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REMARKS ON A SCALAR CURVATURE RIGIDITY THEOREM OF BRENDLE AND MARQUES*

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Abstract. We give an improvement of a scalar curvature rigidity theorem of Brendle and Marques regarding geodesic balls in $\mathbb{S}^n$. The main result is that Brendle and Marques' theorem holds on a geodesic ball larger than that specified in [2].

Key words. Scalar curvature, mean curvature, Min-Oo's conjecture.

AMS subject classifications. Primary 53C20; Secondary 53C24.

1. Introduction. In a recent paper [2], Brendle and Marques proved the following theorem on scalar curvature rigidity of geodesic balls in the standard n-dimensional sphere $\mathbb{S}^n$.

THEOREM 1.1 (Brendle and Marques [2]). *Let $\Omega = B(\delta) \subset \mathbb{S}^n$ be a closed geodesic ball of radius δ with*

$$\cos\delta \geq \frac{2}{\sqrt{n+3}}. \tag{1.1}$$

Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$. Suppose g is another metric on Ω with the properties:

- *$R(g) \geq R(\bar{g})$ at each point in Ω*
- *$H(g) \geq H(\bar{g})$ at each point on $\partial\Omega$*
- *g and $\bar{g}$ induce the same metric on $\partial\Omega$*

where $R(g)$, $R(\bar{g})$ are the scalar curvature of g, $\bar{g}$, and $H(g)$, $H(\bar{g})$ are the mean curvature of $\partial\Omega$ in (Ω, g), $(\Omega, \bar{g})$. If $g - \bar{g}$ is sufficiently small in the C^2-norm, then $\varphi^*(g) = \bar{g}$ for some diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_{\partial\Omega} = \mathrm{id}$.

Theorem 1.1 is an interesting rigidity result for domains in $\mathbb{S}^n$ because the corresponding statement is false for $\delta = \frac{\pi}{2}$, which follows from the counterexample to Min-Oo's conjecture ([6]) constructed by Brendle, Marques and Neves in [3]. For an account of the connection of Theorem 1.1 to other rigidity phenomena involving scalar curvature, readers are referred to the recent survey [1] by Brendle.

In this paper, we provide an improvement of Theorem 1.1 by showing that Theorem 1.1 is still valid on geodesic balls strictly *larger* than those specified by (1.1). Precisely, we prove that condition (1.1) in Theorem 1.1 can be replaced by either one of the following weaker conditions:

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(a) $\cos\delta > \zeta$, *where* ζ *is the positive constant given by*

$$\zeta^2 = \frac{4(n+4) - 4\sqrt{2n-1}}{n^2+6n+17}.$$

(b) $\cos\delta > \cos\delta_0$, *where* δ_0 *is the unique zero of the function*

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4\sin^2\delta}$$

where $\alpha(\delta) = \frac{(n+1)}{8n}\left[1 - \left(1 - \frac{n}{2\mu(\delta)}\right)\cos\delta\right]^{-1}$ *and* $\mu(\delta)$ *is the first nonzero Neumann eigenvalue of* $B(\delta)$. *In particular,* δ_0 *satisfies*

$$(\cos\delta_0)^2 < \frac{7n-1}{2n^2+5n-1}. \tag{1.2}$$

We compare the conditions (**a**) and (**b**). It follows from (1.2) that δ_0 in (**b**) satisfies

$$\limsup_{n\to\infty} \frac{(\cos\delta_0)^2}{\frac{4}{n+3}} \le \frac{7}{8}, \tag{1.3}$$

while in (**a**) one has

$$\lim_{n\to\infty} \frac{\frac{4(n+4)-4\sqrt{2n-1}}{n^2+6n+17}}{\frac{4}{n+3}} = 1. \tag{1.4}$$

Therefore, (**b**) gives a better improvement of Theorem 1.1 for large n.

For relatively small n, the following table provides numerical values of ζ and lower estimates of $\cos\delta_0$:

TABLE 1.1
ζ and $\cos\delta_0$ for small n

$n =$	3	4	5	6	7	$\cdots$
$\zeta \approx$	0.6581	0.6130	0.5774	0.5481	0.5233	$\cdots$
$\cos\delta_0 >$	0.6918	0.6511	0.6154	0.5845	0.5576	$\cdots$

where the lower bound of $\cos\delta_0$ follows from Lemma 2.3 (iii) in Section 2. For these listed small values of n, (**a**) is a better improvement of Theorem 1.1.

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2. Rigidity of geodesic balls. Throughout this paper, we let $\Omega = B(\delta) \subset \mathbb{S}^n$ be a (closed) geodesic ball of radius $\delta < \frac{\pi}{2}$, with boundary $\Sigma = \partial B(\delta)$. We denote by $\bar{g}$ the standard metric on $\mathbb{S}^n$, with volume form $d\mathrm{vol}_{\bar{g}}$ (*resp.* $d\sigma_{\bar{g}}$) on Ω (*resp.* Σ). We additionally define $\overline{\nabla}$ and $\Delta_{\bar{g}}$ to be the covariant derivative and Laplace operator of $\bar{g}$, and adopt the convention that the divergence, trace and norm (denoted by $\mathrm{div}(\cdot)$, $\mathrm{tr}(\cdot)$ and $|\cdot|$, respectively) are always computed with respect to $\bar{g}$.

We assume that $g = \bar{g} + h$ is a metric close to $\bar{g}$ (say $|h| \le \frac{1}{2}$ at each point in Ω) and that g and $\bar{g}$ induce the same metric on Σ. The outward unit normal to Σ in $(\Omega, \bar{g})$ is denoted by $\overline{\nu}$, and X is the vector field on Σ dual to the 1-form $h(\cdot, \overline{\nu})|_{T(\Sigma)}$, *i.e.* $\bar{g}(v, X) = h(v, \overline{\nu})$ for any vector v tangent to Σ. Finally, for any function f and vector ν, $\partial_\nu f$ denotes the directional derivative of f along ν.

