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005

MORSE FIELD THEORY*

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Abstract. In this paper we define and study the moduli space of metric-graph-flows in a manifold M. This is a space of smooth maps from a finite graph to M, which, when restricted to each edge, is a gradient flow line of a smooth (and generically Morse) function on M. Using the model of Gromov-Witten theory, with this moduli space replacing the space of stable holomorphic curves in a symplectic manifold, we obtain invariants, which are (co)homology operations in M. The invariants obtained in this setting are classical cohomology operations such as cup product, Steenrod squares, and Stiefel-Whitney classes. We show that these operations satisfy invariance and gluing properties that fit together to give the structure of a topological quantum field theory. By considering equivariant operations with respect to the action of the automorphism group of the graph, the field theory has more structure. It is analogous to a homological conformal field theory. In particular we show that classical relations such as the Adem relations and Cartan formulae are consequences of these field theoretic properties. These operations are defined and studied using two different methods. First, we use algebraic topological techniques to define appropriate virtual fundamental classes of these moduli spaces. This allows us to define the operations via the corresponding intersection numbers of the moduli space. Secondly, we use geometric and analytic techniques to study the smoothness and compactness properties of these moduli spaces. This will allow us to define these operations on the level of Morse-Smale chain complexes, by appropriately counting metric-graph-flows with particular boundary conditions.

Key words. Morse theory, metric graph, cohomology operations, moduli space.

AMS subject classifications. 58E05, 55S05, 55R40.

Introduction. In this paper we construct a moduli space of graphs $|\mathcal{C}_\Gamma|/Aut\Gamma$ associated to a fixed oriented graph Γ. It is built from a category $\mathcal{C}_\Gamma$ in which the objects are graphs and morphisms are homotopy equivalences. We use this moduli space to study families of maps of graphs into a manifold, which allows us to probe the topology of the manifold. The moduli space is described in detail in section 1. For the moment it is best understood by its following properties. To each element of $|\mathcal{C}_\Gamma|$ is associated an oriented, compact metric graph—where edges are given lengths—and an orientation preserving homotopy equivalence from the metric graph to the given graph Γ that collapses edges and vertices. The space $|\mathcal{C}_\Gamma|$ is contractible, and admits a free $Aut(\Gamma)$ action, and hence the quotient is a model for the classifying space,

$$|\mathcal{C}_\Gamma|/Aut\Gamma \simeq BAut\Gamma.$$

In particular when Γ has non-trivial automorphisms $|\mathcal{C}_\Gamma|/Aut\Gamma$ has non-trivial homology.

Given a fixed closed manifold M, we then thicken this moduli space by defining a space $\mathcal{S}_\Gamma$ whose points are pairs, (x, μ), where $x \in |\mathcal{C}_\Gamma|$, and μ is a labeling of the edges of the graph by smooth functions on M. We call $\mathcal{S}_\Gamma$ the space of metric-Morse structures on M, and define the moduli space of such structures to be the quotient space,

$$\mathcal{M}_\Gamma = \mathcal{S}_\Gamma/Aut(\Gamma).$$

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It will be easily seen that in thickening the moduli space $|\mathcal{C}_\Gamma|/Aut\Gamma$ to define the moduli space of structures, $\mathcal{M}_\Gamma$, we did not change the homotopy type, so that $\mathcal{M}_\Gamma \simeq BAut(\Gamma)$. It is for this reason in our notation we suppress the dependence on M of the moduli space $\mathcal{M}_\Gamma$.

We can then define a moduli space $\mathcal{M}_\Gamma(M)$ of metric graph flows in M. This space consists of isomorphism classes of pairs, (σ, γ), where $\sigma \in \mathcal{S}_\Gamma$ is a metric-Morse structure on Γ, and γ is a continuous map from the graph to M, which, when restricted to a given edge, is a gradient flow line of the smooth function labeling that edge with respect to the parameterization of the edge coming from the orientation and metric. Since $\mathcal{M}_\Gamma \simeq BAut(\Gamma)$, we can take as a representative of a homology class of the automorphism group $Aut(\Gamma)$, a family of metric-Morse structures on the graph Γ. When the structures in the moduli space of metric-graph-flows are restricted to vary in a family representing a fixed homology class of the automorphism group, we will have a finite dimensional moduli space. By studying the topology of this moduli space by two different methods (one using algebraic topology, to define Pontrjagin-Thom constructions and induced "umkehr maps" in homology, and the other using geometry and analysis to understand the smoothness, transversality, and compactness properties of these moduli spaces), we obtain Gromov-Witten type invariants of M. For example, the ring structure (cup product) in the cohomology of the manifold arises as such an invariant when the graph is a tree with three edges, and the family of structures is a single point. Further classical invariants such as Steenrod squares and Stiefel- Whitney classes of the manifold arise when we take higher dimensional families of structures representing nontrivial elements of the homology of the automorphism group.

The approach in this paper is designed specifically to deal with *families* of metric-graphs mapping to manifolds. Graphs are the essential objects here. Functions on the manifold are quite peripheral and do not even need to be Morse. The title "Morse field theory" primarily refers to integral flow lines of gradient vector fields on a manifold as well as the Morse complex and cohomology operations defined on the Morse complex.

The original goal of this project was to understand the Gromov-Witten formalism in the setting of Morse theory, where the analysis is considerably easier. In this model, the role of oriented, metric graphs fills the role of oriented surfaces with a conformal class of metric. Maps from these graphs to manifolds that satisfy gradient flow equations fill the role of J-holomorphic maps from a Riemann surface to a symplectic manifold. This project took its original form in the work of M. Betz in his Ph.D thesis [2] written under the direction of the first author, and in the research announcement [3]. Similar constructions were discovered by Fukaya [10] in which he described his beautiful ideas on the A_∞- structure of Morse homotopy. In particular those ideas have been used in the work of Fukaya and Oh regarding deformations of J- holomorphic disks in the cotangent bundle of a manifold [11].

This present paper contains new ideas involving families of metric-Morse structures, as well as constructions of virtual fundamental classes of these moduli spaces, that allow us to define equivariant invariants, investigate their properties, plus provide proofs of old ideas on non-equivariant invariants [2, 3, 10]. As mentioned above, we show how to deal with families both by using algebraic topological methods, and by using geometric and analytical techniques. The algebraic topological techniques allow us to define generalized Pontrjagin-Thom constructions and resulting umkehr maps, which in turn allow the definition of virtual fundamental classes. These techniques are based on the generalized Pontrjagin-Thom constructions defined by the first author and J. Klein in [8]. In particular these techniques allow us to avoid transversality (smoothness) and compactness issues that arise from the geometric viewpoint. However, because the geometric viewpoint is quite important in its own right, in the second half of the paper we study these analytic issues, and prove the appropriate transversality and compactness results. This allows a second definition of the invariants that are defined on the level of Morse-Smale chain complexes, by counting the number of graph flows in a manifold that satisfy appropriate boundary conditions.

The moduli space $\mathcal{M}_\Gamma$ is somewhat analogous to the moduli space $\mathcal{M}_g$ of Riemann surfaces homeomorphic to a given surface, and more generally to $\mathcal{M}_{g,n}$, the space of Riemann surfaces with n marked points, when the graphs come equipped with marked, univalent vertices. A point in the Teichmuller space $\mathcal{T}(\Sigma)$ of a topological surface Σ (with n labeled points) is a pair (Σ', h) where Σ' is a complete hyperbolic surface and $h : \Sigma' \to \Sigma$ is a homeomorphism well-defined up to isotopy. Teichmuller space is contractible and admits an action of the group of isotopy classes of homeomorphisms of Σ, known as the mapping class group of Σ. The quotient of $\mathcal{T}(\Sigma)$ by the mapping class group is the moduli space of hyperbolic structures on Σ, which appears in algebraic geometry as the moduli space $\mathcal{M}_g$ of Riemann surfaces. In our setup, the contractible space $|\mathcal{C}_\Gamma|$ plays the role of Teichmuller space, $Aut\,\Gamma$ plays the role of the mapping class group, although unlike the mapping class group it acts freely, and the metric graph and homotopy equivalence $h : \Gamma' \to \Gamma$ is analogous to the isotopy class of homeomorphism $h : \Sigma' \to \Sigma$.

A further analogy between $\mathcal{M}_\Gamma$ and $\mathcal{M}_{g,n}$ comes from the fact that $\mathcal{M}_{g,n}$ is homotopy equivalent to the moduli space of metric ribbon graphs—finite graphs whose vertices are at least trivalent, and come equipped with a cyclic ordering of (half-)edges at each vertex and lengths on edges—divided by automorphisms [15]. This analogy will be pursued further by the first author in order to describe a Morse theoretic interpretation of string topology, and the relation between string topology operations and J-holomorphic curves in the cotangent bundle [7]. A description of these ideas was given in [6].

A labeling μ of the edges of a graph in $|\mathcal{C}_\Gamma|$ by functions on M is, in some sense, analogous to choosing a compatible almost complex structure J on a symplectic manifold. In both cases the space of choices of these structures is contractible, and each choice allows the definition of the relevant differential equations used to define a point in the moduli space (a J-holomorphic curve in the Gromov-Witten setting, and a gradient graph flow in our setting).

Aside from the study of these moduli spaces of graphs and graph flows, and the resulting definition of the graph invariants (operations), the main result of this paper is that these invariants fit together to define an appropriate field theory. Recall that an n-dimensional *topological quantum field theory* (TQFT) over a ring R assigns to every closed $n-1$- dimensional manifold N, an R-module $Z(N)$ and to every cobordism W from N_1 to N_2, (i.e W is an n-manifold with boundary $\partial W = N_1 \sqcup N_2$), an operation

$$\mu_W : Z(N_1) \to Z(N_2),$$

which is a map of R-modules. This structure is supposed to satisfy certain properties, the most important of which is gluing: If W_1 is a cobordism from N_1 to N_2, and W_2 is a cobordism from N_2 to N_3, $W = W_1 \cup_{N_2} W_2$ is the "glued cobordism" from N_1 to N_3 obtained by identifying the boundary components of W_1 and W_2 corresponding to N_2, then we require

$$\mu_{W_1 \cup_{N_2} W_2} = \mu_{W_2} \circ \mu_{W_1} : Z(N_1) \to Z(N_2) \to Z(N_3).$$

These operations only depend on the diffeomorphism classes of the cobordisms. See [1] for details.

In the simplest case when $n = 1$, we choose to relax the manifold condition, and think of graphs with univalent vertices as defining generalized cobordisms between zero dimensional manifolds. These univalent vertices can be thought to have signs attached to them, according to whether the edge they lie on is oriented via an arrow that points toward or away from the vertex. Alternatively we can think of these univalent vertices as "incoming" or "outgoing".

For a given manifold M, the Morse field theory functor Z_M assigns to each oriented point, $Z_M(\text{point}) = H_*M$. Given a graph Γ with p incoming and q outgoing univalent vertices (i.e a generalized cobordism between p points and q points), as well as a homology class $\alpha \in H_*(\mathcal{M}_\Gamma) = H_*(BAut(\Gamma))$, the graph invariants described above can be viewed as a homology operation

$$q_\Gamma^\alpha : H_*(M)^{\otimes p} \to H_*(M)^{\otimes q}.$$

We prove that these operations satisfy gluing and a certain invariance properties. This is the "Morse field theory" of the title. We remark that it is a well known folk theorem that a 2-dimensional quantum field theory is equivalent to a Frobenius algebra A. That is, A is an algebra over a field k, equipped with a "trace map" $\theta : A \to k$, such that the pairing

$$A \otimes A \xrightarrow{multiply} A \xrightarrow{\theta} k$$

is nonsingular. A well known example of a Frobenius algebra is the homology of a connected, closed, oriented manifold, $H_*(M)$, where the product is the intersection product, and the trace map is the projection onto the H_0 summand. The resulting nondegeneracy is a manifestation of Poincare duality. As we will see, the basic Frobenius algebra of $H_*(M)$ is realized by our Morse field theory, when the homology classes α are simply taken to be the generator $\alpha = 1 \in H_0(B(Aut(\Gamma)))$. In other words, the basic Frobenius algebra structure is the nonequivariant version of our field theory, achieved by choosing a fixed metric-Morse structure on the graph. It is interesting that by choosing families of these structures we obtain operations

$$q_\Gamma : H_*(B(Aut(\Gamma))) \otimes H_*(M)^{\otimes p} \to H_*(M)^{\otimes q}$$

that satisfy the appropriate gluing and invariance properties. Thus we get an extended Frobenius algebra structure on $H_*(M)$, whose operations we prove encompass such classical operations in algebraic topology as Steenrod squares and Stiefel-Whitney classes. This structure is analogous to the structure in 2-dimensional field theory, where given a connected genus g-cobordism between p circles and q circles, one has an operation,

$$\mu : H_*(\mathcal{M}_{g,p+q}) \otimes Z(S^1)^{\otimes p} \to Z(S^1)^{\otimes q}$$

which satisfy gluing laws. Such a field theory is called a homological conformal field theory [14].

We will also prove that the field theoretic properties (invariance and gluing) of our operations force the classical relations among cohomology operations such as the Adem relations and the Cartan formulae.

The organization of this paper is as follows. In sections 1 and 2 we define the moduli spaces of metric graph structures, $\mathcal{M}_\Gamma$, as well as the moduli space of graph flows in a manifold, $\mathcal{M}_\Gamma(M)$. These are described in algebraic topological terms, using categories of graphs, following ideas of Culler-Vogtmann [9], Igusa [13], and Godin [12]. We then describe a generalized Pontrjagin-Thom construction that allows us to define fundamental classes of these moduli spaces, without having to study smoothness or compactness issues. We then define the invariants (the graph operations) in section 3, and prove their field theoretic properties in section 4. In section 5 we describe examples of these invariants, and show how cup products, Steenrod operations, and Stiefel-Whitney classes arise. We also show how the Cartan and Adem formulae follow from the field theoretic properties.

The second half of the paper begins in section 6, where we deal with the geometric and analytic aspects of the moduli spaces, and give a more combinatorial, Morse theoretic description of the graph operations. Transversality, compactness issues, the resulting smoothness of the moduli spaces is studied in sections 6 through 8. The Morse theoretic description of the graph operations is given in section 9, where they are shown to live on the level of the Morse-Smale chain complexes associated to Morse functions. In particular the operations are defined by suitably counting the number of metric graph flows in M that satisfy certain boundary conditions. A geometric proof of a generalized gluing formula is also given.

There are three appendices to the paper. Two cover analytic issues such as regularity and Fredholm properties. The third gives a detailed description of the generalized Pontrjagin-Thom construction needed to define the virtual fundamental classes of the moduli spaces.

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1. Categories of graphs, and the moduli space of metric-Morse structures on a graph. In this section we describe a category of graphs that will be used to define our moduli space of graph flows. As we will show, the geometric realization of this category will consist of graphs equipped with appropriate metrics. The idea of this category was inspired by the work of Culler-Vogtmann [9], and the interpretation of this work due to Igusa [13] and Godin [12].

DEFINITION 1. *Define $\mathcal{C}_{b,p+q}$ to be the category of oriented graphs of first Betti number b, with $p+q$ leaves. More specifically, the objects of $\mathcal{C}_{b,p+q}$ are finite graphs (one dimensional CW-complexes) Γ, with the following properties:*

1. *Each edge of the graph Γ has an orientation.*
2. *Γ has $p+q$ univalent vertices, or "leaves". p of these are vertices of edges whose orientation points away from the vertex (toward the body of the graph). These are called "incoming" leaves. The remaining q leaves are on edges whose orientation points toward the vertex (away from the body of the graph). These are called "outgoing" leaves.*
3. *Γ comes equipped with a "basepoint", which is a nonunivalent vertex.*

For set theoretic reasons we also assume that the objects in this category (the graphs) are subspaces of a fixed infinite dimensional Euclidean space, $\mathbb{R}^\infty$.

A morphism between objects $\phi : \Gamma_1 \to \Gamma_2$ is combinatorial map of graphs (cellular map) that satisfies:

1. *ϕ preserves the orientations of each edge.*
2. *The inverse image of each vertex is a tree (i.e a contractible subgraph).*
3. *The inverse image of each open edge is an open edge.*

4. ϕ preserves the basepoints.

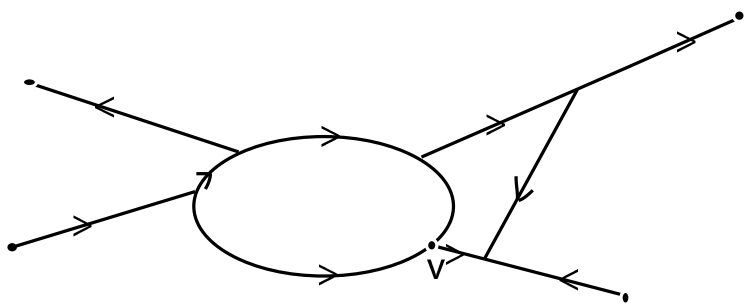

FIG. 1. *An object Γ in $\mathcal{C}_{2,2+2}$*

We observe that by the definition of $\mathcal{C}_{b,p+q}$, a morphism $\phi : \Gamma_1 \to \Gamma_2$ is a basepoint preserving cellular map which is a homotopy equivalence. Given a graph $\Gamma \in \mathcal{C}_{b,p+q}$, we define the automorphism group $Aut(\Gamma)$ to be the group of invertible morphisms from Γ to itself in this category. $Aut(\Gamma)$ is a finite group, as it is a subgroup of the group of permutations of the the edges.

We now fix a graph Γ (an object in $\mathcal{C}_{b,p+q}$), and we describe the category of "graphs over Γ", $\mathcal{C}_\Gamma$. As we will see below, a point in the geometric realization of this category will be viewed as a metric on a generalized subdivision of Γ.

DEFINITION 2. *Define $\mathcal{C}_\Gamma$ to be the category whose objects are morphisms in $\mathcal{C}_{b,p+q}$ with target Γ: $\phi : \Gamma_0 \to \Gamma$. A morphism from $\phi_0 : \Gamma_0 \to \Gamma$ to $\phi_1 : \Gamma_1 \to \Gamma$ is a morphism $\psi : \Gamma_0 \to \Gamma_1$ in $\mathcal{C}_{b,p+q}$ with the property that $\phi_0 = \phi_1 \circ \psi : \Gamma_0 \to \Gamma_1 \to \Gamma$.*

Notice that the identity map $id : \Gamma \to \Gamma$ is a terminal object in $\mathcal{C}_\Gamma$. That is, every object $\phi : \Gamma_0 \to \Gamma$ has a unique morphism to $id : \Gamma \to \Gamma$. This implies that the geometric realization of the category, $|\mathcal{C}_\Gamma|$ is contractible. But notice that the category $\mathcal{C}_\Gamma$ has a free right action of the automorphism group, $Aut(\Gamma)$, given on the objects by composition:

$$Objects\,(\mathcal{C}_\Gamma) \times Aut(\Gamma) \to Objects\,(\mathcal{C}_\Gamma)$$
$$(\phi : \Gamma_0 \to \Gamma) \cdot g \to g \circ \phi : \Gamma_0 \xrightarrow{\phi} \Gamma \xrightarrow{g} \Gamma. \tag{1}$$

This induces a free action on the geometric realization $\mathcal{C}_\Gamma$. We therefore have the following:

PROPOSITION 3. *The orbit space is homotopy equivalent to the classifying space,*

$$|\mathcal{C}_\Gamma|/Aut(\Gamma) \simeq BAut(\Gamma).$$

We now consider the geometric realization of the category $|\mathcal{C}_\Gamma|$. Following an idea of Culler-Vogtmann [9] and Igusa [13], we interpret a point in this space as defining a metric on a generalized subdivision of the graph Γ.

Recall that

$$|\mathcal{C}_\Gamma| = \bigcup_k \Delta^k \times \{\Gamma_k \xrightarrow{\psi_k} \Gamma_{k-1} \xrightarrow{\psi_{k-1}} \Gamma_{k-2} \to \cdots \xrightarrow{\psi_1} \Gamma_0 \xrightarrow{\phi} \Gamma\}/\sim$$

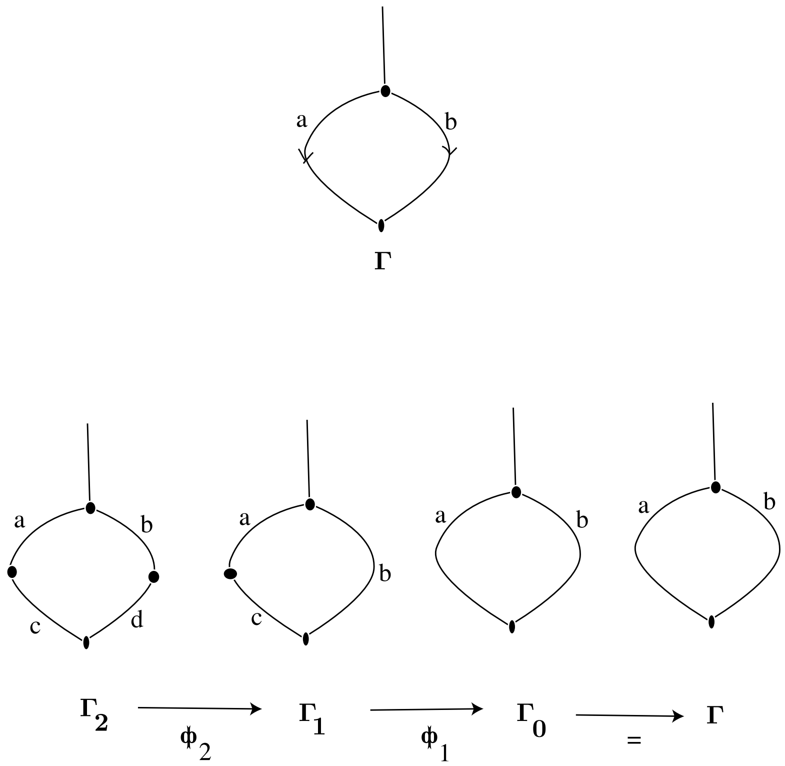

FIG. 2. *A 2-simplex in* $|\mathcal{C}_\Gamma|$.

where the identifications come from the face and degeneracy operations.

Let $(\vec{t}, \vec{\psi})$ be a point in $|\mathcal{C}_\Gamma|$, where $\vec{t} = (t_0, t_1, \cdots, t_k)$ is a vector of positive numbers whose sum equals one, and $\vec{\psi}$ is a sequence of k-composable morphisms in $\mathcal{C}_\Gamma$. Recall that a morphism $\phi_i : \Gamma_i \to \Gamma_{i-1}$ can only collapse trees, or perhaps compose such a collapse with an automorphism. So given a composition of morphisms,

$$\vec{\psi} : \Gamma_k \to \cdots \to \Gamma_0 \to \Gamma$$

we may think of Γ_k is a generalized subdivision of Γ, in the sense that Γ is obtained from Γ_k by collapsing various edges.

