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ELIMINATION ALGEBRAS AND INDUCTIVE ARGUMENTS IN RESOLUTION OF SINGULARITIES*

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Dedicated to Professor Heisuke Hironaka on the occasion of his 80th birthday

Abstract. Over fields of characteristic zero, resolution of singularities is achieved by means of an inductive argument, which is sustained on the existence of the so called hypersurfaces of maximal contact. We report here on an alternative approach which replaces hypersurfaces of maximal contact by generic projections. Projections can be defined in arbitrary characteristic, and this approach has led to new invariants when applied to the open problem of resolution of singularities over arbitrary fields. We show here how projections lead to a form of elimination of one variable using invariants that, to some extent, generalize the notion of discriminant.

This exposition draws special attention on this form of elimination, on its motivation, and its use as an alternative approach to inductive arguments in resolution of singularities. Using techniques of projections and elimination one can also recover some well known results. We illustrate this by showing that the Hilbert-Samuel stratum of a d-dimensional non-smooth variety can be described with equations involving at most d variables.

In addition this alternative approach, when applied over fields of characteristic zero, provides a conceptual simplification of the theorem of resolution of singularities as it trivializes the globalization of local invariants.

Key words. Resolution of singularities, Rees algebras.

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Part 1. Introduction.

1. Resolution: Pairs, Rees algebras, Basic Objects, and Elimination.

A *resolution of singularities* of an algebraic variety X is a proper and birational morphism $\pi : X' \to X$, with X' non-singular. Hironaka's Theorem states that such morphism exists in characteristic zero, and that π can be defined as a composition of blow-ups

$$X \leftarrow X_1 \leftarrow \ldots \leftarrow X_r = X' \tag{1}$$

with closed and smooth centers $C \subset X$, $C_1 \subset X_1, \ldots, C_{r-1} \subset X_{r-1}$.

Constructive resolution of singularities is given by an algorithm that defines an explicit procedure to resolve singularities. Roughly speaking, given X, it provides centers C_i for a sequence as above, defining a resolution. This is usually made by defining a totally ordered set $(\Lambda, \geq)$ and, for any X, an upper semi-continuous function $\Gamma : X \to (\Lambda, \geq)$ that stratifies X in smooth strata. Moreover, the stratum corresponding to the maximum value determines the center C of the blow-up. Thus, a sequence as (1) is constructed by means of upper semi-continuous functions $\Gamma_i : X_i \to (\Lambda, \geq)$ for $i = 0, 1, \ldots, r-1$, so that the stratum corresponding to the maximum value of each Γ_i is smooth, and defines the center C_i.

Over fields of characteristic zero, the upper semi-continuous functions from the previous paragraph are constructed making use of an inductive argument. Induction

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on resolution of singularities is based on the notion of hypersurfaces of maximal contact. These smooth hypersurfaces contain the *worst singularities* of a given variety X, a property that is **stable** under monoidal transformations. In this presentation we discuss an alternative approach. Here hypersurfaces of maximal contact will be replaced by transversal projections and the property of stability under monoidal transformations will be replaced by the **stability of transversality** under monoidal transformations. The very formulation of this alternative form of induction will be given in terms of Rees algebras. So we will reformulate the problem of resolution in terms of Rees algebras.

1.1. Hironaka's reformulation of the problem of resolution. Hironaka reformulated the problem of resolution in terms of pairs, basic objects and resolution of basic objects (we follow the notation in [34] and [35]). In the following paragraphs we give some hints about how the constructive resolutions work. Details and precise definitions and statements will be given in forthcoming sections.

Given a smooth scheme V over a field k, a *pair*, (J,b), is a datum where $J \subset \mathcal{O}_V$ is a non-zero sheaf of ideals, and b is a non-negative integer. A pair (J,b) defines a closed subscheme, $\mathrm{Sing}(J,b)$, which consists of all the points of V where the order of J is at least b.

The blow-up along a smooth closed subscheme $Y \subset \mathrm{Sing}\,(J,b)$, $V \leftarrow V_1$, induces a factorization of the form $J\mathcal{O}_{V_1} = \mathcal{I}(H)^b J_1$, where $\mathcal{I}(H)$ denotes the ideal of definition of the exceptional divisor. The new pair (J_1,b) is said to be *the transform of* (J,b).

A *resolution of a pair* (J,b) is a finite sequence of blow-ups

$$\begin{array}{ccccccc} (J,b)=(J_0,b) & & (J_1,b) & & \dots & & (J_r,b) \\ V=V_0 & \leftarrow & V_1 & \leftarrow & \dots & \leftarrow & V_r \end{array} \tag{2}$$

with closed and smooth centers $Y_i \subset \mathrm{Sing}\,(J_i,b)$ for $i=0,1,\dots,r-1$, so that:

- $\mathrm{Sing}\,(J_r,b)=\emptyset$;
- The exceptional locus of the composite morphism is a union of hypersurfaces, say $H_1,\dots,H_r$, with normal crossings in V_r.

In order to define a resolution, for a sequence of transformation as above, even if $\mathrm{Sing}\,(J_r,b)$ is not empty, we will see that for each $i=1,\dots,r$, there is a well defined factorization of the form $J_i = \mathcal{I}(H_1)^{a_1}\cdots\mathcal{I}(H_i)^{a_i}\overline{J}_i$ for suitable non-negative integers $a_1,\dots,a_i$. The pair (J_i,b) is said to be *within the monomial case* if $\overline{J}_i = \mathcal{O}_{V_i}$, or equivalently, if $J_i = \mathcal{I}(H_1)^{a_1}\cdots\mathcal{I}(H_i)^{a_i}$. In the last case, it is simple to extend the first i steps in (2) to a resolution. In fact, a resolution of a pair is achieved in two steps: first defining a sequence of transformations so as to reach the monomial case, and then extending the sequence to a resolution of the monomial case.

The normal crossings condition on the exceptional hypersurfaces $H_1,\dots,H_r$ leads to the consideration of pairs (J,b) together with *couples* (V,E) with $E=\{H_1,\dots,H_s\}$ a set of smooth hypersurfaces with normal crossing (i.e., their union has only normal crossings in V). All these data are encoded in terms of a *basic object*, $(V,(J,b),E)$. If V is a d-dimensional smooth scheme, then $(V,(J,b),E)$ is said to be a *d-dimensional basic object.*

A smooth closed subscheme $Y \subset V$ is *permissible for* $(V,(J,b),E)$ if $Y \subset \mathrm{Sing}\,(J,b)$ and has normal crossings with E (i.e., with the hypersurfaces of E). If $Y \subset V$ is permissible, then the blow-up at Y, $V \xleftarrow{\pi} V_1$, induces a transformation of the original basic object, $(V_1,(J_1,b),E_1)$, where (J_1,b) is the transform of the pair

(J, b) and $E_1 = E \cup \{\pi^{-1}(Y)\}$. Here the hypersurfaces of E are identified with their strict transform in V_1.

Hironaka reformulates the problem of resolution of singularities as that of *resolution of basic objects*: a resolution of a basic object $(V, (J, b), E)$ is a finite sequence of blow-ups at permissible centers

$$(V, (J, b), E) \leftarrow (V_1, (J_1, b), E_1) \leftarrow \ldots \leftarrow (V_r, (J_r, b), E_r)$$

so that Sing $(J_r, b) = \emptyset$. Hironaka stated that if we knew how to resolve basic objects, then we could desingularize any scheme of finite type over a perfect field.

Fix a hypersurface X in V. By setting $J = \mathcal{I}(X)$ the defining ideal, and letting b be the maximum multiplicity of X, a resolution of $(V, (J, b), \{\emptyset\})$ lowers the maximum multiplicity of X by successive monoidal transformations. For arbitrary algebraic varieties, Hironaka attaches a basic object, say $(V, (J, b), E)$, to the maximum value of the Hilbert-Samuel function. This basic object satisfies two properties: Sing (J, b) is the stratum of maximum value of the function, and finding a resolution of $(V, (J, b), E)$ is equivalent to lowering this maximum value by means of monoidal transformations. On the other hand, he also proves that lowering successively the maximum value of the Hilbert-Samuel function leads to resolution of singularities. This is how desingularization follows from resolution of basic objects.

The basic object $(V, (J, b), E)$ attached to the Hilbert-Samuel function is not unique, and this attachment can be defined only locally. This leads to Hironaka's notion of *weak equivalence* of basic objects: two different basic objects attached to the Hilbert-Samuel function are weakly equivalent (see Section 8). So, strictly speaking, resolution of singularities follows from resolution of basic objects if the latter can be accomplished with some natural compatibility with weak equivalence. This is achieved by the constructive resolution, but only over fields of characteristic zero.

1.2. On constructiveness of Hironaka's resolution. Constructive resolution addresses resolution of basic objects in an explicit manner. Given a d-dimensional basic object

$$(V, (J, b), E)$$

the algorithm of constructive resolution is defined by a string of invariants which are the values of an upper semi-continuous function in a totally ordered set. The key ingredient is a two-coordinates upper semi-continuous function with values on $\mathbb{Q} \times \mathbb{Z}$, ordered lexicographically. This upper semi-continuous function provides:

1. Either a way to choose a canonical permissible center of dimension $(d-1)$;
2. Or a way to reformulate the problem of resolution of $(V, (J, b), E)$ in terms of a $(d-1)$-dimensional basic object $(\overline{V}, (K, e), \overline{E} = \emptyset)$.

Roughly speaking, with this approach, the strategy is either to reduce $(V, (J, b), E)$ to the monomial case, or to use induction and reduce a lower dimensional basic object to the monomial case.

Induction on the dimension is made by restriction to *hypersurfaces of maximal contact*. The existence of these smooth hypersurfaces is only guaranteed when the characteristic is zero.

1.3. Rees algebras and resolution problems. In this paper we report on an alternative form of induction, or say, an alternative way to reduce resolution to a lower dimensional problem, which does not make use of hypersurfaces of maximal contact. This new procedure also reduces the lower dimensional problem to the monomial case, even in positive characteristic. This is done using Rees algebras and their properties. In characteristic zero, the one advantage of this approach is that the *local-global problem in resolution* can be considerably simplified (see [8] and [18]). In positive characteristic, the outcome is weaker than the one required for resolution of singularities, but it has opened the way to new invariants for singularities over perfect fields (see [9], [5], [6]).

A first step for this alternative approach to induction is the reformulation of resolution problems in terms of Rees algebras.

Given a smooth scheme V over a perfect field k, a *sheaf of Rees algebras over V* is a graded sheaf of rings $\mathcal{G} = \oplus_{n\geq 0} I_n W^n$ where for each $n \in \mathbb{N}$, $I_n \subset \mathcal{O}$ is a sheaf of ideals , $I_0 = \mathcal{O}_V$, and $I_k \cdot I_l \subset I_{k+l}$. We also assume that, locally, $\mathcal{G}$ is a finitely generated sheaf of $\mathcal{O}_V$-algebras. The *singular locus of a Rees algebra* is the closed set

$$\text{Sing}\ \mathcal{G} := \{x \in V : \nu_x(I_n) \geq n, \ \text{for all } n \geq 0\}, \tag{3}$$

where ν_x denotes the usual order function in the local regular ring $\mathcal{O}_{V,x}$.

A pair (J, b) defines and can be replaced by a Rees algebra $\mathcal{G}$. To this end, consider the Rees algebra generated in degree b by J. Then the resolution of $(V, (J, b), E)$ can be formulated in terms of a resolution of $(W, \mathcal{G}, E)$.

The analogy of pairs extends to transformations and resolutions. The blow-up at a smooth closed subscheme $Y \subset \text{Sing}\ \mathcal{G}$, $V \leftarrow V_1$, induces a *transform of* $\mathcal{G}$,

$$\mathcal{G}_1 = \oplus_{n\geq 0} \mathcal{I}(H)^{-n} I_n \mathcal{O}_{V_1} W^n$$

where H denotes the exceptional divisor. A smooth closed subscheme $Y \subset V$ is *permissible for* $(V, \mathcal{G}, E)$ if $Y \subset \text{Sing}\ \mathcal{G}$ and has normal crossings with E. A *resolution of a basic object* $(V, \mathcal{G}, E)$ is a finite sequence of blow-ups at permissible centers,

$$(V, \mathcal{G}, E) \leftarrow (V_1, \mathcal{G}_1, E_1) \leftarrow \ldots \leftarrow (V_r, \mathcal{G}_r, E_r),$$

such that $\text{Sing}\ \mathcal{G}_r = \emptyset$.

In the same manner as before, when k is a field of characteristic zero, constructive resolution is essentially addressed making use of both, a two-coordinate upper semicontinuous function, and induction.