2.1. Brendle and Marques' proof. The following weighted integral estimate of $(R(g) - R(\bar{g}))$ and $(H(g) - H(\bar{g}))$ plays a key role in the proof of Theorem 1.1 in [2].

THEOREM 2.1 (Brendle and Marques [2]). *Let* $\Omega = B(\delta)$ *and* $\lambda = \cos r$, *where* r *is the* $\bar{g}$*-distance to the center of* $B(\delta)$. *Assume* $\operatorname{div}(h) = 0$ *where* $h = g - \bar{g}$. *Then*

$$
\begin{aligned}
&\int_\Omega [R(g) - n(n-1)]\lambda \; d\mathrm{vol}_{\bar{g}} + \int_\Sigma (2 - h(\overline{\nu}, \overline{\nu}))[H(g) - H(\bar{g})]\lambda \; d\sigma_{\bar{g}} \\
= &\int_\Omega \left[-\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) - \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right) \right] \lambda \; d\mathrm{vol}_{\bar{g}} \\
&+ \int_\Sigma H(\bar{g}) \left[-\frac{1}{4} h(\overline{\nu}, \overline{\nu})^2 - \frac{n}{2(n-1)}|X|^2 \right] \lambda \; d\sigma_{\bar{g}} \\
&+ \int_\Sigma \left[-h(\overline{\nu}, \overline{\nu})^2 - \frac{1}{2}|X|^2 \right] \partial_{\overline{\nu}} \lambda \; d\sigma_{\bar{g}} + \int_\Omega E(h) \; d\mathrm{vol}_{\bar{g}} + \int_\Sigma F(h) \; d\sigma_{\bar{g}}
\end{aligned}
$$

where $|E(h)| \le C(|h|^3 + |\overline{\nabla} h|^3)$, $|F(h)| \le C\left(|h|^3 + |h|^2 |\overline{\nabla} h|\right)$ *for some constant* C *depending only on* n.

To see how Theorem 1.1 follows from Theorem 2.1, one first pulls back g through a diffeomorphism φ: $\Omega \to \Omega$ with $\varphi|_\Sigma = \mathrm{id}$ such that $\varphi^*(g) - \bar{g}$ is $\bar{g}$-divergence free and $||\varphi^*(g) - \bar{g}||_{W^{2,p}(\Omega)} \le N ||g - \bar{g}||_{W^{2,p}(\Omega)}$ for some $p > n$ and N depending only on Ω ([2, Proposition 11]). Replacing g by $\varphi^*(g)$, one assumes that $\operatorname{div}(h) = 0$, where $h = g - \bar{g}$ and $||h||_{W^{2,p}(\Omega)}$ is small. If $R(g) \ge n(n-1)$ and $H(g) \ge H(\bar{g})$, Theorem 2.1 then implies

$$
\begin{aligned}
&\int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right) \right] \lambda \; d\mathrm{vol}_{\bar{g}} \\
&\quad + \int_\Sigma h(\overline{\nu}, \overline{\nu})^2 \left[\frac{1}{4} H(\bar{g})\lambda + \partial_{\overline{\nu}} \lambda \right] + |X|^2 \left[\frac{n}{2(n-1)} H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}} \lambda \right] d\sigma_{\bar{g}} \\
&\le C ||h||_{C^1(\bar{\Omega})} \int_\Omega \left(|\overline{\nabla} h|^2 + |h|^2\right) d\mathrm{vol}_{\bar{g}}
\end{aligned}
\tag{2.1}
$$

for a constant C independent on h. At Σ, direct calculation shows

$$
\frac{1}{4} H(\bar{g})\lambda + \partial_{\overline{\nu}} \lambda = \frac{(n+3)\cos^2 \delta - 4}{4 \sin \delta}
\tag{2.2}
$$

$$
\frac{n}{2(n-1)} H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}} \lambda = \frac{(n+1)\cos^2 \delta - 1}{2 \sin \delta}.
\tag{2.3}
$$

If $\cos \delta \ge \frac{2}{\sqrt{n+3}}$, then both quantities in (2.2) and (2.3) are nonnegative. Therefore, (2.1) implies $h = 0$ if $||h||_{C^1(\bar{\Omega})}$ is sufficiently small.

2.2. Improvement of Theorem 1.1: approach 1. Let λ and h be given as in Theorem 2.1. Define

$$\begin{aligned} W(h) = & \int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right] \lambda \, d\mathrm{vol}_{\bar{g}} \\ & + \int_\Sigma h(\overline{\nu},\overline{\nu})^2 \left[\frac{1}{4}H(\bar{g})\lambda + \partial_{\overline{\nu}}\lambda\right] + |X|^2 \left[\frac{n}{2(n-1)}H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}}\lambda\right] d\sigma_{\bar{g}}. \end{aligned} \tag{2.4}$$

It is clear from the above Brendle and Marques' proof that Theorem 1.1 holds on a geodesic ball $\Omega = B(\delta)$ provided one can prove

$$W(h) \geq \epsilon \int_\Omega \left(|\overline{\nabla} h|^2 + |h|^2\right) d\mathrm{vol}_{\bar{g}} \tag{2.5}$$

for some positive ϵ independent on h. To show (2.5), the difficulty lies in handling the boundary integral

$$\int_\Sigma h(\overline{\nu},\overline{\nu})^2 \left[\frac{1}{4}H(\bar{g})\lambda + \partial_{\overline{\nu}}\lambda\right] + |X|^2 \left[\frac{n}{2(n-1)}H(\bar{g})\lambda + \frac{1}{2}\partial_{\overline{\nu}}\lambda\right] d\sigma_{\bar{g}}$$

which can be negative if $\cos\delta$ is small.