We use the coordinates $\vec{t}$ of the simplex Δ^k to define a metric on Γ_k as follows. For each edge E of Γ_k, define $k+1$ numbers, $\lambda_0(E), \cdots, \lambda_k(E)$ given by

$$\lambda_i(E) = \begin{cases} 0 & \text{if } E \text{ is collapsed by } \vec{\psi} \text{ in } \Gamma_i \text{, and,} \\ 1 & \text{if } E \text{ is not collapsed in } \Gamma_i \end{cases}$$

We then define the length of the edge E to be

$$\ell(E) = \sum_{i=0}^{k} t_i \lambda_i(E). \tag{2}$$

Notice also that the orientation on the edges and the metric deterimine parameterizations (isometries) of standard intervals to the edges of the graph Γ_k over Γ,

$$\theta_E : [0, \ell(E)] \xrightarrow{\cong} E \tag{3}$$

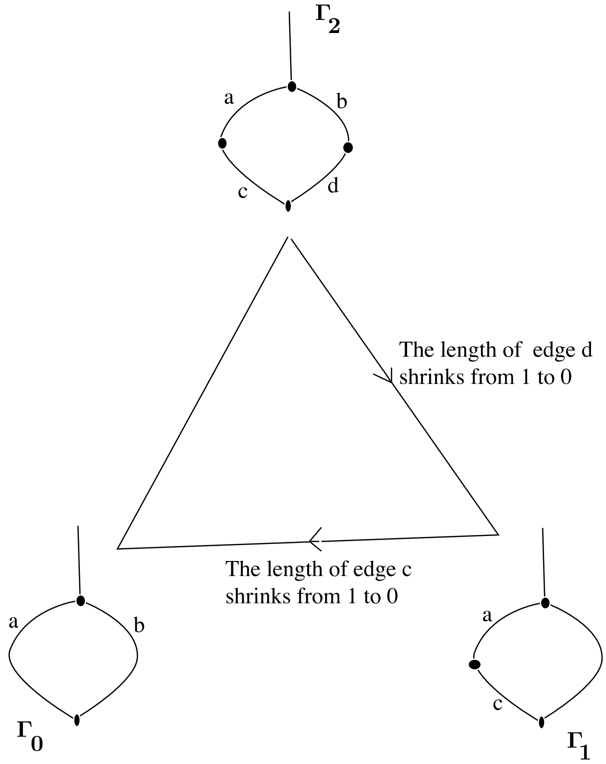

FIG. 3. *A 2-simplex of metrics.*

Thus a point $(\vec{t}, \vec{\psi}) \in |\mathcal{C}_\Gamma|$ determines a metric on a graph Γ_k living over Γ, as well as a parameterization of its edges. In some sense this may be viewed as the analogue in our theory, of the moduli space of Riemann surfaces in Gromov-Witten theory. In that theory, one studies maps from a Riemann surface (an element of moduli space) to a symplectic manifold, which satisfy the Cauchy-Riemann equations (or some perturbation of them) with respect to a choice of a compatible almost complex structure on the symplectic manifold. In our case, we want to study maps from an element of our moduli space, i.e a graph living over Γ, equipped with a metric and parameterization of the edges, to a target manifold M, that satisfies certain *ordinary* differential equations. These differential equations will be the gradient flow equations of smooth functions on M. To define these, we need to impose more structure on our graphs, given by a labeling of the edges of the graph by distinct smooth functions on the manifold. We call such a structure a *Morse labeling* of a graph. We define this precisely as follows.

Let V be a real vector space. Let $F(V, k)$ be the configuration space of k distinct ordered points in V. That is, $F(V, k) = \{(v_1, \cdots, v_k) \in V^k \text{ such that } v_i \neq v_j \text{ if } i \neq j\}$. Recall that if V is infinite dimensional, $F(V, k)$ is contractible.

Throughout the rest of this section we let M be a fixed closed, Riemannian manifold.

DEFINITION 4. *An M-Morse labeling* of a graph $\Gamma \in \mathcal{C}_{b,p+q}$ is a pair $(\phi_0 : \Gamma_0 \to \Gamma,\, c)$, where $\phi_0 : \Gamma_0 \to \Gamma$ is an object of $\mathcal{C}_\Gamma$, and $c \in F(C^\infty(M), e(\Gamma_0))$, where $C^\infty(M)$

is the vector space of smooth, real valued functions on M, and $e(\Gamma_0)$ is the number of edges of Γ_0. We think of the vector of functions making up the configuration c as labeling the edges of Γ_0.

Fixing our manifold M and graph Γ, our goal now is to define the moduli space of metrics and Morse structures (abbreviated "structures") on Γ, $\mathcal{M}_\Gamma$. We do this as follows.

Consider the functor

$$\mu : \mathcal{C}_\Gamma \to \mathit{Spaces}$$

which assigns to a graph over Γ, $\phi_0 : \Gamma_0 \to \Gamma$, the configuration space $F(C^\infty(M), e(\Gamma_0))$. Given a morphism $\psi : \Gamma_1 \to \Gamma_0$, which collapses certain edges and perhaps permutes others, there is an obvious induced map,

$$\mu(\psi) : F(C^\infty(M), e(\Gamma_1)) \to F(C^\infty(M), e(\Gamma_0)).$$

This map projects off of the coordinates corresponding to edges collapsed by ψ, and permutes coordinates corresponding to the permutation of edges induced by ψ.

We can now do a homotopy theoretic construction, called the homotopy colimit (see for example [4]).

Definition 5. *We define the space of metric structures and Morse labelings on G, $\mathcal{S}_\Gamma$, to be the homotopy colimit,*

$$\mathcal{S}_\Gamma = \mathit{hocolim}\,(\mu : \mathcal{C}_\Gamma \to \mathit{Spaces}).$$

The homotopy colimit construction is a simplicial space whose k simplices consist of pairs, $(\vec{f}, \vec{\psi})$, where $\vec{\psi} : \Gamma_k \to \Gamma_{k-1} \to \cdots \Gamma_0 \to \Gamma$ is a k-tuple of composable morphisms in $\mathcal{C}_\Gamma$, and $\vec{f} \in \mu(\Gamma_k)$. That is, $\vec{f}$ is an M - Morse labeling of the edges of Γ_k. So we can think of a point $\sigma \in \mathcal{S}_\Gamma$ as defining a metric on a graph over Γ, together with an M- Morse labeling of its edges.

We now make the following observation.

Lemma 6. *The space of metric-Morse structures $\mathcal{S}_\Gamma$ is contractible with a free $Aut(\Gamma)$ action.*

Proof. The contractibility follows from standard facts about the homotopy colimit construction, considering the fact that both $|\mathcal{C}_\Gamma|$ and $F(C^\infty(M), m)$ are contractible. The free action of $Aut(\Gamma)$ on $|\mathcal{C}_\Gamma|$ extends to an action on $\mathcal{S}_\Gamma$, since $Aut(\Gamma)$ acts by permuting the edges of Γ, and therefore permutes the labels accordingly. □

We now define our moduli space of structures.

Definition 7. *The moduli space of metric structures and M- Morse labelings on G, $\mathcal{M}_\Gamma$, is defined to be the quotient,*

$$\mathcal{M}_\Gamma = \mathcal{S}_\Gamma / Aut(\Gamma).$$

We therefore have the following.

Corollary 8. *The moduli space is a classifying space of the automorphism group,*

$$\mathcal{M}_\Gamma \simeq BAut(\Gamma).$$

2. The moduli space of metric-graph flows in a manifold. Let M be a fixed, smooth, closed n-manifold with a Riemannian metric. Let $\Gamma \in \mathcal{C}_{b,p+q}$ be a graph. In this section we define the moduli space of Γ-flows in M, $\mathcal{M}_\Gamma(M)$, and study its topology. This will be an infinite dimensional space built from the moduli space of metric-Morse structures, $\mathcal{M}_\Gamma$, which in turn has an infinite dimensional homotopy type, since $\mathcal{M}_\Gamma \simeq BAut(\Gamma)$, and $Aut(\Gamma)$ is a finite group. However, given a homology class $\alpha \in H_k(Aut(\Gamma))$, we show how to define a "virtual fundamental class",

$$[\mathcal{M}^\alpha_\Gamma(M)] \in H_q(\mathcal{M}_\Gamma(M))$$

where $q = k + \chi(\Gamma)n$, where $\chi(\Gamma) = 1 - b$ is the Euler characteristic. The smooth structures on these moduli spaces will be studied in later sections. But even without knowledge of this structure, these virtual fundamental classes will be constructed using generalized Pontrjagin-Thom constructions similar to those defined in [8]. These constructions allow us to define invariants in the next section, which we will identify with classical cohomology operations in section 4. Let $\sigma \in \mathcal{S}_\Gamma$ be a metric-Morse structure. Then $\sigma = (\vec{t}, \vec{\psi}, c)$, where $\vec{t} \in \Delta^k$, $\vec{\psi} : \Gamma_k \to \cdots \to \Gamma_0 \to \Gamma$ is a k-simplex in the nerve of $\mathcal{C}_\Gamma$, that is a k-tuple of composable morphisms, and c is a Morse labeling of the edges of Γ_k.

Definition 9. *A metric-Γ-flow in M, is a pair (σ, γ), where $\sigma = (\vec{t}, \vec{\psi}, c) \in \mathcal{S}_\Gamma$ is a metric-Morse structure on Γ, and $\gamma : \Gamma_k \to M$ is a continuous map, smooth on the edges, satisfying the following property. Given any edge E of Γ_k, let $\gamma_E : [0, \ell(E)] \to M$ be the composition*

$$\gamma_E : [0, \ell(E)] \xrightarrow{\theta_E} E \subset \Gamma_k \xrightarrow{\gamma} M,$$

where θ_E is the parameterization of the edge E defined in (3). Then γ_E is required to satisfy the differential equation

$$\frac{d\gamma_E}{dt}(s) + \nabla f_E(\gamma_E(s)) = 0.$$

Here the collection of labeling functions $\{f_E : M \to \mathbb{R} : E \text{ is an edge of } \Gamma\}$ is the configuration $c \in F(C^\infty(M), e(\Gamma))$ determined by the structure σ.

We define the "structure space of metric-graph flows", $\tilde{\mathcal{M}}_\Gamma(M)$, to be the space

$$\tilde{\mathcal{M}}_\Gamma(M) = \{(\sigma, \gamma) \text{ a metric-}\Gamma\text{-flow in } M\}, \tag{4}$$

and the moduli space of graph flows to be the orbit space,

$$\mathcal{M}_\Gamma(M) = \tilde{\mathcal{M}}_\Gamma(M)/Aut(\Gamma).$$

Here the automorphism group $Aut(\Gamma)$ acts on $\tilde{\mathcal{M}}_\Gamma(M)$ by acting on the structure σ as described above.

We have not yet defined the topology on these spaces of flows. To do that we first consider the case when the graph Γ is a tree. That is, Γ is contractible, so $b_1(\Gamma) = 0$.

Proposition 10. *Let Γ be a tree. Then there is an $Aut(\Gamma)$-equivariant bijective correspondence*

$$\begin{aligned}\Psi : \tilde{\mathcal{M}}_\Gamma(M) &\xrightarrow{\cong} \mathcal{S}_\Gamma \times M \\ (\sigma, \gamma) &\to \sigma \times \gamma(v)\end{aligned}$$

where v is the fixed vertex of the graph Γ_k over Γ determined by the structure σ. On the right hand side, $Aut(\Gamma)$ acts on $\mathcal{S}_\Gamma$ as described above, and acts trivially on M.

Proof. This follows from the existence and uniqueness theorem for solutions of ODE's on compact manifolds. The point is that the values of γ on the edges emanating from v are completely determined by $\gamma(v) \in M$, since one has a unique flow line through that point for any of the functions labeling these edges. The value of γ on these edges determines the value of γ on coincident edges (i.e edges that share a vertex) for the same reason. The fact that Ψ is a bijection now follows. The $Aut(\Gamma)$-equivariance of Ψ is immediate. □

We now topologize $\tilde{\mathcal{M}}_\Gamma(M)$ so that $\Psi : \tilde{\mathcal{M}}_\Gamma(M) \to \mathcal{S}(\Gamma)\times M$ is a homeomorphism. We then have the following description of the moduli space of graph flows, when Γ is a tree:

COROLLARY 11. *Let Γ be a tree. Then Ψ induces a homeomorphism,*

$$\Psi : \mathcal{M}_\Gamma(M) \xrightarrow{\cong} \mathcal{S}_\Gamma/Aut(\Gamma) \times M$$

which has the homotopy type of $BAut(\Gamma) \times M$.

For general connected graphs Γ, we analyze the topology of $\mathcal{M}_\Gamma(M)$ in the following way. Let $\sigma \in \mathcal{S}_\Gamma$. A *tree flow* of Γ with respect to the structure σ is a collection $\gamma = \{\gamma_T\}$ where $\gamma_T : T \to M$ is a graph flow on a maximal subtree $T \subset \Gamma_k$. The collection ranges over all maximal subtrees $T \subset \Gamma_k$, and is subject only to the condition that the values at the basepoint are the same:

$$\gamma_{T_1}(v) = \gamma_{T_2}(v)$$

for any two maximal trees $T_1, T_2 \subset \Gamma_k$. (Here $v \in T \subset \Gamma$ is the fixed point vertex.)

We define

$$\tilde{\mathcal{M}}_{tree}(\Gamma, M) = \{(\sigma, \gamma) \, : \, \sigma \in \mathcal{S}_\Gamma, \text{ and } \gamma = \{\gamma_T\} \text{ is a tree flow of } \Gamma \text{ with respect to } \sigma\} \tag{5}$$

and

$$\mathcal{M}_{tree}(\Gamma, M) = \tilde{\mathcal{M}}_{tree}(\Gamma, M)/Aut(\Gamma).$$

Notice that the proof of proposition 10 also proves the following.

THEOREM 12. *For any graph $\Gamma \in \mathcal{C}_{b,p+q}$ there is an $Aut(\Gamma)$ -equivariant bijective correspondence,*

$$\begin{aligned}\Psi : \tilde{\mathcal{M}}_{tree}(\Gamma, M) &\xrightarrow{\cong} \mathcal{S}_\Gamma \times M \\ (\sigma, \gamma) &\to \sigma \times \gamma(v).\end{aligned}$$

We therefore again topologize $\tilde{\mathcal{M}}_{tree}(\Gamma, M)$ so that Ψ is an equivariant homeomorphism. Then

$$\mathcal{M}_{tree}(\Gamma, M) \cong \mathcal{S}_\Gamma/Aut(\Gamma) \times M \simeq BAut(\Gamma) \times M.$$

Consider the inclusion, $\tilde{\rho} : \tilde{\mathcal{M}}_\Gamma(M) \hookrightarrow \tilde{\mathcal{M}}_{tree}(\Gamma, M)$ defined to be the map that sends a graph flow γ to the tree flow obtained by restricting γ to each maximal tree. We then give $\tilde{\mathcal{M}}_\Gamma(M)$ the subspace topology, which makes ρ an equivariant embedding. This defines an embedding $\rho : \mathcal{M}_\Gamma(M) \hookrightarrow \mathcal{M}_{tree}(\Gamma, M)$.

We use this embedding to define virtual fundamental classes of $\mathcal{M}_\Gamma(M)$. Recall that the space $\mathcal{M}_\Gamma(M)$ is infinite dimensional because the moduli space $\mathcal{S}_\Gamma/Aut(\Gamma) \simeq BAut(\Gamma)$ is infinite dimensional. We can "cut down" this moduli space by considering an embedding of a compact manifold of structures, $\tilde{N} \subset \mathcal{S}_\Gamma$. We let $N = \tilde{N}/Aut(\Gamma) \subset \mathcal{S}_\Gamma/Aut(\Gamma) \simeq BAut(\Gamma)$. We can then define the space $\mathcal{M}_\Gamma^N(M) \subset \mathcal{M}_\Gamma(M)$ to be the subspace $\mathcal{M}_\Gamma^N(M) = \{(\sigma, \gamma) \in \tilde{\mathcal{M}}_\Gamma(M)$ such that $\sigma \in \tilde{N})/Aut(\Gamma)\}$. Then the embedding $\rho : \mathcal{M}_\Gamma(M) \hookrightarrow \mathcal{M}_{tree}(\Gamma, M) \cong \mathcal{S}_\Gamma/Aut(\Gamma) \times M$ defines an embedding

$$\rho_N : \mathcal{M}_\Gamma^N(M) \hookrightarrow N \times M. \tag{6}$$

To motivate our construction of the virtual fundamental classes, suppose we know that $\mathcal{M}_\Gamma^N(M)$ is a smooth closed submanifold of $N \times M$ of codimension k. Then the image of its fundamental class $[\mathcal{M}_\Gamma^N(M)] \in H_*(\mathcal{M}_\Gamma^N(M))$ in $H_*(\mathcal{M}_\Gamma(M))$ would be the image under the "umkehr map",

$$H_*(N \times M) \xrightarrow{(\rho_N)_!} H_{*-k}(\mathcal{M}_\Gamma^N(M))) \to H_{*-k}(\mathcal{M}_\Gamma(M))$$

of the product of the fundamental classes $[N] \times [M]$. The umkehr map $(\rho_N)_! : H_*(N \times M) \to H_{*-k}(\mathcal{M}_\Gamma^N(M))$ is Poincare dual to the restriction map in cohomology, $\rho_N^* : H^*(N \times M) \to H^*(\mathcal{M}_\Gamma^N(M))$, induced by the embedding $\rho_N : \mathcal{M}_\Gamma^N(M) \hookrightarrow N \times M$. In particular the fundamental class $[\mathcal{M}_\Gamma^N(M)] \in H_{*-k}(\mathcal{M}_\Gamma(M))$ only depends on the homology class represented by the manifold $[N] \in H_*(\mathcal{S}_\Gamma/Aut(\Gamma)) \cong H_*(BAut(\Gamma))$.

To define our "virtual fundamental class", we avoid the question of whether $\mathcal{M}_\Gamma^N(M)$ can be given a smooth structure (we address this question in a later section), by directly defining the umkehr map

$$\rho_! : H_*(BAut(\Gamma) \times M) = H_*(\mathcal{S}_\Gamma/Aut(\Gamma) \times M) \to H_{*-bn}(\mathcal{M}_\Gamma(M)) \tag{7}$$

where $b = b_1(\Gamma)$, and $n = dim\, M$. Once we have this map, then given $\alpha \in H_q(BAut(\Gamma))$, the virtual fundamental class $[\mathcal{M}_\Gamma^\alpha(M)]$ is defined by

$$[\mathcal{M}_\Gamma^\alpha(M)] = \rho_!(\alpha \times [M]) \in H_{q-(b-1)n}(\mathcal{M}_\Gamma(M)). \tag{8}$$

The rest of this section will be devoted to defining the umkehr map $\rho_!$. The existence of this map follows from a construction that is used to give a proof of a general existence theorem for umkehr maps by the first author and J. Klein in [8]. This construction is based on the existence of "Pontrjagin-Thom collapse maps". We recall that given a smooth embedding of compact manifolds, $e : N \hookrightarrow M$ of codimension k, the umkehr map $e_! : H_*(N) \to H_{*-k} * (M)$ can be computed via the Pontrjagin-Thom collapse map,

$$\tau_e : M \to M/M - \eta_e$$

where $N \subset \eta_e$ is a tubular neighborhood. This quotient space is the one point compactification of the tubular neighborhood, which is homeomorphic to the Thom space of the normal bundle, N^{ν_e}. The umkehr map is then given by the composition,

$$e_! : H_*(M) \xrightarrow{(\tau_e)_*} H_*(N^{\nu_e}) \xrightarrow[\cong]{\cap u} H_{*-k}(N)$$

where the last map is the cap product with the Thom class, yielding the Thom isomorphism.

To apply this construction in our setting, we need to produce an open neighborhood η_ϵ of the embedding $\rho : \mathcal{M}_\Gamma(M) \hookrightarrow \mathcal{M}_{tree}(\Gamma, M) \cong \mathcal{S}_\Gamma/Aut(\Gamma) \times M$, that is homeomorphic to the total space of an appropriate normal bundle, ν_ρ. We now define these objects.

Let $T \subset \Gamma$ be a maximal tree. We define a map $p_T : \tilde{\mathcal{M}}_{tree}(\Gamma, M) \to M^{2b}$ as follows. Since T is a maximal tree, the complement $\Gamma - T$ consists of $b = b_1(\Gamma)$ open edges, $e_1^T, \cdots, e_b^T$. Now let $\phi : \Gamma_0 \to \Gamma$ be an object in $\mathcal{C}_\Gamma$. Since the inverse image under ϕ of an edge is an edge, then $\phi^{-1}(e_i^T) = e_i^T(\Gamma_0)$ is an edge, and the tree $T(\Gamma_0) = \phi^{-1}(T) \subset \Gamma_0$ has complement $\Gamma_0 - T(\Gamma_0)$ given by the b open edges $e_i^T(\Gamma_0), i = 1, \cdots, b$. The edges $e_i^T(\Gamma_0)$ are oriented, so they have source and target vertices, $s_i^T(\Gamma_0)$, and $t_i^T(\Gamma_0)$.

Now let (σ, γ) be a point in $\tilde{\mathcal{M}}_{tree}(\Gamma, M)$. So $\sigma = (\vec{t}, \vec{\psi}, c) \in \mathcal{S}_\Gamma$, and $\gamma = \{\gamma_{T_j} : T_j \to M\}$, where the T_j's are the maximal trees in Γ_k, and γ_{T_j} is a graph flow on the tree T_j with respect to the structure σ. Let $T_1 = T(\Gamma_k) = \phi_k^{-1}(T) \subset \Gamma_k$.

Consider the graph flow $\gamma_{T_1} : T_1 \to M$, and let x_i be the image of the source vertex,

$$x_i = \gamma_{T_1}(s_i^T(\Gamma_k)) \in M. \tag{9}$$

Now consider the image of the target vertex, $\gamma_{T_1}(t_i^T(\Gamma_k)) \in M$. The existence and uniqueness theorem for solutions of ODE's says there is a unique map $\alpha_i : e_i^T(\Gamma_k) \to M$ which is graph flow with respect to the structure σ, satisfying the initial condition, $\alpha_i(t_i^T(\Gamma_k) = \gamma_{T_1}(t_i^T(\Gamma_k)) \in M$. We then define $y_i \in M$ to be the image of the source vertex under the map α_i:

$$y_i = \alpha_i(s_i^T(\Gamma_k)) \in M.$$

Notice that the tree flow γ is induced from a flow on the full graph Γ if and only if $x_i = y_i$ for all $i = 1, \cdots, b$. Said another way, we have defined a map

$$\begin{aligned} p_T : \tilde{\mathcal{M}}_{tree}(\Gamma, M) &\to (M^2)^b \\ (\sigma, \gamma) &\to (x_1, y_1), \cdots, (x_b, y_b) \end{aligned} \tag{10}$$

where the following diagram is a pullback square:

$$\begin{array}{ccc} \tilde{\mathcal{M}}_\Gamma(M) & \xrightarrow[\hookrightarrow]{\rho} & \tilde{\mathcal{M}}_{tree}(\Gamma, M) \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle p_T} \\ M^b & \xrightarrow[\Delta^b]{\hookrightarrow} & (M^2)^b. \end{array} \tag{11}$$

Here $\Delta : M \to M^2$ is the diagonal. We now define our tubular neighborhood and normal bundle. Give M a Riemannian metric.

