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008

NOETHER'S PROBLEM AND UNRAMIFIED BRAUER GROUPS*

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Abstract. Let k be any field, G be a finite group acting on the rational function field $k(x_g : g \in G)$ by $h \cdot x_g = x_{hg}$ for any $h, g \in G$. Define $k(G) = k(x_g : g \in G)^G$. Noether's problem asks whether $k(G)$ is rational (= purely transcendental) over k. It is known that, if $\mathbb{C}(G)$ is rational over $\mathbb{C}$, then $B_0(G) = 0$ where $B_0(G)$ is the unramified Brauer group of $\mathbb{C}(G)$ over $\mathbb{C}$. Bogomolov showed that, if G is a p-group of order p^5, then $B_0(G) = 0$. This result was disproved by Moravec for $p = 3, 5, 7$ by computer calculations. We will prove the following theorem. Theorem. Let p be any odd prime number, G be a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10} in R. James's classification of groups of order p^5.

Key words. Noether's problem, rationality problem, unramified Brauer groups, Bogomolov multipliers, rationality, retract rationality.

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1. Introduction. Let k be any field and G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms so that $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $k(G)$ the fixed field $k(x_g : g \in G)^G$. Noether's problem asks whether $k(G)$ is rational (= purely transcendental) over k. It is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions [Sw; Sa1; GMS, 33.1, p. 86]. Noether's problem for abelian groups was studied by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to Swan's paper for a survey of this problem [Sw].

On the other hand, just a handful of results about Noether's problem are obtained when the groups are not abelian. It is the case even when G is a p-group.

Before stating the results on Noether's problem for non-abelian p-groups, we recall some relevant definitions.

DEFINITION 1.1. Let $k \subset K$ be an extension of fields. K is rational over k (for short, k-rational) if K is purely transcendental over k. K is stably k-rational if $K(y_1, \ldots, y_m)$ is rational over k for some $y_1, \ldots, y_m$ such that $y_1, \ldots, y_m$ are algebraically independent over K. When k is an infinite field, K is said to be retract k-rational if there is a k-algebra A contained in K such that (i) K is the quotient field of A, (ii) there exist a non-zero polynomial $f \in k[X_1, \ldots, X_n]$ (where $k[X_1, \ldots, X_n]$ is the polynomial ring) and k-algebra homomorphisms $\varphi \colon A \to k[X_1, \ldots, X_n][1/f]$ and $\psi \colon k[X_1, \ldots, X_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$. (See [Sa2; Ka4] for details.) It is not difficult to see that "k-rational" ⇒ "stably k-rational" ⇒ "retract k-rational".

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DEFINITION 1.2. Let $k \subset K$ be an extension of fields. The notion of the unramified Brauer group of K over k, denoted by $\mathrm{Br}_{v,k}(K)$, was introduced by Saltman [Sa3]. By definition, $\mathrm{Br}_{v,k}(K) = \bigcap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$ where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R.

LEMMA 1.3 (Saltman [Sa3; Sa4]). *If k is an infinite field and K is retract k-rational, then the natural map $\mathrm{Br}(k) \to \mathrm{Br}_{v,k}(K)$ is an isomorphism. In particular, if k is an algebraically closed field and K is retract k-rational, then $\mathrm{Br}_{v,k}(K) = 0$.*

THEOREM 1.4 (Bogomolov, Saltman [Bo; Sa5, Theorem 12]). *Let G be a finite group, k be an algebraically closed field with $\gcd\{|G|, \mathrm{char}\, k\} = 1$. Let μ denote the multiplicative subgroup of all roots of unity in k. Then $\mathrm{Br}_{v,k}(k(G))$ is isomorphic to the group $B_0(G)$ defined by*

$$B_0(G) = \bigcap_A \mathrm{Ker}\{\mathrm{res}_G^A : H^2(G, \mu) \to H^2(A, \mu)\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

Note that $B_0(G)$ is a subgroup of $H^2(G, \mu)$ (where $\gcd\{|G|, \mathrm{char}\, k\} = 1$). Since $H^2(G, \mu) \simeq H_2(G)$, which is the Schur multiplier of G (see [Kar]), we will call $B_0(G)$ the Bogomolov multiplier of G, following the convention in [Ku]. Because of Theorem 1.4 we will not distinguish $B_0(G)$ and $\mathrm{Br}_{v,k}(k(G))$ when k is algebraically closed and $\gcd\{|G|, \mathrm{char}\, k\} = 1$. In this situation, $B_0(G)$ is canonically isomorphic to $\bigcap_A \mathrm{Ker}\{\mathrm{res}_G^A : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$, i.e. we may replace the coefficient μ by $\mathbb{Q}/\mathbb{Z}$ in Theorem 1.4.

Using the unramified Brauer groups, Saltman and Bogomolov are able to establish counter-examples to Noether's problem for non-abelian p-groups.

THEOREM 1.5. *Let p be any prime number, k be any algebraically closed field with $\mathrm{char}\, k \neq p$.*

(1) (Saltman [Sa3]) *There is a group G of order p^9 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not retract k-rational. Thus $k(G)$ is not k-rational.*

(2) (Bogomolov [Bo]) *There is a group G of order p^6 such that $B_0(G) \neq 0$. Thus $k(G)$ is not k-rational.*

For p-groups of small order, we have the following result.

THEOREM 1.6 (Chu and Kang [CK]). *Let p be any prime number, G be a p-group of order $\leq p^4$ and of exponent e. If k is a field satisfying either* (i) $\mathrm{char}\, k = p$, *or* (ii) *k contains a primitive e-th root of unity, then $k(G)$ is k-rational.*

Because of the above Theorems 1.5 and 1.6, we may wonder what happens to non-abelian p-groups of order p^5.

THEOREM 1.7 (Chu, Hu, Kang and Prokhorov [CHKP]). *Let G be a group of order* 32 *and of exponent e. If k is a field satisfying either* (i) $\mathrm{char}\, k = 2$, *or* (ii) *k contains a primitive e-th root of unity, then $k(G)$ is k-rational. In particular, $B_0(G) = 0$.*

Working on p-groups, Bogomolov developed a lot of techniques and interesting results. Here is one of his results.

THEOREM 1.8. (1) [Bo, Lemma 4.11] *If G is a p-group with $B_0(G) \neq 0$ and $G/[G,G] \simeq C_p \times C_p$, then $p \geq 5$ and $|G| > p^7$.*

(2) [Bo, Lemma 5.6; BMP, Corollary 2.11] *If G is a p-group of order $\leq p^5$, then $B_0(G) = 0$.*

Because of part (2) of the above theorem, Bogomolov proposed to classify all the groups G with $|G| = p^6$ satisfying $B_0(G) \neq 0$ [Bo, page 479].

It came as a surprise that Moravec's recent paper [Mo1] disproved the above Theorem 1.8.

THEOREM 1.9 (Moravec [Mo1, Section 5]). *If G is a group of order 243, then $B_0(G) \neq 0$ if and only if $G = G(243, i)$ with $28 \leq i \leq 30$, where $G(243, i)$ is the i-th group among groups of order 243 in the database of GAP.*

Moravec proves Theorem 1.9 by using computer calculations. No theoretic proof is given. A file of the GAP functions and commands for computing $B_0(G)$ can be found at Moravec's website `www.fmf.uni-lj.si/~moravec/b0g.g`. Recently, using this computer package, Moravec was able to classify all groups G of order 5^5 and 7^5 such that $B_0(G) \neq 0$.

Before stating the main result of this paper, we recall the classification of p-groups of order $\leq p^6$ and introduce the notion of isoclinism.

A list of groups of order 2^5 (resp. 3^5, 5^5, 7^5) can be found in the database of GAP. However the classification of groups of order p^5 dated back to Bagnera (1898), Bender (1927), R. James (1980), etc. [Ba; Be; Ja], although some minor errors might occur in the classification results finished before the computer-aided time. For example, in Bender's classification of groups of order 3^5, one group is missing, i.e. the group $\Delta_{10}(2111)a_2$ which was pointed by [Ja, page 613]. A beautiful formula for the total number of the groups of order p^5, for $p \geq 3$, was found by Bagnera [Ba] as

$$2p + 61 + \gcd\{4, p-1\} + 2\gcd\{3, p-1\}.$$

Note that the above formula is correct only when $p \geq 5$ (see the second paragraph of Section 4).

On the other hand, groups of order 2^n ($n \leq 6$) were classified by M. Hall and Senior [HaS]. There are 267 groups of order 2^6 in total. Groups of order 2^7 were classified by R. James, Newman and O'Brien [JNOB].

DEFINITION 1.10. Two p-groups G_1 and G_2 are called isoclinic if there exist group isomorphisms $\theta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi\colon [G_1, G_1] \to [G_2, G_2]$ such that $\phi([g,h]) = [g', h']$ for any $g, h \in G_1$ with $g' \in \theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$ (note that $Z(G)$ and $[G,G]$ denote the center and the commutator subgroup of the group G respectively).

For a prime number p and a fixed integer n, let $G_n(p)$ be the set of all non-isomorphic groups of order p^n. In $G_n(p)$ consider an equivalence relation: two groups G_1 and G_2 are equivalent if and only if they are isoclinic. Each equivalence class of $G_n(p)$ is called an isoclinism family.

QUESTION 1.11. Let G_1 and G_2 be isoclinic p-groups. Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic ?

According to a private communication from Bogomolov, one should expect an affirmative answer even within larger classes of groups. Our results for groups of order p^5 confirm many cases for these expectations.

After this paper had been submitted, Bogomolov and Böhning posted a paper solving the above question in the affirmative [BB, Theorem 3.2]. Their result makes possible to shorten many proofs of this paper, but we choose to retain our "empirical" proof.

Return to groups of order p^5. If p is an odd prime number, then there are precisely 10 isoclinism families for groups of order p^5; each family is denoted by Φ_i, $1 \le i \le 10$ [Ja, pages 619–621]. As for groups of order 64, there are 27 isoclinism families [JNOB, page 147].

The main result of the present paper is the following theorem.

Theorem 1.12. *Let p be any odd prime number, G be a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10}. Each group G in the family Φ_{10} satisfies the condition $G/[G,G] \simeq C_p \times C_p$. There are precisely 3 groups in this family if $p = 3$. For $p \ge 5$, the total number of non-isomorphic groups in this family is*

$$1 + \gcd\{4, p-1\} + \gcd\{3, p-1\}.$$

Note that, for $p = 3$, the isoclinism family Φ_{10} consists of the groups $\Phi_{10}(2111)a_r$ (where $r = 0, 1$) and $\Phi_{10}(5)$ [Ja, page 621], which are just the groups $G(3^5, i)$ with $28 \le i \le 30$ in the GAP code numbers. This confirms the computation of Moravec [Mo1]. Similarly, when $p = 5$, the isoclinism family Φ_{10} consists of the groups $G(5^5, i)$ with $33 \le i \le 38$; when $p = 7$, the isoclinism family consists of the groups $G(7^5, i)$ with $37 \le i \le 42$. They agree with Moravec's computer results.

We use the computer package provided by Moravec to study groups of order 11^5. We find that, for a group G of order 11^5, $B_0(G) \neq 0$ if and only if $G \simeq G(11^5, i)$ with $39 \le i \le 42$, also confirming the above Theorem 1.12.

It may be interesting to record the computing time to determine $B_0(G)$ for all p-groups of order p^5 with $p = 3, 5, 7, 11$. When $p = 3, 5, 7$, it requires only 20 seconds, one hour and two days respectively. When $p = 11$, it requires more than one month by parallel computing at four cores.

As a corollary of Theorem 1.12, we record the following result.

Theorem 1.13. *Let n be a positive integer and k be a field with $\gcd\{|G|, \operatorname{char} k\} = 1$. If $2^6 \mid n$ or $p^5 \mid n$ for some odd prime number p, then there is a group G of order n such that $B_0(G) \neq 0$. In particular, $k(G)$ is not stably k-rational; when k is an infinite field, $k(G)$ is not even retract k-rational.*

See Theorem 5.7 for another application of Theorem 1.12.

For completeness, we record the result for groups of order 2^6. Recall that there are 267 non-isomorphic groups of order 2^6 and 27 isoclinism families in total [JNOB].

Theorem 1.14 (Chu, Hu, Kang and Kunyavskii [CHKK]). *Let G be a group of order 2^6.*

(1) *$B_0(G) \neq 0$ if and only if G belongs to the 16th isoclinism family, i.e. $G = G(2^6, i)$ where $149 \le i \le 151$, $170 \le i \le 172$, $177 \le i \le 178$, or $i = 182$.*

(2) *If $B_0(G) = 0$ and k is an algebraically closed field with $\operatorname{char} k \neq 2$, then $k(G)$ is rational over k except possibly for groups G belonging to the 13rd isoclinism family, i.e. $G = G(2^6, i)$ with $241 \le i \le 245$.*

Finally we mention a recent result which supplements Moravec's result in Theorem 1.9.