PROPOSITION 2.1. *Let h be any C^2 symmetric (0,2) tensor on $\Omega = B(\delta)$ with* $\operatorname{div}(h) = 0$. *Let $s = \sin\delta$. Given any positive function ϕ on Ω, we have*

$$\begin{aligned} & s\int_\Sigma (\operatorname{tr} h) h(\overline{\nu},\overline{\nu}) d\sigma_{\bar{g}} \\ \leq & \int_\Omega \left[\frac{\phi}{2}\sqrt{1-\lambda^2}|h|^2 + \lambda(\operatorname{tr} h)^2 + \frac{1}{2\phi}\sqrt{1-\lambda^2}|\overline{\nabla}(\operatorname{tr} h)|^2\right] d\mathrm{vol}_{\bar{g}}. \end{aligned} \tag{2.6}$$

In particular, if $h|_{T(\Sigma)} = 0$, *then*

$$\begin{aligned} & s\int_\Sigma h(\overline{\nu},\overline{\nu})^2 d\sigma_{\bar{g}} \\ \leq & \int_\Omega \left[\frac{\phi}{2}\sqrt{1-\lambda^2}|h|^2 + \lambda(\operatorname{tr} h)^2 + \frac{1}{2\phi}\sqrt{1-\lambda^2}|\overline{\nabla}(\operatorname{tr} h)|^2\right] d\mathrm{vol}_{\bar{g}}. \end{aligned} \tag{2.7}$$

Proof. Let ω be the 1-form on Ω given by

$$\omega_k = (\operatorname{tr} h) h_{ik} \overline{\nabla}^i \lambda.$$

Using the fact $\overline{\nabla}_k \overline{\nabla}^i \lambda = -\lambda \delta_k^i$ and the assumption $\operatorname{div}(h) = 0$, we have

$$\overline{\nabla}^k \omega_k = -\lambda(\operatorname{tr} h)^2 + h(\overline{\nabla}\lambda, \overline{\nabla}(\operatorname{tr} h)).$$

At Σ, $\omega(\overline{\nu}) = -s(\operatorname{tr} h)h(\overline{\nu},\overline{\nu})$. It follows from the divergence theorem

$$s\int_\Sigma (\operatorname{tr} h) h(\overline{\nu},\overline{\nu}) d\sigma_{\bar{g}} = \int_\Omega \left[\lambda(\operatorname{tr} h)^2 - h(\overline{\nabla}\lambda, \overline{\nabla}(\operatorname{tr} h))\right] d\mathrm{vol}_{\bar{g}}. \tag{2.8}$$

Given any positive function ϕ on Ω, using the fact $|\overline{\nabla}\lambda|^2 = 1 - \lambda^2$, we have

$$\begin{aligned} -h(\overline{\nabla}\lambda, \overline{\nabla}(\operatorname{tr} h)) \leq & |\overline{\nabla}\lambda|\,|h|\,|\overline{\nabla}(\operatorname{tr} h)| \\ \leq & \sqrt{1-\lambda^2}\left[\frac{\phi}{2}|h|^2 + \frac{1}{2\phi}|\overline{\nabla}(\operatorname{tr} h)|^2\right]. \end{aligned} \tag{2.9}$$

Thus, (2.6) follows from (2.8) and (2.9). If $h|_{T(\Sigma)} = 0$, $h(\overline{\nu}, \overline{\nu}) = \operatorname{tr} h$ at Σ. Therefore, (2.6) implies (2.7). □

THEOREM 2.2. *Let δ be a constant in $(0, \frac{\pi}{2})$. Suppose $\cos\delta > \zeta$, where ζ is the positive constant given by*

$$\zeta^2 = \begin{cases} \frac{2}{n+1} & \text{if } n \le 4 \\ \frac{4(n+4)-4\sqrt{2n-1}}{n^2+6n+17} & \text{if } n \ge 5. \end{cases} \tag{2.10}$$

Then the conclusion of Theorem 1.1 holds on $B(\delta)$.

Proof. Let $c = \cos\delta$. Note that (2.10) implies $c^2 \ge \frac{1}{n+1}$, hence the coefficient of $|X|^2$ in (2.4) is nonnegative. By Theorem 1.1, it suffices to assume $c^2 < \frac{4}{n+3}$. Choosing $\phi = \sqrt{2}$ in Proposition 2.1, we have

$$\begin{aligned} W(h) \ge & \int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right] \lambda \, d\mathrm{vol}_{\bar g} \\ & + \frac{(n+3)c^2 - 4}{4(1-c^2)} \sqrt{2(1-c^2)} \int_\Omega \left(\frac{1}{2}|h|^2 + \frac{1}{4}|\overline{\nabla}(\operatorname{tr} h)|^2\right) d\mathrm{vol}_{\bar g} \\ & + \frac{(n+3)c^2 - 4}{4(1-c^2)} \int_\Omega \lambda (\operatorname{tr} h)^2 d\mathrm{vol}_{\bar g}. \end{aligned} \tag{2.11}$$

We seek conditions on c such that

$$c + \frac{(n+3)c^2 - 4}{4(1-c^2)} \sqrt{2(1-c^2)} > 0 \tag{2.12}$$

and

$$\frac{1}{2} + \frac{(n+3)c^2 - 4}{4(1-c^2)} \ge 0. \tag{2.13}$$

Direct calculation shows that (2.12) (under the assumption $c^2 < \frac{4}{n+3}$) is equivalent to

$$c^2 > \frac{4(n+4) - 4\sqrt{2n-1}}{n^2 + 6n + 17} \tag{2.14}$$

and (2.13) is equivalent to

$$c^2 \ge \frac{2}{n+1}. \tag{2.15}$$

Since

$$\frac{4(n+4) - 4\sqrt{2n-1}}{n^2 + 6n + 17} \ge \frac{2}{n+1} \tag{2.16}$$

precisely when $n \ge 5$, we conclude that (2.5) holds for some $\epsilon > 0$ if (2.10) is satisfied. Theorem 2.2 is proved. □

Theorem 2.2 verifies condition (**a**) in the introduction for $n \ge 5$. The remaining case $n = 3, 4$ in condition (**a**) will be verified in section 2.4.

2.3. Improvement of Theorem 1.1: approach 2. In this section, we give a different approach to estimate the boundary integral of $(\operatorname{tr} h)^2$ in $W(h)$ in terms of the interior integral in $W(h)$. To do so, we use the linearization of the scalar curvature (2.17). Noticing that the integral of $\operatorname{tr} h$ over $B(\delta)$ is close to zero, we apply the Poincaré inequality through an estimate of the first nonzero Neumann eigenvalue of $B(\delta)$ in [5].