Definition 13. *1. For $\epsilon > 0$, let $\eta_\epsilon \subset \tilde{\mathcal{M}}_{tree}(\Gamma, M)$ be the open set containing $\rho(\tilde{\mathcal{M}}_\Gamma(M))$ defined to be the inverse image of the ϵ-neighborhood of the diagonal,*

$$\eta_\epsilon = \{(\sigma, \gamma) \in \tilde{\mathcal{M}}_\Gamma(M) : d(p_T(\sigma, \gamma), \Delta(M)) < \epsilon \quad \text{for every maximal tree } T \subset \Gamma\}$$

where d is the Riemannian distance in $M \times M$. *2. Let* $\nu(\rho) \to \tilde{\mathcal{M}}_\Gamma(M)$ *be the vector bundle defined as follows. Let* $p : \tilde{\mathcal{M}}_\Gamma(M) \to M$ *be the map* $(\sigma, \gamma) \to \gamma(v)$. *This is the right hand factor of the embedding* $\rho : \tilde{\mathcal{M}}_\Gamma(M) \hookrightarrow \tilde{\mathcal{M}}_{tree}(\Gamma, M) \cong \mathcal{S}_\Gamma \times M$. *Define*

$$\nu(\rho) = p^*(\bigoplus_b TM)$$

to be the pullback of the Whitney sum of b*-copies of the tangent bundle.*

We notice that η_e is an $Aut(\Gamma)$-invariant open subspace of $\tilde{\mathcal{M}}_{tree}(\Gamma, M)$ and therefore defines an open neigborhood which, by abuse of notation we also call η_ϵ of the embedding of quotient spaces, $\rho : \mathcal{M}_\Gamma(M) \hookrightarrow \mathcal{M}_{tree}(\Gamma, M)$. Similarly, $\nu(\rho)$ is an invariant bundle over $\tilde{\mathcal{M}}_\Gamma(M)$,and therefore defines a bundle $\nu(\rho) = p^*(\bigoplus_b TM)$ over $\mathcal{M}_\Gamma(M)$. The following theorem will allow us to define a Pontrjagin-Thom collapse map, which as observed above, will allow us to define the umkehr map $\rho_!$. This is a tubular neighborhood theorem for the embedding $\rho : \mathcal{M}_\Gamma(M) \hookrightarrow \mathcal{M}_{tree}(\Gamma, M)$. Its proof is rather technical, so we leave it to the appendix.

Theorem 14. *For* $\epsilon > 0$ *sufficiently small, there is a homeomorphism* $\Theta : \eta_\epsilon \xrightarrow{\cong} \nu(\rho)$ *taking* $\mathcal{M}_\Gamma(M)$ *to the zero section.*

The homeomorphism Θ then defines a homeomorphism of the quotient space to the Thom space,

$$\Theta : \mathcal{M}_{tree}(\Gamma, M)/(\mathcal{M}_{tree}(\Gamma, M) - \eta_\epsilon) \to \mathcal{M}_\Gamma(M)^{\nu(\rho)}$$

and so we have a Pontrjagin-Thom collapse map,

$$\tau_\rho : \mathcal{S}_\Gamma/Aut(\Gamma) \times M \cong \mathcal{M}_{tree}(\Gamma, M) \xrightarrow{project} \mathcal{M}_{tree}(\Gamma, M)/(\mathcal{M}_{tree}(\Gamma, M) - \eta_\epsilon) \xrightarrow{\Theta} \mathcal{M}_\Gamma(M)^{\nu(\rho)}. \tag{12}$$

Assuming M is oriented, this defines an umkehr map,

$$\rho_! : H_*(BAut(\Gamma) \times M) \cong H_*(\mathcal{S}_\Gamma/Aut(\Gamma) \times M) \xrightarrow{\tau_\rho} H_*(\mathcal{M}_\Gamma(M)^{\nu(\rho)}) \xrightarrow{Thom\, iso} H_{*-b\cdot n}(\mathcal{M}_\Gamma(M)). \tag{13}$$

We are now ready to define virtual fundamental classes of these moduli spaces.

Definition 15. *Let* $\alpha \in H_q(BAut(\Gamma); k)$, *where* k *is a coefficient field. Define the virtual fundamental class,* $[\mathcal{M}^\alpha(\Gamma, M)] \in H_{q+\chi(\Gamma)n}(\mathcal{M}_\Gamma(M); k)$ *to be the image of* $\alpha \otimes [M]$ *under the umkehr map*

$$\rho_! : H_*(BAut(\Gamma); k) \otimes H_*(M; k) \to H_{*-bn}(\mathcal{M}_\Gamma(M); k).$$

Notice that since $1 - b$ is the Euler characteristic $\chi(\Gamma)$, we have that the virtual fundamental class associated to a homology class α of degree q lies in degree, $q+\chi(\Gamma)n$,

$$[\mathcal{M}^\alpha(\Gamma, M)] \in H_{q+\chi(\Gamma)n}(\mathcal{M}_\Gamma(M); k).$$

These virtual fundamental classes, and more generally the umkehr map $\rho_!$, will allow us to define cohomology operations yielding the Morse Field Theory described in the introduction. We define and study these operations in the next section.

3. Graph operations. In this section we describe Gromov-Witten type operations induced by our moduli spaces of graphs and their virtual vector bundles. We actually describe two types of operations induced by a graph Γ, the first, q^0_Γ, is equivariant with respect to the *bordered* automorphism group, $Aut_0(\Gamma)$, which consists of those automorphisms $g \in Aut(\Gamma)$ that fix the marked points (univalent vertices). These operations are the directly analogous to Gromov- Witten operations. We then show how these operations can be extended to operations q_Γ that are equivariant with respect to the full automorphism group.

Let Γ be an object in $\mathcal{C}_{b,p+q}$, and M a closed, n- dimensional manifold. In what follows we consider homology and cohomology with coefficients in an arbitrary but fixed field k. We begin by defining the operations,

$$q^0_\Gamma : H_*(BAut_0(\Gamma)) \otimes H_*(M)^{\otimes p} \to H_*(M)^{\otimes q} \tag{14}$$

which raises total dimension by $\chi(\Gamma)n - np$ where $\chi(\Gamma)$ is the Euler characteristic of the graph Γ ($\chi(\Gamma) = 1 - b$), b is the first Betti number of Γ, and p and q are the number of incoming and outgoing marked points of Γ respectively.

Let $\mathcal{M}_0(\Gamma, M) = \tilde{\mathcal{M}}_\Gamma(M)/Aut_0(\Gamma) \simeq BAut_0(\Gamma)$. Consider the evaluation maps $ev_{in} : \tilde{\mathcal{M}}_\Gamma(M) \to M^p$ and $ev_{out} : \tilde{\mathcal{M}}_\Gamma(M) \to M^q$ that evaluate a graph flow on the incoming and outgoing marked points, respectively. Since automorphisms in $Aut_0(\Gamma)$ preserve these marked points, they descend to give maps $ev_{in} : \mathcal{M}_0(\Gamma, M) \to M^p$ and $ev_{out} : \mathcal{M}_0(\Gamma, M) \to M^q$. Let ev be the product map,

$$ev = ev_{in} \times ev_{out} : \mathcal{M}_0(\Gamma, M) \to M^p \times M^q.$$

Let $\alpha \in H_r(BAut_0(\Gamma)) = H_r(\mathcal{M}_0(\Gamma, M))$. As we did in the last section (8) we can define a virtual fundamental class

$$[\mathcal{M}^\alpha_0(\Gamma, M)] = \rho_!(\alpha \times [M]) \in H_{r+n-bn}(\mathcal{M}_0(\Gamma, M)) = H_{r+\chi(\Gamma)n}(\mathcal{M}_0(\Gamma, M)).$$

Consider the Gromov-Witten type invariant,

$$\bar{q}^0_\Gamma(\alpha) : H^*(M)^{\otimes p} \otimes H^*(M)^{\otimes q} \to k$$
$$x \otimes y \to \langle ev^*(x \otimes y), [\mathcal{M}^\alpha_0(\Gamma, M)]\rangle.$$

Notice that $\bar{q}^0_\Gamma(\alpha)$ can only be nonzero if the total dimension of $x \otimes y$ is $r + \chi(\Gamma)n$. We may think of $\bar{q}^0_\Gamma(\alpha)$ as an element of homology,

$$\bar{q}^0_\Gamma(\alpha) = ev_*([\mathcal{M}^\alpha_0(\Gamma, M)]) \in H_*(M^p) \otimes H_*(M^q)$$

of total dimension $r + \chi(\Gamma)n$. By applying Poincare duality to the left hand tensor factor, this defines a class

$$q^0_\Gamma(\alpha) \in H^{np-*}(M^p) \otimes H_*(M^q) \cong Hom(H_*(M)^{\otimes p}; H_*(M)^{\otimes q})$$

which raises total dimension by $r+\chi(\Gamma)n-np$. We have therefore defined an operation

$$q^0_\Gamma : H_*(BAut_0(\Gamma)) \otimes H_*(M)^{\otimes p} \to H_*(M)^{\otimes q} \tag{15}$$

which raises total dimension by $\chi(\Gamma)n - np$.

We now describe an extension of the operation q^0_Γ to an operator on equivariant homology,

$$q_\Gamma : H^{Aut(\Gamma)}_*(M^p) \to H^{Aut(\Gamma)}_{*+\chi(\Gamma)n-np}(M^q).$$

Here $Aut(\Gamma)$ acts on M^p via the permutation action determined by the homomorphism $Aut(\Gamma) \to \Sigma_p$ that sends an automorphism to the induced permutation of the p-incoming marked points. The $Aut(\Gamma)$ action on M^q is defined similarly. The sense in which the operation q_Γ will extend q^0_G, is the following. Since an element $g \in Aut_0(\Gamma)$ lies in the kernel of the homomorphism $Aut(\Gamma) \to \Sigma_p$ its action on M^p is trivial. Therefore the inclusion $Aut_0(\Gamma) \subset Aut(\Gamma)$ induces a map of homotopy orbit spaces,

$$BAut_0(\Gamma) \times M^p \to EAut(\Gamma) \times_{Aut(\Gamma)} M^p$$

and therefore an induced map in homology, $H_*(BAut_0(\Gamma)) \otimes H_*(M)^{\otimes p} \to H^{Aut(\Gamma)}_*(M^p)$. The compatibility of the operators q^0_Γ and q_Γ is that the following diagram commutes:

$$\begin{array}{ccc} H_*(BAut_0(\Gamma)) \otimes H_*(M)^{\otimes p} & \xrightarrow{q^0_\Gamma} & H_*(M^q) \\ \downarrow & & \downarrow \\ H^{Aut(\Gamma)}_*(M^p) \xrightarrow[q_\Gamma]{} & H^{Aut(\Gamma)}_*(M^q) \longrightarrow & H^{\Sigma_q}_*(M^q). \end{array} \tag{16}$$

We now define the graph operation q_Γ. As above, consider the evaluation map

$$ev_{in} : \tilde{\mathcal{M}}_\Gamma(M) \to M^p,$$

which evaluates a graph flow on the p incoming marked points. This map is $Aut(\Gamma)$ equivariant, where as above, $Aut(\Gamma)$ acts on M^p by permuting the coordinates according to the homomorphism $Aut(\Gamma) \to \Sigma_p$. Taking homotopy orbit spaces, we get a map

$$ev_{in} : \tilde{\mathcal{M}}_\Gamma(M)/Aut(\Gamma) = \mathcal{M}_\Gamma(M) \to EAut(\Gamma) \times_{Aut(\Gamma)} M^p.$$

We similarly have a map $ev_{out} : \mathcal{M}_\Gamma(M) \to EAut(\Gamma) \times_{Aut(\Gamma)} M^q$. Notice that up to homotopy, the map ev_{in} factors as the composition,

$$\begin{aligned} ev_{in} : \mathcal{M}_\Gamma(M) &\xrightarrow{\rho} \mathcal{M}_{tree}(\Gamma, M) \cong \mathcal{S}_\Gamma(M)/Aut(\Gamma) \times M \\ &\simeq BAut(\Gamma) \times M \xrightarrow{\Delta^p} EAut(\Gamma) \times_{Aut(\Gamma)} M^p. \end{aligned} \tag{17}$$

Here $\Delta^p : M \to M^p$ is the p-fold diagonal, which maps M to the fixed points of the $Aut(\Gamma)$-action on M^p. Therefore by applying homotopy orbit spaces, we have an induced map $\Delta^p : BAut(\Gamma) \times M \to EAut(\Gamma) \times_{Aut(\Gamma)} M^p$. Δ^p is a codimension $n(p-1)$ embedding, and so there is a Pontrjagin-Thom map to the Thom space of the normal bundle,

$$\tau_{\Delta^p} : EAut(\Gamma) \times_{Aut(\Gamma)} M^p \to (BAut(\Gamma) \times M)^{\nu(\Delta^p)}.$$

As described in the previous section, such a map induces an umkehr map in homology,

$$(\Delta^p)_! : H_*(EAut(\Gamma) \times_{Aut(\Gamma)} M^p) \to H_{*-n(p-1)}(BAut(\Gamma) \times M).$$

Because of the factoring of ev_{in} in (17), we can then define the umkehr map $(ev_{in})_!$ as the composition of umkehr maps,

$$\begin{aligned}(ev_{in})_! : H_*(EAut(\Gamma) \times_{Aut(\Gamma)} M^p) &\xrightarrow{(\Delta^p)_!} H_{*-n(p-1)}(BAut(\Gamma) \times M) \\ &\xrightarrow{\rho_!} H_{*-n(p-1)-bn}(\mathcal{M}_\Gamma(M)) \\ &= H_{*+\chi(\Gamma)n-np}(\mathcal{M}_\Gamma(M)).\end{aligned}$$

We now define the operation q_Γ as follows.

DEFINITION 16. *Define*

$$q_\Gamma : H_*^{Aut(\Gamma)}(M^p) \to H_{*+\chi(\Gamma)n-np}^{Aut(\Gamma)}(M^q)$$

to be the composition

$$\begin{aligned}q_\Gamma : H_*(EAut(\Gamma) \times_{Aut(\Gamma)} M^p) &\xrightarrow{(ev_{in})_!} H_{*+\chi(\Gamma)n-np}(\mathcal{M}_\Gamma(M)) \\ &\xrightarrow{ev_{out}} H_{*+\chi(\Gamma)n-np}(EAut(\Gamma) \times_{Aut(\Gamma)} M^q).\end{aligned}$$

We now observe the following property relating the operations q_Γ and q_Γ^0.

PROPOSITION 17. *The operation q_Γ extends q_Γ^0 in the sense that it makes diagram (16) commute.*

Proof. Let $\alpha \in H_*(BAut_0(\Gamma))$, $\beta \in H_*(M^p)$, and $x \in H^*(M^q)$ be in the image of $H^*_{\Sigma_q}(M^q) \to H^*(M^q)$. Then by definition,

$$\langle x \,,\, q_\Gamma^0(\alpha \otimes \beta)\rangle = \langle ev_{out}^*(x) \cup ev_{in}^*(D\beta) \,,\, \rho_!(\alpha \otimes [M])\rangle, \tag{18}$$

where $D : H_*(M^p) \to H^{np-*}(M^p)$ is Poincare duality. On the other hand, by the definition of q_Γ,

$$\langle x \,,\, q_\Gamma(\alpha \otimes \beta)\rangle = \langle ev_{out}^*(x) \,,\, \rho_!(\alpha \otimes \Delta_!^p(\beta)\rangle.$$

But by the commutativity of the diagram

$$\begin{array}{ccc} H_*(M^p) & \xrightarrow{\Delta_!^p} & H_*(M) \\ {\scriptstyle D}\downarrow & & \uparrow{\scriptstyle \cap [M]} \\ H^{np-*}(M^p) & \xrightarrow[(\Delta^p)^*]{} & H^{np-*}(M) \end{array}$$

this quantity is equal to

$$\langle ev_{out}^*(x) \,,\, \rho_!(\alpha \otimes (\Delta^p)^*(D\beta) \cap [M])\rangle. \tag{19}$$

Now the dual umkehr map,

$$\rho^! : H^*(\mathcal{M}_0(\Gamma, M)) \to H^{*+bn}(BAut_0(\Gamma) \times M)$$

is a map of $H^*(\mathcal{M}_\Gamma(M))$-modules. This implies that $\rho_!(\alpha \otimes (\Delta^p)^*(D\beta) \cap [M]) = \rho_!(\alpha \otimes [M]) \cap \rho^*((\Delta^p)^*(D\beta))$. Thus quantity (19) is equal to

$$\langle ev_{out}^*(x) \,,\, \rho_!(\alpha \otimes [M]) \cap \rho^*((\Delta^p)^*(D\beta))\rangle. \tag{20}$$

Now by the definition of the evaluation map ev_{in}, the following diagram commutes:

$$\begin{array}{ccccc} H^*(M^p) & \xrightarrow{(\Delta^p)^*} & H^*(M) & \xrightarrow{\hookrightarrow} & H^*(M \times BAut_0(\Gamma)) \\ {\scriptstyle ev_{in}}\downarrow & & & & \downarrow{\scriptstyle \rho^*} \\ H^*(\mathcal{M}_0(\Gamma, M) & & \xrightarrow[=]{} & & H^*(\mathcal{M}_0(\Gamma, M). \end{array}$$

So quantity (20) is equal to

$$\langle ev_{out}^*(x)\, ,\, \rho_!(\alpha \otimes [M]) \cap ev_{in}^*(D\beta) = \langle ev_{out}^*(x) \cup ev_{in}^*(D\beta)\, ,\, \rho_!(\alpha \otimes [M])\rangle,$$

which is the same as the quantity in equation (18). ▯

We end this section with the observation that given any group homomorphism $\theta : G \to Aut(\Gamma)$, the above constructions and arguments using the moduli space $\mathcal{M}_\Gamma^G(M) = EG \times_\theta \tilde{\mathcal{M}}_\Gamma(M)$, allow us to construct an umkehr map,

$$(ev_{in})_! : H_*^G(M^p) \to H_{*+\chi(\Gamma)n-np}(\mathcal{M}_\Gamma^G(M)),$$

which in turn allows the definition of an operation defined on G-equivariant homology,

$$q_\Gamma^G : H_*^G(M^p) \to H_*^G(M^q) \tag{21}$$

that is natural with respect to homomorphisms between groups living over $Aut(\Gamma)$. That is if $\theta_1 : G_1 \to Aut(\Gamma)$ and $\theta_2 : G_2 \to Aut(\Gamma)$ are homomorphisms and $f : G_1 \to G_2$ is a group homomorphism such that $\theta_2 \circ f = \theta_1$, then then the following diagram commutes:

$$\begin{array}{ccc} H_*^{G_1}(M^p) & \xrightarrow{q_\Gamma^{G_1}} & H_*^{G_1}(M^q) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_*^{G_2}(M^p) & \xrightarrow[q_\Gamma^{G_2}]{} & H_*^{G_2}(M^q) \end{array} \tag{22}$$

4. Field theoretic properties of the graph operations. In this section we describe the two properties, invariance and gluing, that imply the assignment to a graph Γ the operation q_Γ defines a field theory. We begin with the invariance property. Roughly this says that a morphism

$$\phi : \Gamma_1 \to \Gamma_2$$

in $\mathcal{C}_{b,p+q}$ takes the operation q_{Γ_1} to q_{Γ_2}. We state this more precisely as follows. Let $G_1 < Aut(\Gamma)_1$ and $G_2 < Aut(\Gamma)_2$ be subgroups.

Definition 18. *We say that a morphism $\phi : \Gamma_1 \to \Gamma_2$ is G_1-G_2 equivariant, if for every $g_1 \in G_1$ there exists a unique $g_2 \in G_2$ such that the following composite morphisms are equal:*

$$\phi \circ g_1 = g_2 \circ \phi : \Gamma_1 \to \Gamma_2.$$

In this setting, ϕ determines a homomorphism,

$$\begin{aligned} \phi_* : G_1 &\to G_2 \\ g_1 &\to g_2. \end{aligned}$$

Furthermore, one easily checks that the homomorphism $\phi_* : \Gamma_1 \to \Gamma_2$ lives over the identity in the symmetric groups. That is, the following diagram commutes:

$$\begin{array}{ccc} G_1 & \longrightarrow & \Sigma_p \times \Sigma_q \\ {\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle =} \\ G_2 & \longrightarrow & \Sigma_p \times \Sigma_q \end{array}$$

where the two horizontal maps assign to an automorphism the induced permutation of the incoming and outgoing leaves. The commutativity of this diagram then says that ϕ induces maps of equivariant homology,

$$\phi_* : H_*^{G_1}(M^p) \to H_*^{G_2}(M^p) \quad \text{and} \quad \phi_* : H_*^{G_1}(M^q) \to H_*^{G_2}(M^q).$$

THEOREM 19. *(Invariance) Let $\phi : \Gamma_1 \to \Gamma_2$ be a morphism in $\mathcal{C}_{g,p+q}$, $G_1 < Aut(\Gamma)_1$ and $G_2 < Aut(\Gamma)_2$ be such that ϕ is G_1-G_2 equivariant. Then the following diagram commutes:*

$$\begin{array}{ccc} H_*^{G_1}(M^p) & \xrightarrow{q_{\Gamma_1}^{G_1}} & H_*^{G_1}(M^q) \\ {\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ H_*^{G_2}(M^p) & \xrightarrow{q_{\Gamma_2}^{G_2}} & H_*^{G_2}(M^q) \end{array}$$

Proof. The proof of this theorem is immediate from the definitions, using the naturality of the Pontrjagin-Thom collapse maps (and thus umkehr maps). We leave the details of this argument to the reader. □

We now discuss a gluing relation held by these operations. In the nonequivariant setting, a gluing relation is proved using analytic techniques below. Here we describe and prove a gluing relation in the general equivariant setting. Let Γ_1 be a graph with p incoming marked points and q outgoing marked points. Let Γ_2 be a graph with q incoming and r outgoing marked points. Say $\Gamma_1 \in \mathcal{C}_{b_1,p+q}$, and $\Gamma_2 \in \mathcal{C}_{b_2,q+r}$. By identifying the the q outgoing leaves (univalent vertices) of Γ_1 to the q incoming leaves of Γ_2, this defines a "glued" graph, $\Gamma_1 \# \Gamma_2 \in \mathcal{C}_{b_1+b_2+q-1,p+r}$.

Let Γ_1 and Γ_2 be as above. Consider the homomorphisms

$$\rho_{out} : Aut(\Gamma_1) \to \Sigma_q \quad \rho_{in} : Aut(\Gamma_2) \to \Sigma_q$$

defined by the induced permutations of the outgoing and incoming leaves, respectively. Let $Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2)$ be the fiber product of these homomorphisms. That is,

$$Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2) \subset Aut(\Gamma_1) \times Aut(\Gamma_2)$$

is the subgroup consisting of those (g_1, g_2) with $\rho_{out}(g_1) = \rho_{in}(g_1)$. Let

$$p_1 : Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2) \to Aut(\Gamma_1) \quad \text{and} \quad p_2 : Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2) \to Aut(\Gamma_2)$$

be the projection maps. There is also an obvious inclusion as a subgroup of the automorphism group of the glued graph,

$$\iota : Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2) \hookrightarrow Aut(\Gamma_1 \# \Gamma_2)$$

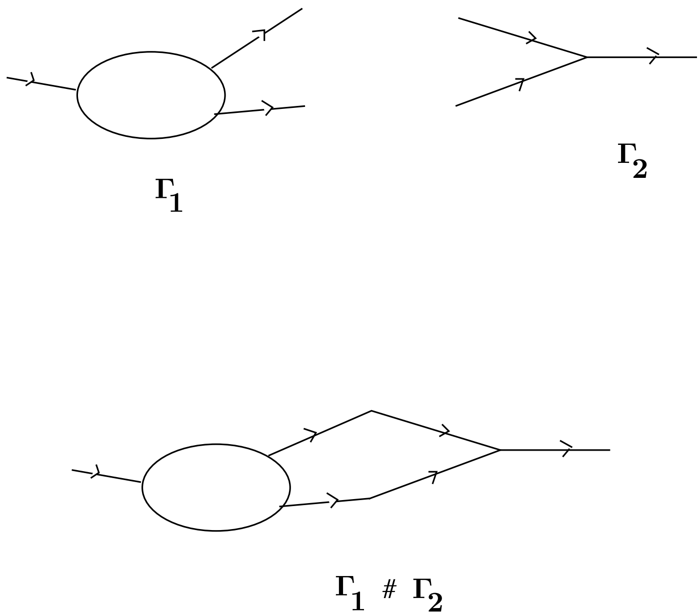

FIG. 4. $\Gamma_1 \# \Gamma_2$

which realizes $Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2)$ as the subgroup of $Aut(\Gamma_1 \# \Gamma_2)$ consisting of automorphisms that preserve the subgraphs, Γ_1 and Γ_2. Similarly, for any pair of homomorphisms, $\theta_1 : G_1 \to Aut(\Gamma_1)$ and $\theta_2 : G_2 \to Aut(\Gamma_2)$, we have an induced homomorphism

$$\theta_1 \times \theta_2 : G_1 \times_{\Sigma_q} G_2 \to Aut(\Gamma_1) \times_{\Sigma_q} Aut(\Gamma_2) \hookrightarrow Aut(\Gamma_1 \# \Gamma_2).$$

We then have the following gluing theorem.

