb{C}) \\ & (z, g) \to zg \end{aligned} \tag{29}$$

where Z is the centre of $GL(2;\mathbb{C})$. Because Z commutes with all elements, we can generate the appropriate subgroup Γ by concatenating the generators of the subgroup of Z with that of $SL(2;\mathbb{C})$.

Subsequently, all the non-cyclic (the cyclic ones are the $\mathbb{Z}_{n(p)}$) finite subgroups Γ of $GL(2;\mathbb{C})$ may be obtained from the quadruple

$$\Gamma := (G_1, N_1; G_2, N_2)$$

where $N_1 \triangleleft G_1 \subset Z$ and $N_2 \triangleleft G_2 \subset SL(2;\mathbb{C})$ with G_2 not cyclic. Furthermore, the factor groups are isomorphic as

$$\phi : G_1/N_1 \xrightarrow{\sim} G_2/N_2 \ .$$

Under these conditions, we can construct Γ explicitly as

$$\Gamma = \psi(G_1 \times_\phi G_2)$$

where $G_1 \times_\phi G_2 := \{(g_1, g_2) \in (G_1, G_2) | \bar{g_2} = \phi(\bar{g_2})\}$ with $\bar{g_i}$ being the residue class associate to g_i under the quotient G_i/N_i. In fact the conjugacy class of $\Gamma \subset GL(2;\mathbb{C})$ is independent of the chosen isomorphism ϕ and so the quadruple suffices to denote the group.

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