Theorem 1.15 (Chu, Hoshi, Hu and Kang [CHHK]). *Let G be a group of order 3^5 and of exponent e. If k is a field containing a primitive e-th root of unity and $B_0(G) = 0$, then $k(G)$ is rational over k except possibly for groups $G \in \Phi_7$, i.e. $G = G(3^5, i)$ with $56 \leq i \leq 60$.*

We explain briefly the idea of the proof of Theorem 1.12. Let G be a group of order p^5 where p is an odd prime number. To show that $B_0(G) = 0$, we apply Theorems 3.3–3.6 or some "standard" techniques. For the proof of $B_0(G) = 0$ when G belongs to the isoclinism family Φ_6, we use the 7-term cohomology exact sequence in [DHW] (see also [Hu1], [Hu2], [Hu3]), see Theorems 5.4 and 5.6. We remark that, for many cases in Sections 4 and 5, we prove not only $B_0(G) = 0$, but also $k(G)$ is retract k-rational or the $k(G)$'s are k-isomorphic for the groups G belonging to the same isoclinism family. Moravec has another proof showing that $B_0(G) = 0$ when G is a group of order p^5 not belonging to the isoclinism family Φ_{10} [Mo2].

On the other hand, to show that $B_0(G) \neq 0$, we find suitable generators and relations for G. It turns out that $B_0(G) \neq 0$ if some relations are satisfied (see Lemma 2.2). All the groups in the isoclinism family Φ_{10} satisfy these relations. Lemma 2.2 relies on the 5-term exact sequence of Hochschild and Serre [HS]

$$\begin{aligned} 0 &\to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \\ &\to H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

where ψ is the inflation map. The crux of showing $B_0(G) \neq 0$ is to prove that the image of ψ is non-zero and is contained in $B_0(G)$.

The paper is organized as follows. In Section 2, we prove that $B_0(G) \neq 0$ if G belongs to the isoclinism family Φ_{10}. Then we give a proof of Theorem 1.13. Section 3 contains some rationality criteria or previous results for showing $B_0(G) = 0$. Section 4 is devoted to the proof of $B_0(G) = 0$ if G belongs to the isoclinism family Φ_i where $1 \leq i \leq 9$ and $i \neq 6$. The case of Φ_6 is postponed till Section 5. In our proof, we check all of the groups in every isoclinism family Φ_i for $1 \leq i \leq 10$. The reader should be aware that such a proof can be shortened, because it suffices to check only one group in each isoclinism family by Bogomolov-Böhning's Theorem [BB, Theorem 3.2].

Standing notations. Throughout this paper, k is a field, ζ_n denotes a primitive n-th root of unity. Whenever we write $\zeta_n \in k$ (resp. $\gcd\{n, \operatorname{char} k\} = 1$), it is understood that either $\operatorname{char} k = 0$ or $\operatorname{char} k = l > 0$ with $l \nmid n$. When k is an algebraically closed field, μ denotes the set of all roots of unity, i.e. $\mu = \{\alpha \in k\backslash\{0\} : \alpha^n = 1$ for some integer n depending on $\alpha\}$. If G is a group, $Z(G)$ and $[G, G]$ denote the center and the commutator subgroup of G respectively. If $g, h \in G$, we define $[g, h] = g^{-1}h^{-1}gh \in G$. When N is a normal subgroup of G and $g \in G$, the element $\bar{g} \in G/N$ denotes the image of g in the quotient group G/N. The exponent of G is defined as $\operatorname{lcm}\{\operatorname{ord}(g) : g \in G\}$ where $\operatorname{ord}(g)$ is the order of the element g. We denote by C_n the cyclic group of order n. A group G is called a bicyclic group if it is either a cyclic group or a direct product of two cyclic groups. When we write cohomology groups $H^q(G, \mu)$ or $H^q(G, \mathbb{Q}/\mathbb{Z})$, it is understood that μ and $\mathbb{Q}/\mathbb{Z}$ are trivial G-modules.

For emphasis, recall that the field $k(G)$ was defined in the first paragraph of this section. The group $G(n, i)$ is the i-th group among the groups of order n in GAP. The version of GAP we refer to in this paper is GAP4, Version: 4.4.12 [GAP]. All the groups G in this paper are finite.

2. Groups in the isoclinism family Φ_{10}. We start with a general lemma.

LEMMA 2.1. *Let G be a finite group, N be a normal subgroup of G. Assume that* (i) $\mathrm{tr}\colon H^1(N,\mathbb{Q}/\mathbb{Z})^G \to H^2(G/N,\mathbb{Q}/\mathbb{Z})$ *is not surjective where* tr *is the transgression map, and* (ii) *for any bicyclic subgroup A of G, the group AN/N is a cyclic subgroup of G/N. Then $B_0(G) \neq 0$.*

Proof. Consider the Hochschild–Serre 5-term exact sequence

$$\begin{aligned} 0 &\to H^1(G/N,\mathbb{Q}/\mathbb{Z}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^1(N,\mathbb{Q}/\mathbb{Z})^G \\ &\xrightarrow{\mathrm{tr}} H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z}) \end{aligned}$$

where ψ is the inflation map [HS].

Since tr is not surjective, we find that ψ is not the zero map. Thus $\mathrm{Image}(\psi) \neq 0$.

We will show that $\mathrm{Image}(\psi) \subset B_0(G)$. By definition, it suffices to show that, for any bicyclic subgroup A of G, the composite map $H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(A,\mathbb{Q}/\mathbb{Z})$ becomes the zero map where res is the restriction map. Consider the following commutative diagram

$$\begin{tikzcd} H^2(G/N,\mathbb{Q}/\mathbb{Z}) \arrow[r, "\psi"] \arrow[d, "\psi_0"'] & H^2(G,\mathbb{Q}/\mathbb{Z}) \arrow[r, "\mathrm{res}"] & H^2(A,\mathbb{Q}/\mathbb{Z}) \\ H^2(AN/N,\mathbb{Q}/\mathbb{Z}) \arrow[rr, "\widetilde{\psi}", "\simeq"'] & & H^2(A/A\cap N,\mathbb{Q}/\mathbb{Z}) \arrow[u, "\psi_1"'] \end{tikzcd}$$

where ψ_0 is the restriction map, ψ_1 is the inflation map, $\widetilde{\psi}$ is the natural isomorphism.

Since AN/N is cyclic, write $AN/N \simeq C_m$ for some integer m. It is well-known that $H^2(C_m,\mathbb{Q}/\mathbb{Z}) = 0$ (see, e.g., [Kar, page 37, Corollary 2.2.12]). Hence ψ_0 is the zero map. Thus $\mathrm{res}\circ\psi\colon H^2(G/N,\mathbb{Q}/\mathbb{Z}) \to H^2(A,\mathbb{Q}/\mathbb{Z})$ is also the zero map.

As $\mathrm{Image}(\psi) \subset B_0(G)$ and $\mathrm{Image}(\psi) \neq 0$, we find that $B_0(G) \neq 0$. □

LEMMA 2.2. *Let $p \geq 3$ and G be a p-group of order p^5 generated by f_i where $1 \leq i \leq 5$. Suppose that, besides other relations, the generators f_i satisfy the following conditions:*

(i) $f_4^p = f_5^p = 1$, $f_5 \in Z(G)$,
(ii) $[f_2, f_1] = f_3$, $[f_3, f_1] = f_4$, $[f_4, f_1] = [f_3, f_2] = f_5$, $[f_4, f_2] = [f_4, f_3] = 1$, *and*
(iii) $\langle f_4, f_5\rangle \simeq C_p \times C_p$, $G/\langle f_4, f_5\rangle$ *is a non-abelian group of order p^3 and of exponent p.*

Then $B_0(G) \neq 0$.

REMARK. If $p = 2$ and G/N is a non-abelian group of order 8, then $H^2(G/N,\mathbb{Q}/\mathbb{Z}) = 0$ or C_2 [Kar, page 138, Theorem 3.3.6]. Thus $\mathrm{tr}\colon H^1(N,\mathbb{Q}/\mathbb{Z})^G \to H^2(G/N,\mathbb{Q}/\mathbb{Z})$ in Lemma 2.2 may become surjective. This is the reason why we assume $p \geq 3$ in this lemma.

Proof. Choose $N = \langle f_4, f_5\rangle$. We will check the conditions in Lemma 2.1 are satisfied. Thus $B_0(G) \neq 0$.

Step 1. Since $N \simeq C_p \times C_p$, we find that $H^1(N,\mathbb{Q}/\mathbb{Z}) \simeq C_p \times C_p$.

Define $\varphi_1, \varphi_2 \in H^1(N,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(N,\mathbb{Q}/\mathbb{Z})$ by $\varphi_1(f_4) = 1/p$, $\varphi_1(f_5) = 0$, $\varphi_2(f_4) = 0$, $\varphi_2(f_5) = 1/p$. Clearly $H^1(N,\mathbb{Q}/\mathbb{Z}) = \langle\varphi_1, \varphi_2\rangle$.

The action of G on φ_1, φ_2 are given by ${}^{f_1}\varphi_1(f_4) = \varphi_1(f_1^{-1}f_4f_1) = \varphi_1(f_4f_5) = \varphi_1(f_4) + \varphi_1(f_5) = 1/p$, ${}^{f_1}\varphi_1(f_5) = \varphi_1(f_1^{-1}f_5f_1) = \varphi_1(f_5) = 0$. Thus ${}^{f_1}\varphi_1 = \varphi_1$. Similarly, ${}^{f_1}\varphi_2(f_4) = 1/p$, ${}^{f_1}\varphi_2(f_5) = 1/p$ and ${}^{f_1}\varphi_2 = \varphi_1 + \varphi_2$.

For any $\varphi \in H^1(N, \mathbb{Q}/\mathbb{Z}) = \langle \varphi_1, \varphi_2 \rangle \simeq C_p \times C_p$, write $\varphi = a_1\varphi_1 + a_2\varphi_2$ for some integers $a_1, a_2 \in \mathbb{Z}$ (modulo p). Since ${}^{f_1}\varphi = {}^{f_1}(a_1\varphi_1 + a_2\varphi_2) = a_1({}^{f_1}\varphi_1) + a_2({}^{f_1}\varphi_2) = (a_1 + a_2)\varphi_1 + a_2\varphi_2$, we find that ${}^{f_1}\varphi = \varphi$ if and only if $a_2 = 0$, i.e. $\varphi \in \langle \varphi_1 \rangle$. On the other hand, it is easy to see that ${}^{f_2}\varphi_1 = \varphi_1 = {}^{f_3}\varphi_1$ and therefore $\varphi_1 \in H^1(N, \mathbb{Q}/\mathbb{Z})^G$. We find $H^1(N, \mathbb{Q}/\mathbb{Z})^G = \langle \varphi_1 \rangle \simeq C_p$.

By [Le, Proposition 6.3; Kar, page 138, Theorem 3.3.6], since G/N is a non-abelian group of order p^3 and of exponent p, we find $H^2(G/N, \mathbb{Q}/\mathbb{Z}) \simeq C_p \times C_p$. Thus $\mathrm{tr}\colon H^1(N, \mathbb{Q}/\mathbb{Z})^G \to H^2(G/N, \mathbb{Q}/\mathbb{Z})$ is not surjective. Hence the first condition of Lemma 2.1 is verified.

Step 2. We will verify the second condition of Lemma 2.1, i.e. for any bicyclic subgroup A of G, AN/N is a cyclic group.

Before the proof, we list the following formulae which are consequences of the commutator relations, i.e. relations (ii) of this lemma. The proof of these formulae is routine and is omitted.

For $1 \le i, j \le p-1$, $f_4^i f_1^j = f_1^j f_4^i f_5^{ij}$, $f_3^i f_2^j = f_2^j f_3^i f_5^{ij}$, and

$$f_3^i f_1^j = f_1^j f_3^i f_4^{ij} f_5^{i\cdot\binom{j}{2}}, \quad f_2^i f_1^j = f_1^j f_2^i f_3^{ij} f_4^{i\cdot\binom{j}{2}} f_5^{i\cdot\binom{j}{3}+\binom{i}{2}\cdot j}$$

where $\binom{a}{b}$ denotes the binomial coefficient when $a \ge b \ge 1$ and we adopt the convention $\binom{a}{b} = 0$ if $1 \le a < b$.

Moreover, in G/N, $(\bar{f}_1^j \bar{f}_2^i)^e = \bar{f}_1^{ej} \bar{f}_2^{ei} \bar{f}_3^{\binom{e}{2}\cdot ij}$ for $1 \le i, j \le p-1$, $1 \le e \le p$.

Step 3. Let $A = \langle h_1, h_2 \rangle$ be a bicyclic subgroup of G. We will show that AN/N is cyclic in G/N.

Since AN/N is abelian and G/N is not abelian, we find that AN/N is a proper subgroup of G/N which is of order p^3.

If $|AN/N| \le p$, then AN/N is cyclic. From now on, we will assume AN/N is an order p^2 subgroup and try to find a contradiction.