THEOREM 20. *Let Γ_1, Γ_2, $\theta_1 : G_1 \to Aut(\Gamma_1)$, and $\theta_2 : G_2 \to Aut(\Gamma_2)$ be as above. Then the composition of the graph operations*

$$q_{\Gamma_2}^{G_1 \times_{\Sigma_q} G_2} \circ q_{\Gamma_1}^{G_1 \times_{\Sigma_q} G_2} : H_*^{G_1 \times_{\Sigma_q} G_2}(M^p) \to H_*^{G_1 \times_{\Sigma_q} G_2}(M^q) \to H_*^{G_1 \times_{\Sigma_q} G_2}(M^r)$$

is equal to the graph operation for the glued graph,

$$q_{\Gamma_1 \# \Gamma_2}^{G_1 \times_{\Sigma_q} G_2} : H_*^{G_1 \times_{\Sigma_q} G_2}(M^p) \to H_*^{G_1 \times_{\Sigma_q} G_2}(M^r).$$

Proof. For the sake of ease of notation, we leave off the superscript $G_1 \times_{\Sigma_q} G_2$ in the following description of moduli spaces and graph operations. We wish to prove that $q_{\Gamma_1 \# \Gamma_2} = q_{\Gamma_2} \circ q_{\Gamma_1}$.

Consider the restriction maps,

$$\mathcal{M}_{\Gamma_1}(M) \xleftarrow{r_1} \mathcal{M}_{\Gamma_1 \# \Gamma_2}(M) \xrightarrow{r_2} \mathcal{M}_{\Gamma_2}(M).$$

given by restricting a graph flow on $\Gamma_1 \# \Gamma_2$ to Γ_1 or Γ_2, respectively. Notice that the following is a pullback square of fibrations,

$$\begin{array}{ccc} \mathcal{M}_{\Gamma_1\#\Gamma_2}(M) & \xrightarrow{r_2} & \mathcal{M}_{\Gamma_2}(M) \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle ev^2_{in}} \\ \mathcal{M}_{\Gamma_1}(M) & \xrightarrow[ev^1_{out}]{} & E(G_1 \times_{\Sigma_q} G_2) \times_{G_1\times_{\Sigma_q}G_2} M^q. \end{array}$$

Here the superscripts of the evaluation maps are meant to represent the graph moduli space on which they are defined. By the naturality of the Pontrjagin-Thom collapse maps used to define the umkehr maps (see in the proof of theorem 14 as well as the more general setup described in [8]), we have the following relation:

$$r_2 \circ (r_1)_! = (ev^2_{in})_! \circ ev^1_{out} : H_*^{G_1\times_{\Sigma_q}G_2}(M^q) \to H_*(\mathcal{M}_{\Gamma_1\#\Gamma_2}(M)). \tag{23}$$

Notice furthermore that we have commutative diagrams

$$\begin{array}{ccc} \mathcal{M}_{\Gamma_1\#\Gamma_2}(M) & \xrightarrow{ev^{1,2}_{in}} & E(G_1 \times_{\Sigma_q} G_2) \times_{G_1\times_{\Sigma_q}G_2} M^p \\ {\scriptstyle r_1}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{M}_{\Gamma_1}(M) & \xrightarrow[ev^1_{in}]{} & E(G_1 \times_{\Sigma_q} G_2) \times_{G_1\times_{\Sigma_q}G_2} M^p \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}_{\Gamma_1\#\Gamma_2}(M) & \xrightarrow{ev^{1,2}_{out}} & E(G_1 \times_{\Sigma_q} G_2) \times_{G_1\times_{\Sigma_q}G_2} M^r \\ {\scriptstyle r_2}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{M}_{\Gamma_2}(M) & \xrightarrow[ev^2_{out}]{} & E(G_1 \times_{\Sigma_q} G_2) \times_{G_1\times_{\Sigma_q}G_2} M^r. \end{array}$$

The first of these diagrams implies, by the naturality of the Pontrjagin-Thom collapse maps, that

$$(ev^{1,2}_{in})_! = (r_1)_! \circ (ev^1_{in})_! : H_*^{G_1\times_{\Sigma_q}G_2}(M^p)) \to H_*(\mathcal{M}_{\Gamma_1\#\Gamma_2}(M)). \tag{24}$$

These naturality properties allow us to calculate:

$$\begin{aligned} &q_{\Gamma_1\#\Gamma_2} \\ &= ev^{1,2}_{out} \circ (ev^{1,2}_{in})_!, \quad \text{by definition} \\ &= ev^{1,2}_{out} \circ (r_1)_! \circ (ev^1_{in})_!, \quad \text{by (24)} \\ &= ev^2_{out} \circ r_2 \circ (r_1)_! \circ (ev^1_{in})_!, \quad \text{by the commutativity of the second diagram above,} \\ &= ev^2_{out} \circ (ev^2_{in})_! \circ ev^1_{out} \circ (ev^1_{in})_!, \quad \text{by (23)} \\ &= q_{\Gamma_2} \circ q_{\Gamma_1}, \quad \text{by definition.} \end{aligned}$$

This completes the proof of this theorem. □

5. Examples. In this section we give some examples of the equivariant operations q_Γ.

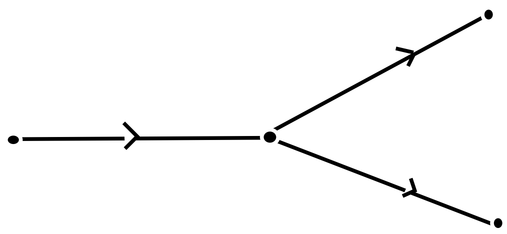

FIG. 5. Γ_1

5.1. The "Y"-graph and the Steenrod squares. Let Γ_1 be the graph:

This graph is a tree with one incoming and two outgoing leaves. The automorphism group is the group of order 2: $Aut(\Gamma) = \mathbb{Z}/2$. The operation q_{Γ_1} is therefore a homomorphism,

$$q_{\Gamma_1} = ev_{out} \circ (ev_{in})_! : H_*(B\mathbb{Z}/2) \otimes H_*(M) \to H_*(\mathcal{M}_{\Gamma_1}(M)) \to H_*^{\mathbb{Z}/2}(M \times M).$$

Since Γ_1 is a tree, $\mathcal{M}_{\Gamma_1}(M) \simeq B\mathbb{Z}/2 \times M$, and clearly $ev_{in} : \mathcal{M}_{\Gamma_1}(M) \to B(\mathbb{Z}/2) \times M$ is homotopic to the identity. This means $(ev_{in})_!$ is the identity homomorphism, and so $q_{\Gamma_1} = ev_{out}$. But as identified earlier, $ev_{out} : \mathcal{M}_{\Gamma_1}(M) \simeq B(\mathbb{Z}/2) \times M \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} M \times M$ is homotopic to the equivariant diagonal map. Thus

$$q_{\Gamma_1} : H_*(B(\mathbb{Z}/2)) \otimes H_*(M) \to H_*^{\mathbb{Z}/2}(M \times M)$$

is the equivariant diagonal.

Consider the dual map in cohomology with $\mathbb{Z}/2$-coefficients:

$$(q_{\Gamma_1})^* : H^*_{\mathbb{Z}/2}(M \times M) \to H^*(B\mathbb{Z}/2) \otimes H^*(M).$$

This is Steenrod's equivariant cup product map [18]. Indeed if we considered the nonequivariant operation (associated to the homomorphism $\{id\} \hookrightarrow Aut(\Gamma) = \mathbb{Z}/2$), then the operation

$$(q^{id}_{\Gamma_1})^* : H^*(M) \otimes H^*(M) \to H^*(M)$$

is the cup product homomorphism. In the $\mathbb{Z}/2$-equivariant setting, recall that Steenrod defined the Steenrod squaring operations Sq^j in terms of the equivariant cup product map in the following way. Let $\alpha \in H^q(M; \mathbb{Z}/2)$. So $\alpha \otimes \alpha$ represents a well defined class in $H^{\mathbb{Z}/2}(M \times M; \mathbb{Z}/2)$. Then

$$(q_{\Gamma_1})^*(\alpha \otimes \alpha) = \sum_{j=0^2 q} a^j \otimes Sq^{2q-j}(\alpha). \tag{25}$$

Here $a \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2) = H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator.

5.2. The Cartan and Adem formulas. We now describe how the Cartan and Adem formulas for the Steenrod squares follow from the field theoretic properties (invariance and gluing) of the graph operations. Consider the following graph, Γ_2:

Notice that the automorphism group, $Aut(\Gamma)_2 \cong \Sigma_2 \int \Sigma_2$, the wreath product of the symmetric group with itself. It sits in a short exact sequence, $1 \to \Sigma_2 \times \Sigma_2 \to \Sigma_2 \int \Sigma_2 \to \Sigma_2 \to 1$. We will view this group as a subgroup of the symmetric group, $\Sigma_2 \int \Sigma_2 \hookrightarrow \Sigma_4$. Consider the subgroup $\tau : \mathbb{Z}/2 \hookrightarrow \Sigma_2 \int \Sigma_2$ defined by the permutation,

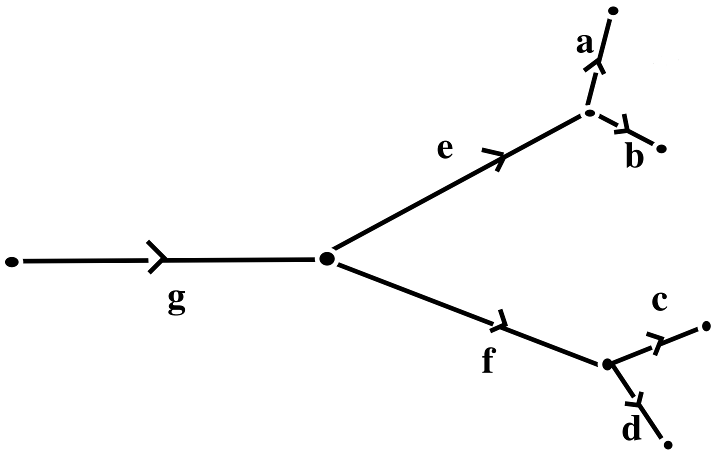

FIG. 6. Γ_2

$(a,b,c,d) \to (b,a,d,c)$. We consider the graph operation in cohomology with $\mathbb{Z}/2$-coefficients:

$$q^{\tau}_{\Gamma_2} : H^*_{\tau}(M^4) \to H^*(B(\mathbb{Z}/2)) \otimes H^*(M),$$

where H^*_{τ} is the $\mathbb{Z}/2$-equivariant cohomology determined by the embedding τ.

Notice that Γ_2 is the graph obtained by gluing two copies of the Y-graph Γ_1, each having a single incoming leaf, to the two outgoing leaves of a third Y-graph Γ_1.

By the gluing formula (theorem (20)) and the description of q_{Γ_1} above in terms of the (equivariant) cup product, then if $\alpha \in H^q(M)$, $\beta \in H^r(M)$, then

$$(26)\quad q^{\tau}_{\Gamma_2}(\alpha \otimes \alpha \otimes \beta \otimes \beta) = \sum_{i+s+t=q+r} a^i \otimes Sq^s(\alpha) \cup Sq^t(\beta) \in H^*(B(\mathbb{Z}/2)) \otimes H^*(M).$$

We now use the invariance property (theorem(19)) to understand this operation in another way. Let Γ_3 be the following graph:

Here $Aut(\Gamma_3) = \Sigma_4$, the symmetric group. Consider the morphism,

$$\theta : \Gamma_2 \to \Gamma_3$$

obtained by collapsing the edges e and f in figure 6 and then permuting the two internal outgoing leaves. That is, on the level of edges,

$$\theta : g \to e, \quad a \to a, \quad b \to c, \quad c \to b, \quad d \to d.$$

θ sends the involution τ on Γ_2 to the involution σ on Γ_3 defined by the inclusion $\sigma : \mathbb{Z}/2 \hookrightarrow \Sigma_4$, given by the permutation, $(a,b,c,d) \to (c,d,a,b)$. So by the invariance property, the following diagram commutes:

$$(27)\quad \begin{array}{ccc} H^*_{\tau}(M^4) & \xrightarrow{q^{\tau}_{\Gamma_2}} & H^*(B(\mathbb{Z}/2)) \otimes H^*(M) \\ {\scriptstyle \theta}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} \\ H^*_{\sigma}(M^4) & \xrightarrow[q^{\sigma}_{\Gamma_3}]{} & H^*(B(\mathbb{Z}/2)) \otimes H^*(M). \end{array}$$

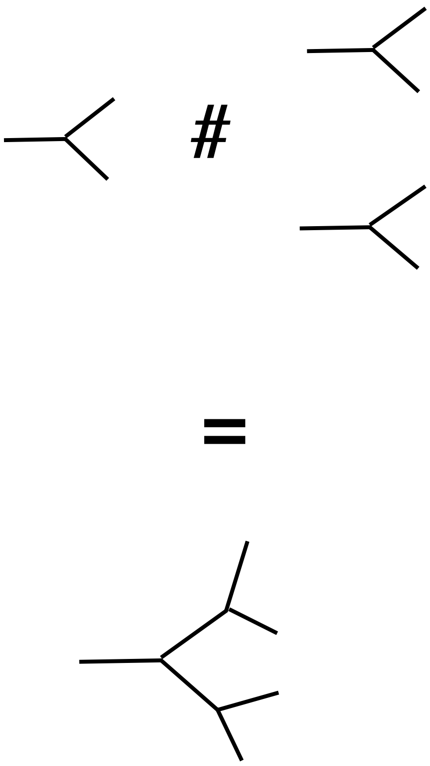

FIG. 7.

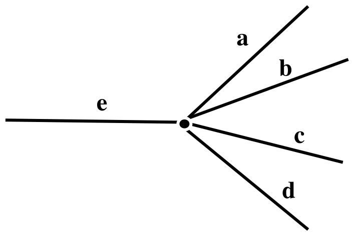

FIG. 8. Γ_3

Now $\theta(\alpha \otimes \alpha \otimes \beta \otimes \beta) = \alpha \otimes \beta \otimes \alpha \otimes \beta \in H^*_\sigma(M^4)$. Thus we know from the invariance property and formula (26), that

$$q^\sigma_{\Gamma_3} = \sum_{i+s+t=q+r} a^i \otimes Sq^s(\alpha) \cup Sq^t(\beta). \tag{28}$$

On the other hand, consider the morphism

$$\phi : \Gamma_2 \to \Gamma_3 \tag{29}$$

that also collapses edges e and f but maps edges a, b, c, and d, to a, b, c, and d respectively. Since the image of $\sigma : \mathbb{Z}/2 \hookrightarrow \Sigma_4$ lies in $\Sigma_2 \int \Sigma_2$, the invariance property implies

$$q^\sigma_{\Gamma_2} = q^\sigma_{\Gamma_3} : H^*_\sigma(M^4) \to H^*(B(\mathbb{Z}/2)) \otimes H^*(M).$$

But by using figure 7 the gluing formula (theorem 20) implies that

$$\begin{aligned} q^{\sigma}_{\Gamma_2}(\alpha \otimes \beta \otimes \alpha \otimes \beta) &= q_{\Gamma_1}(\alpha\beta \otimes \alpha\beta) \\ &= \sum_i a^i \otimes Sq^{q+r-i}(\alpha\beta), \quad \text{by (25).} \end{aligned} \tag{30}$$

Comparing this to formula (28) yields the Cartan formula,

$$Sq^m(\alpha\beta) = \sum_{u+v=m} Sq^u(\alpha)Sq^v(\beta).$$

For the Adem relations, the graph operations don't give us new calculational techniques, but they do supply an interesting perspective on what calculations are necessary. Namely, the Adem relations are relations involving iterates of Steenrod squaring operations. From the graph point of view, the gluing formula tells us that these operations come from considering the graph Γ_2 given in figure 6. As pointed out above, the automorphism group of Γ_2 is the wreath product, $Aut(\Gamma_2) = \Sigma_2 \int \Sigma_2$. In cohomology, the graph operation is a homomorphism,

$$q^*_{\Gamma_2} : H^*_{\Sigma_2 \int \Sigma_2}(M^4) \to H^*(B(\Sigma_2 \int \Sigma_2)) \otimes H^*(M),$$

and the relevant calculation is $q^*_{\Gamma_2}(\alpha^{\otimes 4})$ for $\alpha \in H^*(M)$. Now consider the morphism $\phi : \Gamma_2 \to \Gamma_3$ described above. As remarked above, $Aut(\Gamma)_4 = \Sigma_4$. Moreover, in the language of theorem (19), ϕ is $\Sigma_2 \int \Sigma_2 - \Sigma_4$ equivariant. Therefore by the invariance property, the following diagram commutes:

$$\begin{array}{ccc} H^*_{\Sigma_2 \int \Sigma_2}(M^4) & \xrightarrow{q^*_{\Gamma_2}} & H^*(B(\Sigma_2 \int \Sigma_2)) \otimes H^*(M) \\ \phi \Big\uparrow & & \iota\otimes 1 \Big\uparrow \\ H^*_{\Sigma_4}(M^4) & \xrightarrow{q^*_{\Gamma_3}} & H^*(B(\Sigma_4)) \otimes H^*(M) \end{array}$$

where $\iota : \Sigma_2 \int \Sigma_2 \hookrightarrow \Sigma_4$ is the inclusion as a subgroup. But since $\alpha^{\otimes 4}$ lies in the image of $\phi : H^*_{\Sigma_4}(M^4) \to H^*_{\Sigma_2 \int \Sigma_2}(M^4)$, we have that $q^*_{\Gamma_2}(\alpha^{\otimes 4})$ is the image of $q^*_{\Gamma_3}(\alpha^{\otimes 4})$ under the map

$$\iota^* \otimes 1 : H^*(B(\Sigma_4)) \otimes H^*(M) \to H^*(B(\Sigma_2 \int \Sigma_2)) \otimes H^*(M).$$

Now any approach to the Adem relations involves computing the relative cohomologies of $\iota : \Sigma_2 \int \Sigma_2 \hookrightarrow \Sigma_4$, and in particular, the relative equivariant cohomologies of the permutation action on M^4. However from this perspective, the reasons these calculations are forced upon us, are the gluing and invariance properties of the graph operations.

5.3. Stiefel-Whitney classes. Consider the following graph, Γ_4: In this case the automorphism group $Aut(\Gamma_4) \cong \mathbb{Z}/2$. Also, since there is just one incoming leaf, the operation q_{Γ_4} taken with $\mathbb{Z}/2$-coefficients is a map,

$$H_*(B(\mathbb{Z}/2)) \otimes H_*(M) \to \mathbb{Z}/2.$$

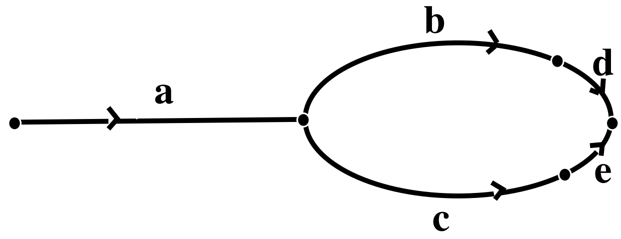

FIG. 9. Γ_4

Or, equivalently, $q_{\Gamma_4} \in H^*(B(\mathbb{Z}/2)) \otimes H^*(M)$. The following identifies this graph operation.

THEOREM 21.

$$\begin{aligned} q_{\Gamma_4} &= \sum_{i=0}^{n} a^i \otimes w_{n-i}(M) \\ &\in H^*(B(\mathbb{Z}/2)) \otimes H^*(M) \end{aligned} \tag{31}$$

where, $w_j(M) \in H^j(M)$ is the j^{th}-Stiefel-Whitney class of the tangent bundle of M, and as above, $a \in H^1(B(\mathbb{Z}/2))$ is the generator.

Proof. Let $T \subset \Gamma_4$ be the tree obtained by removing the edges d and e in figure 9 above. T has the same automorphism group, $Aut(T) = \mathbb{Z}/2$. By restricting a Γ_4-graph flow to T, one obtains an embedding,

$$\mathcal{M}_{\Gamma_4}(M) \xrightarrow[\hookrightarrow]{\rho} \mathcal{M}_T(M) \cong (\mathcal{S}_T(M)/\mathbb{Z}/2) \times M \simeq B\mathbb{Z}/2 \times M.$$

By definition (16) the operation q_{Γ_4} is given by the image of the umkehr map in cohomology,

$$q_{\Gamma_4} = \rho^!(1) \in H^n(B(\mathbb{Z}/2) \times M).$$

To understand this class, notice that the tree T has one incoming and two outgoing leaves. Evaluating a graph flow on T at the two outgoing leaves defines a map

$$ev_{out} : \mathcal{M}_T(M) \to E(\mathbb{Z}/2) \times_{\mathbb{Z}/2} M \times M$$

which is homotopic to the equivariant diagonal, $\Delta : B(\mathbb{Z}/2) \times M \to E(\mathbb{Z}/2) \times_{\mathbb{Z}/2} M \times M$. Furthermore, from (11), the following diagram is a homotopy cartesian square:

$$\begin{array}{ccccc} \mathcal{M}_{\Gamma_4}(M) & \xrightarrow[\hookrightarrow]{\rho} & \mathcal{M}_T(M) \cong (\mathcal{S}(T,M)/\mathbb{Z}/2) \times M & \xrightarrow{\simeq} & B\mathbb{Z}/2 \times M \\ \delta \downarrow & & & & \downarrow \Delta \\ B(\mathbb{Z}/2) \times M & & \xrightarrow[\Delta]{} & & E(\mathbb{Z}/2) \times_{\mathbb{Z}/2} M \times M \end{array}$$

By the naturality of the Pontrjagin-Thom collapse map and the resulting umkehr map in cohomology, this homotopy cartesian square implies that

$$\rho^! \circ \delta^* = \Delta^* \circ \Delta^! : H^*(B(\mathbb{Z}/2)) \otimes M \to H^*(B(\mathbb{Z}/2)) \otimes M.$$

So

$$q_{\Gamma_4} = \rho^!(1) = \rho^! \circ \delta^*(1) = \Delta^* \circ \Delta^!(1).$$

But by standard properties of umkehr maps, $\Delta^* \circ \Delta^!(1)$ is the mod 2 Euler class of the normal bundle of the equivariant diagonal embedding, $\Delta : B\mathbb{Z}/2 \times M \hookrightarrow E(\mathbb{Z}/2) \times_{\mathbb{Z}/2} M \times M$. Since the normal bundle of the (nonequivariant) diagonal $\Delta : M \to M \times M$ is the tangent bundle, $p : TM \to M$, the normal bundle of the equivariant diagonal is the equivariant tangent bundle,

$$E(\mathbb{Z}/2) \times_{\mathbb{Z}/2} TM \xrightarrow{1 \times p} B(\mathbb{Z}/2) \times M,$$

where $\mathbb{Z}/2$ acts fiberwise on TM by multiplication by -1. The mod 2 Euler class is the n^{th}-Stiefel-Whitney class of this bundle, which is given by the sum, $\sum_{i=0}^{n} a^i \otimes w_{n-i}(TM)$.