However Rees algebras can be **naturally** enriched via the use of differential operators, a fact that has some advantages over the use of pairs:

(i) For instance the basic object $(V, (J, b), E)$ gives rise to the Rees algebra $\mathcal{O}_V[JW^b]$, where W is a dummy variable that helps us keeping track of the grading. This Rees algebra can be enlarged to a new algebra by using the action of differential operators, say $\mathbb{D}\text{iff}\mathcal{G} = \mathcal{O}_V \oplus I_1 W \oplus I_2 W^2 \oplus \ldots$. Then it can be shown that for a point $x \in \text{Sing}(J, b)$, a hypersurface H containing x is of maximal contact if and only if $\mathcal{I}(H_1) \subset I_1$ locally in a neighborhood of x.

(ii) As we will see, differential operators play a role in elimination (in arbitrary characteristic). And elimination opens the way to a new form of induction which does not make use of hypersurfaces of maximal contact. This new form of induction is approached in the context of Rees algebras, and opens some hopes (and new invariants) for the problem of resolution over perfect fields. The starting point in this approach is the notion of elimination algebra.

1.4. Elimination algebras. To fix ideas, suppose we are given a finitely generated smooth k-algebra, S, and a monic polynomial $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in S[Z]$. We are interested in the closed set of n-fold points of the hypersurface $\{f(Z) = 0\}$ in Spec $(S[Z])$, say Υ_n. Observe that Υ_n is the singular locus of the Rees algebra generated by $f(Z)$ over $S[Z]$ in degree n. Set

$$\mathcal{G} = S[Z][f(Z)W^n]$$

as graded subalgebra in $S[Z][W]$. Let $B = S[Z]/\langle f(Z)\rangle$ and consider the natural morphisms β : Spec $(S[Z]) \to$ Spec (S) and $\overline{\beta}$: Spec $(B) \to$ Spec (S). Zariski's Multiplicity formula for projections asserts that Υ_n is mapped bijectively to its image via $\overline{\beta}(\Upsilon)$ (cf. [41]). In Section 3 we will see how to construct a Rees algebra on S, hence independent of the variable Z whose singular locus contains $\overline{\beta}(\Upsilon_n)$. We will refer to it as an *elimination algebra associated to* $\mathcal{G}$, (denoted by $\mathcal{R}_{\mathcal{G}}$), and it will be described as certain polynomial expressions in terms of the coefficients of $f(Z)$. As indicated, in general

$$\beta(\Upsilon_n) \subset \text{Sing } \mathcal{R}_{\mathcal{G}}, \tag{4}$$

which is an equality if the characteristic is zero or coprime with n (see [36, Theorem 1.16] or Theorem 3.4 below). In order to get an equality in (4) with full generality we need to consider a Rees algebra larger than $S[Z][f(Z)W^n]$ (actually, saturating it by using differential operators will do the work). Details regarding this matter will be given in forthcoming sections, but the ideas exposed in the previous lines already indicate the general strategy: assign to a given basic object

$$(V = \text{Spec}(S[Z]), \mathcal{G} = S[Z][f(Z)W^n], E = \{\emptyset\})$$

another in lower dimension, namely $(\text{Spec}(S), \mathcal{R}_{\mathcal{G}}, E = \{\emptyset\})$.

In the next section, Section 2, we give a more detailed explanation about the way induction on the dimension works (in characteristic zero) using the language of pairs. Here *hypersurfaces of maximal contact* play a central role. In Section 3 we explain the use of smooth general projections as an alternative to the use of hypersurfaces of maximal contact, which is valid in arbitrary characteristic. Smooth general projections are very easy to construct. A look at the first lines of the Appendix will clarify this point.

The remainder of the paper is divided in three parts. Part 2 is devoted to presenting Rees algebras from scratch. The main invariants associated to a Rees algebra are the order at a point and the τ-invariant. These will be treated, respectively, in Sections 4 and 7. In Section 6 we will formulate resolution in terms of Rees algebras, and in Section 8 the notion of weak equivalence will be discussed (this parallels Hironaka's notion of weak equivalence for idealistic exponents). Special attention will be drawn on Rees algebras that are closed by the action of differential operators, and these are studied in Section 5.

In Part 3 elimination algebras are introduced. We review their main properties, briefly explain how to construct them, and give a description of their behavior under blow-ups (Section 10). Section 11 is dedicated to presenting some of the results of our research team in resolutions problems following this approach.

To conclude, in Part 4 we state and prove Proposition 11.4, using techniques of elimination: *the maximum Hilbert-Samuel stratum of a d-dimensional (non-smooth, reduced) scheme X can be described in terms of equations involving at most d variables.*

2. On constructive resolution: Main invariants and maximal contact. The main invariant in resolution problems is the order of an ideal in a smooth scheme. Given a non-zero sheaf of ideals on a smooth scheme V over a perfect field k, say $J \subset \mathcal{O}_V$, an upper semi-continuous function can be defined,

$$\begin{array}{rccc} \nu(J): & V & \to & \mathbb{Z} \\ & x & \to & \nu_x(J), \end{array}$$

by assigning to each point the order of the stalk J_x in the local regular ring $\mathcal{O}_{V,x}$ (which we denote by $\nu_x(J)$). Recall that the order of J at $\mathcal{O}_{V,x}$ is the highest integer n with $J \subset m_x^n$, where m_x denotes the maximal ideal of $\mathcal{O}_{V,x}$.

It is not hard to see that, in general, the order by itself is not sharp enough to provide a good stratification in regular subvarieties on an arbitrary variety (consider for instance $z^2 - (x^3 - y^2)^2 = 0$ in the affine space; the highest value of the previous order-function is achieved along a singular curve). It is at this point where the use of differential operators and induction on the dimension of the ambient space come into play as useful tools to sharpening the order function.

2.1. Order and differential operators. Let $x \in V$ be a closed point, fix a a regular system of parameters $\{x_1, \dots, x_d\}$ in $\mathcal{O}_{V,x}$, and consider the completion $\hat{\mathcal{O}}_{V,x} = k'[[x_1, \dots, x_d]]$, where k' denotes the residue field at the point. The order of an element $f \in \mathcal{O}_{V,x}$ can be computed by looking at its series expansion in $k'[[x_1, \dots, x_d]]$, which in turn is connected to the action of differential operators on f. To be more precise, consider the morphism:

$$\begin{array}{rccc} Tay: & k'[[x_1, \dots, x_d]] & \longrightarrow & k'[[x_1, \dots, x_d, T_1, \dots, T_d]] \\ & x_i & \to & x_i + T_i. \end{array} \tag{5}$$

For $f \in k'[[x_1, \dots, x_d]]$, $Tay(f(x)) = \sum_{\alpha \in \mathbb{N}^d} g_\alpha T^\alpha$, with $g_\alpha \in k'[[x_1, \dots, x_d]]$. For each $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ define

$$\begin{array}{rccc} \Delta^\alpha: & k'[[x_1, \dots, x_d]] & \to & k'[[x_1, \dots, x_d]] \\ & f & \to & \Delta^\alpha(f) = g_\alpha. \end{array} \tag{6}$$

It turns out that for $f \in \mathcal{O}_{V,x} \subset \hat{\mathcal{O}}_{V,x}$, $\Delta^\alpha(f) \in \mathcal{O}_{V,x}$, i.e.,

$$\Delta^\alpha_{|\mathcal{O}_{V,x}} : \mathcal{O}_{V,x} \to \mathcal{O}_{V,x}$$

and that $Diff_k^{(r)}(V)$, the sheaf of $k-$differential operators of order r, is generated by $\{\Delta^\alpha : \alpha \in \mathbb{N}^d, 0 \leq |\alpha| = \alpha_1 + \dots + \alpha_d \leq r\}$ locally in a neighborhood of x. It is here where we require k to be perfect.

Given a sheaf of ideals, J, and a positive integer r, define:

$$Diff_k^{(r)}(J) = \langle \Delta^\alpha(f); f \in J; 0 \leq |\alpha| \leq r-1 \rangle = \langle D(f); f \in J; D \in Diff_k^{(r)}(V) \rangle,$$

which contains J. It is not hard to check that

$$V(Diff_k^{(b-1)}(J)) = \{x \in V\ ; \nu_x(J) \geq b\},$$

is the set of points where J has order at least b, and that

$$\begin{aligned} &\nu_x(Diff_k^{(1)}(J)) \geq \nu_x(J) - 1\ ; \nu_x(Diff_k^{(2)}(J)) \\ &\qquad \geq \nu_x(J) - 2; \dots \nu_x(Diff_k^{(\nu_x(J)-1)}(J)) \geq 1. \end{aligned} \tag{7}$$

In addition, if the characteristic of the base field is zero then

$$\begin{aligned}\nu_x(Diff_k^{(1)}(J)) = \nu_x(J) - 1;\ \nu_x(Diff_k^{(2)}(J))& \\ = \nu_x(J) - 2;\ \dots;\ \nu_x(Diff_k^{(\nu_x(J)-1)}(J)) = 1.&\end{aligned} \tag{8}$$

This means, that, if the characteristic is zero, then locally, in a neighborhood of x, the ideal $Diff_k^{(\nu_x(J)-1)}(J)$ contains an element of order one at x. So the points of highest order are locally contained in smooth hypersurfaces, classically referred to as *hypersurfaces of maximal contact*. These hypersurfaces contain the points of maximum order of an ideal, and they have the property that this containment is preserved by blow-ups at suitably defined smooth centers until the maximum order of the ideal drops. This is the starting point for induction when the characteristic is zero.

EXAMPLE 2.2. Consider the following surface in the affine space Spec $(k[x, y, z])$,

$$X := \{z^2 - (x^3 - y^2)^2 = 0\}.$$

The maximum order of the ideal $\langle z^2 - (x^3 - y^2)^2\rangle$ is 2, and the closed set of points of maximum order, $C := \{x^3 - y^2 = 0, z = 0\}$, is contained in the smooth surface $\{z = 0\}$. Now consider the ideal $\langle (x^3 - y^2)^2\rangle$ in the affine plane, and identify C with its restriction to $z = 0$. Observe that if the characteristic is different from 2, then $z \in \langle Diff_k^{(1)}(z^2 - (x^3 - y^2)^2)\rangle$. Thus, $\langle (x^3 - y^2)^2\rangle$ has order 2 in $C \setminus (0, 0)$, and its maximum order is 4, which is attained at $(0, 0)$. The combination of all this information already defines a smooth stratifying function for X,

$$\Gamma : X \to (\{\mathbb{N} \cup \infty\} \times \{\mathbb{N} \cup \infty\}, \leq_{\text{lex}}),$$

where

$$\Gamma(x) = \begin{cases} (1, \infty) & \text{if} \quad x \in X \setminus C \\ (2, 2) & \text{if} \quad x \in C \setminus (0, 0, 0) \\ (2, 4) & \text{if} \quad x = (0, 0, 0). \end{cases}$$

The objective now is to define a sequence of monoidal transformations along the two-fold points of the hypersurface and of its successive strict transforms, so as to eliminate all points of multiplicity two. In other words, our goal is to find a resolution of the basic object (Spec $(k[x, y, z]), (\langle z^2 - (x^3 - y^2)^2\rangle, 2), E = \emptyset)$. This problem can be, somehow, reformulated within the hypersurface $\overline{V} := \{z = 0\}$, where we consider the ideal $\langle (x^3 - y^2)^2\rangle$, and the basic object (Spec $(k[x, y]), (\langle (x^3 - y^2)^2\rangle, 2), \overline{E} = \emptyset)$. Taking into account the law of transformation of pairs and basic objects, there is strong link between the resolutions of the two basic objects:

> CLAIM. *In order to find a sequence of blow-ups to eliminate the points of order two of $\langle z^2 - (x^3 - y^2)^2\rangle$, it is enough to find a sequence of blow-ups to lowering the order of the ideal $\langle (x^3 - y^2)^2\rangle \subset k[x, y]$ below 2. The surface $\{z = 0\}$ and its strict transforms contain the points of order 2 of the strict transforms of $z^2 - (x^3 - y^2)^2$ until the order drops below 2.*

Equivalently, finding a resolution of the three-dimensional basic object

$$(Spec\ (k[x,y,z]), (\langle z^2 - (x^3 - y^2)^2 \rangle, 2), E = \emptyset)$$

is equivalent to finding a resolution of the two-dimensional basic object

$$(Spec\ (k[x,y]), (\langle (x^3 - y^2)^2 \rangle, 2), \overline{E} = \emptyset).$$

Thus, $\{z = 0\}$ is a hypersurface of maximal contact. The ideal $\langle (x^3 - y^2)^2 \rangle$ is usually referred to as the *coefficient ideal*.