In G/N, write $\bar{h}_1 = \bar{f}_1^{a_1} \bar{f}_2^{a_2} \bar{f}_3^{a_3}$, $\bar{h}_2 = \bar{f}_1^{b_1} \bar{f}_2^{b_2} \bar{f}_3^{b_3}$ for some integers a_j, b_j (recall that $G/N = \langle \bar{f}_1, \bar{f}_2, \bar{f}_3 \rangle$ and $A = \langle h_1, h_2 \rangle$). After suitably changing the generators h_1 and h_2, we will show that there are only three possibilities: $(\bar{h}_1, \bar{h}_2) = (\bar{f}_2, \bar{f}_3)$, $(\bar{f}_1 \bar{f}_3^{a_3}, \bar{f}_2 \bar{f}_3^{b_3})$, $(\bar{f}_1 \bar{f}_2^{a_2}, \bar{f}_3)$ for some integers a_2, a_3, b_3.

Suppose $\bar{h}_1 = \bar{f}_1^{a_1} \bar{f}_2^{a_2} \bar{f}_3^{a_3}$ and $\bar{h}_2 = \bar{f}_1^{b_1} \bar{f}_2^{b_2} \bar{f}_3^{b_3}$ as above. If $a_1 = b_1 = 0$, then $\langle \bar{h}_1, \bar{h}_2 \rangle = \langle \bar{f}_2, \bar{f}_3 \rangle$. Thus after changing the generating elements h_1, h_2, we may assume that $\bar{h}_1 = \bar{f}_2$, $\bar{h}_2 = \bar{f}_3$. This is the first possibility.

If $a_1 \not\equiv 0$ or $b_1 \not\equiv 0 \pmod{p}$, we may assume $1 \le a_1 \le p-1$. Find an integer e such that $1 \le e \le p-1$ and $a_1 e \equiv 1 \pmod{p}$. Use the formulae in Step 2, we get $\bar{h}_1^e = \bar{f}_1 \bar{f}_2^{c_2} \bar{f}_3^{c_3}$. Since $\langle h_1, h_2 \rangle = \langle h_1^e, h_2 \rangle$, without loss of generality, we may assume that $\bar{h}_1 = \bar{f}_1 \bar{f}_2^{a_2} \bar{f}_3^{a_3}$ (i.e. $a_1 = 1$ from the beginning).

Since $\langle h_1, h_2 \rangle = \langle h_1, (h_1^{b_1})^{-1} h_2 \rangle$, we may assume $\bar{h}_1 = \bar{f}_1 \bar{f}_2^{a_2} \bar{f}_3^{a_3}$ and $\bar{h}_2 = \bar{f}_2^{b_2} \bar{f}_3^{b_3}$.

In the case $1 \le b_2 \le p-1$, take an integer e' with $1 \le e' \le p-1$ and $b_2 e' \equiv 1 \pmod{p}$. Use the generating set $\langle h_1, h_2^{e'} \rangle$ for A. Thus we may assume $\bar{h}_1 = \bar{f}_1 \bar{f}_3^{a_3}$, $\bar{h}_2 = \bar{f}_2 \bar{f}_3^{b_3}$. This is the second possibility.

If $b_2 \equiv 0 \pmod{p}$, then $\bar{h}_1 = \bar{f}_1 \bar{f}_2^{a_2} \bar{f}_3^{a_3}$, $\bar{h}_2 = \bar{f}_3^{b_3}$. If $b_3 = 0$, then AN/N is cyclic. Thus $b_3 \not\equiv 0 \pmod{p}$. Changing the generators again, we may assume $\bar{h}_1 = \bar{f}_1 \bar{f}_2^{a_2}$, $\bar{h}_2 = \bar{f}_3$. This is the third possibility.

Step 4. We will show that all three possibilities in Step 3 lead to contradiction.

Suppose $\bar{h}_1 = \bar{f}_2$, $\bar{h}_2 = \bar{f}_3$. Write $h_1 = f_2 f_4^{a_4} f_5^{a_5}$, $h_2 = f_3 f_4^{b_4} f_5^{b_5}$. Since $h_1 h_2 = h_2 h_1$, we get $f_2 f_4^{a_4} f_3 f_4^{b_4} = f_3 f_4^{b_4} f_2 f_4^{a_4}$ (because $f_5 \in Z(G)$). Rewrite this equality with the help of the formulae in Step 2. We get $f_2 f_3 f_4^{a_4+b_4} = f_2 f_3 f_4^{a_4+b_4} f_5$, which is a contradiction.

Suppose $\bar{h}_1 = \bar{f}_1 \bar{f}_3^{a_3}$, $\bar{h}_2 = \bar{f}_2 \bar{f}_3^{b_3}$. In G/N, we have $\bar{h}_1 \bar{h}_2 = \bar{h}_2 \bar{h}_1$, but it is obvious the two elements $\bar{f}_1 \bar{f}_3^{a_3}$, $\bar{f}_2 \bar{f}_3^{b_3}$ do not commute. Done.

Suppose $\bar{h}_1 = \bar{f}_1 \bar{f}_2^{a_2}$, $\bar{h}_2 = \bar{f}_3$. Write $h_1 = f_1 f_2^{a_2} f_4^{a_4} f_5^{a_5}$, $h_2 = f_3 f_4^{b_4} f_5^{b_5}$. Use the fact $h_1 h_2 = h_2 h_1$. It is easy to find a contradiction. ◻

Theorem 2.3. *Let p be an odd prime number and G be a group of order p^5 belonging to the isoclinism family Φ_{10}. Then $B_0(G) \neq 0$.*

Proof. Apply Lemma 2.2. It suffices to show that G satisfies conditions (i), (ii), (iii) in Lemma 2.2.

Case 1. $p = 3$.

It is routine to verify that the groups $\Phi_{10}(1^5)$, $\Phi_{10}(2111)a_0$, $\Phi_{10}(2111)a_1$ in [Ja, page 621] are isomorphic to $G(3^5, 28)$, $G(3^5, 29)$, $G(3^5, 30)$ respectively. All these three groups $G(3^5, i)$ with $28 \le i \le 30$ can be defined as

$$G(3^5, i) = \langle f_1, f_2, f_3, f_4, f_5 \rangle, \quad Z(G(3^5, i)) = \langle f_5 \rangle,$$
$$[f_2, f_1] = f_3,\ [f_3, f_1] = f_4,\ [f_4, f_1] = [f_3, f_2] = f_5,\ [f_4, f_2] = [f_4, f_3] = 1$$

with additional relations

$$\begin{array}{ll} f_1^3 = f_4^3 = f_5^3 = 1,\ f_2^3 = f_4^{-1},\ f_3^3 = f_5^{-1} & \text{for } G(3^5, 28), \\ f_4^3 = f_5^3 = 1,\ f_1^3 = f_5,\ f_2^3 = f_4^{-1},\ f_3^3 = f_5^{-1} & \text{for } G(3^5, 29), \\ f_4^3 = f_5^3 = 1,\ f_1^3 = f_5^{-1},\ f_2^3 = f_4^{-1},\ f_3^3 = f_5^{-1} & \text{for } G(3^5, 30). \end{array}$$

Case 2. $p \ge 5$.

The group $G = \Phi_{10}(1^5)$ in [Ja, page 621] is defined as

$$G = \langle f_1, f_2, f_3, f_4, f_5 \rangle, \quad Z(G) = \langle f_5 \rangle,$$
$$f_i^p = 1 \text{ for } 1 \le i \le 5,$$
$$[f_2, f_1] = f_3,\ [f_3, f_1] = f_4,\ [f_4, f_1] = [f_3, f_2] = f_5,\ [f_4, f_2] = [f_4, f_3] = 1.$$

The group $G = \Phi_{10}(2111)a_r$ in [Ja, page 621] is defined as

$$G = \langle f_1, f_2, f_3, f_4, f_5 \rangle, \quad Z(G) = \langle f_5 \rangle,$$
$$f_1^p = f_5^{\alpha^r},\ f_i^p = 1 \text{ for } 2 \le i \le 5,$$
$$[f_2, f_1] = f_3,\ [f_3, f_1] = f_4,\ [f_4, f_1] = [f_3, f_2] = f_5,\ [f_4, f_2] = [f_4, f_3] = 1$$

where α is the smallest positive integer which is a primitive root (mod p) and $0 \le r \le \gcd\{3, p-1\} - 1$.

The group $G = \Phi_{10}(2111)b_r$ in [Ja, page 621] is defined as

$$G = \langle f_1, f_2, f_3, f_4, f_5 \rangle, \quad Z(G) = \langle f_5 \rangle,$$
$$f_2^p = f_5^{\alpha^r},\ f_1^p = f_i^p = 1 \text{ for } 3 \le i \le 5,$$
$$[f_2, f_1] = f_3,\ [f_3, f_1] = f_4,\ [f_4, f_1] = [f_3, f_2] = f_5,\ [f_4, f_2] = [f_4, f_3] = 1$$

where α is the smallest positive integer which is a primitive root (mod p) and $0 \leq r \leq \gcd\{3, p-1\} - 1$. □

REMARK. In the proof of [Bo, Lemma 5.6, page 478], Bogomolov tried to prove that there do not exist p-groups G of order p^5 with $B_0(G) \neq 0$. He assumed that the commutator group $[G,G]$ was abelian and discussed three situations when the order of $G/[G,G]$ was p^2, p^3, or $\geq p^4$ (in general, if G is a non-abelian group of order p^5, then $[G,G]$ is abelian, since G has an abelian normal subgroup of order p^3 by a theorem of Burnside). The case when $G/[G,G] = p^2$ was reduced to [Bo, Lemma 4.11, page 478] (see the first part of Theorem 2.3). But this lemma is disproved in the proof of the above theorem.

Proof of Theorem 1.13. Suppose that $p^5 \mid n$ for some odd prime number p. Write $n = p^5 m$. By Theorem 2.3 choose a group G_0 of order p^5 satisfying $B_0(G_0) \neq 0$. Define $G = G_0 \times C_m$.

We will prove that $k(G)$ is not stably k-rational (resp. not retract k-rational if k is infinite). Suppose not. Assume that $k(G)$ is stably k-rational (resp. retract k-rational if k is infinite). Then so is $\bar{k}(G)$ over $\bar{k}$ where $\bar{k}$ is the algebraic closure of k. In particular, $\bar{k}(G)$ is retract $\bar{k}$-rational. Since $G = G_0 \times C_m$, by [Sa1, Theorem 1.5; Ka4, Lemma 3.4], we find that $\bar{k}(G_0)$ is retract $\bar{k}$-rational. This implies $B_0(G) = 0$ by Lemma 1.3. A contradiction.

In case $2^6 \mid n$, the proof is similar by applying Theorem 1.5. □

3. Some reduction theorems. We recall several known results in this section.

THEOREM 3.1 (Ahmad, Hajja and Kang [AHK, Theorem 3.1]). *Let L be any field, $L(x)$ the rational function field in one variable over L, and G a finite group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$ where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G \backslash L^G\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property $L(x)^G = L^G(f)$.*

THEOREM 3.2 (Hajja and Kang [HK, Theorem 1]). *Let G be a finite group acting on $L(x_1, \ldots, x_n)$, the rational function field in n variables over a field L. Suppose that*
(i) *for any $\sigma \in G$, $\sigma(L) \subset L$,*
(ii) *the restriction of the action of G to L is faithful,*
(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over L.

Then there exist elements $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ so that $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

THEOREM 3.3 (Fischer [Sw, Theorem 6.1]). *Let G be a finite abelian group of exponent e, and let k be a field containing a primitive e-th root of unity. Then $k(G)$ is rational over k.*

THEOREM 3.4 (Kang and Plans [KP, Theorem 1.3]). *Let k be any field, G_1 and G_2 be two finite groups. If $k(G_1)$ and $k(G_2)$ are rational over k, then so is $k(G_1 \times G_2)$ over k.*

THEOREM 3.5. *Let k be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is a cyclic group, and* (ii) *k contains a primitive e-th root of unity where $e = \exp(G)$.*

(1) (Bogomolov [Bo, Lemma 4.9]) *If k is algebraically closed, then $B_0(G) = 0$.*

(2) (Kang [Ka4, Theorem 5.10]) *If k is an infinite field, then $k(G)$ is retract k-rational. In particular, $B_0(G) = 0$.*

(3) (Kang [Ka2, Theorem 2.2]) *If $\mathbb{Z}[\zeta_n]$ is a unique factorization domain where $n = |G/H|$, then $k(G)$ is rational over k.*

THEOREM 3.6 (Kang [Ka3, Theorem 1.8]). *Let $n \geq 3$ and G be a non-abelian group of order p^n such that G has a cyclic subgroup of index p^2. If k is a field containing a primitive p^{n-2}-th root of unity, then $k(G)$ is rational over k.*

THEOREM 3.7. *Let L be any field containing a field k, $L(x)$ be the rational function field of one variable over L.*

(1) (Saltman [Sa2, Proposition 3.6; Ka4, Lemma 3.4]) *If k is an infinite field, then L is retract k-rational if and only if so is $L(x)$ over k.*

(2) (Saltman [Sa4, Section 2; Ka4, Theorem 3.2]) *The natural map* $\mathrm{Br}_{v,k}(L) \to \mathrm{Br}_{v,k}(L(x))$ *is an isomorphism.*

The following is an elementary result in group theory, whose proof is omitted.