This completes the proof of this theorem. □

5.4. Miscellaneous. We conclude this section with a few miscellaneous remarks about examples. Here we work nonequivariantly (i.e we take q^1_Γ where $1 \in G$ is the trivial subgroup).

- Consider the operation q^1_Γ with field coefficients. Then rather than a homomorphism $q^1_\Gamma : H_*(M)^{\otimes p} \to H_*(M)^{\otimes q}$, we may think of q^1_Γ as living in the tensor product, $\bigotimes_p H^*(M) \otimes \bigotimes_q H_*(M)$. It is shown in section 9 below (corollary 43), that if one changes the orientation of an edge connected to a univalent vertex, one changes the invariant by Poincare duality on that factor.
- The previous remark shows that when one lets $\tilde{\Gamma}_1$ be the Y-graph as in figure 5, except that the orientations of all three edges are reversed, then the operation

$$q^1_{\tilde{\Gamma}_1} : H_*(M) \otimes H_*(M) \to H_*(M)$$

  is the intersection pairing.
- Consider the graph below with two incoming univalent vertices. Then the op-

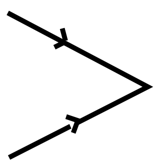

FIG. 10. Γ_0

  eration $q^1_{\Gamma_0} : H_*(M) \otimes H_*(M) \to k$ is the nondegenerate intersection pairing. Thus the Frobenius algebra structure of $H_*(M)$ is encoded in the the Morse field theory structure.

6. Transversality. We now give a differential topological construction of the graph invariants q_Γ defined in section 3. Throughout this section, and the rest of the paper, we will only be using the automorphism group $Aut_0(\Gamma)$ introduced in section 3, that consists of those automorphisms that preserve the univalent vertices (leaves). Since we will be using this group exclusively throughout the remainder of the paper, we ease notation by simply writing $Aut(\Gamma)$ for $Aut_0(\Gamma)$, $\mathcal{M}_\Gamma(M)$ for $\tilde{\mathcal{M}}_\Gamma(M)/Aut_0(\Gamma)$.

Giving this alternative definition of the graph operations involves studying the smoothness properties of the moduli spaces. This is the main goal of this section. Our plan for this section is the following.

We will consider the "graph flow map"

$$\begin{array}{rcccl} \Phi & : & \mathcal{P}_\Gamma(M) & \to & \mathcal{P}_\Gamma(TM) \\ & & \gamma & \mapsto & \dfrac{d\gamma_E}{dt} + \nabla f_E(\gamma(t)) \end{array}$$

for each edge E of the metric graph Γ_k, where $(\Gamma_k, f_E) \in \mathcal{S}_\Gamma$ and where $\mathcal{P}_\Gamma(M)$ (which will be defined carefully below) is a space consisting of pairs (Γ_k, γ), where Γ_k is a graph over Γ (i.e an object in $\mathcal{C}_\Gamma$), and $\gamma : \Gamma_k \to M$ is a map. The map Φ is a section of the vector bundle $\mathcal{P}_\Gamma(TM)$ over $\mathcal{P}_\Gamma(M)$ with fibres given by sections of $\gamma^*(TM)$.

The universal moduli space of metric-graph flows can therefore be thought of as the quotient by $Aut(\Gamma)$ of the zero set of the section Φ

$$\mathcal{M}_\Gamma(M) = \tilde{\mathcal{M}}_\Gamma(M)/Aut(\Gamma), \quad \tilde{\mathcal{M}}_\Gamma(M) = \Phi^{-1}(0) \subset \mathcal{P}_\Gamma(M).$$

We will show that it is a smooth, orientable manifold by an application of the implicit function theorem. Furthermore, we will show that the projection map

$$\begin{array}{c} \mathcal{M}_\Gamma(M) \\ \downarrow \pi \\ \mathcal{M}_\Gamma \end{array}$$

is smooth, and has virtual codimension (i.e the dimension of $\mathcal{M}_\Gamma$ minus the dimension of $\mathcal{M}_\Gamma(M)$) equal to $-\dim M \cdot \chi(\Gamma)$. Thus for any submanifold $N \subset \mathcal{M}_\Gamma$ transverse to the map π, the space

$$\mathcal{M}_\Gamma^N(M) = \pi^{-1}(N)$$

is a smooth manifold of dimension $\dim M \cdot \chi(\Gamma) + \dim N$.

The evaluation map $ev_v : \mathcal{M}_\Gamma^N(M) \to M$ of a graph flow at a univalent vertex $v \in \Gamma$ allows one to cut down the moduli space further. Given a Morse function f on M associate to an outgoing univalent vertex $v \in \Gamma$ a critical point a_v of f with stable manifold $\mathcal{W}^s(a_v) \subset M$. Then we will see that $N \subset \mathcal{M}_\Gamma$ can be chosen transverse to the map $\pi : \mathcal{M}_\Gamma(M) \to \mathcal{M}_\Gamma$, and so that $ev_v(\mathcal{M}_\Gamma^N(M))$ intersects $\mathcal{W}^s(a_v)$ transversely in M. This will imply that

$$\mathcal{M}_\Gamma^N(M; a_v) = \mathcal{M}_\Gamma^N(M) \cap ev_v^{-1}(\mathcal{W}^s(a_v))$$

is a smooth manifold of dimension $\dim M \cdot \chi(\Gamma) + \dim N - \text{index}(a_v)$. By repeated application of this on a collection of critical points $\vec{a} = \{a_v\}$ of f labeled by the univalent vertices of Γ, one can choose N to get a smooth manifold $\mathcal{M}_\Gamma^N(M; \vec{a})$ of dimension

$$\dim \mathcal{M}_\Gamma^N(M; \vec{a}) = \dim \mathcal{M}_\Gamma^N(M) - \sum_{v\ incoming} (\dim M - \text{index}(a_v)) - \sum_{v\ outgoing} \text{index}(a_v)$$

where the two sums are taken over the set of incoming and outgoing univalent vertices, respectively.

In section 7 we will prove that the zero dimensional moduli spaces $\mathcal{M}_\Gamma^N(M; \vec{a})$ are compact and hence one can count the number of points in the moduli space to get invariants of the manifold. (We will study more general compactness issues in section 8.) These invariants take their values in formal sums of critical points of f and can be interpreted as homology classes in the Morse chain complex of f. This will lead to a differential topology construction of the invariants q_Γ which we do in section 9.

The simple purpose of this section is to prove that transversality can be arranged. This is a generalisation of the fact that Morse-Smale functions exist. We will use the Sard-Smale theorem so we must first put a Banach manifold structure on the universal moduli space $\mathcal{M}_\Gamma(M)$.

6.1. Mapping spaces. To study the flow map

$$\Phi : \mathcal{P}_\Gamma(M) \to \mathcal{P}_\Gamma(TM)$$

we put a Banach manifold structure on the spaces of maps $\mathcal{P}_\Gamma(M)$ and $\mathcal{P}_\Gamma(TM)$, then linearise Φ, and prove regularity. When the moduli spaces are finite-dimensional an index calculation gives the dimension.

Continuous maps from a graph Γ to a compact manifold M are best understood when one equips Γ and M with metrics. More precisely, equip M with a smooth Riemannian metric and take an oriented metric graph $\Gamma_k \to \Gamma$ homotopy equivalent to Γ. (Strictly speaking we are considering a point in the geometric realization of $|\mathcal{C}_\Gamma|$, and interpreting it as a metric-graph over Γ as discussed in section 1.) The mapping space $\mathcal{P}_\Gamma(M)$ consists of continuous maps with square integrable derivative of all metric graphs Γ_k homotopy equivalent to Γ as follows.

Definition 22. *For an oriented metric graph Γ_k define $\mathcal{P}_{\Gamma_k}(M)$ to be the subset of continuous maps $\Gamma_k \to M$ with square integrable derivative*

$$\mathcal{P}_{\Gamma_k}(M) = \left\{ \gamma : \Gamma_k \to M \;\middle|\; \gamma \text{ continuous}, \int_{\Gamma_k} \left| \frac{d\gamma}{dt} \right|^2 dt < \infty \right\}.$$

Put the $W^{1,2}$ metric on $\mathcal{P}_{\Gamma_k}(M)$ to give it a Banach manifold structure, i.e. take *continuous* sections s of $\mathcal{V} = \gamma^* TM$ satisfying

$$\|s\|^2 = \int_{\Gamma_k} \left(\left| \frac{ds}{dt} \right|^2 + |s|^2 \right) dt < \infty.$$

Note that the Sobolev embedding theorem

$$W^{1,2}(E, \mathcal{V}) \subset C^0(E, \mathcal{V})$$

on the interior of edges shows that the requirement of continuity on s can be stated more weakly as continuity at vertices.

We wish to take the union of $\mathcal{P}_{\Gamma_k}(M)$ over all $\sigma = (\Gamma_k, \vec{f}) \in \mathcal{S}_\Gamma$. Before doing this, we need a Banach manifold structure on $\mathcal{S}_\Gamma$. This is achieved by building up $\mathcal{S}_\Gamma$ from finite dimensional manifolds, so that the Banach manifold structure is simply obtained by taking finite objects in $\mathcal{S}_\Gamma$.

In the construction of $\mathcal{S}_\Gamma$, take $V \subset C^\infty(M)$ to be an N-dimensional vector space and allow only Γ_k for k less than N. That is, g_k is a point on the k-skeleton of $|\mathcal{C}_\Gamma|/Aut(\Gamma)$, for $k < N$. (We have unnecessarily chosen the bound on k to coincide with the dimension of V.) Then $\mathcal{S}_\Gamma$ is built up out of the union of $\mathcal{S}_\Gamma^{(N)} \subset \mathcal{S}_\Gamma$.

Definition 23.

- *Define*

$$\mathcal{P}_\Gamma(M) = \bigcup_{(\Gamma_k, \vec{f}) \in \mathcal{S}_\Gamma} \{\Gamma_k, \vec{f}\} \times \mathcal{P}_{\Gamma_k}(M)$$

and equip it with the topology induced from $\mathcal{S}_\Gamma$ and $\mathcal{P}_{\Gamma_k}(M)$.

- *Define the vector bundle $\mathcal{P}_\Gamma(TM) \to \mathcal{P}_\Gamma(M)$ so that for each $\gamma \in \mathcal{P}_\Gamma(M)$ the fibre over γ consists of L^2 maps.*

$$(\mathcal{P}_\Gamma(TM))_\gamma = \left\{ (\gamma, \xi) : \Gamma_k \to TM \;\middle|\; \int_{\Gamma_k} |\xi|^2 dt < \infty \right\}.$$

6.2. Surjectivity. In the previous section we proved that the space $\mathcal{P}_\Gamma(M)$ is a manifold. We next show that 0 is a regular value of Φ.

Theorem 24. *$\Phi : \mathcal{P}_\Gamma(M) \to \mathcal{P}_\Gamma(TM)$ intersects the zero section transversally.*

Proof. The tangent space at a point $(\Gamma_k, \vec{f}, \gamma) \in \mathcal{P}_\Gamma(M)$ is given by

$$T_{(\Gamma_k, \vec{f}, \gamma)} \mathcal{P}_\Gamma(M) = T_{(\Gamma_k, \vec{f})} \mathcal{S}_\Gamma \oplus W^{1,2}(\Gamma_k, \mathcal{V})$$

where $\mathcal{V} = \gamma^*(TM)$ is a vector bundle over Γ. (If M is orientable then $\mathcal{V}$ is trivial so $W^{1,2}(\mathbb{R}^n)$ suffices.) The linearisation of Φ decomposes into $D\Phi = (I, D_1 + D_{\Gamma_k})$ where I is the identity on the $T\mathcal{S}_\Gamma$ part and

$$D_1 + D_{\Gamma_k} : T_{(\Gamma_k, \vec{f})} \mathcal{S}_\Gamma \oplus W^{1,2}(\Gamma_k, \mathcal{V}) \to L^2(\Gamma_k, \mathcal{V}) \; .$$

We must show that for all points of the universal moduli space $(\Gamma_k, \vec{f}, \gamma) \in \mathcal{M}_\Gamma(M)$, $D\Phi_{(\Gamma_k, \vec{f}, \gamma)}$ is surjective and has a right inverse.

A tangent vector in $T_{(\Gamma_k, \vec{f}, \gamma)} \mathcal{P}_\Gamma(M)$ is given by a triple $(\lambda, \vec{h}, s)$ where $\lambda = \{\lambda_E\}$ is the infinitesimal change in the length of E, $\vec{h} = \{h_E\}$ is the infinitesimal change in the smooth function labeling E, and $s = \{s_E\}$ is a section of the vector bundle $\mathcal{V} = \gamma^* TM$ over Γ.

$$\begin{aligned} \gamma_E &\mapsto \gamma_E + \epsilon s_E \\ l_E &\mapsto l_E + \epsilon \lambda_E \\ f_E &\mapsto f_E + \epsilon h_E. \end{aligned}$$

To linearise $\Phi(\Gamma_k, \vec{f}, \gamma) = \{\dot{\gamma_E} + \nabla f_E(\gamma_E)\}$ assume for the moment that $l_E > 0$ and reparametrise E by $\tau \in [0,1]$ so $t = \tau l_E$. Then $\ell_E d\gamma/dt = d\gamma/d\tau$ and $\ell_E \left(d\gamma_E/dt + \nabla f_E(\gamma_E) \right) = d\gamma_E/d\tau_E + \ell_E \nabla f_E(\gamma_E)$.

$$\begin{aligned} &\ell_E \left(\Phi + \epsilon \Delta \Phi \right) \\ &= \frac{d}{d\tau_E}(\gamma_E + \epsilon s_E) + (\ell_E + \epsilon \lambda_E) \nabla (f_E + \epsilon h_E) \\ &= \frac{d}{d\tau_E}\gamma_E + \ell_E \nabla f_E(\gamma_E) + \epsilon(\frac{d}{d\tau_E} s_E + \ell_E \nabla\nabla f_E \cdot s_E + \lambda_E \nabla f_E + \ell_E \nabla h_E) \\ &= \ell_E \left(\frac{d}{dt_E}\gamma_E + \nabla f_E(\gamma_E) + \epsilon(\frac{d}{dt_E} s_E + \nabla\nabla f_E \cdot s_E + \frac{\lambda_E}{\ell_E} \nabla f_E + \nabla h_E) \right). \end{aligned}$$

hence for $\ell_E > 0$

$$D_1(\lambda, \vec{h}) + D_{\Gamma_k} s = (\frac{\lambda_E}{\ell_E} \nabla f_E + \nabla h_E) + (\dot{s}_E + \nabla\nabla f_E \cdot s_E).$$

For the case $\ell_E = 0$ we first need to understand more about the cokernel of D_{Γ_k}. Since $L^2(\Gamma, \mathcal{V})$ is a Hilbert space the cokernel of D_{Γ_k} can be identified with the orthogonal complement of its image. Thus

$$\mathrm{coker}\ \mathrm{D}_{\Gamma_k} = \{\mathrm{r} \in \mathrm{L}^2(\Gamma, \mathcal{V}) | \langle \mathrm{r}, \mathrm{D}_{\Gamma_k} \phi \rangle = 0 \text{ for all } \phi \in \mathrm{C}_0^\infty(\Gamma)\}$$

which gives good local behaviour of an element r of the cokernel on the interior of an edge and at a vertex.

LEMMA 25. *On the interior of any edge $E \subset \Gamma_k$, an element $r \in \mathrm{coker}\ \mathrm{D}_{\Gamma_k}$ is smooth and satisfies*

$$\dot{r}_E - (\nabla\nabla f_E)^T \cdot r_E = 0. \tag{32}$$

At a vertex $v \in \Gamma_k$, r is free to be discontinuous up to the codimension 1 condition

$$\sum_{E \ni v} (-1)^{v(E)} r_E(v) = 0 \tag{33}$$

where $v(E) = 0$ (or 1) when E is incoming (outgoing).

Proof. The first part of the lemma is standard so we defer that to an appendix. To prove (33) consider $\phi \in C_0^\infty(\Gamma)$ whose support lies in a neighbourhood of the vertex $v \in \Gamma$. Then

$$\begin{aligned} 0 &= \int_\Gamma \langle r, \dot{\phi} + A\phi \rangle dt \\ &= \sum_{E \ni v} (-1)^{v(E)} r_E(v)\phi(v) - \int_\Gamma \langle \dot{r}, \phi \rangle dt + \int_\Gamma \langle A^T r, \phi \rangle dt \\ &= \sum_{E \ni v} (-1)^{v(E)} r_E(v)\phi(v). \end{aligned}$$

□

When $\ell_E = 0$ we can make sense of $D_1(\lambda, \vec{h})$ weakly in L^2 as follows. Along an edge E, $\nabla f_E \in \ker D_{\Gamma_k}$ since it gives an infinitesimal change in parametrisation. Thus, for any $r_E \in \mathrm{coker}\ \mathrm{D}_{\Gamma_k}$,

$$\frac{d}{dt} \langle r_E(t), \nabla f_E(t) \rangle = \langle \dot{r}_E - (\nabla\nabla f_E)^T \cdot r_E, \nabla f_E(t) \rangle + \langle r_E(t), D_{\Gamma_k} \nabla f_E(t) \rangle = 0$$

so

$$\langle r_E(t), \nabla f_E(t) \rangle_2 = \ell_E \langle r_E(0), \nabla f_E(0) \rangle$$

since $r_E(0)$ makes sense (as a one-sided limit.) Therefore,

$$\langle r_E, D_1(\lambda, 0) \rangle_2 = \frac{\lambda_E}{\ell_E} \langle r_E, \nabla f_E \rangle_2 = \lambda_E \langle r_E(0), \nabla f_E(0) \rangle \tag{34}$$

which makes sense when $\ell_E = 0$.

When building up $\mathcal{S}_\Gamma$ from finite-dimensional $V \subset C^\infty(M)$, choose V to be generated by $\{f_1, \dots, f_N\}$ such that $\{\nabla f_1, \dots, \nabla f_N\}$ span T_xM at every point $x \in M$. At a vertex $v \in \Gamma_k$, identify Γ_k with $\Gamma_{k+d} \to \Gamma_k$ a metric graph with $d = \dim M$ extra edges that contract to v and associate to these edges smooth functions with gradients spanning $T_{\gamma(v)}M$. Then an infinitesimal increase in the length of one of the zero length edges E' at v gives any direction $\nabla f_{E'}(0)$ and thus from (34) $\lambda_{E'}\langle r_{E'}(0), \nabla f_{E'}(0)\rangle = 0$ implies $r_E(0) = r_{E'}(0) = 0$ and since r_E satisfies an ODE this implies that $r_E(t) \equiv 0$ so $D\Phi$ is onto.

It is proven in an appendix that D_{Γ_k} is Fredholm and it is a standard fact that this implies $D\Phi$ has a right inverse. □

THEOREM 26. *The universal moduli space of graph flows $\mathcal{M}_\Gamma(M)$ is a smooth Banach manifold. The projection map*

$$\pi : \mathcal{M}_\Gamma(M) \to \mathcal{M}_\Gamma$$

has virtual codimension $-\dim M \cdot \chi(\Gamma)$. Its cover $\tilde{\mathcal{M}}_\Gamma(M)$ inherits a natural coorientation from an orientation of M.

Proof. Since Φ intersects the zero section transversally, from the implicit function theorem it follows that $\tilde{\mathcal{M}}_\Gamma(M) = \Phi^{-1}(0)$ is a manifold and since $Aut(\Gamma)$ acts freely on $\tilde{\mathcal{M}}_\Gamma(M)$ so too is $\mathcal{M}_\Gamma(M) = \tilde{\mathcal{M}}_\Gamma(M)/Aut(\Gamma)$.

The projection map

$$\pi : \tilde{\mathcal{M}}_\Gamma(M) \to \mathcal{S}_\Gamma$$

is a Fredholm map between Banach manifolds since it is clear that $\ker D\pi = \ker D_{\Gamma_k}$ and with a little more thought one can see that D_1 induces an isomorphism between coker $\mathrm{D}\pi$ and coker D_{Γ_k}. Since D_{Γ_k} is Fredholm (see the appendix), $D\pi$ is Fredholm with index equal to index D_{Γ_k}.

LEMMA 27. *The index of the operator D_{Γ_k} is given by*

$$\text{index } D_{\Gamma_k} = \dim\ M \cdot \chi(\Gamma).$$

Proof. The index remains unchanged on the continuous families of operators

$$D_{\Gamma_k}(\lambda) = \frac{d}{dt} + \lambda \nabla\nabla f,\ \lambda \in [0,1]$$

so we may replace D_{Γ_k} by $\frac{d}{dt}$ which is differentiation on $\gamma^*(TM)$, an $\mathbb{R}^d$ bundle over Γ. The operator $\frac{d}{dt}$ is well-defined even if $\gamma^*(TM)$ is non-trivial since we choose trivialisations so that $\gamma^*(TM)$ is the sum of a trivial bundle and a non-trivial line bundle with transition function multiplication by -1. Thus the transition function commutes with $\frac{d}{dt}$. By Lemma 25 $\frac{d}{dt}$ is self-adjoint up to boundary terms and elements r of the cokernel also satisfy $\frac{d}{dt}r = 0$.

If the bundle $\gamma^*(TM)$ is trivial then ker $\frac{d}{dt}$ consists of constant sections so it has dimension $\dim M$. Elements of the cokernel are constant along edges but may be discontinuous at vertices, satisfying a codimension one condition there. The cokernel is spanned by sections constant around a cycle in Γ representing a generator of $H_1(\Gamma)$ and zero outside the cycle. Thus it has dimension $\dim M \cdot b_1(\Gamma)$ and index $D_{\Gamma_k} = \dim\ M \cdot \chi(\Gamma)$.