LEMMA 2.1. *Let $\Omega \subset \mathbb{S}^n$ be a closed domain with smooth boundary Σ. Let $\bar{g}$ be the standard metric on $\mathbb{S}^n$ and $g = \bar{g} + h$ be another smooth metric on Ω such that g, $\bar{g}$ induce the same metric on Σ and $\operatorname{div} h = 0$. Suppose $|h|$ is very small, say $|h| \leq \frac{1}{2}$ at every point.*

(i) Given any smooth function f on Ω, one has

$$\begin{aligned}&\int_\Omega f(\operatorname{tr} h)\Delta_{\bar{g}}(\operatorname{tr} h) + (n-1)f(\operatorname{tr} h)^2 \ d\mathrm{vol}_{\bar{g}}\\ =& \int_\Omega f(\operatorname{tr} h)\left[R(\bar{g}) - R(g)\right] d\mathrm{vol}_{\bar{g}} + E(h,f)\end{aligned}$$

where

$$|E(h,f)| \leq C||f||_{C^1(\overline{\Omega})}\left(\int_\Omega \left(|h|^3 + |\overline{\nabla} h|^3\right) d\mathrm{vol}_{\bar{g}} + \int_\Sigma |h|^2|\overline{\nabla} h| d\sigma_{\bar{g}}\right)$$

for a positive constant C depending only on $(\Omega, \bar{g})$.

(ii)

$$\begin{aligned}\int_\Omega (\operatorname{tr} h) d\mathrm{vol}_{\bar{g}} = -\frac{1}{n-1}&\left(\int_\Omega \left[R(g) - R(\bar{g})\right] d\mathrm{vol}_{\bar{g}}\right.\\ &\left. +2\int_\Sigma \left[H(g) - H(\bar{g})\right] d\sigma_{\bar{g}}\right) + F(h)\end{aligned}$$

where

$$|F(h)| \leq C\left(\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar{g}} + \int_\Sigma (|h|^2 + |h||\overline{\nabla} h|) d\sigma_{\bar{g}}\right)$$

for a positive constant C depending only on $(\Omega, \bar{g})$.

Proof. Since $\operatorname{div}(h) = 0$ and $\operatorname{Ric}(\bar{g}) = (n-1)\bar{g}$, h satisfies

$$-\Delta_{\bar{g}}(\operatorname{tr} h) - (n-1)(\operatorname{tr} h) = DR_{\bar{g}}(h), \tag{2.17}$$

where $DR_{\bar{g}}(\cdot)$ denotes the linearization of the scalar curvature at $\bar{g}$. By [2, Proposition 4] (also see [5, Lemma 2.1]), one knows

$$\begin{aligned}R(g) - R(\bar{g}) =& DR_{\bar{g}}(h) - \frac{1}{2}DR_{\bar{g}}(h^2) + \langle h, \overline{\nabla}^2(\operatorname{tr} h)\rangle\\ &- \frac{1}{4}\left(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr}_{\bar{g}} h)|^2\right) + \frac{1}{2}h^{ij}h^{kl}\overline{R}_{ikjl}\\ &+ E(h) + \overline{\nabla}_i(E_1^i(h))\end{aligned} \tag{2.18}$$

where h^2 is the $\bar{g}$-square of h, i.e. $(h^2)_{ik} = \bar{g}^{jl}h_{ij}h_{kl}$, $E(h)$ is a function and $E_1(h)$ is a vector field on Ω satisfying

$$|E(h)| \leq C(|h||\overline{\nabla} h|^2 + |h|^3), \quad |E_1(h)| \leq C|h|^2|\overline{\nabla} h|$$

for a positive constant C depending only on n. Multiplying (2.17) by $f(\operatorname{tr} h)$ and integrating by parts, (i) follows from (2.18).

To prove (ii), we integrate (2.17) on Ω to get

$$-(n-1)\int_\Omega (\operatorname{tr} h) d\mathrm{vol}_{\bar g} = \int_\Omega DR_{\bar g}(h) d\mathrm{vol}_{\bar g} + \int_\Sigma \partial_{\overline{\nu}}(\operatorname{tr} h)\, d\sigma_{\bar g}. \tag{2.19}$$

Let $DH_{\bar g}(h)$ denote the linearization of the mean curvature of Σ at $\bar g$. Direct calculation (see [2, Proposition 5] or [4, (34)]) shows

$$2DH_{\bar g}(h) = \partial_{\overline{\nu}}(\operatorname{tr} h) - \operatorname{div} h(\overline{\nu}) - \operatorname{div}_\Sigma X. \tag{2.20}$$

Since $\operatorname{div}(h) = 0$, (2.20) implies

$$\int_\Sigma \partial_{\overline{\nu}}(\operatorname{tr} h)\, d\sigma_{\bar g} = 2\int_\Sigma DH_{\bar g}(h) d\sigma_{\bar g}. \tag{2.21}$$

By [2, Proposition 5], one has

$$|H(g) - H(\bar g) - DH_{\bar g}(h)| \le C(|h|^2 + |h||\overline{\nabla} h|) \tag{2.22}$$

for a positive constant C depending only on n. (ii) now follows from (2.18)-(2.22) and integration by parts on Ω. ∎

We will make use of the first nonzero Neumann eigenvalue of $B(\delta)$, which we denote by $\mu(\delta)$. The next lemma on $\mu(\delta)$ was proved in [5, Lemma 3.1].

LEMMA 2.2 ([5]). *Let $\mu(\delta)$ be the first nonzero Neumann eigenvalue of $B(\delta)$ (with respect to $\bar g$). Then*

(i) $\mu(\delta)$ is a strictly decreasing function of δ on $(0, \frac{\pi}{2}]$;

(ii) for any $0 < \delta < \frac{\pi}{2}$,

$$\mu(\delta) > n + \frac{(\sin\delta)^{n-2}\cos\delta}{\int_0^\delta (\sin t)^{n-1} dt} > \frac{n}{(\sin\delta)^2}.$$

Using $\mu(\delta)$, we have the following estimate of $\int_\Sigma (\operatorname{tr} h)^2 d\sigma_{\bar g}$.