LEMMA 3.8. *Let G be a finite p-group. If H is a normal subgroup of G and $H \neq \{1\}$, then $H \cap Z(G) \neq \{1\}$.*

LEMMA 3.9. *Let G be a finite p-group, $Z(G)$ be its center. Let $\theta : G \to GL(W)$ be a linear representation of G where W is a finite-dimensional vector space over some field k. Assume that, for any $g \in Z(G) \backslash \{1\}$, $\theta(g) \neq 1$. Then θ is a faithful representation of G, i.e. θ is injective.*

Proof. Let $N = \mathrm{Ker}(\theta)$. If $N \neq \{1\}$, then $N \cap Z(G) \neq \{1\}$ by Lemma 3.8. It follows that there is some $g \in Z(G) \backslash \{1\}$ with $\theta(g) = 1$. A contradiction. □

We recall the definitions of G-lattices and purely monomial actions.

DEFINITION 3.10. Let G be a finite group. A G-lattice M is a finitely generated $\mathbb{Z}[G]$-module which is $\mathbb{Z}$-free as an abelian group, i.e. $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot x_i$ with a $\mathbb{Z}[G]$-module structure.

If k is a field and $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot x_i$ is a G-lattice, define $k(M) = k(x_1, \ldots, x_n)$ the rational function field over k with G acting by k-automorphisms defined as follows: For any $\sigma \in G$, if $\sigma \cdot x_j = \sum_{1 \leq i \leq n} a_{ij} x_i$ in M, then $\sigma \cdot x_j = \prod_{1 \leq i \leq n} x_i^{a_{ij}}$ in $k(M)$. The action of G on $k(M)$ is called a purely monomial k-action [HKK, Definition 1.15]. The fixed field of $k(M)$ under the G-action is denoted by $k(M)^G$.

THEOREM 3.11 (Barge [Bar]). *Let G be a finite group, k be an algebraically closed field with* $\gcd\{|G|, \mathrm{char}\, k\} = 1$. *The following two statements are equivalent,*

(i) *all the Sylow subgroups of G are bicyclic.*

(ii) $\mathrm{Br}_{v,k}(k(M)^G) = 0$ *for all G-lattices M.*

Proof. In [Bar], the above theorem is proved for the case $k = \mathbb{C}$ but the arguments there work in the general case.

Here is an alternative proof for the direction "(i) $\Rightarrow$ (ii)" of the above theorem: apply [Sa5, Theorem 12]. □

4. $B_0(G) = 0$ for the groups not belonging to Φ_6 and Φ_{10}. Let p be an odd prime number and G be a group of order p^5 belonging to the isoclinism family Φ_i where $1 \le i \le 9$. We will show that $B_0(G) = 0$ in this section and the next section.

We adopt the classification of groups of order p^5 by R. James [Ja]. For groups of order p^5, there are in total 10 isoclinism families Φ_i where $1 \le i \le 10$ [Ja, pages 619–621]. When $p \ge 5$, the numbers of groups in the family Φ_i where $1 \le i \le 10$ are

$$7,\ 15,\ 13,\ p+8,\ 2,\ p+7,\ 5,\ 1,\ \gcd\{3,p-1\}+2,\ \gcd\{4,p-1\}+\gcd\{3,p-1\}+1$$

respectively. The same numbers hold true for groups of order 3^5 except for Φ_6 and Φ_{10}. The numbers of groups of order 3^5 in Φ_6 and Φ_{10} are 7 and 3 respectively.

We call the attention of the reader to two conventions of James's paper [Ja]. First the notation $\alpha_{i+1}^{(p)}$ is not α_{i+1}^{p} in general; it is defined as $\alpha_{i+1}^{(p)} = \alpha_{i+1}^{p}\alpha_{i+2}^{\binom{p}{2}}\cdots\alpha_{i+k}^{\binom{p}{k}}\cdots\alpha_{i+p}$ where $\alpha_{i+2},\ldots,\alpha_{i+p}$ are suitably defined [Ja, p. 614, lines 8–10]. In particular, for the groups of order p^5 with $p \ge 5$ defined in [Ja, pages 619–621], $\alpha_{i+1}^{(p)} = \alpha_{i+1}^{p}$. On the other hand, when $p = 3$, the relations $\alpha_1^{(3)} = \alpha_2^{(3)} = \alpha_3^{(3)} = \alpha_4^{(3)} = 1$ for the group $\Phi_9(2111)a$ in [Ja, page 621] are equivalent to the relations $\alpha_1^3 = \alpha_3^{-1}\alpha_4$, $\alpha_2^3 = \alpha_4^{-1}$ and $\alpha_3^3 = \alpha_4^3 = 1$. The second convention of [Ja] is that all relations of the form $[\alpha,\beta] = 1$ are omitted from the list [Ja, p. 614, lines 11–12].

Theorem 4.1. *Let p be an odd prime number and G be a group of order p^5 and of exponent e. If k is an infinite field containing a primitive e-th root of unity and G belongs to the isoclinism family Φ_i where $1 \le i \le 4$ or $8 \le i \le 9$, then $k(G)$ is retract rational over k. In particular, $B_0(G) = 0$.*

Proof. If G belongs to the isoclinism family Φ_i where $1 \le i \le 4$ or $8 \le i \le 9$, it is not difficult (from the list of [Ja, pages 619–621]) to find an abelian normal subgroup H such that G/H is cyclic. Thus $k(G)$ is retract k-rational and $B_0(G) = 0$ by Theorem 3.5. But we can say more about $k(G)$.

Step 1. The groups in Φ_1 are abelian groups. If $G \in \Phi_1$, then $k(G)$ is k-rational by Theorem 3.3.

Step 2. Some groups in Φ_2 are direct products. If $G \in \Phi_2$ and $G \simeq G_1 \times G_2$ with $|G_1|, |G_2| < |G|$, then both $k(G_1)$ and $k(G_2)$ are k-rational by Theorem 1.6. Thus $k(G)$ is k-rational by Theorem 3.4.

For the other groups $G \in \Phi_2$, it is easy to verify that $G/Z(G) \simeq C_p \times C_p$. Let $\bar{g}$ be an element of order p in $G/Z(G)$ and g be a preimage of $\bar{g}$ in G. Then $H = \langle Z(G), g\rangle$ is abelian and normal in G with $G/H \simeq C_p$. By Theorem 3.5, $k(G)$ is retract k-rational.

Step 3. If G belongs to Φ_3 or Φ_4, it is not difficult to show that G contains an abelian normal subgroup of index p by checking the list provided in [Ja, page 620].

Alternatively, we may use the fact asserted in Bender's paper [Be, p.69]: If G is a group of order p^5 (where $p \ge 3$) with $|Z(G)| = p^2$ and $|[G,G]| \le p^2$, then G contains an abelian normal subgroup of index p. Assuming this fact, since $|Z(G)| = |[G,G]| = p^2$ (if $G \in \Phi_3$ and G is not a direct product) and $|Z(G)| = |[G,G]| = p^2$ (if $G \in \Phi_4$), we are done.

In either case, apply Theorem 3.5. We find that $k(G)$ is retract k-rational.

Step 4. If $G \in \Phi_8$, the family Φ_8 consists of only one group $G \simeq C_{p^3} \rtimes C_{p^2}$. Apply Theorem 3.6. We find $k(G)$ is k-rational.

Step 5. If $G \in \Phi_9$, check the list of the generators and relations of these groups in [Ja, p.621]. We find that these groups G are generated by elements f_0, f_1, f_2, f_3, f_4 and, besides other relations, they satisfy the relations

$$\begin{aligned}
&[f_i, f_0] = f_{i+1} \text{ for } 1 \le i \le 3,\\
&[f_i, f_j] = 1 \quad\ \text{ for } 1 \le i, j \le 4, \text{ and}\\
&\{f_1, f_2, f_3, f_4\} \text{ generates a subgroup of index } p.
\end{aligned}$$

Define $H = \langle f_1, f_2, f_3, f_4\rangle$. It follows that H is an abelian normal subgroup of index p. Apply Theorem 3.5. □

The following theorem is essentially due to Barge [Bar]. We include a proof for the convenience of the reader.

THEOREM 4.2. *Let $G = A \rtimes G_0$ be a finite group where A and G_0 are subgroups of G such that* (i) *A is an abelian normal subgroup of G with G_0 acting on A, and* (ii) *all the Sylow subgroups of G_0 are bicyclic. If k is an algebraically closed field with* $\gcd\{|G|, \operatorname{char} k\} = 1$, *then* $\operatorname{Br}_{v,k}(k(G)) = 0$.

Proof. Step 1. Let $V = \bigoplus_{g\in G} k\cdot x(g)$ with the G-action defined by $g\cdot x(h) = x(gh)$ for any $g, h \in G$. Then $k(G) = k(x(g) : g \in G)^G$ by definition.

Consider a subspace $W = \bigoplus_{\tau\in A} k \cdot x(\tau)$. Since A is abelian, the action of A on W can be diagonalized. Explicitly, there is a linear change of variables of W with $W = \bigoplus_{1\le i\le n} k\cdot x_i$ (where $n = |A|$) such that, for all $\tau \in A$, $\tau\cdot x_i \in k\cdot x_i$ for $1 \le i \le n$. Thus we may write $\tau \cdot x_i = \chi_i(\tau)x_i$ where $\chi_i : A \to k^\times$ is a linear character of A.

For any $h \in G_0$, define $W(h) = \bigoplus_{\tau\in A} k \cdot x(h\tau)$. Since $x(h\tau) = h \cdot x(\tau)$, we find that $W(h) = h(W) = h(\bigoplus_{1\le i\le n} k \cdot x_i) = \bigoplus_{1\le i\le n} k \cdot (h \cdot x_i)$. Note that $\tau \cdot (h \cdot x_i) = h(h^{-1}\tau h) \cdot x_i = \chi_i(h^{-1}\tau h)(h \cdot x_i)$ for any $\tau \in A$, any $h \in G_0$.

Write $y_i(g) = g \cdot x_i$ for any $g \in G_0$, any $1 \le i \le n$. It follows that $k(x(g) : g \in G) = k(y_i(g) : 1 \le i \le n, g \in G_0)$. The action of G on $y_i(g)$ is given as follows: For all $\tau \in A$, $g, h \in G_0$, $1 \le i \le n$, we have

$$\tau \cdot y_i(g) = \chi_i(g^{-1}\tau g)y_i(g), \quad h \cdot y_i(g) = y_i(hg).$$

It remains to show that $\operatorname{Br}_{v,k}(k(y_i(g) : 1 \le i \le n, g \in G_0)^G) = 0$.

Step 2. Define a G_0-lattice $N = \bigoplus_{g\in G_0,1\le i\le n} \mathbb{Z} \cdot y_i(g)$ with

$$h \cdot y_i(g) = y_i(hg)$$

for any $h, g \in G_0$.

Let us choose $\tau_1, \dots, \tau_m \in A$ such that $A = \langle\tau_1, \dots, \tau_m\rangle$. Let ζ be a root of unity such that $\langle\chi_i(\tau) : \tau \in A, 1 \le i \le n\rangle = \langle\zeta\rangle$. Regard $\langle\zeta\rangle^m := \langle\zeta\rangle \times \cdots \times \langle\zeta\rangle$ (the direct product of m copies of $\langle\zeta\rangle$) as a $\mathbb{Z}[G_0]$-module where the action of G_0 is trivial. Define a morphism $\Phi\colon N \to \langle\zeta\rangle^m$ of $\mathbb{Z}[G_0]$-modules by $\Phi\big(\sum_{g\in G_0,1\le i\le n} a_{i,g}y_i(g)\big) = \Big(\frac{\tau_1(Y)}{Y}, \frac{\tau_2(Y)}{Y}, \dots, \frac{\tau_m(Y)}{Y}\Big)$ where $Y = \prod_{g\in G_0,1\le i\le n} y_i(g)^{a_{i,g}} \in k(y_i(g) : 1 \le i \le n, g \in G_0)$.

Define $M = \operatorname{Ker}(\Phi)$. Clearly M is a G_0-lattice.

It is easy to see that $k(y_i(g) : 1 \le i \le n, g \in G_0)^A = k(M)$, i.e. if $M = \bigoplus_{1\le l\le e} \mathbb{Z} \cdot z_l$, then $k(y_i(g) : 1 \le i \le n, g \in G_0)^A = k(z_1, z_2, \dots, z_e)$ where each z_l is a monomial in $y_i(g)$'s.

Moreover, $k(y_i(g) : 1 \leq i \leq n, g \in G_0)^G = \{k(y_i(g) : 1 \leq i \leq n, g \in G_0)^A\}^{G_0} = k(M)^{G_0}$. The group G_0 acts on $k(M)$ by purely monomial k-automorphisms (see Definition 3.10). Applying Theorem 3.11, we find that $\mathrm{Br}_{v,k}(k(M)^{G_0}) = 0$. Hence the result. □

REMARK. Saltman shows that, if $G = A \rtimes G_0$ where A is abelian normal such that (i) $\gcd\{|A|, |G_0|\} = 1$, and (ii) both $k(A)$ and $k(G_0)$ are retract k-rational, then $k(G)$ is also retract k-rational [Sa1, Theorem 3.5; Ka4, Theorem 3.5].