If the bundle $\gamma^*(TM)$ is non-trivial then $\ker \frac{d}{dt}$ has dimension $\dim M - 1$ since it is trivial on the non-trivial sub-line bundle of $\gamma^*(TM)$. This is because there has to be a cycle in Γ on which there is an odd number of transition functions given by multiplication by -1. But then since any element of the kernel is a constant c, we must have $c = -c = 0$. To see that the cokernel has dimension $\dim M \cdot b_1(\Gamma) - 1$ it is enough to consider the non-trivial sub-line bundle of $\gamma^*(TM)$ since the argument above takes care of the trivial sub-bundle. Note that $H_1(\Gamma)$ can be generated by $b_1(\Gamma)$ cycles such that $\gamma^*(TM)$ is trivial around all but one cycle and it has an odd number of transition functions given by multiplication by -1 along one cycle. (To see this, take $b_1(\Gamma)$ cycles in Γ that generate $H_1(\Gamma)$. If the bundle is non-trivial on two cycles α and β in the generating set, then replace α and β by $\alpha+\beta$ and β. Continue this until the bundle is non-trivial on only one generator.) Again the cokernel is spanned by sections constant around a cycle in the generating set of $H_1(\Gamma)$, hence it is zero on the non-trivial cycle and constant on the other $b_1(\Gamma) - 1$ cycles. As before index $D_{\Gamma_k} = \dim M \cdot \chi(\Gamma)$. □

The normal bundle of $\pi : \tilde{\mathcal{M}}_\Gamma(M) \to \mathcal{S}_\Gamma$ at $\sigma = (\Gamma_k, \vec{f}) \in \mathcal{S}_\Gamma$ is canonically isomorphic to $(\text{coker } \mathrm{D}\phi)^* = (\text{coker } \mathrm{D}\Gamma_\mathrm{k})^*$. A coorientation of $\tilde{\mathcal{M}}_\Gamma(M)$ in $\mathcal{S}_\Gamma$ which is an orientation of its normal bundle is thus a section of the line bundle $\wedge^{\max}(\text{coker } \mathrm{D}\Phi)^*$ over $\tilde{\mathcal{M}}_\Gamma(M)$. This line bundle coincides with the determinant line bundle

$$\det D\Phi = \wedge^{\max} \ker D\Phi \otimes \wedge^{\max}(\text{coker } \mathrm{D}\Phi)^*$$

since $D\Phi$ is surjective. The determinant line bundle extends to a locally trivial line bundle over all of $\mathcal{P}_\Gamma(M)$, and since $\mathcal{P}_\Gamma(M)$ is contractible the determinant line bundle is globally trivial . Thus $\pi(\tilde{\mathcal{M}}_\Gamma(M))$ is coorientable. A coorientation is *canonically* determined from an orientation on M via the evaluation map. □

The following is a corollary of what we have just proved. It is the main result of this section. It gives the smoothness of the finite dimensional moduli spaces.

THEOREM 28. *For a generic submanifold with boundary $N \subset \mathcal{M}_\Gamma$, the moduli space $\mathcal{M}_\Gamma^N(M)$ is a manifold with boundary, and has dimension*

$$\dim \mathcal{M}_\Gamma^N(M) = \dim M \cdot \chi(\Gamma) + \dim N.$$

Proof. A strong version of the Sard-Smale theorem guarantees that that any submanifold of $\mathcal{S}_\Gamma$ can be perturbed to an arbitrarily close submanifold N that is transverse to $\pi : \mathcal{M}_\Gamma(M) \to \mathcal{M}_\Gamma$, with its boundary transverse to $\pi(\mathcal{M}_\Gamma(M))$. Hence $\mathcal{M}_\Gamma^N(M) = \pi^{-1}(N)$ is a manifold with boundary. The dimension formula follows immediately from the codimension of $\pi(\mathcal{M}_\Gamma(M))$ in $\mathcal{M}_\Gamma$. (In the case $b_1(\Gamma) = 0$, this means that $\mathcal{M}_\Gamma(M)$ maps onto $\mathcal{S}_\Gamma$ with fibre of dimension $\dim M$.) □

REMARK. The moduli space $\mathcal{M}_{g,n}(M, \beta)$ of stable maps from genus g curves with n marked points to a variety M (or symplectic manifold) $\gamma : \Sigma \to M$ with image representing $\beta = [\gamma(\Sigma)] \in H_2(M)$ uses the entire parameter space $N = \mathcal{M}_{g,n}$ and has (real) dimension

$$\dim \mathcal{M}_{g,n}^N(M) = \frac{1}{2}\dim M \cdot \chi(\Sigma) + \dim N + 2\langle c_1(M), [\gamma(\Sigma)]\rangle.$$

The analogue of Lemma 27 is the Riemann-Roch formula given in terms of *complex* dimensions

$$\dim H^0(\gamma^*(TM)) - \dim H^1(\gamma^*(TM)) = \frac{1}{2}\dim \gamma^*(TM) \cdot \chi(\Sigma) + \text{degree } \gamma^*(TM).$$

Both Riemann-Roch and Lemma 27 are index theorems relating the index of a differential operator to topological information. The topological term $\langle c_1(M), [\gamma(\Sigma)]\rangle$ specifies different components of the moduli space and is detected in the dimension formula. Similarly, connected components of the moduli space of graph flows have constant $\langle w_1(M), [\gamma(\Gamma)]\rangle$. One might expect different dimensions for different connected components, however the term $\langle w_1(M), [\gamma(\Gamma)]\rangle$ is not detected in the dimension formula, although curiously it does appear in the calculation of the dimension.

7. Zero dimensional moduli spaces and counting. Given a Morse function f on M, the *Morse complex* of f is a chain complex generated by the critical points of f, with boundary maps obtained from counting gradient flows. Using this description of the homology of M, we will show how the graph operations defined earlier can be defined geometrically on the chain level as maps between formal linear combinations of critical points of f. The graph moduli spaces are ideal for defining such chain level maps.

The stable and unstable manifolds of critical points of f represent homology and cohomology classes on M and they intersect the image of the moduli space of graph flows under the evaluation map. This will allow us to give a realisation of the umkehr maps defined in section 3 from tensor products of the homology and cohomology of M to to the homology of the moduli space of graph flows.

The intersection of the image of the evaluation map with stable and unstable manifolds will be interpreted in terms of a moduli space of graph flows for a *non-compact* graph. Non-compact edges will map to gradient flows of the Morse function f. As we will see, the Morse condition—that the critical points of f are non-degenerate—arises because the gradient flows live on a non-compact graph. Until now the degeneracy of critical points of a smooth function on M has been of no concern to the construction of the moduli spaces because only compact graphs have been used.

For the remainder of the paper we work with the non-compact graph $\tilde{\Gamma}$, obtained from Γ by adding, for each univalent vertex $v \in \Gamma$, a non-compact edge E_v oriented incoming or outgoing according to whether v is incoming or outgoing. A graph flow is a continuous map $\gamma : \tilde{\Gamma} \to M$ which is the previously defined graph flow on Γ, and on non-compact edges it is the gradient flow of the Morse function f.

To define the moduli space of graph flows we now specify a collection of critical points $\vec{a} = \{a_v\}$ of the Morse function f and require that the gradient flow of the non-compact edge E_v, of $\tilde{\Gamma}$ converges to the critical point a_v. The graph flow map is defined on appropriate path spaces (defined below) and is given by:

$$\Phi(\gamma) = \frac{d\gamma_E}{dt} + \nabla f_E(\gamma(t))$$

where $E \subset \tilde{\Gamma}_k$ varies over all edges of Γ_k and non-compact edges of $\tilde{\Gamma}_k$. In this notation f_E is the restriction of f to the edge E.

The universal moduli space of graph flows of $\tilde{\Gamma}$ is notated by

$$\begin{array}{c} \tilde{\mathcal{M}}_\Gamma(M;\vec{a}) = \Phi^{-1}(0) \\ \downarrow \pi \\ \mathcal{S}_\Gamma \end{array}$$

where Γ encodes $\tilde{\Gamma}$ through its oriented univalent vertices. Notice that the space of structures remains $\mathcal{S}_\Gamma$, (i.e it has not changed even though we are now working with the enlarged graph $\tilde{\Gamma}$), because there is a fixed function f labeling all the noncompact edges.

Most of the results for compact graphs generalise to these particular non-compact graphs. The following theorem encapsulates these generalisations.

THEOREM 29. *For a generic submanifold with boundary $N \subset \mathcal{M}_\Gamma$, the moduli space of graph flows $\mathcal{M}^N_\Gamma(M;\vec{a}) = \pi^{-1}(N)/Aut(\Gamma) \subset \mathcal{M}_\Gamma(M;\vec{a})$ is a manifold with boundary, of dimension*

$$\dim \mathcal{M}^N_\Gamma(M;\vec{a}) = \dim \mathcal{M}^N_\Gamma(M) - \sum_{v\ incoming} (\dim M - \mathrm{index}(a_v)) - \sum_{v\ outgoing} \mathrm{index}(a_v).$$

REMARKS. 1. The moduli space $\mathcal{M}^N_\Gamma(M;\vec{a})$ is in general not compact.

2. There is a canonical orientation on $\tilde{\mathcal{M}}^N_\Gamma(M;\vec{a})$ induced by an orientation on M.

Proof. To prove the theorem we must define the Banach manifold structure on the mapping spaces, construct the universal moduli space of graph flows, prove that the projection π to the structure space is Fredholm, calculate its index and prove regularity. Except for the proof that the operator is Fredholm, these results require only small adjustments to the compact graph case.

For an oriented metric graph Γ_k with univalent vertices attach a half-line E_v to each univalent vertex v to get the non- compact graph $\tilde{\Gamma}_k$. The non-compact edge is oriented according to the orientation of v and this is realised in the parametrisation of incoming E_v by $t \in (-\infty, 0]$ and outgoing E_v by $t \in [0, \infty)$.

DEFINITION 30. *Define $\mathcal{P}_{\Gamma_k}(M;\vec{a})$ to be the subset of continuous maps from $\tilde{\Gamma}_k \to M$, that converge on non-compact edges to a_i, with square integrable derivative*

$$\mathcal{P}_{\Gamma_k}(M;\vec{a}) = \left\{ \gamma : \Gamma_k \to M \;\middle|\; \gamma \text{ continuous}, \lim_{t\to(-)\infty} \gamma_{E_v}(t) = a_v, \int_{\tilde{\Gamma}_k} \left|\frac{d\gamma}{dt}\right|^2 dt < \infty \right\}$$

and for any section s of the vector bundle $\mathcal{V} = \gamma^*TM$ over $\tilde{\Gamma}_k$ define its norm using the $W^{1,2}$ metric

$$\|s\|^2 = \int_{\tilde{\Gamma}_k} \left(\left|\frac{ds}{dt}\right|^2 + |s|^2 \right) dt.$$

This Banach manifold contains the solutions to the graph flow equation since on a non-compact (outgoing, say) edge E associated to the critical point a_v,

$$\int_E \left|\frac{d\gamma}{dt}\right|^2 dt = -\int_0^\infty \left\langle \frac{d\gamma}{dt}, \nabla f_E \right\rangle dt = -\int_0^\infty \frac{df_E}{dt} dt = f_E(\gamma(v)) - f_E(a_v) < \infty.$$

Then $\mathcal{P}_\Gamma(M;\vec{a})$ is defined as a union of $\mathcal{P}_{\Gamma_k}(M;\vec{a})$ in the same way that $\mathcal{P}_\Gamma(M)$ is defined. Similarly define $\mathcal{P}_\Gamma(TM;\vec{a})$. The graph flow map $\Phi : \mathcal{P}_\Gamma(M;\vec{a}) \to \mathcal{P}_\Gamma(TM;\vec{a})$ defines the universal moduli space as its zero set:

$$\mathcal{M}_\Gamma(M;\vec{a}) = \Phi^{-1}(0) \subset \mathcal{P}_\Gamma(M;\vec{a}).$$

Regularity of Φ at 0 requires the following minor adjustments to the compact case. The proof of Theorem 24 shows that any element r of the cokernel of $D\Phi$ must vanish on Γ_k, the compact part of $\tilde{\Gamma}_k$. But on a non-compact edge E_v, r is determined, via the codimension 1 condition (33) at v, by its values on compact edges containing v

and hence it vanishes on E_v and so vanishes everywhere on $\tilde{\Gamma}_k$. Thus $D\Phi$ is onto. It has a right inverse since $D_{\tilde{\Gamma}_k}$ is Fredholm (see the appendix).

There is no change to the proof of a canonical coorientation on $\mathcal{M}_\Gamma(M;\vec{a})$. As before, the virtual codimension of π follows from an index calculation. The projection π to the parameter space is Fredholm, since $D_{\tilde{\Gamma}_k}$ is Fredholm, and index $D\pi =$ index $D_{\tilde{\Gamma}_k}$.

Lemma 31.

$$\text{index } D_{\tilde{\Gamma}_k} = \dim M \cdot \chi(\Gamma) - \sum_{v>0}(\dim M - \text{index}(a_v)) - \sum_{v<0}\text{index}(a_v).$$

Proof. Choose trivialisations of $\mathcal{V} = \gamma^*TM$ over $\tilde{\Gamma}_k$ so that transition functions are simply multiplication by ± 1. With respect to these local trivialisations $D_{\tilde{\Gamma}_k} = \frac{d}{dt} + A(t)$ is well-defined since the operator commutes with multiplication by ± 1. The index remains unchanged under continuous deformations of $D_{\tilde{\Gamma}_k}$ although we cannot deform $A(t)$ to zero as in the compact case, because at infinity $A(t)$ looks like the Hessian of the Morse function f at each critical point and hence it is invertible there. However, we may deform $A(t)$ so that it is diagonal, zero on $\Gamma_k \subset \tilde{\Gamma}_k$ and constant outside of a compact subset of $\tilde{\Gamma}_k$ that contains Γ_k.

For $A = \text{diag}(\lambda_1(t), ..., \lambda_d(t))$ we can explicitly solve the system for the kernel:

$$\dot{s}_i = -\lambda_i(t)s(t),\ i = 1, ..., d\ .$$

Since $\lambda_i(t) = \lambda_i^v$ is constant near infinity along $E_v \subset \tilde{\Gamma}$ then $s(t) \sim e^{-\lambda_i t}$ near infinity. Thus, $s \in W^{1,2}(\mathbb{R}^+, \mathbb{R})$ only when $\lambda_i^v < 0$ (respectively, $\lambda_i^v > 0$) when E_v is incoming (respectively, outgoing). If the ith eigenvalue does not satisfy this condition for a single v then the solution must vanish on E_v and hence by continuity on all of $\tilde{\Gamma}$. We see then that the dimension of the kernel is given by the number of $\lambda_i(t)$ with $\lambda_i^v < 0$ for all v oriented positively (E_v incoming) and $\lambda_i^v > 0$ for all v oriented negatively (E_v outgoing.)

For the cokernel we use $-A^T$ so negate each λ_i^v. Then $r_i \in W^{1,2}(\mathbb{R}^+, \mathbb{R})$ when $\lambda_i^v > 0$ (respectively, $\lambda_i^v < 0$) when E_v is incoming (respectively, outgoing). It is no longer true that if r_i vanishes along one edge then it vanishes on all of $\tilde{\Gamma}$. For each i we get a contribution to the cokernel from each incoming (outgoing) edge E_v with $\lambda_i^v > 0 (< 0)$.

In order to calculate

$$\text{index } D_{\tilde{\Gamma}_k} = \dim\ker D_{\tilde{\Gamma}_k} - \dim\text{coker } \mathrm{D}_{\tilde{\Gamma}_k}$$

change the index of a critical point and observe the change in index $D_{\tilde{\Gamma}_k}$. For an incoming edge E_v change index(a_v) to index$(a_v) - 1$, so take $\lambda_i^v < 0$ and send it to $-\lambda_i^v$. Either λ_i^v contributes to the kernel (it cannot contribute to the cokernel) then $-\lambda_i^v$ cannot contribute to the cokernel and we lose 1 from index $D_{\tilde{\Gamma}_k}$, or λ_i^v does not contribute to the kernel in which case $-\lambda_i^j$ does contribute to the cokernel and we again lose 1 from index $D_{\tilde{\Gamma}_k}$. A similar argument shows that on an outgoing edge E_v, the change index$(a_v) \mapsto$ index$(a_v) + 1$ affects the change index $D_{\tilde{\Gamma}_k} \mapsto$ index $D_{\tilde{\Gamma}_k} - 1$. Thus

$$\text{index } D_{\tilde{\Gamma}_k} = \sum_{v>0}\text{index}(a_v) - \sum_{v<0}\text{index}(a_v) + \text{ constant.}$$

To determined the constant, suppose that index$(a_v) = \dim M$ (i.e. $\lambda_i^v < 0$ for all i) for each incoming E_v and index$(a_v) = 0$ (i.e. $\lambda_i^v > 0$ for all i) for each outgoing E_v. Then the non-compact edges make no contribution to the cokernel and there is no obstruction to the kernel. Hence the index is the same as that for the compact graph, i.e.

$$\text{index } D_{\tilde{\Gamma}_k} = \text{index } D_{\Gamma_k} = \dim M \cdot \chi(\Gamma)$$

and the constant agrees with the statement of the lemma. (In terms of the graph flow, we have just seen that when incoming and outgoing edges converge respectively to maxima and minima of f, locally it is as if there is no critical point restriction.) □

This completes the proof of the theorem. □

To the collection $\vec{a}$ of l critical points of f associate the product of stable and unstable manifolds $W(\vec{a}) \subset M^l$

$$W(\vec{a}) = \prod_{v>0} \mathcal{W}^u(a_v) \times \prod_{v<0} \mathcal{W}^s(a_v).$$

Now consider the evaluation map on the univalent vertices, $ev : \mathcal{M}_\Gamma^N(M) \to M^{p+q}$(we are assuming p incoming leaves and q outgoing leaves). It is clear that

$$\mathcal{M}_\Gamma^N(M;\vec{a}) = \mathcal{M}_\Gamma^N(M) \cap ev^{-1}(W(\vec{a})).$$

In the introduction to Section 6, we claimed that N can be chosen so that $ev(\mathcal{M}_\Gamma^N(M))$ intersects $W(\vec{a})$ transversally. The proof of this does not give a new proof that $\mathcal{M}_\Gamma^N(M;\vec{a})$ is a manifold since it uses the proof of that fact, although it is a more intuitive way of seeing the manifold structure and its dimension and it will be used in the compactness arguments.

Lemma 32. *If $\gamma \in \mathcal{M}_\Gamma^N(M;\vec{a})$ is a regular point of the flow map Φ, then $ev(\mathcal{M}_\Gamma^N(M))$ intersects $W(\vec{a})$ transversally in M^{p+q}.*

Proof. If $ev(\mathcal{M}_\Gamma^N(M))$ does not intersect $W(\vec{a})$ transversally at $\vec{x} \in M^{p+q}$ then there is a vector $\vec{\xi} \in T_{\vec{x}}M^{p+q}$ orthogonal to the tangent spaces of $ev(\mathcal{M}_\Gamma^N(M))$ and $W(\vec{a})$. Take a non-zero component of $\vec{\xi}$ in one factor M of M^{p+q}, corresponding to the univalent vertex $v \in \Gamma$. Along the non-compact edge E_v parametrised by $t \in [0,\infty)$ solve the equation $\dot{r}(t) - (\nabla\nabla f)^T \cdot r(t) = 0$ with $r(0) = \xi$. Since ξ is orthogonal to $ev(\mathcal{M}_\Gamma^N(M))$ at x, $r(t)$ decays at infinity and lives in L^2. Put $r = 0$ on the rest of the graph $\tilde{\Gamma}_k$.

Since D_{Γ_k} is surjective, $r \in \text{im } D_{\Gamma_k}$ so $r = D_{\tilde{\Gamma}_k}s$ for some $s \in W^{1,2}(\tilde{\Gamma}_k)$. Now

$$\int_{\tilde{\Gamma}_k} \langle r, r\rangle dt = \int_0^\infty \langle r, D_{\tilde{\Gamma}_k}s\rangle dt = \int_0^\infty d/dt\langle r, s\rangle dt = \langle \xi, s(0)\rangle.$$

But $r \equiv 0$ on Γ_k so $D_{\Gamma_k}s = 0$ so $s(0) \in T_x ev(\mathcal{M}_\Gamma^N(M))$ and $\langle \xi, s(0)\rangle = 0$ which is a contradiction. □

The following corollary is a generalisation of the Morse-Smale condition.

Corollary 33. *Any submanifold $N \subset \mathcal{M}_\Gamma(M)$ can be perturbed so that $ev(\mathcal{M}_\Gamma^N(M))$ intersects $W(\vec{a})$ transversally* for all *collections of critical points $\vec{a}$ of f.*

Proof. For each collection of critical points $\vec{a}$ of f, the proof of Theorem 29 supplies a universal moduli space together with a map to the parameter space π :

$\mathcal{M}_\Gamma(M;\vec{a}) \to \mathcal{S}_\Gamma$. For a given $\vec{a}$, the Sard-Smale theorem allows one to make an arbitrarily small deformation of a submanifold with boundary $N_0 \subset \mathcal{M}_\Gamma$ to N_1 that is transverse to $\pi(\mathcal{M}_\Gamma(M;\vec{a}))$. Take another collection $\vec{a'}$ and again apply the Sard-Smale theorem to choose a deformation N_2 of N_1 small enough so that it remains transverse to $\pi(\mathcal{M}_\Gamma(M;\vec{a}))$ and so that it is also transverse to $\pi(\mathcal{M}_\Gamma(M;\vec{a'}))$. Take the finite list of all collections of critical points $\vec{a}$ labeled by a given set of univalent vertices of Γ, and update $N_0, N_1, N_2, \ldots$ to get a finite sequence that finishes at $N \subset \mathcal{M}_\Gamma$ simultaneously transverse to all the spaces, $\pi(\mathcal{M}_\Gamma(M;\vec{a}))$. □

8. Compactness. The graph moduli spaces are non-compact due to the non-compact edges of the graph. This will imply, as we will see, the non-compactness and gluing issues essentially reduce to these same issues for spaces of gradient flows of a Morse function.

8.1. Piecewise graph flows. We begin by recalling the natural compactification of the space of gradient flow lines converging to two fixed critical points of the Morse function f.

The space of flow-lines of the Morse function f from the critical point a to the critical point b can be viewed as using the noncompact graph $\Gamma = \mathbb{R}$ which has a one-dimensional space of translational symmetries. The moduli space of flows is the quotient space

$$\mathcal{M}(a,b) = \mathcal{M}_{\mathbb{R}}(M;a,b)/\mathbb{R}.$$

Notice that $\mathcal{M}_{\mathbb{R}}(M;a,b)$ is the intersection of the unstable manifold of a with the stable manifold of b, $\mathcal{M}_{\mathbb{R}}(M;a,b) = W_a^u \cap W_b^s$.

Now assume that M is equipped with a metric so that $f : M \to \mathbb{R}$ satisfies the Morse-Smale condition. This says that the intersections of stable and unstable manifolds are all transverse. Recall the partial ordering on the set of critical points in this setting, $a \geq b$ if there is a gradient flow connecting a and b, i.e $\mathcal{M}(a,b) \neq \emptyset$.

Define the space of *piecewise flow lines* connecting critical points a and b by:

$$\overline{\mathcal{M}}(a,b) = \bigcup_{a=a_0 \geq a_1 \geq \ldots \geq a_j = b} \mathcal{M}(a,a_1) \times \mathcal{M}(a_1,a_2) \times \ldots \times \mathcal{M}(a_{j-1},b)$$

where the union is taken over all nonincreasing finite sequences of critical points. For example, $a \geq b$ implies $\mathcal{M}(a,b) \subset \overline{\mathcal{M}}(a,b)$.

Since f satisfies the Morse-Smale condition, $a > b$ implies that $f(a) > f(b)$ and index $a >$ index b. The result is that $\overline{\mathcal{M}}(a,b)$ is compact, which is a simple equicontinuity argument, and that it contains $\mathcal{M}(a,b)$ as an open dense subset. This is often expressed as a gluing theorem since it implies the existence of true flows arbitrarily close to piecewise flows. A uniqueness part of gluing further implies that $\overline{\mathcal{M}}(a,b)$ is a manifold with corners. For our purposes it is sufficient to consider at most 1-dimensional moduli spaces. In the one dimensional $\overline{\mathcal{M}}(a,b)$ is a 1-manifold with boundary. In particular a deleted neighbourhood of any boundary component in $\overline{\mathcal{M}}(a,b)$ is a connected, open interval.