So induction comes into play once more, since we have translated our initial problem that involves three variables, into another expressible in one variable less: the law of transformations of pairs and basic objects is so that finding a resolution of $(\text{Spec } (k[x,y,z]), (\langle z^2 - (x^3 - y^2)^2 \rangle, 2), E = \{\emptyset\})$ is equivalent to finding a resolution of the 2-dimensional basic object $(\text{Spec } (k[x,y]), (\langle (x^3 - y^2)^2 \rangle, 2), \overline{E} = \{\emptyset\})$, since there are commutative diagrams of permissible monoidal transformations and restrictions:

$$\begin{array}{ccccccc}
\begin{array}{c}(\text{Spec } (k[x,y,z]), E) \\ (\langle z^2 - (x^3 - y^2)^2 \rangle, 2)\end{array} & \leftarrow & \begin{array}{c}(V_1^{(3)}, E_1 = \{H_1\}) \\ (\langle z_1 - x_1^2(x_1 - y_1^2)^2 \rangle, 2)\end{array} & \leftarrow & \begin{array}{c}(V_2^{(3)}, E_2 = \{H_1, H_2)\}) \\ (\langle z_2 - x_2^2 y_2^2 (x_2 - y_2)^2 \rangle, 2)\end{array} & \leftarrow & \begin{array}{c}V_3^{(3)} \\ \dots\end{array} \\
\cup & & \cup & & \cup & & \cup \\
\begin{array}{c}(\text{Spec } (k[x,y]), \overline{E}) \\ (\langle (x^3 - y^2)^2 \rangle, 2)\end{array} & \leftarrow & \begin{array}{c}(V_1^{(2)}, \overline{E}_1 = \{\overline{H}_1\}), \\ (\langle x_1^2(x_1 - y_1^2)^2 \rangle, 2)\end{array} & \leftarrow & \begin{array}{c}(V_2^{(2)}, \overline{E}_2 = \{\overline{H}_1, \overline{H}_2\}) \\ (\langle x_2^2 y_2^2 (x_2 - y_2)^2 \rangle, 2)\end{array} & \leftarrow & \begin{array}{c}V_3^{(2)} \\ \dots\end{array}
\end{array}$$

Here the first monoidal transformation is the blow-up at $(0,0,0)$; $V_1^{(3)}$ denotes the affine chart Spec $k\,[x_1, y_1, z_1]$ (with $x_1 = x, y_1 = \frac{y}{x}$ and $z_1 = \frac{z}{x}$) and H_1 denotes the exceptional divisor. The second blow-up is the monoidal transformation at the origin of $V_1^{(3)}$, with $V_2^{(3)} = \text{Spec}\,[x_2, y_2, z_2]$ (with $x_2 = \frac{x_1}{y_1}, y_2 = y_1$ and $z_2 = \frac{z_1}{y_1}$) and H_2 is the exceptional divisor.

The second row in the diagram corresponds to the transformation of the the two-dimensional object $(\text{Spec } (k[x,y]), \overline{E})$ obtained by restricting to $z = 0$, and considering the pair constructed from the coefficient ideal: $(\langle (x^3 - y^2)^2 \rangle, 2)$. The third blow-up would be the monoidal transformation with center the curve, $z_2 = 0, x_2 - y_2 = 0$. This would lead to a two-dimensional basic object $(V_3^{(2)}, (\overline{J}_3^{(2)}, 2), \overline{E}_3)$ that is within the monomial case. At this point, a resolution follows from a combinatorial argument.

This example already illustrates the general strategy for constructive resolution in characteristic zero: force the basic object constructed by induction (in a lower dimensional space) to fall within the monomial case. This procedure is referred to as a *simplification of singularities* or a *reduction to the monomial case*, which we discuss below.

General strategy for resolution of singularities in characteristic zero. To fix ideas, assume that X is a hypersurface contained in some smooth d-dimensional scheme V, and let Υ_n be the closed subset of points of maximum multiplicity, say n, of X (i.e., the points where $\mathcal{I}(X)$ has maximum order n). Associate to this closed set the d-dimensional basic object $(V, (\mathcal{I}(X), n), E = \{\emptyset\})$, so that Sing $(\mathcal{I}(X), n) = \Upsilon_n$. It can be shown that Υ_n is locally contained in some smooth hypersurface of maximal contact $\overline{V}$, and that it is possible to attach to the previous d-dimensional basic object, at least locally, another of dimension one less over $\overline{V}$. The procedure of constructive resolution applied to X proceeds by induction on the dimension, in two steps. The first step consists of the definition of a suitable sequence of blow-ups along smooth centers that leads to a simplification of Υ_n. This is usually referred to as a *reduction to the monomial case*, meaning that, locally, Υ_n can be described in terms of the ideal of a divisor with normal crossings support contained in some smooth lower dimensional scheme. In the second step, the monomial case is treated and a lowering of the maximum multiplicity of X follows from a combinatorial argument. The simplification of Υ_n is obtained by blowing-up the stratum of maximum value of an upper semi-continuous function defined on X and its strict transforms. This stratifying function is defined in terms of the order of certain ideals, obtained via an inductive argument using maximal contact.

Hypersurfaces of maximal contact and positive characteristic. In characteristic zero, hypersurfaces of maximal contact are closely related to Abhyankar's notion of Tschirnhausen transform (see [1], [2], and [3]). In J. Giraud's work, hypersurfaces of maximal contact arise using techniques that involve differential operators on smooth schemes over fields of characteristic zero ([19]).

J. Giraud also introduced methods involving differential operators in his attempt to extend some of the arguments in resolution to the case of positive characteristic (cf. [20]). However, in positive characteristic hypersurfaces of maximal contact may not exist (see for instance [22] and [33]), and hence, the inductive arguments valid in characteristic zero cannot be extended to this context.

3. Elimination vs. maximal contact. Suppose that we are interested in studying the maximum multiplicity locus Υ_n of a hypersurface X embedded in a smooth scheme V of dimension $d+1$. By the Weierstrass Preparation Theorem, it can be assumed that, in a neighborhood of a point $x \in \Upsilon_n$, the defining equation of X is given by a monic polynomial. This requires a generic smooth projection (see Appendix). So, there is no lost of generality if we work in the following setting: let S is a d-dimensional smooth ring of finite type over a perfect field k, consider

$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_n \in S[Z]$$

and denote by Υ_n the set of points in $\{f(Z)=0\}$ with multiplicity n. Note that Υ_n is the singular locus of the basic object $(\mathrm{Spec(S[Z])}, (f(Z), n), E = \emptyset)$.

> Υ_n *is a closed subset in the* $d+1$*-dimensional smooth scheme* *Spec*$(S[Z])$.

Notice that there is a natural smooth projection β, a finite restricted map $\overline{\beta}$, and a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}\,(S[Z]/\langle f(Z)\rangle) & \hookrightarrow & \mathrm{Spec}(S[Z]) \\ \overline{\beta} \searrow & & \beta \downarrow \\ & & \mathrm{Spec}(S). \end{array} \tag{9}$$

Now, recall that $\overline{\beta}$ is *purely ramified over a point* $x \in$ *Spec*(S) if the *geometric fiber* over x consists of a unique point. Since $S[Z]/\langle f(Z)\rangle$ is a free S-module of rank n, Zariski's projection formula for multiplicities ensures that $\overline{\beta}$ is purely ramified over any point in $\beta(\Upsilon_n)$ (see [41, Corollary 1, p. 299]), and as a consequence Υ_n can be identified with its image under β in $\mathrm{Spec}(S)$. In this sense:

There is a bijection between the points in Υ_n and $\beta(\Upsilon_n)$. The closed subset $\beta(\Upsilon_n)$ is contained in Spec(S); in particular it can be described by an ideal in S.

Moreover, we would like to describe it as the singular locus of a basic object on the smooth d-dimensional scheme $\mathrm{Spec}(S)$. This is a first step towards induction.

To fix ideas, consider a degree two polynomial, $f(Z) = Z^2 + a_1 Z + a_2$ with $a_1, a_2 \in S$. In this case the discriminant, $a_1^2 - 4a_2 \in S$ describes the image under $\overline{\beta}$ of the purely ramified locus in $\mathrm{Spec}(S)$. Observe that to describe the image of the two-fold points under β, we have to use a sharper argument, and interpret $a_1^2 - 4a_2 \in S$ as a weighted homogeneous polynomial of degree 2 (provided that we assign weight one to a_1 and weight two to a_2). If the characteristic of S is different from 2, then the image of the two fold points is precisely the closed subset of $\mathrm{Spec}(S)$ where the discriminant has order at least two. If the characteristic is 2, then the image of the two-fold points is exactly the closed subset of $\mathrm{Spec}(S)$ where the discriminant has order at least two, and all the partial derivatives of a_2 have order at least one (see Section 10 for precise statements and a justification of this fact). It is at this point, when we need to make use of *weighted equations*, where the language of Rees algebras comes in handy, as we will see in forthcoming sections. As for this example, the points where the discriminant has order at least two will be expressed as the singular locus of a suitably defined Rees algebra.

Another example is the particular case in which $f(Z)$ factors in $S[Z]$ as a product of linear forms, i.e.,

$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 = (Z - \alpha_1) \cdots (Z - \alpha_n),$$

with $\alpha_1, \ldots, \alpha_n \in S$. Then $\{f(Z) = 0\} = \cup_i \{Z - \alpha_i = 0\}$, $\Upsilon_n = \{\cap_i (Z - \alpha_i)\} = 0$, and

$$\begin{aligned} \beta(\Upsilon_n) &= \{x \in \mathrm{Spec}\ (S) : \alpha_i(x) = \alpha_j(x), \forall i, j\} \\ &= \{x \in \mathrm{Spec}\ (S) : (\alpha_i - \alpha_j)(x) = 0, \forall i, j\}. \end{aligned} \tag{10}$$

Therefore, $\beta(\Upsilon_n)$ is the closed set of points where the $\alpha_i - \alpha_j$ have order at least one at the local ring of a point in Spec (S). It is not hard to see that $\alpha_i - \alpha_j$ is invariant by changes of the form $Z \to Z - \alpha$. This closed set has also a natural interpretation as the singular locus of a Rees algebra over S, namely of $S[(\alpha_i - \alpha_j)W]_{i<j}$, viewed as a graded subalgebra of $S[W]$, whose singular locus coincides with $\beta(\Upsilon_n)$.

In the general case, the problem of finding equations that describe the image in $\mathrm{Spec}(S)$ of Υ_n, in terms of the coefficients of $f(Z)$, and for an arbitrary n, is first treated in the universal case.

3.1. The universal elimination algebra. [36, Section 1] Consider the polynomial ring in n variables $k[Y_1, \ldots, Y_n]$, with coefficients in a ring k, and the *universal polynomial* of degree n,

$$F_n(Z) = (Z - Y_1) \cdots (Z - Y_n) = Z^n - s_{n,1} Z^{n-1} + \ldots + (-1)^n s_{n,n} \in k[Y_1, \ldots, Y_n, Z],$$

where for $i = 1, \ldots, n$, $s_{n,i} \in k[Y_1, \ldots, Y_n, Z]$ denotes the i-th symmetric polynomial in n variables.

The diagram

$$\begin{array}{ccc} \mathrm{Spec}\ (k[s_{n,1}, \ldots, s_{n,n}][Z]/\langle F_n(Z)\rangle) & \hookrightarrow & \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}][Z]) \\ \overline{\alpha} \searrow & & \alpha \downarrow \\ & & \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}]) \end{array} \tag{11}$$

illustrates the universal situation, and (9) is a specialization of this case where:

$$
\begin{array}{rrcl} \Theta: & k[s_{n,1},\dots,s_{n,n}] & \longrightarrow & S \\ & (-1)^i s_{n,i} & \to & a_i. \end{array} \tag{12}
$$

Our motivation for this discussion is to find equations in the coefficients of F_n that describe the image of the n-fold points of $F_n = 0$. By Zariski's multiplicity formula for finite projections, we start by looking for equations in the coefficients that describe the purely ramified locus of the morphism.

Since we look for equations in the coefficients of F_n, we will be considering elements in

$$k[Y_1,\dots,Y_n]^{S_n} = k[s_{n,1},\dots,s_{n,n}].$$

Note that $k[s_{n,1},\dots,s_{n,n}] \subset k[Y_1,\dots,Y_n]$ is an inclusion of graded rings since the action of S_n in $k[Y_1,\dots,Y_n]$ preserves the grading.

On the other hand, notice that the purely ramified locus does not vary under changes of variable of the form

$$uZ - \alpha, \tag{13}$$

with $\alpha, u \in S$ and u invertible. This change of variable can be seen as the composition of a translation, $Z - \beta$ followed by multiplication by a unity. We will start by considering the translations of the variable Z.

It is not hard to see that $k[Y_i - Y_j]_{1\le i,j\le n}$ are functions that are invariant by translations of the variable Z. Since S_n acts linearly on $k[Y_i - Y_j]_{1\le n}$, clearly $k[Y_i - Y_j]^{S_n}_{1\le i,j\le n}$ are functions on the coefficients (a subring of $k[Y_1,\dots,Y_n]^{S_n} = k[s_{n,1},\dots,s_{n,n}]$) that are also invariant under translations of Z. It requires some extra work to prove that indeed $k[Y_i - Y_j]^{S_n}_{1\le i,j\le n}$ are precisely all the polynomials in $k[Y_1,\dots,Y_n]$ that are invariants by both, the action of the permutation group, S_n, and the translations on the variable Z (see [36] for a proof of this fact).