Now we turn to groups belonging to the isoclinism family Φ_5 for groups of order p^5.

DEFINITION 4.3. Let p be an odd prime number. The isoclinism family Φ_5 for groups of order p^5 consists of two groups: $\Phi_5(2111)$ and $\Phi_5(1^5)$ (see [Ja, page 620]). These two groups are defined as follows.

For $G = \Phi_5(2111)$, $G = \langle f_i : 1 \leq i \leq 5\rangle$ with $Z(G) = \langle f_5\rangle$ and relations

$$[f_1, f_2] = [f_3, f_4] = f_5,\ [f_1, f_3] = [f_2, f_3] = [f_1, f_4] = [f_2, f_4] = 1,$$
$$f_1^p = f_5,\ f_i^p = 1 \text{ for } 2 \leq i \leq 5.$$

For $G = \Phi_5(1^5)$, $G = \langle f_i : 1 \leq i \leq 5\rangle$ with $Z(G) = \langle f_5\rangle$ and relations

$$[f_1, f_2] = [f_3, f_4] = f_5,\ [f_1, f_3] = [f_2, f_3] = [f_1, f_4] = [f_2, f_4] = 1,$$
$$f_i^p = 1 \text{ for } 1 \leq i \leq 5.$$

Note that both $\Phi_5(2111)$ and $\Phi_5(1^5)$ are extra-special p-groups.

THEOREM 4.4. *Let p be an odd prime number and G belong to the isoclinism family Φ_5 for groups of order p^5. Then $B_0(G) = 0$.*

Proof. Choose an algebraically closed field k with $\operatorname{char} k \neq p$ (in particular, we may choose $k = \mathbb{C}$). If $\mathrm{Br}_{v,k}(k(G)) = 0$, then $B_0(G) = 0$ by Theorem 1.4. Hence we will show that $\mathrm{Br}_{v,k}(k(G)) = 0$ by using Theorem 4.2.

For $G = \Phi_5(2111)$ or $\Phi_5(1^5)$, write $G = A \rtimes G_0$ where $A = \langle f_1, f_3, f_5\rangle$ and $G_0 = \langle f_2, f_4\rangle$. Conditions (i), (ii), (iii) in Theorem 4.2 are satisfied. Hence we may apply Theorem 4.2. Done. □

Now we consider groups in the isoclinism family Φ_7. Since the relations for $p = 3$ and $p \geq 5$ are not the same (due to the notation $\alpha_1^{(p)} = 1$), we define these groups separately.

DEFINITION 4.5. The isoclinism family Φ_7 for groups of order 3^5 consists of five groups: $G = G(3^5, i)$ where $56 \leq i \leq 60$ and $G(3^5, i)$ is the GAP code number. These groups G are defined by $G = \langle f_i : 1 \leq i \leq 5\rangle$ with $Z(G) = \langle f_5\rangle$, common relations

$$[f_2, f_1] = f_4,\ [f_3, f_2] = [f_4, f_1] = f_5,\ [f_3, f_1] = [f_4, f_2] = [f_4, f_3] = 1,$$

but with extra relations

(1) for $G = G(3^5, 56) : f_i^3 = 1$ for $1 \leq i \leq 5$;
(2) for $G = G(3^5, 57) : f_2^3 = f_5,\ f_1^3 = f_i^3 = 1$ for $3 \leq i \leq 5$;
(3) for $G = G(3^5, 58) : f_2^3 = f_5^2,\ f_1^3 = f_i^3 = 1$ for $3 \leq i \leq 5$;
(4) for $G = G(3^5, 59) : f_1^3 = f_2^{-3} = f_5,\ f_i^3 = 1$ for $3 \leq i \leq 5$;
(5) for $G = G(3^5, 60) : f_3^3 = f_5,\ f_1^3 = f_2^3 = f_4^3 = f_5^3 = 1$.

Note that, in the notation of [Ja, page 621], the GAP groups $G(3^5, i)$, $56 \le i \le 60$, correspond to $\Phi_7(2111)b_1$, $\Phi_7(2111)b_\nu$, $\Phi_7(1^5)$, $\Phi_7(2111)a$ and $\Phi_7(2111)c$ respectively.

THEOREM 4.6. *If G is a group belonging to the isoclinism family Φ_7 for groups of order 3^5, then $B_0(G) = 0$.*

Proof. The proof is the same as that of Theorem 4.8 except for $G = G(3^5, 59)$ and $G = G(3^5, 60)$. Write $G = A \rtimes G_0$ where $G_0 \simeq C_3 \times C_3$, and

(i) if $G = G(3^5, 56)$, $A = \langle f_2, f_4, f_5\rangle$, $G_0 = \langle f_1, f_3\rangle$;

(ii) if $G = G(3^5, 57)$ or $G(3^5, 58)$, $A = \langle f_2, f_4\rangle$, $G_0 = \langle f_1, f_3\rangle$.

It can be shown as before that $B_0(G) = 0$ when $G = G(3^5, 56), G(3^5, 57)$ or $G(3^5, 58)$. It remains to prove that $B_0(G) = 0$ for $G = G(3^5, 59)$ and $G = G(3^5, 60)$. We will indicate only the proof for $G = G(3^5, 59)$. The case $G(3^5, 60)$ is almost the same.

Step 1. Let η be a primitive 9th root of unity and $\zeta = \eta^3$. We will construct a faithful 9-dimensional representation of $G = G(3^5, 59)$ over k, which may be embedded into the regular representation of G. The method is similar to that of Step 1 in the proof of Theorem 4.2.

Let $A = \langle f_1, f_3\rangle = \langle f_1, f_3, f_5\rangle \simeq C_9 \times C_3$ act on the 1-dimensional space $k \cdot X$ by $f_1 \cdot X = \eta X$, $f_3 \cdot X = X$. It follows that $f_5 \cdot X = \zeta X$.

The above action defines a linear character $\rho : A \to k^\times$. The induced representation can be written explicitly as follows.

Define $V = \bigoplus_{1\le i\le 9} k \cdot x_i$ where $x_1 = X$, $x_2 = f_4 \cdot X$, $x_3 = f_4^2 \cdot X$, $x_4 = f_2 \cdot X$, $x_5 = f_2 f_4 \cdot X$, $x_6 = f_2 f_4^2 \cdot X$, $x_7 = f_2^2 \cdot X$, $x_8 = f_2^2 f_4 \cdot X$, $x_9 = f_2^2 f_4^2 \cdot X$. The action of G on x_i is given by

$$
\begin{aligned}
f_1 &: x_1 \mapsto \eta x_1,\ x_2 \mapsto \eta^7 x_2,\ x_3 \mapsto \eta^4 x_3,\ x_4 \mapsto \eta^4 x_6,\ x_5 \mapsto \eta x_4,\ x_6 \mapsto \eta^7 x_5,\\
&\quad x_7 \mapsto \eta^7 x_8,\ x_8 \mapsto \eta^4 x_9,\ x_9 \mapsto \eta x_7,\\
f_2 &: x_1 \mapsto x_4 \mapsto x_7 \mapsto \zeta^2 x_1,\ x_2 \mapsto x_5 \mapsto x_8 \mapsto \zeta^2 x_2,\ x_3 \mapsto x_6 \mapsto x_9 \mapsto \zeta^2 x_3,\\
f_3 &: x_1 \mapsto x_1,\ x_2 \mapsto x_2,\ x_3 \mapsto x_3,\ x_4 \mapsto \zeta x_4,\ x_5 \mapsto \zeta x_5,\ x_6 \mapsto \zeta x_6,\\
&\quad x_7 \mapsto \zeta^2 x_7,\ x_8 \mapsto \zeta^2 x_8,\ x_9 \mapsto \zeta^2 x_9,\\
f_4 &: x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1,\ x_4 \mapsto x_5 \mapsto x_6 \mapsto x_4,\ x_7 \mapsto x_8 \mapsto x_9 \mapsto x_7,\\
f_5 &: x_i \mapsto \zeta x_i \text{ for } 1 \le i \le 9.
\end{aligned}
$$

By Lemma 3.9, it is a faithful representation of G. This representation can be embedded into the regular representation of G, because it is an irreducible representation of G.

Apply Theorem 3.2. We find that $k(G)$ is rational over $k(x_i : 1 \le i \le 9)^G$.

Step 2. Define $u_1 = x_4/x_1$, $u_2 = x_7/x_4$, $u_3 = x_2/x_1$, $u_4 = x_3/x_2$, $u_5 = x_5/x_4$, $u_6 = x_6/x_5$, $u_7 = x_8/x_7$, $u_8 = x_9/x_8$. Apply Theorem 3.1. We find that $k(x_i : 1 \le i \le 9)^G = k(u_i : 1 \le i \le 8)^G(u_0)$ for some element u_0 fixed by the action of G.

We conclude that $k(G)$ is rational over $k(u_i : 1 \le i \le 8)^G$.

By Theorem 3.7 and Theorem 1.4, it follows that $B_0(G) \simeq \mathrm{Br}_{v,k}(k(u_i : 1 \le i \le 8)^G)$.

Step 3. Now consider the group $H = G(3^5, 58)$. We will repeat the procedure of Step 1 and Step 2 for H.

Namely, define $B = \langle f_1, f_3, f_5\rangle \simeq C_3 \times C_3 \times C_3$. Let B act on $k \cdot Y$ by $f_1 \cdot Y = f_3 \cdot Y = Y$, $f_5 \cdot Y = \zeta Y$.

Construct the induced representation $W = \bigoplus_{1\le i\le 9} k\cdot y_i$ where $y_1 = Y$, $y_2 = f_4\cdot Y$, $y_3 = f_4^2 \cdot Y$, $y_4 = f_2 \cdot Y$, $y_5 = f_2 f_4 \cdot Y$, $y_6 = f_2 f_4^2 \cdot Y$, $y_7 = f_2^2 \cdot Y$, $y_8 = f_2^2 f_4 \cdot Y$, $y_9 = f_2^2 f_4^2 \cdot Y$. The actions of f_2, f_3, f_4, f_5 on W are the same as those on V (just replace x_i's by y_i's), but

$$\begin{aligned} f_1 : x_1 \mapsto x_1,\ x_2 \mapsto \zeta^2 x_2,\ x_3 \mapsto \zeta x_3,\ x_4 \mapsto \zeta x_6,\ x_5 \mapsto x_4,\ x_6 \mapsto \zeta^2 x_5, \\ x_7 \mapsto \zeta^2 x_8,\ x_8 \mapsto \zeta x_9,\ x_9 \mapsto x_7. \end{aligned}$$

The coincidence of the group actions can be explained as follows. The relations of $G(3^5, 59)$ and $G(3^5, 58)$ are almost the same except for $f_1^3 = f_5$ in $G(3^5, 59)$ and $f_1^3 = 1$ in $G(3^5, 58)$.

Step 4. Define $v_1 = y_4/y_1$, $v_2 = y_7/y_4$, $v_3 = y_2/y_1$, $v_4 = y_3/y_2$, $v_5 = y_5/y_4$, $v_6 = y_6/y_5$, $v_7 = y_8/y_7$, $v_8 = y_9/y_8$. Similar to Step 2, we get that $k(H)$ is rational over $k(v_i : 1 \le i \le 8)^H$ and $B_0(H) \simeq \mathrm{Br}_{v,k}(k(v_i : 1 \le i \le 8)^H)$.

Compare the actions of G on $u_1, \ldots, u_8$ with the actions of H on $v_1, \ldots, v_8$. We find they are the same!

Thus $k(u_i : 1 \le i \le 8)^G \simeq k(v_i : 1 \le i \le 8)^G$ over k.

Hence $B_0(G) \simeq \mathrm{Br}_{v,k}(k(u_i : 1 \le i \le 8)^G) \simeq \mathrm{Br}_{v,k}(k(v_i : 1 \le i \le 8)^H) \simeq B_0(H)$. But $B_0(H) = 0$ has been proved at the beginning. Hence $B_0(G) = 0$. □

Definition 4.7. Let p be a prime number and $p \ge 5$. The isoclinism family Φ_7 for groups of order p^5 consists of five groups: $G = \Phi_7(2111)a$, $\Phi_7(2111)b_1$, $\Phi_7(2111)b_\nu$ (where $2 \le \nu \le p-1$ and ν is a fixed quadratic non-residue modulo p), $\Phi_7(2111)c$ and $\Phi_7(1^5)$ (see [Ja, page 621]). These groups G are defined by $G = \langle f_i : 0 \le i \le 4\rangle$ with $Z(G) = \langle f_3\rangle$, common relations

$$[f_1, f_0] = f_2,\ [f_2, f_0] = [f_1, f_4] = f_3,\ [f_4, f_0] = [f_2, f_1] = [f_4, f_2] = 1,$$

but with extra relations

(1) for $G = \Phi_7(2111)a : f_0^p = f_3,\ f_i^p = 1$ for $1 \le i \le 4$;
(2) for $G = \Phi_7(2111)b_1 : f_1^p = f_3,\ f_0^p = f_i^p = 1$ for $2 \le i \le 4$;
(3) for $G = \Phi_7(2111)b_\nu : f_1^p = f_3^\nu,\ f_0^p = f_i^p = 1$ for $2 \le i \le 4$;
(4) for $G = \Phi_7(2111)c : f_4^p = f_3,\ f_i^p = 1$ for $0 \le i \le 3$;
(5) for $G = \Phi_7(1^5) : f_i^p = 1$ for $0 \le i \le 4$.