PROPOSITION 2.2. *Let $\Omega = B(\delta)$ and $\mu(\delta)$ be the first nonzero Neumann eigenvalue of $B(\delta)$. Let $g = \bar g + h$ be a smooth metric on $B(\delta)$ such that g, $\bar g$ induce the same metric on Σ and $\operatorname{div}(h) = 0$. Suppose $|h|$ is small, say $|h| \le \frac{1}{2}$ at every point. Let $c = \cos\delta$ and $s = \sin\delta$. Then*

$$\begin{aligned} s\int_\Sigma (\operatorname{tr} h)^2 d\sigma_{\bar g} \le\ & 2\left[1 - c\left(1 - \frac{n}{2\mu(\delta)}\right)\right]\int_\Omega \lambda|\overline{\nabla}(\operatorname{tr} h)|^2\, d\mathrm{vol}_{\bar g} \\ & - 2\int_\Omega (\lambda - c)(\operatorname{tr} h)(R(g) - R(\bar g))\, d\mathrm{vol}_{\bar g} \\ & + C||h||_{C^1}\left[\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2\, d\sigma_{\bar g}\right] \\ & + C\left[\int_\Omega (R(g) - R(\bar g))\, d\mathrm{vol}_{\bar g} + 2\int_\Sigma (H(g) - H(\bar g))\, d\sigma_{\bar g}\right]^2 \end{aligned}$$

for some positive constant C depending only on $(\Omega, \bar g)$ and c.

Proof. Integrating by parts, using the fact $\lambda = c$ at Σ and $\Delta_{\bar g}\lambda = -n\lambda$ on Ω, we have

$$\begin{aligned}&\int_\Sigma (\operatorname{tr} h)^2 \partial_{\overline{\nu}}\lambda \, d\sigma_{\bar g}\\ &= \int_\Omega (\operatorname{tr} h)^2 \Delta_{\bar g}\lambda - (\lambda - c)\Delta_{\bar g}(\operatorname{tr} h)^2 \, d\mathrm{vol}_{\bar g}\\ &= \int_\Omega -n\lambda(\operatorname{tr} h)^2 - 2(\lambda - c)[(\operatorname{tr} h)\Delta_{\bar g}(\operatorname{tr} h) + |\overline{\nabla}(\operatorname{tr} h)|^2] d\mathrm{vol}_{\bar g}.\end{aligned} \tag{2.23}$$

Choosing $f = \lambda - c$ in Lemma 2.1(i), we have

$$\begin{aligned}&\int_\Omega (\lambda - c)(\operatorname{tr} h)\Delta_{\bar g}(\operatorname{tr} h) \, d\mathrm{vol}_{\bar g}\\ &= \int_\Omega -(n-1)(\lambda - c)(\operatorname{tr} h)^2 - (\lambda - c)(\operatorname{tr} h)\,[R(g) - R(\bar g)]\, d\mathrm{vol}_{\bar g} + E_2(h)\end{aligned} \tag{2.24}$$

where

$$|E_2(h)| \le C\left(\int_\Omega \left(|h|^3 + |\overline{\nabla} h|^3\right) d\mathrm{vol}_{\bar g} + \int_\Sigma |h|^2 |\overline{\nabla} h| d\sigma_{\bar g}\right)$$

for some constant C depending on $(\Omega, \bar g)$ and c. It follows from (2.23) and (2.24) that

$$\begin{aligned}\int_\Sigma (\operatorname{tr} h)^2 \partial_{\overline{\nu}}\lambda \, d\sigma_{\bar g} = &\int_\Omega \left[(n-2)(\operatorname{tr} h)^2 - 2|\overline{\nabla}(\operatorname{tr} h)|^2\right] \lambda \, d\mathrm{vol}_{\bar g}\\ &+ 2c\int_\Omega \left[|\overline{\nabla}(\operatorname{tr} h)|^2 - (n-1)(\operatorname{tr} h)^2\right] d\mathrm{vol}_{\bar g}\\ &+ 2\int_\Omega (\lambda - c)(\operatorname{tr} h)\,[R(g) - R(\bar g)]\, d\mathrm{vol}_{\bar g} - 2E_2(h).\end{aligned} \tag{2.25}$$

Since $\lambda \ge c$ on Ω, (2.25) implies

$$\begin{aligned}\int_\Sigma (\operatorname{tr} h)^2 \partial_{\overline{\nu}}\lambda \, d\sigma_{\bar g} \ge &- 2\int_\Omega |\overline{\nabla}(\operatorname{tr} h)|^2 \lambda d\mathrm{vol}_{\bar g} + 2c\int_\Omega \left[|\overline{\nabla}(\operatorname{tr} h)|^2 - \frac{n}{2}(\operatorname{tr} h)^2\right] d\mathrm{vol}_{\bar g}\\ &+ 2\int_\Omega (\lambda - c)(\operatorname{tr} h)\,[R(g) - R(\bar g)]\, d\mathrm{vol}_{\bar g} - 2E_2(h).\end{aligned}$$