Let us add a dummy variable W, and define $\mathcal{U}$ as the $k[s_{n,1},\dots,s_{n,n}][Z]$-subalgebra of $k[s_{n,1},\dots,s_{n,n}][Z][W]$ generated by F_n in degree n, namely:

$$k[s_{n,1},\dots,s_{n,n}][Z][F_nW^n].$$

Then define the *universal elimination algebra* $\mathcal{R}_{\mathcal{U}}$ associated to $\mathcal{U}$ as

$$k[H_{m_1},\dots,H_{m_r}] := k[Y_i - Y_j]^{S_n}_{1\le i,j\le n}, \tag{14}$$

where each H_{m_i} is a homogeneous polynomial in degree m_i, for $i = 1,\dots,r$. Note that each H_{m_i} is also a weighted homogeneous polynomial in $s_{n,1},\dots,s_{n,n}$ where $s_{n,i}$ is homogeneous of degree i in the variables $Y_1,\dots,Y_n$ for $i = 1,\dots,n$. For instance, in the case of the universal degree two polynomial, the elimination algebra is generated by the discriminant in degree two (i.e., $n = 2$). We will see later in this paper how these graded algebras relate to Rees algebras, the latter being defined as direct sums of ideals (see Theorem 3.4).

REMARK 3.2. Observe that the elements H_{m_i} are invariant under changes of the form $Z - \beta$, and are weighted homogeneous on the s_{n_i}. As a consequence, the images of H_{m_i} in S generate an ideal that is invariant under changes of the form $Z \to uZ$, for an invertible element $u \in S$, and thus invariant under changes of the form $uZ - \alpha$ as in (13).

Recently J. Schicho has implemented a program to calculate the weighted homogeneous polynomials H_{m_i} that generate $k[Y_i - Y_j]^{S_n}$. Differential operators also have an interpretation in terms of invariants.

Remark 3.3. Notice that in the universal case changes of the form $Z - \beta$ can be expressed as

$$F_n(Z + T) = (Z - (Y_1 - T)) \cdots (Z - (Y_n - T)) \in k[Y_1 - T, \dots, Y_n - T]^{S_n}[Z],$$

and that

$$k[Y_1 - T, \dots, Y_n - T]^{S_n}[Z] = k[F_n(Z), \Delta^\alpha(F_n(Z)); 1 \leq \alpha \leq n-1]$$

where $\Delta^\alpha(F_n(Z))$ denotes the α-th derivative of $F_n(Z)$ relative to Z. Therefore since

$$Y_i - Y_j = (Y_i - T) - (Y_j - T)$$

we have that

$$k[Y_i - Y_j]^{S_n}_{1 \leq i,j \leq n} \subset k[F_n(Z), \Delta^\alpha(F_n(Z)); 1 \leq \alpha \leq n-1]. \tag{15}$$

Furthermore, denote by $\Delta^\alpha(F_n(Y_1))$ the image of $\Delta^\alpha(F_n(Z))$ in

$$k[s_{n,1}, \dots, s_{n,n}][Y_1] = k[s_{n,1}, \dots, s_{n,n}][Z]/\langle F_n(Z)\rangle.$$

It can be shown that

$$k[H_{m_1}, \dots, H_{m_r}] = k[\Delta^\alpha(F_n(Y_1)); 1 \leq \alpha \leq n-1] \cap k[s_{n,1}, \dots, s_{n,n}]$$

as subrings of $k[s_{n,1}, \dots, s_{n,n}][Y_1]$ (see [36, Corollary 1.10]).

With the previous notation, the following theorem can be proven:

Theorem 3.4. *[36, Theorem 1.16] Let S be a k-algebra, let $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_{n-1}Z + a_n \in S[Z]$ and consider a commutative diagram*

$$\begin{array}{ccc} Spec(S[Z]/\langle f(Z)\rangle) & \hookrightarrow & Spec(S[Z]) \\ \overline{\beta} \searrow & & \beta \downarrow \\ & & Spec(S). \end{array} \tag{16}$$

as in (9). Let Υ_n denote the set of n-fold points of $\{f(Z) = 0\} \subset Spec(S[Z])$. Consider the morphism defined by specialization,

$$\begin{array}{ccc} k[s_{n,1}, \dots, s_{n,n}] & \longrightarrow & S \\ s_{n,i} & \to & (-1)^i a_i \end{array}$$

which gives rise to the elimination algebra associated to $f(Z)$, say,

$$\mathcal{R}_{f(Z)} = S[H_{m_j}(a_1, \dots, a_n)W^{m_j}; j = 1, \dots, r] \subset S[W],$$

where m_j denotes the degree of the weighted homogeneous polynomial $H_{m_j}(s_{n,1}, \dots, s_{n,n})$ (see (14)).

Then:

i) The closed set $V(H_{m_j}(a_1,\dots,a_n); j=1,\dots,r) \subset Spec(S)$ *is the set of points where* $\overline{\beta}$ *is purely ramified.*

ii) If S *is regular, then*

$$\beta(\Upsilon_n) \subset \cap_{1\leq j\leq r}\{x \in Spec(S) : \nu_x(H_{m_j}(a_1,\dots,a_n)) \geq m_j\}, \tag{17}$$

where ν_x *denotes the order in the regular local ring of* x. *If in addition, the characteristic of* S *is zero, or if* n *is coprime with the characteristic, then the inclusion in (17) is an equality.*

3.5. The universal elimination algebra for a finite number of polynomials. Note that $\mathcal{R}_{f(Z)} = S[H_{m_j}(a_1,\dots,a_n)W^{m_j}; j=1,\dots,r] \subset S[W]$, is a Rees algebra over S. Moreover the right hand side of (17) is the singular locus of this algebra (see (3)). The inclusion (17) is, in general, strict when n is a multiple of the characteristic: in this case, just considering the equation $f(Z)$ falls too short to provide a good characterization of $\beta(\Upsilon_n)$. This can be remedied by adding some extra information. In general, we will be working with a hypersurface X embedded in smooth scheme V over a perfect field k. Then, locally, in an étale neighborhood of a point of maximum multiplicity of X, one can assume a setting like the one of Theorem 3.4 after applying Weirstrass Preparation Theorem. In this case, we will be considering $f(Z)$ together with all the k-differential operators (up to order $n-1$) acting on it. Then an *elimination algebra* associated to all these data can be defined over S (see [36, Definition 1.42] for full details; some hints will be given in forthcoming paragraphs). In terms of algebras, this leads to considering algebras saturated by the action of k-differential operators, or in short, in working with absolute differential Rees algebras.

The previous discussion motivates the study of elimination algebras for algebras generated by more than one polynomial. This is done in full detail in [36]. For instance, given a smooth local ring S over a perfect field k; and a finite number of monic polynomials,

$$f_i(Z) = Z^{n_i} + a_1^i Z^{n_i-1} + \dots + a_{n_i}^i \in S[Z] \; i=1,\dots,s$$

define a subalgebra of $S[Z][W]$ of the form

$$S[Z][f_i(Z)W^{n_i}, i=1,\dots,s].$$

Similarly, in the universal setting we can work with a finite number of polynomials

$$F_{n_i} = (Z-Y_{i,1})\cdots(Z-Y_{i,n_i}) \in k[Y_{1,1},\dots,Y_{1,n_1},Y_{2,1},\dots,Y_{2,n_2},\dots,Y_{s,1},\dots,Y_{s,n_s},Z],$$

for $i=1,\dots,s$, and then a *universal elimination algebra* can be defined (see [36, Definition 1.42]). Via a specialization morphism, in a similar manner as in (12), an elimination algebra can be constructed for $S[Z][f_i(Z)W^{n_i}, i=1,\dots,s]$.

3.6. Towards equality in (17). To get an equality in (17) we need to go one step beyond and consider algebras saturated by the action of differential operators relative to the base field k. Define $\mathcal{G}$ as the graded algebra generated by $f(Z)$ in degree n, and the action on $f(Z)$ of all higher order differential operators of degree at most n (see Section 5). This will ultimately lead us, in the universal case, to algebras as those studied in (15). The singular locus of $\mathcal{G}$ is still the set of $n-$fold points of

$f(Z) = 0$. In this setting, an elimination algebra $\mathcal{R}_{\mathcal{G}}$ can be associated to $\mathcal{G}$ (see Section 10), and formula (15) ensures the existence of a natural inclusion $\mathcal{R}_{\mathcal{G}} \subset \mathcal{G}$.

Assume that $\mathcal{R}_{\mathcal{G}}$ is generated by $G_1, \dots, G_s$ in degrees $m_1, \dots, m_s$ over S, i.e.,

$$\mathcal{R}_{\mathcal{G}} = S[G_1 W^{m_1}, G_2 W^{m_2}, \dots, , G_s W^{m_s}](\subset S[W]).$$

By definition its singular locus is

$$\mathrm{Sing}\ \mathcal{R}_{\mathcal{G}} = \cap_j \{\overline{x} \in \mathrm{Spec}\ S : \nu_{\overline{x}}(G_j) \geq m_j : j = 1, \dots, s\}.$$

Then, for algebras saturated by k-differential operators, it can be shown that

$$\beta(\mathrm{Sing}\ \mathcal{G}) = \mathrm{Sing}\ \mathcal{R}_{\mathcal{G}},$$

i.e., Sing $\mathcal{R}_{\mathcal{G}}$ is the projection of the maximum multiplicity locus Υ_n of $f(Z) = 0$ (cf. [36, Corollary 4.12]).

The starting point in the previous statement was the case in which an algebra of the form $S[Z][f(Z)W^n]$ was considered, and it was later saturated by the action of k-linear differential operators. In general, given an arbitrary (finitely generated) k-differential graded algebra $\mathcal{G}$ in an n-dimensional smooth scheme, and a sufficiently general smooth projection to an $(n-1)$-dimensional smooth scheme,

$$\begin{array}{cccc} \beta: & V & \to & \overline{V} \\ & x & \to & \overline{x}, \end{array}$$

an elimination algebra $\mathcal{R}_{\mathcal{G}}$ can be defined locally, in a neighborhood of each point x of its singular locus (see [36, Definition 1.42] for the complete construction and full details; or see Section 10 in this paper for a brief sketch). When $\mathcal{G}$ is closed under the action of differential operators (i.e. when $\mathcal{G}$ is a Differential Rees algebra) then

$$\beta(\mathrm{Sing}\ \mathcal{G}) = \mathrm{Sing}\ \mathcal{R}_{\mathcal{G}},$$

(see [36, Corollary 4.12]).

3.7. Compatibility of elimination with monoidal transformations. With the same notation as in the previous paragraph, it can be shown that if $Y \subset \mathrm{Sing}\ \mathcal{G}$ is a smooth center, then $\beta(Y) \subset \mathrm{Sing}\ \mathcal{R}_{\mathcal{G}}$ is also a smooth center. Then the blow-up in V with center Y, say $V \leftarrow V'$, induces a blow-up of $\overline{V}$ with center $\beta(Y)$, say $\overline{V} \leftarrow \overline{V}'$. There is a law of transformation of Rees algebras which induces a commutative diagram of smooth projections, elimination algebras and blow-ups in a neighborhood of $x \in \mathrm{Sing}\ \mathcal{G}$:

$$\begin{tikzcd} \begin{array}{c}(V,x)\\ \mathcal{G}\end{array} \arrow[d, "\beta"] & \begin{array}{c}(U \subset V', x')\\ \mathcal{G}'\end{array} \arrow[l, "\pi"'] \arrow[d, "\beta'"] \arrow[dl, phantom, "\circlearrowleft"] \\ \begin{array}{c}(\overline{V}, \overline{x})\\ \mathcal{R}_{\mathcal{G}}\end{array} & \begin{array}{c}(\overline{V}', \overline{x}')\\ \mathcal{R}'_{\mathcal{G}},\end{array} \arrow[l, "\overline{\pi}"'] \end{tikzcd}$$

where $\mathcal{R}'_{\mathcal{G}}$ is an elimination algebra for $\mathcal{G}'$, and U is a suitable neighborhood of x', a point in V' mapping to x (cf. [9, Theorem 9.1] or Theorem 10.8 in this paper). It is worth noticing here that, even if the starting setting is that $\beta(\text{Sing } \mathcal{G}) = \text{Sing } \mathcal{R}_{\mathcal{G}}$, there is a difference of the outcome depending on the characteristic of the base field. When the characteristic is zero, it can be proved that $\beta'(\text{Sing } \mathcal{G}') = \text{Sing } \mathcal{R}'_{\mathcal{G}}$, while in positive characteristic only the inclusion $\beta'(\text{Sing } \mathcal{G}') \subset \text{Sing } \mathcal{R}'_{\mathcal{G}}$ holds (in general).

In this sense the selection of a suitable local projection generalizes the idea of restricting to a hypersurface of maximal contact, as both are somehow stable by suitable chosen monoidal transformations. Thus, we replace pairs by Rees algebras, and restrictions to hypersurfaces of maximal contact by smooth projections; coefficient ideals are replaced by elimination algebras (see 1.1, 1.2 and Example 2.2, specially the argument after the Claim for some hints about coefficient ideals).