Theorem 4.8. *Let p be a prime number and $p \ge 5$. If G belongs to the isoclinism family Φ_7 for groups of order p^5, then $B_0(G) = 0$.*

Proof. The proof is similar to that of Theorem 4.4 by applying Theorem 4.2. Except for groups $G = \Phi_7(2111)a$ or $\Phi_7(2111)c$, we write $G = A \rtimes G_0$ for suitable subgroups A and G_0. Here are the subgroups we choose.

If $G = \Phi_7(2111)b_1$ or $\Phi_7(2111)b_\nu$, $A = \langle f_1, f_2\rangle$, $G_0 = \langle f_0, f_4\rangle$.

If $G = \Phi_7(1^5)$, $A = \langle f_0, f_3, f_4\rangle$, $G_0 = \langle f_1, f_2\rangle$.

When $G = \Phi_7(2111)a$ or $\Phi_7(2111)c$, it is not difficult to show that $\mathbb{C}(G)$ is isomorphic to $\mathbb{C}(H)$ where H is any one of the groups $H = \Phi_7(2111)b_1$, $\Phi_7(2111)b_\nu$ or $H = \Phi_7(1^5)$ by the same method as in the proof of Theorem 4.6. We outline the case of $G = \Phi_7(2111)a$ and $H = \Phi_7(1^5)$ as follows. The situation for $G = \Phi_7(2111)c$ and $H = \Phi_7(1^5)$ is almost the same.

Step 1. Denote by η a primitive p^2-th root of unity and $\zeta = \eta^p$. We will construct a faithful p^2-dimensional representation of $G = \Phi_7(2111)a$ over k.

In the group G, define $A = \langle f_0, f_4\rangle = \langle f_0, f_3, f_4\rangle \simeq C_{p^2} \times C_p$. Let it act on the 1-dimensional space $k \cdot X$ by $f_0 \cdot X = \eta X$, $f_4 \cdot X = X$. It follows that $f_3 \cdot X = \zeta X$.

The above action defines a linear character $\rho : A \to k^\times$. The induced representation can be written as $V = \bigoplus_{0 \le i,j \le p-1} k \cdot x_{i,j}$ where $x_{i,j} = f_1^i f_2^j \cdot X$.

Step 2. Similarly, in the group $H = \Phi_7(1^5)$, define $B = \langle f_0, f_3, f_4\rangle \simeq C_3 \times C_3 \times C_3$. Let B act on $k \cdot Y$ by $f_0 \cdot Y = f_4 \cdot Y = Y$, $f_3 \cdot Y = \zeta Y$.

Construct the induced representation $W = \bigoplus_{0 \le i,j \le p-1} k \cdot x_{i,j}$ where $x_{i,j} = f_1^i f_2^j \cdot Y$; here we use the same notation $x_{i,j}$ as in Step 1 on purpose.

Step 3. It is easy to verify the actions of G and H on these $x_{i,j}$. For both the groups G and H, we have

$$f_1 : x_{i,j} \mapsto x_{i+1,j},\ f_2 : x_{i,j} \mapsto x_{i,j+1},\ f_3 : x_{i,j} \mapsto \zeta x_{i,j},\ f_4 : x_{i,j} \mapsto \zeta^{-i} x_{i,j}.$$

The actions of f_0 are different. For the group G, $f_0(x_{i,j}) = \zeta^{i-j}\eta\, x_{i,j-i}$; for the group H, $f_0(x_{i,j}) = \zeta^{i-j}\, x_{i,j-i}$.

The action of G on $\mathbb{P}(V)$ and that of H on $\mathbb{P}(W)$ become the same. Hence the result. ◻

5. $B_0(G) = 0$ for the groups belonging to Φ_6. Let p be an odd prime number. Throughout this section g is the smallest positive integer which is a primitive root modulo p, and ν is the smallest positive integer which is a quadratic non-residue modulo p.

Definition 5.1. Let p be an odd prime number. The isoclinism family Φ_6 for groups of order p^5 consists of the groups $G = \Phi_6(221)a$, $\Phi_6(221)b_r$ (where $1 \le r \le (p-1)/2$), $\Phi_6(221)c_r$ (where $r = 1$ or ν), $\Phi_6(221)d_0$, $\Phi_6(221)d_r$ (where $1 \le r \le (p-1)/2$), $\Phi_6(2111)a$ (this group exists only for $p \ge 5$), $\Phi_6(2111)b_r$ (where $r = 1$ or ν; these groups exist only for $p \ge 5$), and $\Phi_6(1^5)$. When $p \ge 5$, there are $p + 7$ such groups; when $p = 3$, there are 7 such groups (see [Ja, pages 620–621]). These groups G are defined by $G = \langle f_1, f_2, f_0, h_1, h_2\rangle$ with $Z(G) = \langle h_1, h_2\rangle$, common relations

$$[f_1, f_2] = f_0,\ [f_0, f_1] = h_1,\ [f_0, f_2] = h_2,\ f_0^p = h_1^p = h_2^p = 1,$$

but with extra relations

(1) for $G = \Phi_6(221)a : f_1^p = h_1,\ f_2^p = h_2$;
(2) for $G = \Phi_6(221)b_r : f_1^p = h_1^k,\ f_2^p = h_2$ where $k = g^r$;
(3) for $G = \Phi_6(221)c_r : f_1^p = h_2^{-r/4},\ f_2^p = h_1^r h_2^r$;
(4) for $G = \Phi_6(221)d_0 : f_1^p = h_2,\ f_2^p = h_1^\nu$;
(5) for $G = \Phi_6(221)d_r : f_1^p = h_2^k,\ f_2^p = h_1 h_2$ where $4k = g^{2r+1} - 1$;
(6) for $G = \Phi_6(2111)a : f_1^p = h_1,\ f_2^p = 1$;
(7) for $G = \Phi_6(2111)b_r : f_1^p = 1,\ f_2^p = h_1^r$;
(8) for $G = \Phi_6(1^5) : f_1^p = 1,\ f_2^p = 1$.

(Note that whenever the exponent of h_2 is fractional, it is understood that it is taken modulo p, which is the order of h_2.)

Before proving $B_0(G) = 0$ for the groups G in Definition 5.1, we recall two results in group cohomology.

THEOREM 5.2 (Dekimpe, Hartl and Wauters [DHW], Huebschmann [Hu1], [Hu2], [Hu3]). *Let G be a finite group, N a normal subgroup of G. Then the Hochschild–Serre spectral sequence gives rise to the following 7-term exact sequence*

$$\begin{aligned}0 &\to H^1(G/N,\mathbb{Q}/\mathbb{Z}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^1(N,\mathbb{Q}/\mathbb{Z})^G \to H^2(G/N,\mathbb{Q}/\mathbb{Z})\\ &\to H^2(G,\mathbb{Q}/\mathbb{Z})_1 \to H^1(G/N,H^1(N,\mathbb{Q}/\mathbb{Z})) \xrightarrow{\lambda} H^3(G/N,\mathbb{Q}/\mathbb{Z})\end{aligned}$$

where $H^2(G,\mathbb{Q}/\mathbb{Z})_1 = \mathrm{Ker}\{H^2(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(N,\mathbb{Q}/\mathbb{Z})\}$ and λ is defined as follows. Choose a section $u\colon G/N \to G$ and define a 2-cocycle $\varepsilon\colon G/N \times G/N \to N$ satisfying $u(\tau)u(\tau') = \varepsilon(\tau,\tau')u(\tau\tau')$ for any $\tau,\tau' \in G/N$. For each 1-cocycle $\gamma\colon G/N \to H^1(N,\mathbb{Q}/\mathbb{Z})$, the map λ is defined by

$$\begin{aligned}\lambda\colon H^1(G/N,H^1(N,\mathbb{Q}/\mathbb{Z})) &\longrightarrow H^3(G/N,\mathbb{Q}/\mathbb{Z})\\ \gamma &\longmapsto \lambda(\gamma) = c\end{aligned}$$

where $c : G/N \times G/N \times G/N \to \mathbb{Q}/\mathbb{Z}$ is the 3-cocycle defined as $c(\tau_1,\tau_2,\tau_3) = ({}^{u(\tau_1\tau_2)}\gamma(\tau_3))(\varepsilon(\tau_1,\tau_2))$ for all $\tau_1,\tau_2,\tau_3 \in G/N$.

Proof. See [DHW] for details.

The formula for λ is summarized in [DHW, page 21, formula (6)]. If $\gamma\colon G/N \to H^1(N,M)$ is a 1-cocycle where M is a G-module, $[\gamma]$ denotes its cohomology class in $H^1(G/N,H^1(N,M))$ in the paper [DHW]. The image $\lambda([\gamma]) \in H^3(G/N,M^N)$ is represented by a 3-cocycle $c\colon G/N \times G/N \times G/N \to M^N$ which is given on [DHW, page 21]. Note that the definition of $-\delta^0\colon M \to \mathrm{Der}(N,M)$ can be found on [DHW, page 14].

When M is a trivial G-module, $-\delta^0$ is a zero map and therefore the map $F'\colon G/N \times G/N \to M$ on [DHW, page 21] can be chosen to be a zero map. Consequently, $c(q_1,q_2,q_3) = ({}^{s_1(q_1q_2)}s_2D(q_3))(F_1(q_1,q_2))$ for any $q_1,q_2,q_3 \in G/N$. This is our formula when $M = \mathbb{Q}/\mathbb{Z}$. ∎

THEOREM 5.3. *Let p be a prime number, $C_p = \langle\sigma\rangle$ and M be a C_p-module. For any 1-cocycle $\beta\colon C_p \to M$, the following map*

$$\begin{aligned}\Phi : H^1(C_p,M) &\longrightarrow H^3(C_p,M)\\ \beta &\longmapsto \Phi(\beta) = \gamma\end{aligned}$$

is a group isomorphism where $\gamma : C_p \times C_p \times C_p \to M$ is a 3-cocycle defined as

$$\gamma(\sigma^i,\sigma^j,\sigma^l) = \begin{cases} 0 & \text{if } 0 \le i+j \le p-1\\ \left({}^{\sigma^{i+j}}\beta\right)(\sigma^l) & \text{if } i+j \ge p\end{cases}$$

where $0 \le i,j,l \le p-1$.

Proof. By [Se, page 149, Theorem 14], the 2-cocycle $\alpha : C_p \times C_p \to \mathbb{Z}$ defined as

$$\alpha(\sigma^i,\sigma^j) = \begin{cases} 0 & \text{if } 0 \le i+j \le p-1\\ 1 & \text{if } i+j \ge p\end{cases}$$

represents a "fundamental" cohomology class in $H^2(C_p, \mathbb{Z})$ such that, for any C_p-module M, the map

$$\begin{aligned}\Phi : H^1(C_p, M) &\longrightarrow H^3(C_p, M)\\ \beta &\longmapsto \Phi(\beta) = \alpha \cup \beta\end{aligned}$$

is an isomorphism where $\alpha \cup \beta$ is the cup product. It is easy to check that $\alpha \cup \beta = \gamma$ where γ is defined in the statement of this theorem. ☐

THEOREM 5.4. *Let G be the group $\Phi_6(221)a$ in Definition 5.1. Then $B_0(G) = 0$.*

Proof. Step 1. Write $G = \langle f_1, f_2, f_0, h_1, h_2\rangle$. Choose $N = \langle f_1, f_0, h_1, h_2\rangle$; N is a normal subgroup of G. We will apply Theorem 5.2 to the group extension $1 \to N \to G \to G/N \to 1$.

Since $G/N = \langle \bar{f_2}\rangle \simeq C_p$, we find that $H^2(G/N, \mathbb{Q}/\mathbb{Z}) = 0$ [Kar, page 37, Corollary 2.2.12]). By Theorem 5.2, we obtain the following exact sequence

$$0 \to H^2(G, \mathbb{Q}/\mathbb{Z})_1 \to H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) \xrightarrow{\lambda} H^3(G/N, \mathbb{Q}/\mathbb{Z}).$$

Step 2. Note that $B_0(G)$ is a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})_1$.

For, consider the restriction map $\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(N, \mathbb{Q}/\mathbb{Z})$. It induces a map $\mathrm{res} : B_0(G) \to B_0(N)$ such that the following diagram commutes

$$\begin{array}{ccc} B_0(G) & \longrightarrow & B_0(N) \\ \downarrow & & \downarrow \\ H^2(G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow[\mathrm{res}]{} & H^2(N, \mathbb{Q}/\mathbb{Z}). \end{array}$$

Since N is a p-group of order p^4, $k(N)$ is k-rational for any algebraically closed field k with $\operatorname{char} k \neq p$ by Theorem 1.6. It follows that $B_0(N) \simeq \mathrm{Br}_{v,k}(k(N)) = 0$ by Lemma 1.3 and Theorem 1.4. Hence $B_0(G)$ is contained in the kernel of the map $\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(N, \mathbb{Q}/\mathbb{Z})$. That is, $B_0(G)$ is a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})_1$.