By analogy, we define the space of piecewise graph flows by

$$\overline{\mathcal{M}}_\Gamma^N(M;\vec{a}) = \bigcup_{\vec{b}} \mathcal{M}_\Gamma^N(M;\vec{b}) \times \prod_{v\ incoming} \overline{\mathcal{M}}(a_v,b_v) \times \prod_{v\ outgoing} \overline{\mathcal{M}}(b_v,a_v)$$

where the union is taken over all collections of critical points $\vec{b}$ labeling the univalent vertices of Γ. Notice that the restriction of such piecewise graph flow γ to a compact edge is a gradient flow of the function labeling that edge, and when restricted to a noncompact edge, it is a piecewise flow line.

Proposition 34. *When N is compact, $\overline{\mathcal{M}}_\Gamma^N(M;\vec{a})$ is compact.*

Proof. For any $(\Gamma_k, \vec{f}) \in N$, the gradient vector fields ∇f_E along the edge $E \subset \tilde{\Gamma}_k$ are bounded and uniformly continuous, uniformly in N, since M is compact. (As usual, we express the Morse function f by f_E for any non-compact edge of $\tilde{\Gamma}_k$.)

Hence the space of maps $\overline{\mathcal{M}}_\Gamma^N(M;\vec{a})$ is an equicontinuous family since the derivatives $d\gamma_E/dt = -\nabla f_E$ are uniformly bounded. Let $\{\gamma^j\} \subset \overline{\mathcal{M}}_\Gamma^N(M;\vec{a})$ be a sequence of piecewise graph flows. Take any univalent vertex $v \in \tilde{\Gamma}$, or any point on a non-compact edge labeled by its parameter $T < 0$ ($T > 0$) for an incoming (outgoing) edge E of the metric graph. Both of these give well-defined choices of points in any metric graph in N. Since M is compact, the sequence $\gamma^j(v)$, or $\gamma^j(T)$, has a convergent subsequence converging to a point $x \in M$. By differentiating ∇f_E over M one gets a uniform C^2 bound on the $\{\gamma^j\}$ and thus the limit of the subsequence satisfies the flow equation. Thus the flow from the limit point x is a uniform limit of the subsequence of graph flows. It may be a graph flow or a gradient flow of f. As we choose different points on non-compact edges, we get different gradient flows of f that are also uniform limits of a subsequence of graph flows.

So the limit of a sequence of piecewise flows is locally a flow and to prove that it is itself a piecewise flow it remains to show that the limit is a continuous map from $\tilde{\Gamma}_k$ to M. Canonically parametrise $\{\gamma^j\}$ by $s = f(\gamma^j(t))$ so they satisfy $d\gamma^j(s)/ds + \nabla f/|\nabla f| = 0$. Again one gets a uniform bound on $d\gamma^j(s)/ds$ so by equicontinuity the limit of the subsequence is a continuous map from $\tilde{\Gamma}_k$ to M and hence a piecewise flow. □

Remark. In the above proof it is clear that the that non-compactness of the moduli space of graph flows arises due to the non-compact edges of the graph. We say that a sequence *bubbles* along a non-compact edge if its limit is not a smooth flow there.

Corollary 35. *For generic choice of N, if* $\dim \mathcal{M}_\Gamma^N(M;\vec{a}) = 0$ *then* $\mathcal{M}_\Gamma^N(M;\vec{a})$ *is compact.*

Proof. Choose f to be Morse-Smale and N as in Corollary 33 so that all moduli spaces $\mathcal{M}_\Gamma^N(M;\vec{b})$ are manifolds of the expected dimension. Suppose that a sequence of graph flows bubbles along an incoming edge E_v and converges to a piecewise graph flow. Since f is Morse-Smale, $\mathcal{M}(a_v, b_v)$ is non-empty only if index $a_v >$ index b_v. But then

$$\dim \mathcal{M}_\Gamma^N(M;\vec{b}) < \dim \mathcal{M}_\Gamma^N(M;\vec{a}) = 0$$

so by transversality $\mathcal{M}_\Gamma^N(M;\vec{b})$ is empty, contradicting the claim that the sequence bubbles. The same argument works for an outgoing edge. Thus no bubbling can occur and $\overline{\mathcal{M}}_\Gamma^N(M;\vec{a}) = \mathcal{M}_\Gamma^N(M;\vec{a})$. □

Theorem 36. *For generic choice of N, if* $\dim \mathcal{M}_\Gamma^N(M;\vec{a}) = 1$ *then* $\overline{\mathcal{M}}_\Gamma^N(M;\vec{a})$ *is a 1-manifold with boundary* $\bigcup_v \bigcup_{b_v} \mathcal{M}_\Gamma^N(M;\vec{b}) \times \mathcal{M}(a_v, b_v)$ *where for each v* index $b_v =$ index $a_v \pm 1$, *and it contains* $\mathcal{M}_\Gamma^N(M;\vec{a})$ *as an open dense subset.*

Proof. The same argument as in the proof of Corollary 35 shows that for a 1-dimensional moduli space $\mathcal{M}_\Gamma^N(M;\vec{a})$, any sequence $\{\gamma^j\} \subset \mathcal{M}_\Gamma^N(M;\vec{a})$ bubbles at most once. If a sequence bubbles along the incoming edge E_v then its limit is given by the pair (γ,μ) satisfying

(i) $\gamma \in \mathcal{M}_\Gamma^N(M;\vec{b})$,

(ii) $\mu \in \mathcal{M}(a_v, b_v)$ uniquely defined up to rescaling,

(iii) index $b_v =$ index $a_v - 1$ so dim $\mathcal{M}_\Gamma^N(M;\vec{b}) = 0$.

Conversely, to prove the theorem we need to show that any (γ,μ) satisfying (i), (ii) and (iii) is a unique end of $\mathcal{M}_\Gamma^N(M;\vec{a})$. The same argument will apply to an outgoing edge.

We follow the approach in [5]. The idea is to find a manifold with boundary $\mathcal{P}$ and a smooth manifold $\mathcal{N}$ that lie inside a common ambient space, such that the broken flow (γ,μ) maps to a point in both these manifolds. If $\mathcal{P}$ and $\partial\mathcal{P}$ intersect $\mathcal{N}$ transversely then (γ,μ) is a unique end of the 1-dimensional intersection $\mathcal{P}\cap\mathcal{N}$. More is proven in [5] for higher-dimensional moduli spaces, where $\mathcal{P}$ is a product of manifolds with boundary, so a manifold with corners, hence the transversal intersection inherits a structure of a manifold with corners.

Put $f(b_v) = c$. Choose $\epsilon > 0$ small enough so that c is the only critical value in $[c-\epsilon, c+\epsilon]$. Define

$$M^{\pm} = f^{-1}(c \pm \epsilon) \subset M$$

and

$$\mathcal{P} \subset M^+ \times M^-$$

by pairs (x^+, x^-) that flow to the same point $x \in f^{-1}(c)$ under the forward, respectively backward, gradient flow (possibly flowing for infinite time.)

Let $\vec{a}(-v)$ be $\vec{a}$ with a_v removed. Define W_v^s to be all those points of M that flow under the gradient flow of f to $ev_v(\mathcal{M}_\Gamma^N(M;\vec{a}(-v))$ and

$$\mathcal{N} = W_{a_v}^u \cap M^+ \times W_v^s \cap M^- \subset M^+ \times M^-.$$

The "stable manifold" W_v^s is a manifold of dimension $d -$ index $a_v + 2$ for $d = \dim M$ so $\mathcal{N}$ is a d dimensional manifold.

It is proven in [5] that $\mathcal{P}$ is a $d-1$ dimensional manifold with boundary that intersects $\mathcal{N}$ transversally inside the $2(d-1)$ dimensional manifold $M^+ \times M^-$. The critical point b_v is contained inside the intersection $\mathcal{P}\cap\mathcal{N}$ and a neighbourhood of b_v in $\mathcal{P}\cap\mathcal{N}$ is a 1-manifold K with boundary b_v.

The arguments in [5] require the Morse function f to be Morse-Smale, and we must choose either a metric on M that is standard near critical points of f, or replace the gradient flow with a Morse-like vector field on M. If we choose the latter, the analysis in Section 7 does not change since it depends only on the fact that $\nabla\nabla f$ is invertible at infinity and this is still true of Morse-like vector fields. Thus, in our adaption of the arguments in [5] we will require the same conditions on f and replace the gradient vector field on external edges by a Morse-like vector field.

Finally, we will prove that the analogues of stable and unstable manifolds for a graph flow intersect stable and unstable manifolds of f transversally. This is a slight adjustment of Corollary 33 which shows that the image of the moduli space of graph flows under the evaluation map, $ev(\mathcal{M}_\Gamma^N)$, intersects the stable and unstable manifolds of f transversally.

First notice that the stable manifold for a graph flow, W_v^s, constructed from $ev_v(\mathcal{M}_\Gamma^N(M;\vec{a}(-v))$, is a moduli space of graph flows as follows. For $\sigma = (\Gamma_k, \vec{f}) \in N$ define $\sigma^+ = (\Gamma_k^+, \vec{f})$ on the graph Γ_k^+ obtained from Γ_k by adding a compact edge E at $v \in \Gamma_k$ oriented inwards and assigning to E the vector field ∇f and length ℓ_E any positive real number. This gives a family of structures N^+ with $\dim N^+ = \dim N + 1$. (Since $\Gamma^+ \to \Gamma$ is a homotopy equivalence the set N^+ is almost a subset of $\mathcal{M}_\Gamma$ except that the lengths of edges do not add to 1.)

The argument in Corollary 33 also shows that for any length ℓ_E on the extra compact edge $E \subset \Gamma_k^+$, for generic choice of N the image of the moduli space of graph flows under the evaluation map at the univalent vertex of E intersects the unstable manifolds of f transversally. Thus, as we vary ℓ_E transversality is unchanged so W_v^s intersects the unstable manifolds of f transversally. Note that $N \subset \mathcal{S}_\Gamma$ is chosen so that all moduli spaces $\mathcal{M}_\Gamma^N(M;\vec{b})$ are manifolds of the expected dimension which is *independent* of ℓ_E.

The same construction works for a negatively oriented vertex v by adding an outward pointing compact edge at v to get Γ_k^- and thus showing that W_v^u intersects the stable manifolds of f transversally. □

Remark. The main ingredient in gluing is the transversality of the intersection of the image of the evaluation map and stable and unstable manifolds of f, which follows from surjectivity of $D\Phi_{\tilde{\Gamma}_k}$. Gluing can be defined directly from surjectivity of $D\Phi_{\tilde{\Gamma}_k}$. One uses the energy functional defined on $\mathcal{P}_{\Gamma_k}(M;\vec{a})$

$$\begin{aligned}\mathcal{E}(\gamma) &= \frac{1}{2}\int_{\tilde{\Gamma}_k}\left(\left|\frac{d\gamma}{dt}\right|^2 + |\nabla f(\gamma)|^2\right)dt \\ &= f(\alpha) - f(\beta) + \frac{1}{2}\int_{\tilde{\Gamma}_k}\left|\frac{d\gamma}{dt} + \nabla f(\gamma)\right|^2 dt\end{aligned}$$

where the first expression shows that $\mathcal{E}$ is non-negative and the second expression shows that $\mathcal{E}$ is minimised by graph flows. A broken flow yields a path with small energy—an approximate flow. The implicit function theorem shows that there is a unique true flow nearby. Details for the case of the Morse complex can be found in [17].

Using the same gluing constructions as in the proof of Theorem 36 we will now show how to remove an edge $E \subset \Gamma$ leaving two marked vertices given by its endpoints. (An endpoint of E must not coincide with an existing marked vertex of Γ. To find such an edge it may be necessary to take an edge $E \subset \Gamma_1 \to \Gamma$ and consider all $\Gamma_k \to \Gamma_1 \to \Gamma$.) The edge E may or may not be separating. We denote $\Gamma - E$ to be the graph, or union of two graphs, with marked vertices those of Γ and the endpoints of E, oriented according to the orientation of E.

Choose N so that $\mathcal{M}_\Gamma^N(M;\vec{a})$ is a smooth zero-dimensional moduli space for all $\vec{a}$ and so that for a given edge $E \subset \Gamma$ the induced structure on $\Gamma - E$ gives a smooth zero-dimensional moduli space for all $\vec{b}$. Here $\vec{b} = (\vec{a}, a^-, a^+)$ is a vector of critical points of f associated to the marked vertices of $\Gamma - E$, and $a^- = a^+$ is named twice because it is used twice. The critical point a^- is associated to the negatively oriented (outgoing) endpoint of E and a^+ is associated to the positively oriented (incoming) endpoint of E. To the edge E, each metric-Morse structure in N should associate the gradient vector field ∇f of the external Morse function.

Theorem 37. *The moduli spaces* $\mathcal{M}_\Gamma^N(M;\vec{a})$ *and* $\bigcup_{(a^-,a^+)} \mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+)$ *are cobordant.*

Proof. Define the one-dimensional moduli space $\mathcal{M}_\Gamma^{N_E}(M;\vec{a})$ using a family N_E of structures with $\dim N_E = \dim N + 1$ as follows. For $\sigma = (\Gamma_k, \vec{f}) \in N$, take the edge $E_k = \phi^{-1}(E) \in \Gamma_k$ where $\phi : \Gamma_k \to \Gamma$ is the homotopy equivalence and assign to it any length $\ell \in [\ell_{E_k}, \infty)$. This gives a family $\tilde{N}_E$ of structures on Γ with $\partial\tilde{N}_E = N \cup N|_{\Gamma-E}$. As in the proof of Theorem 36 the family N_E is not contained in $\mathcal{M}_\Gamma$ since the lengths of edges do not add to 1 so we use an enlargement of $\mathcal{S}_\Gamma$ to allow E to have an arbitrarily large edge length. Inside this space of parameters take an arbitrarily small deformation N_E of $\tilde{N}_E$ that fixes the boundary so that N_E is transversal to $\pi(\mathcal{M}_\Gamma(M;\vec{a}))$ for all $\vec{a}$. Then it immediately follows that $\mathcal{M}_\Gamma^{N_E}(M;\vec{a})$ is a one-dimensional manifold with compact and non-compact ends. At the compact ends it is a manifold with boundary $\mathcal{M}_\Gamma^N(M;\vec{a})$ and we will show that it can be compactified at the non-compact ends so that the 1-manifold gives the cobordance stated in the theorem. In other words

$$\partial\overline{\mathcal{M}}_\Gamma^{N_E}(M;\vec{a}) = \mathcal{M}_\Gamma^N(M;\vec{a}) \cup \bigcup_{(a^-,a^+)} \mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+). \tag{35}$$

The same transversality argument as in the proof of Corollary 35 shows that any sequence $\{\gamma^j\} \subset \mathcal{M}_\Gamma^N(M;\vec{a})$ bubbles at most once along the edge E to give a graph flow in $\mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+)$ for critical point $a^- = a^+$ with index so that $\dim\mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+) = 0$. (The expected dimension is the same as the actual dimension.) As usual, if $\Gamma - E$ is disconnected then $\mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+)$ is the product of moduli spaces for each component of $\Gamma - E$ and by a graph flow in $\mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+)$ we mean a pair of graph flows.

The theorem will be proven if we can show that for any flow in the zero-dimensional moduli space $\mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+)$ there is a unique flow nearby in $\mathcal{M}_\Gamma^{N_E}(M;\vec{a})$. This gluing result follows the proof of Theorem 36 exactly. Once again we construct a manifold with boundary $\mathcal{P}$ and a smooth manifold $\mathcal{N}$ that lie inside a common ambient space, such that a broken graph flow given by a flow in $\mathcal{M}_{\Gamma-E}^N(M;\vec{a},a^-,a^+)$ maps to a point in both these manifolds. In fact $\overline{\mathcal{M}}_\Gamma^{N_E}(M;\vec{a}) = \mathcal{P} \cap \mathcal{N}$ and the intersection will be transverse so $\overline{\mathcal{M}}_\Gamma^{N_E}(M;\vec{a})$ is a 1-manifold with boundary and in particular any broken flow is a unique end of this 1-manifold.

Put $f(a^\pm) = c$ then there is no change to the definition of $\mathcal{P} \subset M^+ \times M^-$ for $M^\pm = f^{-1}(c \pm \epsilon)$. In the definition of $\mathcal{N}$ we now use stable and unstable manifolds of graph flows:

$$\mathcal{N} = W_{v^-}^u \cap M^+ \times W_{v^+}^s \cap M^- \subset M^+ \times M^-$$

where $v^\pm$ are the endpoints of E and $W_{v^-}^u$ and $W_{v^+}^s$ are defined in the proof of Theorem 36. Arguing as in Corollary 33 it can be shown that when N_E is chosen transversally to $\pi(\mathcal{M}_\Gamma(M;\vec{a}))$ the stable and manifolds $W_{v^-}^u$ and $W_{v^+}^s$ intersect transversally so the theorem follows. □

Remark. In the previous two theorems, if the moduli spaces are oriented then the orientation on the 1-manifold agrees with the orientations on the boundary. This is because the orientations are canonically induced from the evaluation map, and the gluing construction also used the evaluation map.

9. Cohomology operations on the Morse chain complex. In this section we represent the homology $H_*(M)$ in terms of the Morse complex of the Morse function $f : M \to \mathbb{R}$ and express the homology operation

$$q_\Gamma : H_*(BAut_0(\Gamma)) \otimes H_*(M)^{\otimes p} \to H_*(M)^{\otimes q}$$

with respect to this representation.

Recall that the Morse complex of a Morse function f is the chain complex of abelian groups

$$C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

generated by the critical points of f, graded by their index. The boundary operator ∂ is defined by counting points in the moduli space of solutions to the gradient flow equation converging to critical points of consecutive degrees. We will give the proof that that this does indeed define a complex, i.e. $\partial \circ \partial = 0$, since an analogous proof is used to show that the graph moduli spaces define homological invariants.

Let a and b be critical points of f of index $k+1$ and k respectively. It follows from the analysis in Section 6 that $\mathcal{M}(a,b)$ is a zero-dimensional oriented compact manifold. Thus it makes sense to count the points, with sign, in $\mathcal{M}(a,b)$. Put $n(a,b) = \#\mathcal{M}(a,b)$ and define the linear operator

$$\partial a = \Sigma n(a,b)b$$

where the sum is over all critical points b of index k.

LEMMA 38.

$$\partial^2 = 0\ .$$

Proof. By linearity

$$\partial\partial a = \Sigma n(a,b)\partial b = \Sigma n(a,b)n(b,c)c$$

where the sum is over all critical points b of index k and c of index $k-1$. We will show that for fixed c the sum $\Sigma n(a,b)n(b,c)c$ over all intermediate critical points b vanishes. By Theorem 36 the compactified one-dimensional moduli space $\overline{\mathcal{M}(a,c)}$ is a manifold with boundary. That is the boundary points, which are piecewise flows, each correspond to a unique edge. Since one-dimensional compact manifolds can only be a finite collection of closed intervals this means that the ends come in pairs. Thus the contributions to $\partial^2(a)$ come in pairs. This immediately gives the vanishing of each component modulo two. Furthermore the orientations on the 1 dimensional moduli space and its boundary agree, meaning that $n(a,b)n(b,c) = -1(+1)$ if that boundary component is oriented negatively (positively). This is because the orientations are defined canonically using the evaluation map and the gluing construction also uses the evaluation map. Thus the boundary points are oriented oppositely so the oriented sum vanishes. □

Choose an $Aut_0(\Gamma)$-invariant submanifold $\tilde{N} \subset \mathcal{S}_\Gamma$ such that the quotient $N = \tilde{N}/Aut_0(\Gamma) \in \mathcal{M}_\Gamma$ is transverse to the image of the universal moduli space. Given the Morse-Smale function f, let $C_*(M,f)$ be the associated Morse-Smale chain complex generated by the critical points, and let $C^*(M,f)$ be the dual cochain complex. The cochains are negatively graded so that the evaluation pairing $C^*(M,f)\otimes C_*(M,f)\to\mathbb{Z}$ is of degree zero.

Define a class q_Γ^N to be an element of the tensor product complex,

$$\bigotimes_{v\ incoming} C^*(M,f) \bigotimes_{v\ outgoing} C_*(M,f)$$

in the following manner. Consider those collections of critical points $\vec{a}$ such that $\dim \mathcal{M}_\Gamma^\sigma(M;\vec{a}) = 0$. These spaces contain a finite number of oriented points which can be counted with sign (if $\mathcal{M}_\Gamma^\sigma(M;\vec{a})$ is oriented—otherwise this is well defined mod 2, and we take coefficients to be $\mathbf{Z}_2$).

Definition 39.

$$q_\Gamma^N = \sum \#\mathcal{M}_\Gamma^N(X;\vec{a})[\vec{a}] \in \bigotimes_{v\ incoming} C^*(M,f) \bigotimes_{v\ outgoing} C_*(M,f).$$

Theorem 36 and the definition of the boundary operator in the Morse-Smale complex yields the following.

Lemma 40.

$$dq = 0.$$

Proof. Recall that the boundary operator on the tensor product of the chain complexes is given by

$$\partial : \bigotimes_{1\le i\le k} C_*(M,f) \to \bigotimes_{1\le i\le k} C_*(M,f)$$

$$(a_1,...,a_n) \mapsto \Sigma_i(a_1,...,\partial_i(a_i),...,a_k)$$

where ∂_i is defined using f_i. Then if we think of q as a map

$$q : \bigotimes_{1\le i\le n_1} C_*(M,f) \to \bigotimes_{n_1+1\le i\le n} C_*(M,f)\ ,$$

the requirement that $dq = 0$ is equivalent to the requirement that q is a chain map: $\partial q = q\partial$. Choose $\vec{a} = (\vec{b},\vec{c})$ so that dim $\mathcal{M}_\Gamma^N(M;\vec{a}) = 1$. We have divided $\vec{a}$ into critical points $\vec{b}$ corresponding to incoming flows and $\vec{c}$ corresponding to outgoing flows. Notice that for $\partial\vec{b} = \Sigma\vec{b}^j$, then $\dim \mathcal{M}_\Gamma^N(M;(\partial\vec{b},\vec{c})) = 0$ so $q(\partial\vec{b}) \in \bigotimes_{v<0} C_*(M,f)$ is obtained by counting piecewise graph flows, containing a piecewise gradient flow along an incoming edge, from $\vec{b}$ to $\vec{c}$, and it takes it values in the module generated by $\vec{c}$. The composition $\partial q(\vec{b}) \in \bigotimes_{v<0} C_*(M,f)$ is given by piecewise graph flows, containing a piecewise gradient flow along an outgoing edge, and takes its values in the same module generated by $\vec{c}$ so it makes sense to compare $q(\partial\vec{b})$ and $\partial q(\vec{b})$. We will show that there is a pairing between the two types of piecewise graph flows which gives $\partial q = q\partial$.

The one-dimensional manifold $\mathcal{M}_\Gamma^\sigma(M;\vec{a})$ is compact with boundary so it is a finite collection of closed intervals. Each boundary point of an interval corresponds to a piecewise graph flow with exactly one external edge not a true gradient flow. This is the key fact behind the proof. If more than one external edge were to break then the true graph flow inside this piecewise graph flow would lie in a moduli space of negative dimension, thus contradicting its existence. These boundary piecewise graph flows are paired by the interval they bound.