Using these techniques, given a non-smooth hypersurface, in arbitrary characteristic, we can define an upper semi-continuous function that stratifies its maximum multiplicity locus in smooth strata (see [9, Theorems 10.1, and 10.2] or Theorems 11.1 and 11.2 below). Moreover, the blowing-up along the maximum stratum of this function leads to a form of simplification of the singularities with maximum multiplicity, which we refer to as the *monomial case* (cf. [37, Section 6], or Corollary 11.3 below). When applied in characteristic zero, it is simple to check that this approach leads to the same resolution invariants as the ones obtained using hypersurfaces of maximal contact and coefficient ideals. A more detailed study of the monomial case is made in [5], where it is also shown how this approach leads to a short proof of resolution of singularities of surfaces over perfect fields.

Part 2. Rees algebras. We will discuss some essential properties of Rees algebras that will be used later in this paper. Special attention will be paid to Rees algebras that are, in some form, saturated, under the action of differential operators. Example 4.5 illustrates a typical geometric setting in which Rees algebras are considered.

For our purposes, the most important invariants associated to a Rees algebra are the order at a point, and the τ-invariant. The order measures the complexity of the singularity, while the τ-invariant provides information about the number of variables that can be eliminated from the problem (and it therefore plays a role in inductive arguments). These are presented in sections 4 and 7 respectively.

Rees algebras will be used to reformulate resolution problems. In this sense, we will be actually working with algebras up to integral closure, and sometimes up to *weak equivalence* (see Section 8). If two Rees algebras are weakly equivalent, then they have the same resolution invariants, and hence they will both undergo the same constructive resolution. This equivalence relation parallels Hironaka's notion of weak equivalence, essential in the context of idealistic exponents (what we call here pairs and basic objects).

4. Rees algebras.

DEFINITION 4.1. Let B be a Noetherian ring, and let $\{I_n\}_{n\geq 0}$ be a sequence of ideals in B satisfying the following conditions:

i. $I_0 = B$;

ii. $I_k \cdot I_l \subset I_{k+l}$.

Then the graded subring $\mathcal{G} = \oplus_{n\geq 0} I_n W^n$ of the polynomial ring $B[W]$ is said to be a *Rees algebra* if it is a finitely generated B-algebra.

A Rees algebra can be described by giving a finite set of generators,

$\{f_{n_1}W^{n_1}, \dots, f_{n_s}W^{n_s}\}$, say,

$$\mathcal{G} = B[f_{n_1}W^{n_1}, \dots, f_{n_s}W^{n_s}] \subset B[W]$$

with $f_{n_i} \in B$ for $i = 1\dots, s$. An element $g \in I_n$ will be of the form $g = F_n(f_{n_1}, \dots, f_{n_s})$ for some weighted homogeneous polynomial in s-variables $F_n(Y_1, \dots, Y_s)$ where Y_i has weight n_i for $i = 1, \dots, s$.

Example 4.2. The typical example of a Rees algebra is the Rees ring of an ideal, $J \subset B$: $\mathcal{G} = \oplus_n J^n W^n$. As a matter of fact, any Rees algebra is, up to integral closure, the Rees ring of an ideal: let $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n \subset B[W]$ be the Rees algebra generated by $\{f_{n_1}W^{n_1}, \dots, f_{n_s}W^{s}\}$ with $f_i \in B$, and let N be a suitable common multiple of all integers n_i, $i = 1, \dots, s$. Then

$$\bigoplus_{k\geq 0} I_N^k W^{kN} \subset \bigoplus_{n\geq 0} I_n W^n$$

is a finite extension of Rees algebras (cf. [38, 2.3] and [36]).

4.3. Rees algebras and integral closure. In many problems concerning resolution of singularities it is natural to consider ideals up to integral closure. For instance two ideals with the same integral closure have the same embedded principalizations (log-resolutions). As another example, and for reasons that will be clarified later in this paper, it will be interesting to consider Rees algebras $\mathcal{G} = \oplus_{n\geq 0} I_n W^n$ with the additional property that

$$I_0 \supset I_1 \supset \dots \supset I_n \dots . \tag{18}$$

For an arbitrary Rees algebra, $\mathcal{G} = \oplus_{n\geq 0} I_n W^n$, define

$$I'_n = \sum_{r\geq n} I_r,$$

and set $\mathcal{L} = \oplus_{n\geq 0} I'_n W^n$. Then $\mathcal{L}$ is contained in the integral closure of $\mathcal{G}$ (cf. [36, Remark 2.2 (2)]), and has the additional property that $I'_k \supset I'_s$ if $s \geq k$. So, up to integral closure, it can always be assumed that a Rees algebra fulfills condition (18).

The notion of Rees algebra extends to schemes in the obvious manner: a sequence of sheaves of ideals $\{I_n\}_{n\geq 0}$ on a scheme V, defines a sheaf of Rees algebras, $\mathcal{G}$, if $I_k \cdot I_l \subset I_{k+l}$ for all non-negative integers k, l, and if there is an affine open cover $\{U_i\}$ of V, such that $\mathcal{G}(U_i) \subset \mathcal{O}_V(U_i)[W]$ is a Rees $\mathcal{O}_V(U_i)$-algebra in the sense of Definition 4.1.

4.4. The singular locus of a Rees algebra. Let V be a smooth scheme over a perfect field k, and let $\mathcal{G} = \oplus_n I_n W^n$ be a sheaf of $\mathcal{O}_V$-Rees algebras. Then the *singular locus of* $\mathcal{G}$, Sing $\mathcal{G}$, is

$$\operatorname{Sing} \mathcal{G} := \bigcap_n \{x \in V : \nu_x(I_n) \geq n, \text{ for all } n \in \mathbb{Z}_{\geq 0}\},$$

where $\nu_x(I_n)$ denotes the order of I_n in the regular local ring $\mathcal{O}_{V,x}$. Observe that Sing $\mathcal{G}$ is a closed subscheme in V. The singular locus of a Rees algebra is well defined up to integral closure: If $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{O}_V[W]$ have the same integral closure in $\mathcal{O}_V[W]$, then Sing $\mathcal{G}_1$ = Sing $\mathcal{G}_2$ (see [38, Proposition 4.4 (1)]).

Example 4.5. Let $X \subset V$ be a hypersurface, and let b be a non-negative integer. Then the singular locus of the Rees algebra generated by $\mathcal{I}(X)$ in degree b, say $\mathcal{O}_V[I(X)W^b](\subset \mathcal{O}_V[W])$, is the closed set of points of multiplicity at least b of X (which may be empty). In the same manner, if $J \subset \mathcal{O}_V$ is an arbitrary non-zero sheaf of ideals, and b is a non-negative integer, then the singular locus of the Rees algebra generated by J in degree b, say $\mathcal{O}_V[JW^b](\subset \mathcal{O}_V[W])$, consists of the points of V where the order of J is at least b.

4.6. The order of a Rees algebra at a point. [15, 6.3] Let $x \in \operatorname{Sing} \mathcal{G} = \bigoplus_{n\geq 0} I_nW^n$, and let $fW^n \in I_nW^n$. Then set

$$\operatorname{ord}_x(f) = \frac{\nu_x(f)}{n} \in \mathbb{Q},$$

where $\nu_x(f)$ denotes the order of f in the regular local ring $\mathcal{O}_{V,x}$. Notice that $\operatorname{ord}_x(f) \geq 1$ since $x \in \operatorname{Sing} \mathcal{G}$. Now define

$$\operatorname{ord}_x\mathcal{G} = \inf\{\operatorname{ord}_x(f) : fW^n \in I_nW^n, n \geq 1\}.$$

If $\mathcal{G}$ is generated by $\{f_{n_1}W^{n_1}, \dots, f_{n_m}W^{n_m}\}$ then it can be shown that

$$\operatorname{ord}_x\mathcal{G} = \min\{\operatorname{ord}_x(f_{n_i}) : i = 1, \dots, m\},$$

and therefore, since $x \in \operatorname{Sing} \mathcal{G}$, $\operatorname{ord}_x\mathcal{G}$ is a rational number that is greater than or equal to one. Furthermore if N is a common multiple of all n_i, then

$$\operatorname{ord}_x\mathcal{G} = \frac{\nu_x(I_N)}{N}.$$

In particular if $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{O}_V[W]$ have the same integral closure, then $\operatorname{ord}_x\mathcal{G}_1 = \operatorname{ord}_x\mathcal{G}_2$ at any point $x \in \operatorname{Sing} \mathcal{G}_1 = \operatorname{Sing} \mathcal{G}_2$ (cf. [15, Proposition 6.4]).

5. Differential Rees algebras. Let V be a smooth scheme over a perfect field k. For any non-negative integer s, denote by $Diff^s_k$ the (locally free) sheaf of k-differential operators of order s.

Definition 5.1. A Rees algebra $\mathcal{G} = \oplus_n I_nW^n$ is said to be a *differential Rees algebra*, if the following conditions hold:

i. For all non-negative integers n there is an inclusion $I_n \supset I_{n+1}$.
ii. There is an affine open covering of V, $\{U_i\}$, such that for any $D \in Diff^r_k(U_i)$ and any $h \in I_n(U_i)$ we have that $D(h) \in I_{n-r}(U_i)$ provided that $n \geq r$.

Sometimes we will refer to differential Rees algebras as *absolute differential Rees algebras.*

Given any Rees algebra $\mathcal{G}$ on a smooth scheme V, there is a natural way to construct the smallest differential Rees algebra containing it: the differential Rees algebra generated by $\mathcal{G}$, $\mathbb{D}\mathrm{iff}(\mathcal{G})$ (see [38, Theorem 3.4]). More precisely, if $\mathcal{G}$ is locally generated on an affine open set U by $\{f_{n_1}W^{n_1}, \dots, f_{n_s}W^s\}$, then it can be shown that $\mathbb{D}\mathrm{iff}(\mathcal{G}(U))$ is generated by

$$\{D(f_{n_i})W^{n'_i-r} : D \in Diff^r_k, 0 \leq r < n'_i \leq n_i, i = 1, \dots, s\}.$$

See [38, Theorem 3.4].

5.2. Differential Rees algebras and singular locus. Sheaves of differentials $Diff_k^r$ are useful to study the order of a sheaf of ideals (see 2.1). Similarly, it can be shown that if $\mathcal{G} = \bigoplus_n I_n W^n$, then,

$$\text{Sing } \mathcal{G} = \cap_{r\geq 0} V(Diff_k^{r-1}(I_r)),$$

(see [38, Definition 4.2], and also [38, Proposition 4.4]). In particular, if $\mathbb{D}\text{iff}(\mathcal{G})$ is the differential Rees algebra generated by a Rees algebra $\mathcal{G}$ then

$$\text{Sing } \mathcal{G} = \text{Sing } \mathbb{D}\text{iff}(\mathcal{G});$$

and moreover, if $x \in \text{Sing } \mathcal{G} = \text{Sing } \mathbb{D}\text{iff}(\mathcal{G})$ then

$$\text{ord}_x \mathcal{G} = \text{ord}_x \mathbb{D}\text{iff}(\mathcal{G})$$

(cf. [15, Proposition 6.4]). Furthermore, if $\mathcal{G}$ is a differential Rees algebra, then $\text{Sing } \mathcal{G} = V(I_r)$ for any positive integer r (see [38, Proposition 4.4]).

5.3. Differential Rees algebras and integral closure. If $\mathcal{G}_1 \subset \mathcal{G}_2$ is a finite extension of differential Rees algebras on a smooth scheme V over a field k, then $\mathbb{D}\text{iff}(\mathcal{G}_1) \subset \mathbb{D}\text{iff}(\mathcal{G}_2)$ is also a finite extension. In general, if $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same integral closure, then so do $\mathbb{D}\text{iff}(\mathcal{G}_1)$ and $\mathbb{D}\text{iff}(\mathcal{G}_2)$ (cf. [38, Section 6]).

6. Rees algebras, permissible transformations, and resolutions. In this section we briefly expose how Rees algebras transform under suitable monoidal transformations, and present the notion of resolution of Rees algebras.

6.1. Transforms of Rees algebras under blow-ups. Let $\mathcal{G} = \oplus_n J_n W^n \subset \mathcal{O}_V[W]$ be a Rees algebra, and let $Y \subset \text{Sing } \mathcal{G}$, be a smooth center. Consider the blow-up at Y, $V \leftarrow V'$, and let $H \subset V'$ be the exceptional divisor. Then for each $n \in \mathbb{N}$,

$$J_n \mathcal{O}_{V'} = \mathcal{I}(H)^n J'_n$$

for some sheaf of ideals $J'_n \subset \mathcal{O}_{V'}$. We define the *weighted transform of* $\mathcal{G}$ in V' as:

$$\mathcal{G}' := \oplus_n J'_n W^n.$$

The next proposition gives a local description of the weak transform of a Rees algebra $\mathcal{G}$ after a permissible monoidal transformation.