If we can show that $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$, then $B_0(G) = 0$ and the proof is finished. Note that $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$ if and only if λ is an injective map by the exact sequence in Step 1.

Step 3. We recall a general fact about $H^1(C_n, M)$.

Let $G = \langle\sigma\rangle \simeq C_n$ and M be a G-module. Define the map $Norm : M \to M$ by $Norm\ (x) = x + \sigma \cdot x + \sigma^2 \cdot x + \cdots + \sigma^{n-1} \cdot x$ for any $x \in M$. It is well-known that $H^1(G, M) \simeq \mathrm{Ker}(Norm)/\,\mathrm{Image}(\sigma - 1)$. We will give an explicit correspondence between these two groups. If $x \in M$ satisfies $Norm\ (x) = 0$, define a normalized 1-cocycle $\beta_x : G \to M$ by $\beta_x(\sigma) = x, \beta_x(\sigma^i) = x + \sigma \cdot x + \sigma^2 \cdot x + \cdots + \sigma^{i-1} \cdot x$ for $0 \le i \le n-1$. It is easy to see that $x \in \mathrm{Image}(\sigma - 1)$ if and only β_x is cohomologously trivial.

Step 4. We will determine $H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z}))$.

To keep the notations clean and transparent, we adopt the multiplicative notation for $\mathbb{Q}/\mathbb{Z}$, i.e. we identify $\mathbb{Q}/\mathbb{Z}$ with all the roots of unity in $\mathbb{C}\backslash\{0\}$. Thus a primitive

p-th root of unity is the element i/p (for some $1 \leq i \leq p-1$) in the additive notation of $\mathbb{Q}/\mathbb{Z}$.

Let ζ be a primitive p-th root of unity. Since $H^1(N, \mathbb{Q}/\mathbb{Z}) \simeq \operatorname{Hom}(N, \mathbb{Q}/\mathbb{Z}) \simeq \operatorname{Hom}(N/[N,N], \mathbb{Q}/\mathbb{Z})$ and $N/[N,N] = \langle \bar{f}_1, \bar{f}_0, \bar{h}_2 \rangle \simeq C_p \times C_p \times C_p$, we find that $H^1(N, \mathbb{Q}/\mathbb{Z}) = \langle \varphi_1, \varphi_0, \psi \rangle$ where these 1-cocycles φ_1, φ_0, ψ are defined as

$$\begin{aligned} \varphi_1(f_1) &= \zeta, \quad \varphi_1(f_0) = \varphi_1(h_1) = \varphi_1(h_2) = 1, \\ \varphi_0(f_0) &= \zeta, \quad \varphi_0(f_1) = \varphi_0(h_1) = \varphi_0(h_2) = 1, \\ \psi(h_2) &= \zeta, \quad \psi(f_1) = \psi(f_0) = \psi(h_1) = 1. \end{aligned}$$

The group G (resp. $G/N = \langle \bar{f}_2 \rangle$) acts on $H^1(N, \mathbb{Q}/\mathbb{Z}) = \langle \varphi_1, \varphi_0, \psi \rangle$. It is easy to verify that

$$^{\bar{f}_2}\varphi_1 = \varphi_1, \quad ^{\bar{f}_2}\varphi_0 = \varphi_1\varphi_0, \quad ^{\bar{f}_2}\psi = \varphi_0\psi.$$

Consider the norm map $Norm \colon H^1(N, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})$ defined by the action of $\bar{f}_2$ (see Step 3).

We find that $H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) \simeq \operatorname{Ker}(Norm)/\operatorname{Image}(\bar{f}_2 - 1) = \langle \varphi_1, \varphi_0, \psi \rangle / \langle \varphi_1, \varphi_0 \rangle$ if $p \geq 5$. But, if $p = 3$, $\operatorname{Ker}(1 + \bar{f}_2 + \bar{f}_2^2) = \langle \varphi_1, \varphi_0 \rangle$.

It follows that

$$H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) = \begin{cases} 0, & \text{if } p = 3; \\ \langle \bar{\psi} \rangle \simeq C_p, & \text{if } p \geq 5. \end{cases}$$

When $p = 3$, we obtain $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$ from the exact sequence in Step 1. Hence $B_0(G) = 0$.

From now on, we assume that $p \geq 5$. By Step 3, the element $\bar{\psi} \in \operatorname{Ker}(Norm)/\operatorname{Image}\,(\bar{f}_2 - 1)$ corresponds to the 1-cocycle $\beta : G/N \to H^1(N, \mathbb{Q}/\mathbb{Z})$ defined as

$$\begin{aligned} &\beta(1) = 1, \quad \beta(\bar{f}_2) = \psi, \\ &\beta(\bar{f}_2^i) = \left({}^{\bar{f}_2}\beta(\bar{f}_2^{i-1}) \right) \beta(\bar{f}_2) = \varphi_1^{\binom{i}{3}} \varphi_0^{\binom{i}{2}} \psi^i \end{aligned}$$

where $1 \leq i \leq p-1$ and $\binom{a}{b}$ is the binomial coefficient with the convention that $\binom{a}{b} = 0$ if $1 \leq a < b$.

Step 5. We will show that $\lambda(\beta) \neq 0$ and finish the proof of $B_0(G) = 0$.

Follow the description of λ in Theorem 5.2. Choose a section $u : G/N \to G$ by $u(1) = 1$, $u(\bar{f}_2^i) = f_2^i$ for $1 \leq i \leq p-1$. It is easy to find the 2-cocycle $\varepsilon : G/N \times G/N \to N$. In fact, if $0 \leq i, j \leq p-1$, then

$$\varepsilon(\bar{f}_2^i, \bar{f}_2^j) = \begin{cases} 1, & \text{if } 0 \leq i + j \leq p-1; \\ h_2, & \text{if } i + j \geq p; \end{cases}$$

the second alternative follows from the fact $f_2^p = h_2$.

Now we will evaluate $\lambda(\beta)$ where β is the 1-cocycle determined in Step 4. Write $c = \lambda(\beta)$. Then, for $0 \leq i, j, l \leq p-1$,

$$c(\bar{f}_2^i, \bar{f}_2^j, \bar{f}_2^l) = \left({}^{u(\bar{f}_2^{i+j})}\beta(\bar{f}_2^l) \right) (\varepsilon(\bar{f}_2^i, \bar{f}_2^j))$$

by Theorem 5.2.

In particular, for $0 \le i \le p-1$, we have

$$\begin{aligned} c(\bar{f}_2, \bar{f}_2^{p-1}, \bar{f}_2^i) &= \left({}^{u(1)}\beta(\bar{f}_2^i)\right)(\varepsilon(\bar{f}_2, \bar{f}_2^{p-1})) = (\beta(\bar{f}_2^i))(h_2) \\ &= \left(\varphi_1^{\binom{i}{3}} \varphi_0^{\binom{i}{2}} \psi^i\right)(h_2) = (\psi(h_2))^i = \zeta^i. \end{aligned}$$

On the other hand, apply Theorem 5.3 for $\Phi : H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^3(G/N, \mathbb{Q}/\mathbb{Z})$. We will find a 1-cocycle $\tilde{\beta} : G/N \to \mathbb{Q}/\mathbb{Z}$ such that $\Phi(\tilde{\beta}) = c \in H^3(G/N, \mathbb{Q}/\mathbb{Z})$. In fact, from Theorem 5.3, $c(\bar{f}_2, \bar{f}_2^{p-1}, \bar{f}_2^i) = \tilde{\beta}(\bar{f}_2^i)$. Thus $\tilde{\beta}(\bar{f}_2^i) = \zeta^i$ for all $0 \le i \le p-1$.

By Step 3, the 1-cocycle $\tilde{\beta} \in H^1(G/N, \mathbb{Q}/\mathbb{Z})$ corresponds to the non-zero element $\bar{\zeta} \in \mathrm{Ker}(Norm)/\,\mathrm{Image}(\bar{f}_2 - 1)$, regarding ζ as an element in $\mathrm{Ker}(Norm)$ where $Norm : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is defined by the action of $\bar{f}_2$ (see Step 3). Hence $\tilde{\beta} \ne 0$ and $\Phi(\tilde{\beta}) = c \ne 0$. Thus λ is injective. □

The proof of the following lemma is routine and is omitted.

Lemma 5.5. *Let G be a group in Definition 5.1. If $0 \le i, j \le p-1$, then $f_0^j f_1^i = f_1^i f_0^j h_1^{ij}$, $f_0^j f_2^i = f_2^i f_0^j h_2^{ij}$, and*

$$f_2^i f_1^j = f_1^j f_2^i f_0^{-ij} h_1^{-i\binom{j}{2}} h_2^{-j\binom{i}{2}}.$$

Theorem 5.6. *Let p be an odd prime number. If G is a group belonging to the isoclinism family Φ_6 for groups of order p^5, then $B_0(G) = 0$.*

Proof. Let k be an algebraically closed field with $\mathrm{char} k \ne p$ (in particular, we may choose $k = \mathbb{C}$). Let $\eta \in k$ be a primitive p^2-th root of unity and $\zeta = \eta^p$. In the following we adopt the notation in Definition 5.1. We will show that the fields $k(G)$ are isomorphic to one another over k for all groups G in the isoclinism family Φ_6. Thus they have isomorphic $Br_{v,k}(k(G)) \simeq B_0(G)$ by Theorem 1.4. Since $B_0(G) = 0$ if $G = \Phi_6(221)a$ by Theorem 5.4, it follows that $B_0(G) = 0$ for all other groups G.

Case 1. $G = \Phi_6(221)a$, $\Phi_6(221)b_r$ (where $1 \le r \le (p-1)/2$), $\Phi_6(2111)a$, $\Phi_6(1^5)$.

Step 1. For these groups G, we have

$$f_1^p = h_1^{e_1}, \quad f_2^p = h_2^{e_2}$$

where $0 \le e_1, e_2 \le p-1$.

We will employ the same method as in Step 1 of the proof in Theorem 4.6.

Consider the subgroups $H_1 = \langle f_1, f_0, h_1, h_2 \rangle$ and $H_2 = \langle f_2, f_0, h_1, h_2 \rangle$ of G. Note that $H_2 = \langle f_2, f_0, h_2 \rangle \times \langle h_1 \rangle \simeq \langle f_2, f_0, h_2 \rangle \times C_p$. Hence we get a linear character of H_2 so that $\langle f_2, f_0, h_2 \rangle$ is the kernel. Explicitly, we may define an action of H_2 on $k \cdot X$ defined by

$$h_1 \cdot X = \zeta X, \quad f_2 \cdot X = f_0 \cdot X = h_2 \cdot X = X.$$

Similarly, define an action of H_1 on $k \cdot Y$ by

$$h_2 \cdot Y = \zeta Y, \quad f_1 \cdot Y = f_0 \cdot Y = h_1 \cdot Y = Y.$$

Construct the induced representations of these linear characters by defining $x_i = f_1^i \cdot X$, $y_i = f_2^i \cdot Y$ for $0 \le i \le p-1$. Thus we get an action of G on $(\bigoplus_{0\le i\le p-1} k\cdot x_i) \oplus (\bigoplus_{0\le i\le p-1} k\cdot y_i)$. With the aid of Lemma 5.5, the action of G is given as follows.

$$\begin{aligned}
&f_1 : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto \zeta^{e_1} x_0,\ y_i \mapsto \zeta^{\binom{i}{2}} y_i,\\
&f_2 : x_i \mapsto \zeta^{-\binom{i}{2}} x_i,\ y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p-1} \mapsto \zeta^{e_2} y_0,\\
&f_0 : x_i \mapsto \zeta^i x_i,\ y_i \mapsto \zeta^i y_i,\\
&h_1 : x_i \mapsto \zeta x_i,\ y_i \mapsto y_i,\\
&h_2 : x_i \mapsto x_i,\ y_i \mapsto \zeta y_i.
\end{aligned}$$

By Lemma 3.9, G acts faithfully on $(\bigoplus_{0\le i\le p-1} k\cdot x_i) \oplus (\bigoplus_{0\le i\le p-1} k\cdot y_i)$. Moreover, this representation may be embedded into the regular representation of G. By Theorem 3.2, we find that $k(G)$ is rational over $k(x_i, y_i : 0 \le i \le p-1)^G$.