There are three types of components of the one-dimensional manifold $\mathcal{M}_\Gamma^\sigma(M;\vec{a})$ and thus three types of pairings of piecewise flows. The first type of component consists of an interval whose two boundary points correspond to piecewise gradient flows both containing a piecewise gradient flow along an incoming edge. The sign, or orientation, given to the piecewise flow is the product of the signs, or orientations, given to the two components of the piecewise flow. But this is the orientation induced from the one-dimensional moduli space. Since the two boundary components of the one-dimensional moduli are oriented oppositely - they are two ends of an oriented interval - the two piecewise graph flows contribute a total of $1-1=0$ to $q(\partial\vec{a})$.

The second type of component consists of an interval whose two boundary points correspond to two piecewise gradient flows both containing a piecewise gradient flow along an incoming edge. It behaves like the first type of component and the two piecewise graph flows contribute $1-1=0$ to $\partial q(\vec{a})$.

The third type of component consists of an interval whose two boundary points correspond to two piecewise gradient flows containing, respectively, a piecewise gradient flow along an incoming edge and a piecewise gradient flow along an outgoing edge. The one-dimensional moduli space gives an oriented cobordism between the two piecewise graph flows, so they contribute, respectively, $q(\partial\vec{a})$ and to $\partial q(\vec{a})$ with the same sign.

We pair piecewise flows arising from the third type of component and cancel pairs of piecewise flows arising from the other two types of components to get $q(\partial\vec{a}) = \partial q(\vec{a})$ and the lemma is proven. □

We shall therefore view q_Γ^N as an element of the associated homology,

$$q_\Gamma^N \in H^*(M)^{\otimes n_1} \otimes H_*(M)^{\otimes n_2}.$$

In fact q_Γ^N is independent of the choice of $N \subset \mathcal{S}_\Gamma$ and only depends on the homology class of N. We prove this in the following proposition.

Proposition 41. *If N_1 and N_2 are homologous, then $q_\Gamma^{N_1} = q_\Gamma^{N_2}$.*

Proof. A cobordism between N_1 and N_2 produces a non-compact 1-dimensional moduli space with boundary. Its compactification has boundary components consisting of the moduli spaces associated to N_1 and N_2 and to broken flows which correspond to compositions with the boundary operator. Thus the compactified 1-dimensional moduli space defines a chain homotopy equivalence between the invariants so on the level of homology $q_\Gamma^{N_1}(M) = q_\Gamma^{N_2}(M)$. □

It is easy to see that q_Γ^N coincides with the algebraic topology version of the invariant defined in section 2. This is because of the standard relationship between umkehr maps and intersection theory of chains.

We end this section by giving an analytic version of the gluing construction in Section 4.

Let Γ_1 and Γ_2 be oriented graphs. Let $\Gamma_{1,2}^{i\#j}$ be the oriented graph obtained by gluing incoming edge i of Γ_1 to outgoing edge j of Γ_2.

Proposition 42.

$$q(\Gamma_{1,2}^{i\#j}, M) = q(\Gamma_1, M)\diamond^{i,j} q(\Gamma_2, M),$$

where $\diamond^{i,j}$ *denotes tensorial contraction of cohomology in the* i*th coordinate with homology in the* j*th coordinate.*

Proof. This uses Theorem 37 repeatedly to glue together any number of edges between Γ_1 and Γ_2. As in the proof of Proposition 41, the compactified 1-dimensional moduli spaces have boundary components consisting of components of the zero-dimensional moduli spaces and broken flows so this gives a chain homotopy equivalence between the invariants. □

COROLLARY 43. *Changing the orientation of a non-compact edge induces the Poincare duality isomorphism on the relevent tensor coordinate of the invariant* $q_\Gamma(M)$.

Proof. Let Γ be a given graph with outgoing edge E. Recall the graph with two incoming univalent vertices discussed in section 5 above. It is pictured in Figure 10. Glue this graph to Γ at E to get a graph we'll call Γ'. By Proposition 42 $q_{\Gamma'}$ is the composition of q_Γ with the Poincare duality isomorphism. Contract the internal glued edge to a point. By Proposition 41 this does not change the invariant. □

One can use the contractible graph with one incoming vertex and one outgoing vertex to get a chain homotopy between the Morse complexes of different Morse functions. By gluing this graph onto the external edges of any other graph using Proposition 42, one sees that the cohomology operations do not depend on the choice of external Morse function. This also follows from the definition of the invariants in Section 3.

10. Appendix: Proof of theorem 14. In this section we give a proof of theorem 14. Let M be a closed n-dimensional manifold with a fixed Riemannian metric. We begin by describing an extension of the bundle $\bigoplus_b TM \to M$ to an $Aut(\Gamma)$-equivariant bundle over $\tilde{\mathcal{M}}_{tree}(\Gamma, M) \cong \mathcal{S}_\Gamma(M) \times M$.

Let $(\sigma,\gamma) \in \tilde{\mathcal{M}}_{tree}(\Gamma, M)$. Let $T \subset \Gamma$ be a fixed maximal subtree, and let $T_1 \subset \Gamma_k$ be the inverse image of T under the composite morphism $\phi_k : \Gamma_k \to \Gamma_{k-1} \to \cdots \to \Gamma_0 \to \Gamma$ determined by the structure σ. Write $p_{T_1}(\sigma,\gamma) = ((x_1,y_1),\cdots(x_b,y_b)) \in (M^2)^b$ as in (10). Recall that $x_i = \gamma_{T_1}(s_i^T(\Gamma_k)) \in M$. We define a vector bundle.

$$\zeta \to \tilde{\mathcal{M}}_{tree}(G, M)$$

to have as its fiber over (σ,γ) the sum of the tangent spaces,

$$\zeta_{(\sigma,\gamma)} = \bigoplus_{i=1}^{b} T_{x_i} M$$

It is clear that the bundle ζ is $Aut(\Gamma)$-equivariant. This is because if $g \in Aut(\Gamma)$, the action of g on the element $(\sigma,\gamma) \in \tilde{\mathcal{M}}_{tree}(\Gamma, M)$, is given by $(g\sigma,\gamma)$, where the structure $g\sigma$ is determined by the morphism $g\phi_k : \Gamma_k \to \Gamma$ given by the composition of the morphism ϕ_k with the automorphism g. $g\phi^{-1}(T)$ is the inverse image under ϕ_k of the tree $gT \subset \Gamma$.

Now let $\epsilon > 0$ be chosen so that for every point x, if $B_\epsilon(T_xM)$ is the ball centered at the origin of radius ϵ, then the exponential map,

$$exp : B_\epsilon(T_xM) \to M$$

is a diffeomorphism onto its image. Let $U_\epsilon(x)$ be this image. Consider a point $(\sigma, \gamma) \in \eta_\epsilon$. Notice that each $y_i \in U_\epsilon(x_i)$. Thus there is a unique $u_i \in B_\epsilon(T_{x_i}M)$ with $exp(u_i) = y_i$. The assignment $(\sigma, \gamma) \to (u_1, \cdots, u_k)$ defines an $Aut(\Gamma)$-equivariant section

$$\theta : \eta_\epsilon \to \zeta$$

of the restriction $\zeta_{|_{\eta_\epsilon}} \to \eta_\epsilon$. For each i, the curve $t \to exp(tu_i)$ in M is a path from x_i at $t = 0$, to y_i at $t = 1$. This is a gradient flow line of the distance function $d_{x_i} : M \to \mathbb{R}$, defined to be the distance from x_i,

$$d_{x_i}(x) = d(x_i, x).$$

This allows us to construct a morphism $\psi_k : \Gamma_{k+1} \to \Gamma_k$ in $\mathcal{C}_\Gamma$, as follows. Let Γ_{k+1} be the graph obtained from Γ_k by replacing each vertex $s_i^T(\Gamma_k)$ with an edge of length 1. The morphism ψ_k collapses each of these edges to a point. If we label these new edges by the functions $d_{x_1}, \cdots, d_{x_k}$, we have now defined a new structure σ'. Notice that the element $(\sigma', p(\sigma, \gamma)) \in \mathcal{S}_\Gamma \times M \cong \tilde{\mathcal{M}}_{tree}(Aut(\Gamma), M)$ lies in the image of $\tilde{\mathcal{M}}_\Gamma(M) \hookrightarrow \tilde{\mathcal{M}}_{tree}(Aut(\Gamma), M)$. This is because the coordinate y_i assigned to the pair $(\sigma', p(\sigma, \gamma))$ is the same as the y_i coordinate for the pair (σ, γ). But the x_i coordinate assigned to the pair $(\sigma', p(\sigma, \gamma))$ is equal to $exp(u_i) = y_i$. Thus the projection

$$p_T(\sigma', p(\sigma, \gamma)) \in (M^2)^b$$

lies in the image of $\Delta^b : M^b \subset (M^2)^b$. By the pullback square (11), the pair $(\sigma', p(\sigma, \gamma))$ lies in the image of $\tilde{\mathcal{M}}_\Gamma(M)$. Sending (σ, γ) to $(\sigma', p(\sigma, \gamma))$ defines an map

$$\pi : \eta_\epsilon \to \tilde{\mathcal{M}}_\Gamma(M).$$

We now show that the section $\theta : \eta_\epsilon \to \zeta$ defines an equivariant lifting $\Theta : \eta_\epsilon \to \nu(\iota)$ making the following diagram commute:

(36)
$$\begin{array}{ccc} \eta_\epsilon & \xrightarrow{\Theta} & \nu(\iota) \\ {=}\downarrow & & \downarrow \\ \eta_\epsilon & \xrightarrow[\pi]{} & \tilde{\mathcal{M}}_\Gamma(M) \end{array}$$

The lifting Θ is defined as follows:

Consider the unique geodesic path in the tree T_1 from $s_i^T(\Gamma_k)$ to the fixed vertex $v \in T_1$. Then its image under the tree flow γ_{T_1} is a parameterized curve from x_i to $\gamma_{T_1}(v) = p(\sigma, \gamma)$ in M. (Recall that $p : \tilde{\mathcal{M}}_\Gamma(M) \to M$ maps (σ, γ) to $\gamma(v)$.) Using the Levi-Civita connection, we define $w_i \in T_{p(\sigma,\gamma)}M$ to be the image of $u_i \in T_{x_i}M$ under the parallel transport operator along this path:

$$\tau_{\gamma_{T_1}} : T_{x_i}M) \xrightarrow{\cong} T_{p(\sigma,\gamma)}M.$$

This construction defines an $Aut(\Gamma)$-equivariant map

(37)
$$\Theta : \eta_\epsilon \to p^*(\bigoplus_b TM) = \nu(\rho)$$
$$(\sigma, \gamma) \to (w_1, \cdots, w_b) \in \bigoplus_b T_{p(\sigma,\gamma)}M$$

making the diagram (36) commute.

We claim that Θ is a homeomorphism. One can see this be directly constructing an inverse map

$$\Theta^{-1} : \nu(\rho) \to \eta_\epsilon.$$

This is constructed as follows. Given $(u_1, \cdots, u_b) \in \bigoplus_b T_{p(\sigma,\gamma)}M$ where $(\sigma, \gamma) \in \tilde{\mathcal{M}}_\Gamma(M)$, one can parallel translate along geodesic paths in $T_1(\Gamma)k)$ to obtain the vector $(w_1, \cdots, w_b) \in B_\epsilon(T_{x_1}M \oplus \cdots \oplus T_{x_b}M$. By scaling these vectors, if necessary, one can consider the curves $t \to exp(-tu_i)$ to define a new structure $\sigma^{''}$ so that the point $(\sigma^{''}, p(\sigma, \gamma))$ lives in $\eta_\epsilon \subset \mathcal{S}(\Gamma) \times M \cong \tilde{\mathcal{M}}_{tree}(\Gamma, M)$. Notice that the coordinates $\{(x_i, y_i)\}$ associated to $(\sigma^{''}, p(\sigma, \gamma))$ are the points $(exp(-u_i), x_i)$ where $(x_1, \cdots, x_b) \in M^b$ is $p_T(\sigma, \gamma)$ as in diagram (11). The assignment $(u_1, \cdots, u_b) \to (\sigma^{''}, p(\sigma, \gamma)$ defines a map $\nu(\rho) \to \eta_\epsilon$ which is easily checked to be inverse to Θ.

Thus $\Theta : \eta_\epsilon \to \nu(\rho)$ is an equivariant homeomorphism, and so induces a homeomorphism on orbit spaces. This completes the proof of theorem (14).

11. Appendix: Regularity.

LEMMA 44. *On the interior of any edge $E \subset \Gamma_k$, an element $r \in$ coker D_{Γ_k} is smooth and satisfies*

$$\dot{r}_E - (\nabla\nabla f_E)^T \cdot r_E = 0. \tag{38}$$

Proof. Take an open interval $I = (t_1, t_2) \subset E$ and choose $\phi \in C_0^\infty(I)$. Trivialise $\mathcal{V} = \gamma^* TM$ over I and put $\nabla\nabla f_E = A(t)$ with respect to this trivialisation. Then $\langle r, \dot{\phi} + A\phi \rangle_2 = 0$. Now $\phi(t) = \int_{t_0}^t \dot{\phi}(\tau) d\tau$ so

$$\int_I \langle r(t), \dot{\phi}(t) \rangle dt + \int_I \langle A^T(t) r(t), \int_{t_0}^t \dot{\phi}(\tau) d\tau \rangle dt = 0$$

and by Fubini's theorem

$$\int_I \langle r(\tau), \dot{\phi}(\tau) \rangle d\tau + \int_I \int_\tau^{t_1} \langle A^T(t) r(t), \dot{\phi}(\tau) \rangle dt d\tau = 0.$$

Thus

$$\int_I \langle r(\tau) - \int_{t_1}^\tau A^T(t) r(t) dt, \dot{\phi}(\tau) \rangle d\tau = 0 \text{ for all } \phi \in C_0^\infty(I) \ .$$

Since $\dot{\phi}$ has mean zero and the set of such functions is dense in $L^2(I)$ we have

$$r(\tau) - \int_{t_1}^\tau A^T(t) r(t) dt = \text{constant} \ .$$

This integral equation supplies us with information about the behaviour of r in I. For a start it says that r is absolutely continuous with derivative equal to the integrand almost everywhere. At points of continuity of A the derivative of r is equal to the integrand. Furthermore, regularity of A gives regularity of r. This can be seen as follows. At a point τ_0 of continuity of A

$$\left| \frac{1}{2\delta} \int_{\tau_0 - \delta}^{\tau_0 + \delta} A^T(t) r(t) dt - A^T(\tau_0) r(\tau_0) \right| \leq \epsilon M$$

where $\epsilon = \sup_{(\tau_0-\delta,\tau_0+\delta)}\{|A(t)|,|r(t)|\}$ tends to zero as δ tends to zero since $A(t)$ and $r(t)$ are continuous at τ_0. This shows that the derivative of r exists there and

$$\dot{r}(\tau_0) = A^T(\tau_0)r(\tau_0) \tag{39}$$

If A is differentiable in a neighbourhood of τ_0 then by (39)

$$\ddot{r}(\tau) = (\dot{A}^T(\tau) + A^T(\tau)^2)r(\tau)$$

in that neighbourhood, and so on. Thus r_E satisfies (38). □

12. Appendix: The Fredholm operator. Let D_{Γ_k} be the linearisation of the graph flow equation along the graph flow $\gamma : \Gamma_k \to M$ of the compact metric graph Γ_k so $D_{\Gamma_k}s = \dot{s}_E + \nabla\nabla f_E \cdot s_E$ for s a section of $\mathcal{V} = \gamma^*TM$.

PROPOSITION 45. *$D_{\Gamma_k} : W^{1,2}(\Gamma_k, \mathcal{V}) \to L^2(\Gamma_k, \mathcal{V})$ is Fredholm.*

Proof. Put $D_{\Gamma_k}s = \dot{s}_E + \nabla\nabla f_E \cdot s_E = \dot{s}_E(t) + A(t)s_E(t)$.

$$\int_{\Gamma_k} |\dot{s} + As|^2 dt = \int_{\Gamma_k} \left(\frac{1}{2}|\dot{s} + 2As|^2 + \frac{1}{2}|\dot{s}|^2 - |As|^2\right) dt \geq \int_{\Gamma_k} \left(\frac{1}{2}|\dot{s}|^2 - |As|^2\right) dt\ .$$

Thus using $|A(t) \cdot s(t)| \leq \|A(t)\| \cdot |s(t)|$ and setting $c_A = \max_{\Gamma_k}\|A(t)\|$, we have

$$\int_{\Gamma_k} |\dot{s} + As|^2 dt \geq \frac{1}{2}\int_{\Gamma} |\dot{s}|^2 - c_A \int_{\Gamma_k} |s|^2 dt\ .$$

Hence there is a $c > 0$ satisfying

$$\int_{\Gamma_k} (|s|^2 + |\dot{s}|^2) dt \leq c \int_{\Gamma_k} (|s|^2 + |\dot{s} + As|^2) dt\ .$$

In other words,

$$\|s\|_{W^{1,2}(\Gamma_k)} \leq c(\|s\|_{L^2(\Gamma_k)} + \|D_{\Gamma_k}s\|_{L^2(\Gamma_k)}). \tag{40}$$

It is a rather standard consequence of (41) that D_{Γ_k} is semi-Fredholm, meaning that it has finite-dimensional kernel and closed range (see [17] for example). This can be seen as follows.

The operator

$$K : W^{1,2}(\Gamma_k) \overset{\text{cpt.}}{\hookrightarrow} L^2(\Gamma_k)$$

is compact by Rellich's lemma. Thus the image under K of any bounded sequence in the kernel of D_{Γ_k} has a convergent subsequence which is necessarily Cauchy. The inequality (41) then implies that the subsequence is Cauchy in $W^{1,2}(\Gamma_k)$. Thus the unit ball in the kernel of D_{Γ_k} is compact, showing that the kernel is finite-dimensional.

To show that the image is closed, consider a bounded sequence $\{s_i\} \subset W^{1,2}(\Gamma_k)$ such that $\{D_{\Gamma_k}s_i\}$ is Cauchy in $L^2(\Gamma_k)$. Choose a subsequence $\{s_{i_j}\}$ such that $\{Ks_{i_j}\}$ is Cauchy in $L^2(\Gamma_k)$. It follows from (41) that $\{s_{i_j}\}$ is Cauchy thus converging to s, say. Since D_{Γ_k} is continuous, $\{D_{\Gamma_k}s_i\}$ converges to $D_{\Gamma_k}s$. In fact, the sequence $\{s_i\}$ can be arranged to be bounded as follows. By the Hahn-Banach theorem there exists a closed subspace $U \subset W^{1,2}(\Gamma_k)$ satisfying

$$\ker D_{\Gamma_k} \oplus U = W^{1,2}(\Gamma_k)\ .$$

Project $\{s_i\}$ onto $\{\tilde{s}_i\} \subset U$. This has to be bounded since otherwise a subsequence of $\{\tilde{s}_i/\|\tilde{s}_i\|\}$ converges to $s \in U$ with $\|s\| = 1$ and $D_{\Gamma_k} s = 0$ in contradiction to the construction of U. Thus D_{Γ_k} has closed range.

To complete the proof of the proposition we must show that coker D_{Γ_k} is finite-dimensional. In Lemma 25 it was shown that elements r of the cokernel satisfy the differential equation $D^*_{\Gamma_k} r = 0$ which is much like the equation $D_{\Gamma_k} s = 0$, the only difference being that r need not be continuous at the vertices of $\tilde{\Gamma}_k$. Nevertheless, as for $\ker D_{\Gamma_k}$ the unit ball in the kernel of $D^*_{\Gamma_k}$ is compact and the dimension of coker D_{Γ_k} is finite. Hence D_{Γ_k} is Fredholm. □

Remark. Since elements of the kernel and cokernel are smooth an alternative proof of finite-dimensionality follows from uniqueness of solutions to ODEs. Still, to prove Fredholmness one must show that the image of D_{Γ_k} is closed and there is no smoothness here to work with. This is why we used standard Banach space arguments rather than the more intuitive uniqueness of solutions to ODEs.

To prove that $D_{\tilde{\Gamma}_k}$ is Fredholm for non-compact $\tilde{\Gamma}_k$ requires a further argument.

Proposition 46. *$D_{\tilde{\Gamma}_k} : W^{1,2}(\tilde{\Gamma}_k, \mathcal{V}) \to L^2(\tilde{\Gamma}_k, \mathcal{V})$ is Fredholm.*

Proof. For any $T > 0$, construct the compact graph Γ_k^T lying between $\Gamma_k \subset \Gamma_k^T \subset \tilde{\Gamma}_k$ by cutting $\tilde{\Gamma}_k$ off at the parameter T on outgoing edges and $-T$ on incoming edges. The proof uses the following lemma.

Lemma 47. *For large enough T, there exists $c = c(T)$ such that*

$$\|s\|_{W^{1,2}(\tilde{\Gamma}_k)} \leq c(\|s\|_{L^2(\Gamma_k^T)} + \|D_{\tilde{\Gamma}_k} s\|_{L^2(\tilde{\Gamma}_k)}). \tag{41}$$

Proof. Put

$$D_{\tilde{\Gamma}_k} = \frac{d}{dt} + \nabla\nabla f_E = \frac{d}{dt} + A(t)$$

with respect to a trivialisation of $\mathcal{V}$ over $\tilde{\Gamma}_k$ with transition functions ± 1. Since f is Morse, $\lim_{t\to\infty} A(t)$ is invertible.

The estimate of $\|s\|_{W^{1,2}(\tilde{\Gamma}_k)}$ breaks into one part near infinity and a compact part. Near infinity, the graph flow equation is just the usual gradient flow equation so we can use a result whose proof can be found in [17]. Given $A(t)$ with $\lim_{t\to\infty} A(t)$ invertible, there are constants $T > 0$, $c_1(T) > 0$ such that

$$\|s\|_{W^{1,2}(\tilde{\Gamma}_k)} \leq c_1(T)\|\dot{s} + A(t)s\|_2 \quad \text{for all } s \in W^{1,2}(\tilde{\Gamma}_k),\ s_{|}\Gamma_k^T = 0.$$

For the compact part use the previous proposition applied to Γ_k^T. □

To put together the part near infinity and the compact part, define a cut-off function $\beta \in C^\infty(\tilde{\Gamma}_k, [0,1])$ with the properties

$$\beta|_{\Gamma_k^T} = 1, \quad \beta(t) = 0 \text{ for } |t| \geq T+1, \text{ and } \dot{\beta}(t) \neq 0 \text{ for } |t| \in (T, T+1).$$

In the following, put $\|\cdot\|_{L^2(\tilde{\Gamma}_k)} = \|\cdot\|_2$ and $\|\cdot\|_{W^{1,2}(\tilde{\Gamma}_k)} = \|\cdot\|_{1,2}$ and only specify the compact graph in the norm. Also choose c_4 and c large enough.

$$\begin{aligned}
\|s\|_{1,2} &= \|\beta s + (1-\beta)s\|_{1,2} \leq \|\beta s\|_{1,2} + \|(1-\beta)s\|_{1,2} \\
&\leq c_4(\|\beta s\|_{L^2(\Gamma_k^T)} + \|D_{\tilde{\Gamma}_k}(\beta s)\|_2 + \|D_{\tilde{\Gamma}_k}((1-\beta)s)\|_2) \\
&\leq c_4(\|\beta s\|_{L^2(\Gamma_k^T)} + 2\|\dot{\beta} s\|_2 + \|D_{\tilde{\Gamma}_k} s\|_2) \\
&\leq c(\|s\|_{L^2(\Gamma_k^T)} + \|D_{\tilde{\Gamma}_k} s\|_2)\ .
\end{aligned}$$

and the proof goes through as in the compact case. □

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