PROPOSITION 6.2. *[15, Proposition 1.6] Let $\mathcal{G} = \oplus_n J_n W^n$ be a Rees algebra on a smooth scheme V over a field k, and let $V \leftarrow V'$ be a permissible transformation. Assume, for simplicity, that V is affine. If $\mathcal{G}$ is generated by $\{g_{n_1} W^{n_1}, \ldots, g_{n_s} W^{n_s}\}$, then its weighted transform $\mathcal{G}'$ is generated by $\{g'_{n_1} W^{n_1}, \ldots, g'_{n_s} W^{n_s}\}$, where g'_{n_i} denotes the weighted transform of g_{n_i} in V' for $i = 1, \ldots, s$.*

6.3. Basic objects and resolutions. A *basic object* is a triple $(V, \mathcal{G}, E)$, where V is a smooth scheme, $\mathcal{G}$ is a Rees algebra and E is a set of smooth hypersurfaces having normal crossings. A smooth closed subscheme $Y \subset \text{Sing } \mathcal{G}$ is a *permissible center* if it has normal crossings with E.

The *transform of a basic object* $(V, \mathcal{G}, E = \{H_1, \ldots, H_r\})$ by a permissible monoidal transformation with center $Y \subset V$, $V \xleftarrow{\pi} V'$, is another basic object $(V', \mathcal{G}', E')$, where $\mathcal{G}'$ denotes the weighted transform of $\mathcal{G}$ in V', and $E' = \{H'_1, \ldots, H'_r\} \cup \pi^{-1}(Y)$, with H'_i the strict transform of H_i in V', for $i = 1, \ldots, r$.

A *resolution of a basic object* $(V, \mathcal{G}, E)$ is a finite sequence of permissible monoidal transformations

$$(V, \mathcal{G}, E) \longleftarrow (V'_1, \mathcal{G}_1, E_1) \longleftarrow \cdots \longleftarrow (V'_s, \mathcal{G}_s, E_s), \tag{19}$$

such that Sing $\mathcal{G}_s = \emptyset$.

Example 6.4. Let $X \subset V$ be a hypersurface with maximum multiplicity n, and let $\mathcal{G}$ be the Rees algebra generated by $\mathcal{I}(X)$ in degree n. Then a resolution of $(V, \mathcal{G}, E = \{\emptyset\})$ lowers the maximum multiplicity of a strict transform of X below n. Observe, that, locally at a point x of multiplicity n, a regular system of parameters can be chosen, and differential operators as in (6) can be defined. Consider the differential Rees algebra generated by $\mathcal{G}$: if f is a defining equation for X, locally at x, then

$$\mathbb{D}\mathrm{iff}\ \mathcal{G} = \mathcal{O}_{V,x}[fW^n, \Delta^\alpha f W^{n-|\alpha|} : \Delta^\alpha \in Diff_k^{(n-1)}].$$

Again, a resolution of $(V, \mathbb{D}\mathrm{iff}\ \mathcal{G}, E = \{\emptyset\})$ lowers the maximum multiplicity of a strict transform of X below n. This follows from the so called Giraud Lemma formulated in Theorem 8.7.

6.5. Relative differential Rees algebras. Relative differential Rees algebras will play a central role in our arguments due to their relation to a form of elimination that we shall discuss in the next sections. The problem of resolution of a Rees algebra can be formulated for a differential Rees algebra. However, the transform of a differential Rees algebra by blow-up is no longer a differential Rees algebra: this property is not stable by blow-ups. We remedy this weakness by introducing relative differential Rees algebras (see [9, Theorem 9.1] or Theorem 10.8 below, which states that this property is stable by blow-ups).

Let $\phi : V^{(d)} \to V^{(e)}$ be a smooth morphism of smooth schemes of dimensions d and e respectively. Then, for any non-negative integer s, the sheaf of relative differential operators of order s, $Diff^s(V^{(d)}/V^{(e)})$, is locally free over $V^{(d)}$. A Rees algebra $\mathcal{G} = \oplus_n I_n W^n \subset \mathcal{O}_{V^{(d)}}[W]$ is said to be a *ϕ-relative differential Rees algebra* or simply a *ϕ-differential Rees algebra* if:

i. For all non-negative integers n there is an inclusion $I_n \supset I_{n+1}$.
ii. There is an affine open covering $\{U_i\}$ of $V^{(d)}$ such that for any $D \in Diff^s(V^{(d)}/V^{(e)})(U_i)$ and any $h \in I_n(U_i)$ we have that $D(h) \in I_{n-s}(U_i)$ provided that $n \geq s$.

We will be particularly interested in the case in which $e = d - 1$ (see Section 10).

7. Simple points and tangent cones. Let $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ be a Rees algebra on a d-dimensional smooth scheme V over a perfect field k. We treat here the notion of *τ-invariant* at a singular point $x \in$ Sing $\mathcal{G}$. Among other things, we will see that the τ-invariant indicates the number of variables which are to be *eliminated*, via elimination algebras, and it is therefore significant in the proof of Proposition 11.4.

Definition 7.1. A point $x \in$ Sing $\mathcal{G}$ is *simple* if $\mathrm{ord}_x \mathcal{G} = 1$ (i.e., if for some $k \geq 1$, $\nu_x(I_k) = k$).

7.2. The tangent cone [36, 4.2]. If $x \in \operatorname{Sing} \mathcal{G}$ is a closed point, then we define the *initial ideal* or *tangent ideal* of $\mathcal{G}$ at x, $\operatorname{In}_x(\mathcal{G})$, as the ideal of $\operatorname{Gr}_{m_x}(\mathcal{O}_{V,x})$ generated by the elements $\operatorname{In}_x(I_n)$ for all $n \geq 1$ (here $\operatorname{Gr}_{m_x}(\mathcal{O}_{V,x}) \simeq k'[Z_1, \dots, Z_d]$ is the graded ring of the local ring $\mathcal{O}_{V,x}$, m_x is the maximal ideal of $\mathcal{O}_{V,x}$, and $\operatorname{In}_x(I_n)$ denotes the image of I_n in m_x^n/m_x^{n+1}). Observe that the tangent ideal is non-zero if and only if x is a simple point. The zero set of the tangent ideal in Spec $(\operatorname{Gr}_{m_x}(\mathcal{O}_{V,x}))$ is the *tangent cone* of $\mathcal{G}$ at x, $\mathcal{C}_{\mathcal{G},x}$.

7.3. The τ-invariant at a simple point [36, 4.2]. When $x \in \operatorname{Sing} \mathcal{G}$ is a simple closed point, and $\mathcal{G}$ is a differential Rees algebra, then the tangent cone is a linear subspace. More precisely, assume that k' is the residue field at x. If k' is a field of characteristic zero, then $\operatorname{In}_x\mathcal{G}$ is generated by linear forms. If k' is a field of positive characteristic p, then there is a sequence of non-negative integers, $e_0 < e_1 < \cdots < e_r$, such that $\operatorname{In}_x\mathcal{G}$ is generated by elements of the form

$$l_1, \dots, l_{s_0}, l_{s_0+1}, \dots, l_{s_1}, \dots, l_{s_{r-1}}, \dots, l_{s_r} \tag{20}$$

where each $l_1, \dots, l_{s_0}$ is a linear combination of powers $Z_i^{p^{e_0}}$; if $t \geq 0$, each

$$l_{s_t+1}, \dots, l_{s_{t+1}}$$

is a linear combination of powers $Z_i^{p^{e_t}}$, and the s_r homogeneous elements in (20) form a regular sequence in $\operatorname{Gr}_{m_x}(\mathcal{O}_{V,x})$. Hence, $\langle l_1, \dots, l_{s_r}\rangle$ defines a subscheme of codimension s_r in $\mathbb{T}_{V,x}$. If k' is a perfect field the radical of this ideal is spanned by linear forms, defining a subspace of codimension s_r in $\mathbb{T}_{V,x}$. The integer s_r is said to be the *τ-invariant of the singularity* and it is denoted by $\tau_{\mathcal{G},x}$, or τ_x if the algebra it refers to is clear from the context.

If $\mathcal{G}$ is not a differential Rees algebra, or if x is not a closed point, then the τ-invariant is also defined: from the algebraic point of view, $\tau_{\mathcal{G},x}$ indicates the minimum number of variables needed to describe $\operatorname{In}_x\mathcal{G}$; from the geometric point of view, $\tau_{\mathcal{G},x}$ is the codimension of the largest linear subspace $\mathcal{L}_{\mathcal{G},x} \subset \mathcal{C}_{\mathcal{G},x}$ such that $u + v \in \mathcal{C}_{\mathcal{G},x}$ for all $u \in \mathcal{C}_{\mathcal{G},x}$ and all $v \in \mathcal{L}_{\mathcal{G},x}$. Furthermore,

$$\tau_{\mathcal{G},x} = \tau_{\mathbb{D}\mathrm{iff}(\mathcal{G}),x}.$$

If $\mathcal{G}$ is a differential Rees algebra then:

$$\mathcal{L}_{\mathcal{G},x} = \mathcal{C}_{\mathcal{G},x};$$

and for an arbitrary Rees algebra $\mathcal{G}$, the inclusion $\mathcal{G} \subset \mathbb{D}\mathrm{iff}(\mathcal{G})$ defines an inclusion $\mathcal{C}_{\mathbb{D}\mathrm{iff}(\mathcal{G}),x} \subset \mathcal{C}_{\mathcal{G},x}$, and:

$$\mathcal{C}_{\mathbb{D}\mathrm{iff}(\mathcal{G}),x} = \mathcal{L}_{\mathcal{G},x}.$$

We shall see that locally at x, Sing $\mathcal{G}$ is included in a complete intersection scheme of codimension $\tau_{\mathcal{G},x}$ (see Section 12, specially the discussion following Definition 12.4). This motivates the following definition.

Definition 7.4. A Rees algebra $\mathcal{G}$ over V is said to be of *codimensional type* $\geq e$ if $\tau_{\mathcal{G},x} \geq e$ for all $x \in \operatorname{Sing} \mathcal{G}$.

If $\mathcal{G}$ is of codimensional type $\geq e$, then the codimension of the closed set Sing $\mathcal{G}$ in V is at least e. Moreover the components of codimension e are smooth and define an open and close set in Sing $\mathcal{G}$ (see [9]).

8. Weak equivalence. Given a Rees algebra, $\mathcal{G} = \oplus_n I_n W^n$, and a positive integer s, it is very natural to ask that $\mathcal{G}$ and $\mathcal{G}_s := \oplus_n I_{sn} W^{sn} \subset \mathcal{G}$ have the same resolutions. To start with, their singular locus are the same, and so is the order at any point in Sing $\mathcal{G} =$ Sing $\mathcal{G}_s$. More precisely: it would be desirable that two algebras with the same integral closure have the same resolution invariants, and hence the same resolution.

The previous question can be formulated in a wider context. For instance, we may want to compare the resolution of an arbitrary Rees algebra $\mathcal{G}$ with that of the differential Rees algebra generated by it, $\mathbb{D}\mathrm{iff}(\mathcal{G})$. Again, it is very natural to require that they both undergo the same constructive resolution. However, in general, $\mathcal{G}$ and $\mathbb{D}\mathrm{iff}(\mathcal{G})$ do not have the same integral closure.

This discussion leads to the notion of *weak equivalence* (see Definition 8.5 below). Two weakly equivalent algebras have the same resolution invariants, and hence have equivalent resolutions. Two Rees algebras with the same integral closure will be weakly equivalent. A fundamental result is that a Rees algebra and the differential Rees algebra generated by it will be weakly equivalent too. This means that algebras can be enriched by the action of differential operators and still they will be undistinguishable from the point of view of resolution of singularities.

Weak equivalence is formulated considering three kinds of transformations of Rees algebras: permissible monoidal transformations (see Section 6), étale extensions, and products with affine spaces.

8.1. Smooth morphisms. We will consider the following pull-backs:

i. If $U \to V$ is a étale then the extension of $\mathcal{G}$ to U is a Rees algebra; if $\mathcal{G}$ is a differential Rees algebra, its extension is a differential Rees algebra too.
ii. If $\phi : T = V \times \mathbb{A}^k \to V$ is the projection, then the pull back $\phi^*\mathcal{G}$ is a Rees algebra. Moreover, if $\mathcal{G}$ is a differential Rees algebra, then so is $\phi^*\mathcal{G}$.

Remark 8.2. Observe that if $\mathcal{G} = \bigoplus I_k W^k$ is a differential $\mathcal{O}_V$-Rees algebra and $\phi : T \to V$ is a smooth morphism of smooth schemes then $\phi^*(\mathcal{G})$ is a differential Rees algebra on T and Sing $\phi^*(\mathcal{G}) = \phi^{-1}(\text{Sing } \mathcal{G})$ (cf. [38, Proposition 5.1, and Theorem 5.4]).

Definition 8.3. Let $\mathcal{G}$ be a sheaf of Rees algebras on a smooth scheme V. A morphism $V' \to V$ is said to be a *transformation of* $\mathcal{G}$ if it is either a permissible monoidal transformation as in Section 6, or a smooth morphism as described in 8.1 (i) or (ii).