Step 2. We will apply Theorem 3.1 to $k(x_i, y_i : 0 \le i \le p-1)^G$. Define $u_i = x_i/x_{i-1}$, $U_i = y_i/y_{i-1}$ for $1 \le i \le p-1$. By applying Theorem 3.1 twice, we get $k(x_i, y_i : 0 \le i \le p-1)^G = k(u_i, U_i : 1 \le i \le p-1)^G(u_0, U_0)$ where u_0, U_0 are fixed by the action of G. The action of G on u_i, U_i is given by

$$\begin{aligned}
&f_1 : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p-1} \mapsto \zeta^{e_1}/(u_1u_2\cdots u_{p-1}),\ U_i \mapsto \zeta^{i-1} U_i,\\
&f_2 : u_i \mapsto \zeta^{-(i-1)} u_i,\ U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p-1} \mapsto \zeta^{e_2}/(U_1U_2\cdots U_{p-1}),\\
&f_0 : u_i \mapsto \zeta u_i,\ U_i \mapsto \zeta U_i.
\end{aligned}$$

Note that $h_1(u_i) = h_2(u_i) = u_i$, $h_1(U_i) = h_2(U_i) = U_i$ for $1 \le i \le p-1$. Thus

$$\begin{aligned}
k(u_i, U_i : 1 \le i \le p-1)^G &= k(u_i, U_i : 1 \le i \le p-1)^{G/\langle h_1, h_2\rangle}\\
&= k(u_i, U_i : 1 \le i \le p-1)^{\langle f_0, f_1, f_2\rangle}.
\end{aligned}$$

Step 3. Define $u_i' = u_i/\eta^{e_1}$, $U_i' = U_i/\eta^{e_2}$ for $1 \le i \le p-1$.
It follows that $k(u_i, U_i : 1 \le i \le p-1) = k(u_i', U_i' : 1 \le i \le p-1)$ and

$$\begin{aligned}
&f_1 : u_1' \mapsto u_2' \mapsto \cdots \mapsto u_{p-1}' \mapsto 1/(u_1'u_2'\cdots u_{p-1}'),\ U_i' \mapsto \zeta^{i-1} U_i',\\
&f_2 : u_i' \mapsto \zeta^{-(i-1)} u_i',\ U_1' \mapsto U_2' \mapsto \cdots \mapsto U_{p-1}' \mapsto 1/(U_1'U_2'\cdots U_{p-1}'),\\
&f_0 : u_i' \mapsto \zeta u_i',\ U_i' \mapsto \zeta U_i'.
\end{aligned}$$

Note that the parameters e_1, e_2 of these groups G disappear in the above action. In conclusion, for any group G in this case, $k(G)$ is rational over $k(u_i', U_i' : 1 \le i \le p-1)^{\langle f_1, f_2, f_0\rangle}$. Thus all these fields $k(G)$ are isomorphic.

Case 2. $G = \Phi_6(221)c_r$ (where $r = 1$ or ν), $\Phi_6(221)d_r$ (where $1 \le r \le (p-1)/2$).
For these groups G, we have

$$f_1^p = h_2^{e_1}, \quad f_2^p = h_1^{e_2} h_2^{e_2}$$

where $1 \le e_1, e_2 \le p-1$. The proof is similar to Step 1 and Step 2 of Case 1.
Find integers e_1', e_2' such that $1 \le e_1', e_2' \le p-1$ and $e_1e_1' \equiv e_2e_2' \equiv 1 \pmod p$.

Consider the subgroups $H_1 = \langle f_1, f_0, h_1, h_2\rangle$, $H_2 = \langle f_2, f_0, h_1, h_2\rangle$ of G. Since $H_2/\langle h_2\rangle = \langle \bar{f}_2, \bar{f}_0\rangle \simeq C_{p^2} \times C_p$, we get a linear character of H_2. Similarly for H_1. More precisely, we have actions of H_2 on $k \cdot X$, and H_1 on $k \cdot Y$ defined by

$$\begin{aligned} &f_2 \cdot X = \eta^{e_2'} X, \quad h_1 \cdot X = \zeta X, \quad f_0 \cdot X = h_2 \cdot X = X, \\ &f_1 \cdot Y = \eta^{e_1'} Y, \quad h_2 \cdot Y = \zeta Y, \quad f_0 \cdot Y = h_1 \cdot Y = Y. \end{aligned}$$

Find the induced representations of G from these two linear characters. Define $x_i = f_1^i \cdot X$, $y_i = f_2^i \cdot Y$ where $0 \le i \le p-1$. Then G acts faithfully on $(\bigoplus_{0\le i\le p-1} k \cdot x_i) \oplus (\bigoplus_{0\le i\le p-1} k\cdot y_i)$. Thus $k(G)$ is rational over $k(x_i, y_i : 1 \le i \le p-1)^G$.

The action of G is given by

$$\begin{aligned} &f_1 : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto x_0,\ y_i \mapsto \eta^{e_1' + p\binom{i}{2}} y_i, \\ &f_2 : x_i \mapsto \eta^{e_2' - p\binom{i}{2}} x_i,\ y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p-1} \mapsto \zeta^{e_2} y_0, \\ &f_0 : x_i \mapsto \zeta^i x_i,\ y_i \mapsto \zeta^i y_i, \\ &h_1 : x_i \mapsto \zeta x_i,\ y_i \mapsto y_i, \\ &h_2 : x_i \mapsto x_i,\ y_i \mapsto \zeta y_i. \end{aligned}$$

Define $u_i = x_i/x_{i-1}$, $U_i = y_i/y_{i-1}$ for $1 \le i \le p-1$. We get $k(x_i, y_i : 1 \le i \le p-1)^G = k(u_i, U_i : 1 \le i \le p-1)^G(u_0, U_0)$ where u_0, U_0 are fixed by G by applying Theorem 3.1 twice. The action of G is given by

$$\begin{aligned} &f_1 : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p-1} \mapsto 1/(u_1 u_2 \cdots u_{p-1}),\ U_i \mapsto \zeta^{i-1} U_i, \\ &f_2 : u_i \mapsto \zeta^{-(i-1)} u_i,\ U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p-1} \mapsto \zeta^{e_2}/(U_1 U_2 \cdots U_{p-1}), \\ &f_0 : u_i \mapsto \zeta u_i,\ U_i \mapsto \zeta U_i. \end{aligned}$$

But the above action is just a special case of the action in Step 2 of Case 1. Hence the result.

Case 3. $G = \Phi_6(221)d_0$.

This group satisfies

$$f_1^p = h_2, \quad f_2^p = h_1^e$$

where $1 \le e \le p-1$. In fact, $e = \nu$.

The proof is the same as for Case 2. Choose an integer e' such that $1 \le e' \le p-1$ and $ee' \equiv 1 \pmod p$.

Consider the subgroups $H_1 = \langle f_1, f_0, h_1, h_2\rangle$, $H_2 = \langle f_2, f_0, h_1, h_2\rangle$. Note that $H_1/\langle h_1\rangle \simeq C_{p^2} \times C_p \simeq H_2/\langle h_2\rangle$. Hence we get vectors X and Y such that

$$\begin{aligned} &f_2 \cdot X = \eta^{e'} X, \quad h_1 \cdot X = X, \quad f_0 \cdot X = h_2 \cdot X = X, \\ &f_1 \cdot Y = \eta Y, \quad h_2 \cdot Y = \zeta Y, \quad f_0 \cdot Y = h_1 \cdot Y = Y. \end{aligned}$$

Construct the induced representation of G on $(\bigoplus_{0\le i\le p-1} k\cdot x_i) \oplus (\bigoplus_{0\le i\le p-1} k\cdot y_i)$

where $x_i = f_1^i \cdot X$, $y_i = f_2^i \cdot Y$ with $0 \le i \le p-1$. It follows that

$$\begin{aligned}
&f_1 : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto x_0,\ y_i \mapsto \eta^{1+p\binom{i}{2}} y_i,\\
&f_2 : x_i \mapsto \eta^{e'-p\binom{i}{2}} x_i,\ y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p-1} \mapsto y_0,\\
&f_0 : x_i \mapsto \zeta^i x_i,\ y_i \mapsto \zeta^i y_i,\\
&h_1 : x_i \mapsto \zeta x_i,\ y_i \mapsto y_i,\\
&h_2 : x_i \mapsto x_i,\ y_i \mapsto \zeta y_i.
\end{aligned}$$

By the same arguments as in Case 2, we solve this case.

Case 4. $G = \Phi_6(2111)b_r$ (where $r = 1$ or ν).
These two groups G satisfy

$$f_1^p = 1 \quad \text{and} \quad f_2^p = h_1^e$$

where $1 \le e \le p-1$.
Choose an integer e' such that $1 \le e' \le p-1$ and $ee' \equiv 1 \pmod p$.
The proof is almost the same as for Case 2.

Consider $H_1 = \langle f_1, f_0, h_1, h_2\rangle$ and $H_2 = \langle f_2, f_0, h_1, h_2\rangle$. Note that $H_1/\langle h_1\rangle \simeq C_p \times C_p \times C_p$ and $H_2/\langle h_2\rangle \simeq C_{p^2} \times C_p$. Thus we get vectors X and Y such that

$$\begin{gathered}
f_2 \cdot X = \eta^{e'} X, \quad h_1 \cdot X = \zeta X, \quad f_0 \cdot X = h_2 \cdot X = X,\\
h_2 \cdot Y = \zeta Y, \quad f_1 \cdot Y = f_0 \cdot Y = h_1 \cdot Y = Y.
\end{gathered}$$

Define $x_i = f_1^i \cdot X$, $y_i = f_2^i \cdot Y$ for $0 \le i \le p-1$. The action of G on $k(x_i, y_i : 0 \le i \le p-1)$ is given by

$$\begin{aligned}
&f_1 : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto x_0,\ y_i \mapsto \zeta^{\binom{i}{2}} y_i,\\
&f_2 : x_i \mapsto \eta^{e'-p\binom{i}{2}} x_i,\ y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p-1} \mapsto y_0,\\
&f_0 : x_i \mapsto \zeta^i x_i,\ y_i \mapsto \zeta^i y_i,\\
&h_1 : x_i \mapsto \zeta x_i,\ y_i \mapsto y_i,\\
&h_2 : x_i \mapsto x_i,\ y_i \mapsto \zeta y_i.
\end{aligned}$$

The remaining part is the same as in Case 2. Hence the result. ☐

Proof of Theorem 1.12. Combine Theorems 2.3, 4.1, 4.4, 4.8, 4.6 and 5.6. ☐

THEOREM 5.7. *Let p be an odd prime number and k be an algebraically closed field with char $k \ne p$. If G is a group belonging to the isoclinism family Φ_{10} for groups of order p^5, then there is a linear representation $G \to GL(V)$ over k satisfying* (i) *$\dim_k V = p^2$, and* (ii) *$k(V)^G$ is not k-rational. In particular, the quotient variety $\mathbb{P}(V)/G$ is not k-rational where $\mathbb{P}(V)$ is the projective space associated to V and the action of G on $\mathbb{P}(V)$ by projective linear automorphisms is induced from the linear representation $G \to GL(V)$.*

On the other hand, if k is an algebraically closed field with char$k \ne 2$ and G is a group belonging to the 16th isoclinism family for groups of order 64, then there is a linear representation $G \to GL(V)$ over k satisfying (i) *$\dim_k V = 8$, and* (ii) *$k(V)^G$ is not k-rational. In particular, the quotient variety $\mathbb{P}(V)/G$ is not k-rational.*

Proof. We will find a faithful representation of the required degree for the group G.

In the first case, when p is odd and G is the group given in the theorem, look into the proof of Theorem 2.3 for the generators and relations of G. The center of G is $\langle f_5 \rangle$. Take $H = \langle f_3, f_4, f_5 \rangle$; H is an abelian group. Choose a linear character $\chi : H \to k^\times$ such that $\chi(f_5) = \zeta_p$ and $\chi(f_3) = \chi(f_4) = 1$. Designate the induced representation of χ (from H to G) by $G \to GL(V)$. It is of degree p^2 and is faithful by Lemma 3.9. Note that $Br_{v,k}(k(V)^G)$ is isomorphic to $Br_{v,k}(k(G))$ by the same arguments as in the proof of Theorem 4.2.

For the projective variety $\mathbb{P}(V)/G$, we use Theorem 3.1 and Lemma 1.3. In fact, if $k(V) = k(x_i : 1 \le i \le p^2)$, then $k(\mathbb{P}(V)) = k(x_i/x_1 : 2 \le i \le p^2)$. By Theorem 3.1, $k(x_i : 1 \le i \le p^2)^G = k(x_i/x_1 : 2 \le i \le p^2)^G(x)$ for some element x. These two fixed fields have isomorphic unramified Brauer groups by Lemma 1.3. Hence the result.

Let now G be of order 64. By [CHKK, Lemma 5.5], find the generators and relations of G. We will discuss only the case $G = G(149)$ and leave the other groups to the reader. When $G = G(149)$, take the abelian subgroup $H = \langle f_2, f_5 \rangle$. Note that $Z(G) = \langle f_2^4, f_5 \rangle$. Construct two linear characters of H, χ_1 and χ_2, by $\chi_1(f_2) = \zeta_8, \chi_1(f_5) = 1$ and $\chi_2(f_2) = 1, \chi_2(f_5) = -1$. Let χ be the direct sum of χ_1 and χ_2. The induced representation is of degree 8. The rest of the proof is the same as above. ∎

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