By the variational characterization of $\mu(\delta)$, we have

$$\int_\Omega |\overline{\nabla}(\operatorname{tr} h)|^2 \, d\mathrm{vol}_{\bar g} \ge \mu(\delta)\left[\left(\int_\Omega (\operatorname{tr} h)^2 \, d\mathrm{vol}_{\bar g}\right) - \frac{1}{V(\bar g)}\left(\int_\Omega (\operatorname{tr} h) \, d\mathrm{vol}_{\bar g}\right)^2\right] \tag{2.26}$$

where $V(\bar g) = \int_\Omega 1 d\mathrm{vol}_{\bar g}$. It follows from Lemma 2.1(ii) and (2.26) that

$$\begin{aligned}&\int_\Omega \left[|\overline{\nabla}(\operatorname{tr} h)|^2 - \frac{n}{2}(\operatorname{tr} h)^2\right] d\mathrm{vol}_{\bar g}\\ &\ge \left(1 - \frac{n}{2\mu(\delta)}\right)\int_\Omega |\overline{\nabla}(\operatorname{tr} h)|^2 \, d\mathrm{vol}_{\bar g}\\ &\quad - C\left[\int_\Omega (R(g) - R(\bar g)) \, d\mathrm{vol}_{\bar g} + 2\int_\Sigma (H(g) - H(\bar g)) \, d\sigma_{\bar g}\right]^2\\ &\quad - C\left[\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar g} + \int_\Sigma (|h|^2 + |h||\overline{\nabla} h| d\sigma_{\bar g})\right]^2\end{aligned} \tag{2.27}$$

for a positive constant C depending only on $(\Omega, \bar{g})$. The lemma now follows from (2.25), (2.27) and the fact $\lambda \leq 1$. □

The following lemma is needed for the statement of Theorem 2.3.

LEMMA 2.3. *On* $(0, \frac{\pi}{2}]$, *define*

$$\alpha(\delta) = \left[1 - \left(1 - \frac{n}{2\mu(\delta)}\right)\cos\delta\right]^{-1} \frac{(n+1)}{8n}$$

and

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4\sin^2\delta}.$$

Then

(i) $\alpha(\delta)$ *is strictly decreasing,* $\lim_{\delta\to 0+}\alpha(\delta) = \infty$ *and* $\alpha(\frac{\pi}{2}) = \frac{n+1}{8n}$.

(ii) $F(\delta)$ *is strictly decreasing,* $\lim_{\delta\to 0+}F(\delta) = \infty$ *and* $F(\frac{\pi}{2}) < 0$. *Hence there is exactly one* $\delta_0 \in (0, \frac{\pi}{2})$ *such that* $F(\delta_0) = 0$.

(iii) $\cos\delta_0 > \kappa$ *where* κ *is the positive root of the equation*

$$2n(n+3)x^2 + (n+1)x + (1-7n) = 0.$$

In particular, $(\cos\delta_0)^2 > \frac{1}{n+1}$.

Proof. (i) follows directly from Lemma 2.2. (ii) follows from (i) and the fact

$$F(\delta) = \alpha(\delta) + \frac{n-1}{4}\frac{1}{\sin^2\delta} - \frac{n+3}{4}.$$

To prove (iii), suppose $\cos\delta_0 = a$. Since $0 < 1 - \frac{n}{2\mu(\delta_0)} < 1$, one has $\left(1 - \frac{n}{2\mu(\delta_0)}\right)\cos\delta_0 < a$ and $\alpha(\delta_0) < \frac{n+1}{8n}\frac{1}{(1-a)}$. Therefore,

$$0 = F(\delta_0) < \frac{n+1}{8n}\frac{1}{(1-a)} + \frac{n-1}{4}\frac{1}{1-a^2} - \frac{n+3}{4}$$

which implies (iii). □

THEOREM 2.3. *Let* $\Omega = B(\delta)$ *be a geodesic ball of radius* δ *in* $\mathbb{S}^n$. *Suppose* $\delta < \delta_0$, *where* δ_0 *is the unique zero in* $(0, \frac{\pi}{2})$ *of the function*

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4\sin^2\delta}$$

where $\alpha(\delta) = \left[1 - \left(1 - \frac{n}{2\mu(\delta)}\right)\cos\delta\right]^{-1}\frac{(n+1)}{8n}$. *Then the conclusion of Theorem 1.1 holds on* Ω.

Proof. Let $W(h)$ be given in (2.4). Let $c = \cos\delta$. Lemma 2.3(iii) shows $c^2 > \frac{1}{n+1}$. Hence, the coefficient of $|X|^2$ in $W(h)$ is nonnegative. By Theorem 1.1, it suffices to assume $c^2 < \frac{4}{n+3}$. Apply Proposition 2.2, we have

$$\begin{aligned} W(h) \geq & \int_\Omega \left[\frac{1}{4}(|\overline{\nabla} h|^2 + |\overline{\nabla}(\operatorname{tr} h)|^2) + \frac{1}{2}\left(|h|^2 + (\operatorname{tr} h)^2\right)\right]\lambda \, d\mathrm{vol}_{\bar{g}} \\ & + \left[\frac{(n+3)c^2 - 4}{4(1-c^2)}\right] 2\left[1 - c\left(1 - \frac{n}{2\mu(\delta)}\right)\right]\int_\Omega |\overline{\nabla}(\operatorname{tr} h)|^2 \lambda \, d\mathrm{vol}_{\bar{g}} \\ & + \hat{E}(h, c), \end{aligned} \tag{2.28}$$

where

$$\begin{aligned}\hat{E}(h,c) = \left[\frac{(n+3)c^2-4}{4(1-c^2)}\right]&\left\{-2\int_\Omega (\lambda - c)(\operatorname{tr} h)(R(g)-R(\bar{g}))d\mathrm{vol}_{\bar{g}}\right.\\ &+ C||h||_{C^1}\left[\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar{g}} + \int_\Sigma |h|^2\, d\sigma_{\bar{g}}\right]\\ &\left.+C\left[\int_\Omega (R(g)-R(\bar{g}))\, d\mathrm{vol}_{\bar{g}} + 2\int_\Sigma (H(g)-H(\bar{g}))\, d\sigma_{\bar{g}}\right]^2\right\}.\end{aligned} \tag{2.29}$$

Since $\delta < \delta_0$, Lemma 2.3 (ii) implies

$$F(\delta) = \alpha(\delta) + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)} > F(\delta_0) = 0.$$

Hence there exists a small constant $\epsilon \in (0,1)$ such that

$$\frac{1}{4}\left(1 + \frac{(1-\epsilon)}{n}\right) + \left[\frac{(n+3)c^2-4}{4(1-c^2)}\right] 2\left[1 - c\left(1 - \frac{n}{2\mu(\delta)}\right)\right] > 0. \tag{2.30}$$