Definition 8.4. A *local sequence of transformations of basic objects* is a sequence of the form:

$$(V, \mathcal{G}, E) = (V_0, \mathcal{G}_0, E_0) \longleftarrow (V_1, \mathcal{G}_1, E_1) \longleftarrow \cdots \longleftarrow (V_s, \mathcal{G}_s, E_s), \tag{21}$$

where for $i = 0, 1, \ldots, s$ each $(V_i, \mathcal{G}_i, E_i) \longleftarrow (V_{i+1}, \mathcal{G}_{i+1}, E_{i+1})$, is a pull-back as in 8.1 (i) or (ii), or a permissible monoidal transformation with center $Y_i \subset$ Sing $\mathcal{G}_i$.

Definition 8.5. Two Rees algebras $\mathcal{G}_i$, $i = 1, 2$, or two basic objects $(V, \mathcal{G}_i, E)$, $i = 1, 2$, are said to be *weakly equivalent* if:

(i) Sing $\mathcal{G}_1 =$ Sing $\mathcal{G}_2$;
(ii) Any local sequence of transformations of one of them (see Definition 8.4), say,

$$(V, \mathcal{G}_i, E) = (V_0, \mathcal{G}_{i,0}, E_0) \longleftarrow (V_1, \mathcal{G}_{i,1}, E_1) \longleftarrow \cdots \longleftarrow (V_s, \mathcal{G}_{i,s}, E_s),$$

defines a local sequence of transformations of the other, and Sing $(\mathcal{G}_{1,j}) =$ Sing $(\mathcal{G}_{2,j})$ for $0 \leq j \leq s$.

8.6. Integral closure, differential operators and weak transforms. [15, 4.1] Assume $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3$ is an inclusion of Rees algebras, where $\mathcal{G}_3$ is the differential Rees algebra spanned by $\mathcal{G}_1$, and let $V \leftarrow V'$ be a permissible monoidal transformation with center $Y \subset$ Sing $\mathcal{G}_1$. Then:

(i) There is an inclusion of weak transforms

$$\mathcal{G}'_1 \subset \mathcal{G}'_2 \subset \mathcal{G}'_3.$$

(ii) The three algebras $\mathcal{G}'_1 \subset \mathcal{G}'_2 \subset \mathcal{G}'_3$ span the same differential Rees algebra.

(iii) If $\mathcal{G}_1 \subset \mathcal{G}_2$ is a finite extension, then $\mathcal{G}'_1 \subset \mathcal{G}'_2$ is a finite extension as well.

The following theorem is derived from the cited result of Hironaka. This fact, and many applications of it, are studied in [18].

THEOREM 8.7. *[28, p. 119], [27] If $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same integral closure then they are weakly equivalent. If $\mathbb{D}iff(\mathcal{G})$ is the differential Rees algebra generated by $\mathcal{G}$ then $\mathcal{G}$ and $\mathbb{D}iff(\mathcal{G})$ are weakly equivalent.*

The last assertion, namely that $\mathcal{G}$ and $\mathbb{D}\mathrm{iff}(\mathcal{G})$ are weakly equivalent is known as Giraud's Lemma.

THEOREM 8.8. *[28, p. 101], [27] If $\mathcal{G}_1$ and $\mathcal{G}_2$ are weakly equivalent, then $ord_x\mathcal{G}_1 = ord_x\mathcal{G}_2$ for each $x \in Sing\ \mathcal{G}_1 = Sing\ \mathcal{G}_2$.*

The following theorem is due to Hironaka:

THEOREM 8.9. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are weakly equivalent, then for each $x \in Sing\ \mathcal{G}_1 = Sing\ \mathcal{G}_2$ there is an equality between their τ-invariants, i.e., $\tau_{\mathcal{G}_1,x} = \tau_{\mathcal{G}_2,x}$.*

9. Rees algebras vs. pairs. The notion of Rees algebra is essentially equivalent to Hironaka's notion of *pair* (see [26]). We assign to a pair (J, b) over a smooth scheme V the Rees algebra:

$$\mathcal{G}_{(J,b)} = \mathcal{O}_V[J^bW^b], \tag{22}$$

which is a graded subalgebra in $\mathcal{O}_V[W]$. It turns out that every Rees algebra over V is a finite extension of $\mathcal{G}_{(J,b)}$ for a suitable pair (J, b) (see [37, Proposition 2.9] for details).

Observe that for $\mathcal{G}_{(J,b)} = \mathcal{O}_V[J^bW^b]$ there is an equality of closed sets

$$\mathrm{Sing}\ (\mathcal{G}_{(J,b)}) = \mathrm{Sing}\ (J, b),$$

and also of functions

$$\mathrm{ord}_{\mathcal{G}_{(J,b)}} = \mathrm{ord}_{(J,b)},$$

where the left-hand side is that defined in 4.6, and $\mathrm{ord}_{(J,b)}(x) = \frac{\nu_x(J)}{b}$.

Hence, up to integral closure, any Rees algebra is equivalent to a pair, and it is not hard to check that the construction of a resolution of a basic object $(V, (J, b), E)$ is equivalent to that of $(V, \mathcal{G}_{(J,b)}, E)$.

We will say that a basic object $(V, \mathcal{G}, E)$ is *monomial* or that it is *within the monomial case* if up to integral closure, $\mathcal{G} = \mathcal{G}_{(J,b)}$, and the basic object $(V, (J, b), E)$ is within the monomial case as in 1.1.

Part 3. Elimination.

10. A local projection and the elimination algebra. Let $V^{(d)}$ be a d-dimensional smooth scheme over a field k, let $\mathcal{G} = \oplus_{n\in\mathbb{N}} I_n W^n$ be a sheaf of Rees algebras and let $x \in \operatorname{Sing} \mathcal{G}$ be a simple point (see Section 7). In the following we will establish the conditions needed to construct:

- A suitable local projection to a $(d-1)$-dimensional smooth scheme, in an étale neighborhood of x,
$$\beta_{d,d-1} : V^{(d)} \to V^{(d-1)},$$
with $\beta_{d,d-1}(x) = x_1$.
- An elimination algebra associated to $\mathcal{G}$ in a neighborhood of x_1,
$$\mathcal{R}_{\mathcal{G},\beta_{d,d-1}} \subset \mathcal{O}_{V^{(d-1)}}[W]$$
that captures algebraic-geometric information about the points in $\operatorname{Sing} \mathcal{G}$ in a neighborhood of x.

This will lead us to a geometric notion of *transversality*, which is shown to be stable under permissible monoidal transformations. This issue, addressed in Theorem 10.8, guarantees the compatibility of elimination with blow-ups, essential in our approach based on a *simplification* of an elimination algebra. This parallels the idea of simplification of the coefficient ideal in the context of maximal contact.

Definition 10.1. Let $\mathcal{G}$ be a Rees algebra on a smooth d-dimensional scheme $V^{(d)}$ over a field k, and let $x \in \operatorname{Sing} \mathcal{G}$ be a simple closed point. We say that a local smooth projection to a $(d-1)$-dimensional (smooth) scheme $V^{(d-1)}$,

$$\begin{array}{cccc} \beta_{d,d-1}: & V^{(d)} & \to & V^{(d-1)} \\ & x & \to & x_1 \end{array}$$

is $\mathcal{G}$-*admissible locally at* x if the following conditions hold:

(i) The point x is not contained in any component of codimension one of $\operatorname{Sing} \mathcal{G}$.
(ii) The Rees algebra $\mathcal{G}$ is a $\beta_{d,d-1}$-relative differential Rees algebra (see Definition 6.5).
(iii) *Transversality:* $\ker d\beta_{d,d-1} \cap \mathcal{C}_{\mathcal{G},x} = \{\mathbb{O}\} \subset \mathbb{T}_{V,x}$.

Remark 10.2. Regarding to condition (i) in the previous definition, we underline that any codimension one component of $\operatorname{Sing} \mathcal{G}$ is smooth in a neighborhood of a simple point (cf. [9, Lemma 13.2]). Moreover, in such a case, the blow-up along $\operatorname{Sing} \mathcal{G}$ defines a resolution of $\mathcal{G}$. Hence, any codimension one component of $\operatorname{Sing} \mathcal{G}$ is a canonical center to blow-up. Regarding to condition (ii), observe that absolute differential Rees algebras are also relative differential Rees algebras for arbitrary smooth maps. However, the condition of being differential is not stable under permissible monoidal transformations; only relative differential Rees algebras are stable under this kind of transformations (see [9, Theorem 9.1] for full details, or Theorem 10.8 below). As for condition (iii) we stress here that almost any smooth local projection, or more generally, almost any smooth morphism defined locally, in a neighborhood of a simple point in the singular locus of a Rees algebra, will fulfill this condition. We refer to 10.3 for more details regarding to this point. In [9, Section 8] it is proven that if conditions (i), (ii) and (iii) hold in a point $x \in \operatorname{Sing} \mathcal{G}$, then they hold in a neighborhood of x in $\operatorname{Sing} \mathcal{G}$.

10.3. Local projections [9, Section 8]. The previous discussion shows that we may assume that Sing $\mathcal{G}$ has no components of codimension 1. Normally we will take as a starting point an absolute differential Rees algebra $\mathcal{G}$ over the smooth scheme $V^{(d)}$. This ensures that $\mathcal{G}$ will be $\beta_{d,d-1}$-differential for any local smooth projection to a $(d-1)$-dimensional smooth scheme,

$$\begin{array}{rccc} \beta_{d,d-1}: & V^{(d)} & \longrightarrow & V^{(d-1)} \\ & x & \to & x_1. \end{array} \tag{23}$$

We now discuss how to construct a local projection satisfying condition (iii) from Definition 10.1. Given a local projection as above, any regular system of parameters $\{y_1, \dots, y_{d-1}\} \subset \mathcal{O}_{V^{(d-1)},x_1}$ extends to a regular system of parameters $\{y_1, \dots, y_{d-1}, y_d\} \subset \mathcal{O}_{V^{(d)},x}$. Notice that condition (iii) in Definition 10.1 holds if and only if $\{\mathrm{In}_x y_1 = 0, \dots, \mathrm{In}_x y_{d-1} = 0\} \subset \mathbb{T}_{V,x}$ is not contained in the tangent cone of $\mathcal{G}$ at x, $\mathcal{C}_{\mathcal{G},x}$. So it suffices to choose a regular system of parameters $\{y_1, \dots, y_{d-1}, y_d\} \subset \mathcal{O}_{V^{(d)},x}$ such that $\{\mathrm{In}_x y_1 = 0, \dots, \mathrm{In}_x y_{d-1} = 0\} \subset \mathbb{T}_{V,x}$ is not contained in $\mathcal{C}_{\mathcal{G},x}$. Note that there is a natural injective map from the ring of polynomials in $(d-1)$-variables with coefficients in k into $\mathcal{O}_{V^{(d)},x}$, and localizing we get an inclusion of regular local rings,

$$\begin{array}{rcl} k[Y_1, \dots, Y_{d-1}]_{\langle Y_1, \dots, Y_{d-1} \rangle} & \longrightarrow & \mathcal{O}_{V^{(d)},x} \\ Y_i & \to & y_i. \end{array}$$

This is one way to produce a local projection as (23), to a $(d-1)$-dimensional regular scheme, satisfying condition (iii) in Definition 10.1.

10.4. The elimination algebra $\mathcal{R}_{\mathcal{G}_{\beta_{d,d-1}}}$. Fix a locally admissible projection in a neighborhood of a simple closed point $x \in$ Sing $\mathcal{G}$, as in Definition 10.1,

$$\begin{array}{rccc} \beta_{d,d-1}: & V^{(d)} & \to & V^{(d-1)} \\ & x & \to & x_1. \end{array}$$

Then an *elimination algebra*

$$\mathcal{R}_{\mathcal{G}_{\beta_{d,d-1}}} \subset \mathcal{O}_{V^{(d-1)},x_1}[W]$$

can be defined (see [36, 1.25, Definitions 1.42 and 4.10]). To do so, first note that there is a positive integer n, and an element $f \in I_n$ of order n at $\mathcal{O}_{V^{(d)},x}$ which is transversal to $\beta_{d,d-1}$ at x. Then construct a monic polynomial $f(Z) \in I_n$ in a suitable étale neighborhood of x (this follows from Weirstrass Preparation Theorem). It can be checked that, up to integral closure, we may assume that $\mathcal{G}$ is as in 3.5, for $S = \mathcal{O}_{V^{(d-1)},x_1}$, and suitable monic polynomials $f_i(Z)$, $i = 1, \dots, s$. In particular, $\mathcal{G}$ is locally (and up to integral closure) the pull-back of a universal algebra as in 3.5; so an elimination algebra $\mathcal{R}_{\mathcal{G}_{\beta_{d,d-1}}} \subset \mathcal{O}_{V^{(d-1)},x_1}[W]$ can be defined by a specialization morphism in a similar manner as in (12).