By (2.28) and (2.30), using the fact $|\overline{\nabla} h|^2 \geq \frac{1}{n}|\overline{\nabla}(\operatorname{tr} h)|^2$, we have

$$W(h) \geq \frac{1}{4}\epsilon c \int_\Omega (|\overline{\nabla} h|^2 + |h|^2)\, d\mathrm{vol}_{\bar{g}} + \hat{E}(h,c). \tag{2.31}$$

Now suppose $R(g) - R(\bar{g}) \geq 0$, $H(g) - H(\bar{g}) \geq 0$ and $||h||_{W^{2,p}(\Omega)}$ is sufficiently small. It follows from Theorem 2.1, (2.29) and (2.31) that

$$\begin{aligned}&\frac{1}{2}\int_\Omega [R(g)-R(\bar{g})]\lambda\, d\mathrm{vol}_{\bar{g}} + \frac{1}{2}\int_\Sigma [H(g)-H(\bar{g})]\lambda\, d\sigma_{\bar{g}}\\ &\leq \epsilon \int_\Omega (|\overline{\nabla} h|^2 + |h|^2)\, d\mathrm{vol}_{\bar{g}}\\ &\quad + C||h||_{C^1}\left[\int_\Omega \left(|h|^2 + |\overline{\nabla} h|^2\right) d\mathrm{vol}_{\bar{g}} + \int_\Sigma |h|^2\, d\sigma_{\bar{g}}\right].\end{aligned} \tag{2.32}$$

for some positive constant C independent of h. We can then proceed as in [2]: since $||h||_{L^2(\Sigma)} \leq C||h||_{W^{1,2}(\Omega)}$, one knows the terms in the last line in (2.32) is bounded by $C||h||_{C^1(\overline{\Omega})}||h||_{W^{1,2}(\Omega)}$. Therefore, if $||h||_{W^{2,p}(\Omega)}$ is sufficiently small, (2.32) implies h must vanish identically. This completes the proof of Theorem 2.3. □

We give some lower estimates of δ_0 which are relatively more explicit.

PROPOSITION 2.3. *δ_0 in Theorem 2.3 satisfies*
(i) $\delta_0 > \tilde{\delta}_0$ where $\tilde{\delta}_0$ is the unique zero in $(0, \frac{\pi}{2})$ of the equation

$$\left[1 - \left(1 - \frac{n}{2\tilde{\mu}(\delta)}\right)\cos\delta\right]^{-1} \frac{n+1}{8n} + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)} = 0$$

where $\tilde{\mu}(\delta) = n + \dfrac{(\sin\delta)^{n-2}\cos\delta}{\int_0^\delta (\sin t)^{n-1} dt}$.

(ii) $\cos\delta_0 < \tilde{\kappa}$ *where* $\tilde{\kappa}$ *is the unique zero in* $(0,1)$ *of the equation*

$$n(n+3)x^4 + n(n+3)x^3 + 2n(n+1)x^2 + (1-3n)x - 7n + 1 = 0.$$

(iii) $(\cos\delta_0)^2 < \dfrac{7n-1}{2n^2+5n-1}$.

Proof. By Lemma 2.2 (ii), $\mu(\delta_0) > \tilde{\mu}(\delta_0)$. Hence,

$$\left[1 - \left(1 - \frac{n}{2\tilde{\mu}(\delta_0)}\right)\cos\delta_0\right]^{-1}\frac{n+1}{8n} + \frac{(n+3)\cos^2\delta_0 - 4}{4(1-\cos^2\delta_0)} < 0. \tag{2.33}$$

Note that $\tilde{\mu}(\delta)$ is strictly decreasing in $(0, \frac{\pi}{2}]$. As in the proof of Lemma 2.3(ii), we know the function

$$\left[1 - \left(1 - \frac{n}{2\tilde{\mu}(\delta)}\right)\cos\delta\right]^{-1}\frac{n+1}{8n} + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)}$$

is strictly decreasing and has a unique zero $\tilde{\delta}_0$ in $(0, \frac{\pi}{2})$. Hence, (i) follows from (2.33).

The proof of (ii) is similar to that of (i) except we replace the lower bound $\mu(\delta) > \tilde{\mu}(\delta)$ by a weaker lower bound $\mu(\delta_0) > \frac{n}{(\sin\delta_0)^2} = \frac{n}{1-(\cos\delta_0)^2}$.

(iii) follows from the fact

$$\frac{n+1}{8n} + \frac{(n+3)\cos^2\delta - 4}{4(1-\cos^2\delta)} < 0.$$

∎

Theorem 2.3 and Proposition 2.3 (iii) verify condition (**b**) in the introduction.

2.4. A combined approach. It remains to confirm the case $n = 3, 4$ in condition (**a**). To do so, we combine the two methods leading to Theorem 2.2 and Theorem 2.3.

Theorem 2.4. *Suppose* $3 \le n \le 4$, *Theorem 1.1 is true on* $B(\delta)$ *if*

$$\cos\delta > \left(\frac{4(n+4) - 4\sqrt{2n-1}}{n^2+6n+17}\right)^{\frac{1}{2}} \approx \begin{cases} 0.6581, & n = 3 \\ 0.6130, & n = 4. \end{cases} \tag{2.34}$$

Proof. Let $c = \cos\delta$. (2.34) implies $c^2 > \frac{1}{n+1}$. By (2.11), we have $W(h) \ge Y(h)$ where

$$\begin{aligned} Y(h) = &\left[c + \frac{(n+3)c^2 - 4}{4(1-c^2)}\sqrt{2(1-c^2)}\right]\int_\Omega \left(\frac{1}{2}|h|^2 + \frac{1}{4}|\overline{\nabla}(\operatorname{tr} h)|^2\right) d\mathrm{vol}_{\bar{g}} \\ &+ \left[\frac{1}{2} + \frac{(n+3)c^2 - 4}{4(1-c^2)}\right]\int_\Omega \lambda(\operatorname{tr} h)^2 d\mathrm{vol}_{\bar{g}} + \frac{c}{4}\int_\Omega |\overline{\nabla} h|^2 \, d\mathrm{vol}_{\bar{g}}. \end{aligned}$$

As before, we always assume $c^2 < \frac{4}{n+3}$. Then (2.34) implies (2.12), i.e.

$$c + \frac{(n+3)c^2 - 4}{4(1-c^2)}\sqrt{2(1-c^2)} > 0. \tag{2.35}$$

To continue, we only need to assume $\frac{1}{2}+\frac{(n+3)c^2-4}{4(1-c^2)}<0$. (If $n\geq 5$, this term would automatically be nonnegative by (2.16).)

Given any constants $\theta,\tau\in(0,1)$, using the fact $|\overline{\nabla}h|^2\geq\frac{1}{n}|\overline{\nabla}(\operatorname{tr}h)|^2$, $|h|^2\geq\frac{1}{n}(\operatorname{tr}h)^2$, $\lambda\leq 1$ and applying (2.26) as in Theorem 2.3, we have

(2.36)

$$\begin{aligned}
Y(h)\geq&\int_\Omega\left\{\frac{\theta c}{4}|\overline{\nabla}h|^2+\frac{1}{4}\left[\frac{1-\theta}{n}c+c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]|\overline{\nabla}(\operatorname{tr}h)|^2\right.\\
&+\tau\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{|h|^2}{2}+\frac{1-\tau}{n}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{(\operatorname{tr}h)^2}{2}\\
&\left.+\left[1+\frac{(n+3)c^2-4}{2(1-c^2)}\right]\frac{(\operatorname{tr}h)^2}{2}\right\}\,d\mathrm{vol}_{\bar g}\\
\geq&\,\epsilon\left(\int_\Omega|\overline{\nabla}h|^2+|h|^2d\mathrm{vol}_{\bar g}\right)\\
&+\left\{\frac{1}{2}\left[\frac{(n+1)-\theta}{n}c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\mu(\delta)+\frac{1-\tau}{n}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\right.\\
&\left.+\left[1+\frac{(n+3)c^2-4}{2(1-c^2)}\right]\right\}\left(\int_\Omega\frac{(\operatorname{tr}h)^2}{2}\,d\mathrm{vol}_{\bar g}\right)+E(h)
\end{aligned}$$

where $\epsilon=\min\left\{\frac{\theta c}{4},\frac{\tau}{2}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\right\}>0$, $\mu(\delta)$ is the first nonzero Neumann eigenvalue of $B(\delta)$, and $E(h)$ is an error term satisfying

$$\begin{aligned}
|E(h)|\leq&C\left[\int_\Omega(R(g)-R(\bar g))\,d\mathrm{vol}_{\bar g}+2\int_\Sigma(H(g)-H(\bar g))\,d\sigma_{\bar g}\right]^2\\
&+C\left[\int_\Omega\left(|h|^2+|\overline{\nabla}h|^2\right)d\mathrm{vol}_{\bar g}+\int_\Sigma(|h|^2+|h||\overline{\nabla}h|)d\sigma_{\bar g}\right]^2
\end{aligned}$$

with C depending only on $B(\delta)$.

We claim that θ and τ can be chosen so that the coefficient of

$$\int_\Omega\frac{(\operatorname{tr}h)^2}{2}d\mathrm{vol}_{\bar g}$$

above is positive. To see this, let

$$\begin{aligned}
F_n(c)=&\frac{1}{2}\left[\frac{n+1}{n}c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\mu(\delta)+\frac{1}{n}\left[c+\frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\\
&+\left[1+\frac{(n+3)c^2-4}{2(1-c^2)}\right].
\end{aligned}$$

By (2.35) and the eigenvalue estimate $\mu(\delta)>\frac{n}{(\sin\delta)^2}$ (Lemma 2.2 (ii)), one has

$$F_n(c)>G_n(c)$$

where

$$G_n(c) = \frac{1}{2}\left[\frac{n+1}{n}c + \frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{n}{1-c^2} + \frac{1}{n}\left[c + \frac{(n+3)c^2-4}{2\sqrt{2(1-c^2)}}\right] + \left[1 + \frac{(n+3)c^2-4}{2(1-c^2)}\right].$$

When $n = 3$ and 4, $G_3(c)$ and $G_4(c)$ are respectively given by

$$G_3(c) = \frac{1}{2}\left[\frac{4}{3}c + \frac{6c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{3}{1-c^2} + \frac{1}{3}\left[c + \frac{6c^2-4}{2\sqrt{2(1-c^2)}}\right] + \left[1 + \frac{6c^2-4}{2(1-c^2)}\right],$$

$$G_4(c) = \frac{1}{2}\left[\frac{5}{4}c + \frac{7c^2-4}{2\sqrt{2(1-c^2)}}\right]\frac{4}{1-c^2} + \frac{1}{4}\left[c + \frac{7c^2-4}{2\sqrt{2(1-c^2)}}\right] + \left[1 + \frac{7c^2-4}{2(1-c^2)}\right].$$

Using *Mathematica*, one verifies that

$$G_3(c) > 0 \quad \text{if } 0.6378 < c < 1 \tag{2.37}$$

and

$$G_4(c) > 0 \quad \text{if } 0.5933 < c < 1. \tag{2.38}$$

In particular, this shows that $G_n(c) > 0$ is guaranteed by (2.34) for $n = 3, 4$.

Therefore, there exist small positive constants θ, τ such that the coefficient of $\int_\Omega \frac{(\operatorname{tr} h)^2}{2} d\mathrm{vol}_{\bar g}$ in (2.36) is positive. For these θ and τ, we have

$$W(h) \geq Y(h) \geq \epsilon\left(\int_\Omega |\overline{\nabla} h|^2 + |h|^2 d\mathrm{vol}_{\bar g}\right) + E(h).$$

Arguing as in the proof of Theorem 2.3 (the part following (2.31)), we conclude that Theorem 1.1 holds on such a $B(\delta)$. □

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