10.5. Elimination algebras and their properties. The elimination algebra depends on the projection $\beta_{d,d-1}$ but it can be shown that it does not depend on the choice of the element f, once the projection is fixed. Elimination algebras satisfy the following properties:

1. The inclusion $\beta^*_{d,d-1} : \mathcal{O}_{V^{(d-1)},x_1} \to \mathcal{O}_{V^{(d)},x}$ induces an inclusion of Rees algebras $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}} \subset \mathcal{G}$ ([36, Theorem 4.13]).

2. If $\mathcal{G}$ is a differential Rees algebra, then so is $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$.

3. There is an inclusion of closed subsets

$$\beta_{d,d-1}(\text{Sing }\mathcal{G}) \subset \text{Sing }\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$$

with equality if $\mathcal{G}$ is an absolute differential Rees algebra (cf. [36, Corollary 4.12]).

4. The order of $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$ at x_1 does not depend on the projection, in other words, $\text{ord}_{x_1}\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$ is independent of $\beta_{d,d-1}$ (see [36, Theorem 5.5]).

5. With the same notation as in 10.4, consider the following diagram:

$$\begin{array}{rcl} \mathcal{O}_{V^{(d)},x}[W] & \xrightarrow{\gamma^*} & \mathcal{O}_{V^{(d)},x}/\langle f_n\rangle[W] \simeq \mathcal{O}_{V^{(d-1)},x_1}[Z]/\langle F(Z)\rangle[W] \\ \beta^*_{d,d-1}\uparrow & \nearrow & \\ \mathcal{O}_{V^{(d-1)},x_1}[W] & & \end{array}$$

where γ^* denotes the natural restriction. Then the image of $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$ in the quotient ring $\mathcal{O}_{V^{(d-1)},x_1}[Z]/\langle F(Z)\rangle[W]$ is contained in $\gamma^*(\mathcal{G})$, and they both have the same integral closure (see [36, Theorem 4.11]). This Theorem also says that if an inclusion of Rees algebras $\mathcal{G} \subset \mathcal{G}'$ is finite, then $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}} \subset \mathcal{R}_{\mathcal{G}',\beta_{d,d-1}}$ is finite too.

6. If $\mathcal{G}$ is a differential Rees algebra, then $\tau_{\mathcal{R}_{\mathcal{G},\beta_{d,d-1}},x_1} = \tau_{\mathcal{G},x} - 1$ (cf. [4, Theorem 6.4]).

10.6. Notation. In what follows, given a Rees algebra $\mathcal{G} = \mathcal{G}^{(d)}$ on a d-dimensional smooth scheme $V^{(d)}$, we will refer to an elimination algebra as $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$ if we need to emphasize the projection, or just as $\mathcal{G}^{(d-1)} \subset \mathcal{O}_{V^{(d-1)}}[W]$ if the choice of the projection is not relevant in the discussion. So, in general, if $\mathcal{G}$ is of codimensional type $\geq e(\geq 1)$ in a neighborhood of x (i.e., if $\tau_{\mathcal{G},x} \geq e$ in $U \subset \text{Sing }\mathcal{G}$) then we can expect to iterate the arguments in 10.3 e-times. In that case a sequence of local projections can be defined:

$$\begin{array}{ccccccc} V^{(d)} & \xrightarrow{\beta_{d,d-1}} & V^{(d-1)} & \to & \dots & \xrightarrow{\beta_{d-(e-1),d-e}} & V^{(d-e)} \\ x = x_0 & \to & x_1 & \to & \dots & \to & x_e, \end{array}$$

which by composition induces a local projection from $V^{(d)}$ to some $(d-e)$-dimensional smooth space $V^{(d-e)}$. In this way, by iteration, we can define elimination algebras

$$\mathcal{G}^{(d-1)} \subset \mathcal{O}_{V^{(d-1)}}[W], \dots, \mathcal{G}^{(d-e)} \subset \mathcal{O}_{V^{(d-e)}}[W]$$

if for each $i = 1, \dots, e$, the projection

$$\beta_{d-(i-1),d-i} : V^{(d-(i-1))} \to V^{(d-i)}$$

is $\mathcal{G}^{(d-(i-1))}$-admissible locally at x_{i-1}. By [36, Corollary 4.12], there is an inclusion of closed subsets

$$\beta_{d-(i-1),d-i}(\text{Sing }\mathcal{G}^{(d-(i-1))}) \subset \text{Sing }\mathcal{G}^{(d-i)},$$

which is an equality when $\mathcal{G}^{(d-(i-1))}$ is a differential Rees algebra for $i = 1, \dots, e$. This motivates the next definition.

Definition 10.7. Let $\mathcal{G}^{(d)}$ be a Rees algebra on a smooth d-dimensional scheme $V^{(d)}$, and let $x \in \operatorname{Sing} \mathcal{G}$ be a simple point with $\tau_{\mathcal{G}^{(d)},x} \geq e (\geq 1)$. We will say that a local projection to a smooth $(d-e)$-dimensional smooth scheme

$$\begin{array}{rccc} \beta_{d,d-e}: & V^{(d)} & \to & V^{(d-e)} \\ & x & \to & x_e \end{array}$$

is *locally $\mathcal{G}$-admissible at x* if it factors as a sequence of local $\mathcal{G}^{(d-i)}$-admissible projections as in Definition 10.1,

$$\begin{array}{ccccccc} (V^{(d)}, x_0 = x) & \overset{\beta_{d,d-1}}{\longrightarrow} & (V^{(d-1)}, x_1) & \to & \dots & \overset{\beta_{d-(e-1),d-e}}{\longrightarrow} & (V^{(d-e)}, x_e) \\ \mathcal{G}^{(d)} & & \mathcal{G}^{(d-1)} & & \dots & & \mathcal{G}^{(d-e)}, \end{array}$$

where for $i = 1, \dots, e$, each $\mathcal{G}^{(d-i)} \subset \mathcal{O}_{V^{(d-i)},x_i}[W]$ is the elimination algebra of $\mathcal{G}^{(d-(i-1))} \subset \mathcal{O}_{V^{(d-(i-1))},x_{i-1}}[W]$, and $\beta_{d-(i-1),d-i}(x_{i-1}) = x_i$.

The following Theorem establishes a natural from of stability for admissible projections under blow-ups.

Theorem 10.8. *[9, Theorem 9.1] Let $\mathcal{G}^{(d)}$ be a Rees algebra on a smooth d-dimensional scheme $V^{(d)}$ and let $x \in \operatorname{Sing} \mathcal{G}^{(d)}$ be a simple point (i.e., $\tau_{\mathcal{G}^{(d)},x} \geq 1$). Suppose that a local $\mathcal{G}^{(d)}$-admissible projection is given, defining an elimination algebra:*

$$\begin{array}{rccc} \beta_{d,d-1}: & (V^{(d)}, x) & \longrightarrow & (V^{(d-1)}, x_1) \\ & \mathcal{G}^{(d)} & & \mathcal{G}^{(d-1)}. \end{array}$$

Let $Y \subset \operatorname{Sing} \mathcal{G}^{(d)}$ be a permissible center. Then, locally in a neighborhood of x:

(i) The closed set $\beta_{d,d-1}(Y) \subset \operatorname{Sing} \mathcal{G}^{(d-1)} \subset V^{(d-1)}$ is a permissible center for $\mathcal{G}^{(d-1)}$.

(ii) Given the monoidal transformations on $V^{(d)}$ with center Y, say $V^{(d)} \leftarrow V^{(d)'}$, and on $V^{(d-1)}$ with center $\beta_{d,d-1}(Y)$, say $V^{(d-1)} \leftarrow V^{(d-1)'}$, there is a projection $\beta'_{d,d-1}$ defined in a suitable open set, and a commutative diagram of projections and weighted transforms:

$$\begin{array}{ccc} (V^{(d)}, x) & \xleftarrow{\pi^{(d)}} & U \subset V^{(d)'} \\ \mathcal{G}^{(d)} & & \mathcal{G}^{(d)'} \\ \downarrow \beta_{d,d-1} & \circlearrowleft & \downarrow \beta'_{d,d-1} \\ (V^{(d-1)}, x_1) & \xleftarrow{\pi^{(d-1)}} & V^{(d-1)'} \\ \mathcal{G}^{(d-1)} & & \mathcal{G}^{(d-1)'}. \end{array}$$

Furthermore, if $x' \in \operatorname{Sing} \mathcal{G}^{(d)'} \neq \emptyset$ maps to x, then,

(a) The projection $V^{(d)'} \to V^{(d-1)'}$ is $\mathcal{G}^{(d)'}$-admissible locally at x' (see Definition 10.1). In particular $\mathcal{G}^{(d)'}$ is a $\beta'_{d,d-1}$-relative-differential Rees algebra, defining an elimination algebra $\mathcal{G}'^{(d-1)}$.

(b) Let $x'_1 = \beta'_{d,d-1}(x')$. Locally in an open neighborhood of x'_1, there is a natural inclusion $\mathcal{G}^{(d-1)'} \subset \mathcal{G}'^{(d-1)}$ which is an equality up to integral closure.

11. Elimination and resolution invariants. As indicated in sections 2 and 9, Rees algebras parallel Hironaka's notion of idealistic exponents:

- Local smooth projections parallel local restrictions to hypersurfaces of maximal contact;
- Elimination algebras play the role of coefficient ideals.

In fact, the theory of idealistic exponents can be embedded in that of Rees algebras. The following theorems say that, using elimination algebras one can define resolution invariants that lead to smooth stratifying functions.

THEOREM 11.1. *[9, Theorem 10.1] Let $V^{(d)}$ be a d-dimensional scheme smooth over a perfect field k, let $\mathcal{G}^{(d)} \subset \mathcal{O}_{V^{(d)}}[W]$ be a differential Rees algebra, let $x \in Sing\, \mathcal{G}^{(d)}$ be a simple closed point, and let $m \leq \tau_{\mathcal{G},x}$. Consider two different $\mathcal{G}^{(d)}$-admissible local projections to some $(d-m)$-dimensional smooth schemes with their corresponding elimination algebras:*

(24)
$$\begin{array}{lccc} \beta_{1_{d,d-m}}: & (V^{(d)},x) & \longrightarrow & (V_1^{(d-m)}, x_{m,1}) \\ & \mathcal{G}^{(d)} & \to & \mathcal{G}_1^{(d-m)} \end{array} \qquad \begin{array}{lccc} \beta_{2_{d,d-m}}: & (V^{(d)},x) & \longrightarrow & (V_2^{(d-m)}, x_{m,2}) \\ & \mathcal{G}^{(d)} & \to & \mathcal{G}_2^{(d-m)}. \end{array}$$

Then:

$$ord_{x_{m,1}} \mathcal{G}_1^{(d-m)} = ord_{x_{m,2}} \mathcal{G}_2^{(d-m)}.$$

Moreover, if $V^{(d)} \leftarrow V^{(d)'}$ is a composition of permissible monoidal transformations, $x' \in Sing\, \mathcal{G}^{(d)'}$ a closed point dominating x, and

$$\begin{array}{ccc} (V^{(d)},x) & \longleftarrow & (U \subset V^{(d)'}, x') \\ \mathcal{G}^{(d)} & & \mathcal{G}^{(d)'} \\ \Big\downarrow & \circlearrowleft & \Big\downarrow \beta'_{d,d-m} \\ (V_j^{(d-m)}, x_{m,j}) & \longleftarrow & (V_j^{(d-m)'}, x'_{m,j}) \\ \mathcal{G}_j^{(d-m)} & & \mathcal{G}_j^{(d-m)'} \end{array}$$

is the corresponding commutative diagram of elimination algebras and admissible projections for $j = 1, 2$, then

$$ord_{x'_{m,1}} \mathcal{G}_1^{(d-m)'} = ord_{x'_{m,2}} \mathcal{G}_2^{(d-m)'},$$

where for $j = 1, 2$, $x'_{m,j} = \beta'_{j_{d,d-m}}(x')$ and $\beta'_{j_{d,d-m}} : U \to V_j^{(d-m)'}$ is the induced projection map as in Theorem 10.8.

THEOREM 11.2. [9, Theorem 13.1] *Let $\mathcal{G}^{(d)}$ be a differential Rees algebra on a smooth d-dimensional scheme $V^{(d)}$ over a field k. Let $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$ and let*

$$I_d = \underbrace{\mathbb{Q}^* \times \mathbb{Q}^* \times \ldots \times \mathbb{Q}^*}_{d-times}$$

ordered lexicographically. Then there is an upper semi-continuous function,

$$\gamma_{\mathcal{G}^{(d)}} : Sing\, \mathcal{G}^{(d)} \to I_d$$

such that:

(i) The level sets of $\gamma_{\mathcal{G}^{(d)}}$ stratify Sing $\mathcal{G}^{(d)}$ in smooth locally closed strata.

(ii) If k is a field of characteristic zero then $\gamma_{\mathcal{G}^{(d)}}$ coincides with the resolution function used for resolution of singularities in characteristic zero.

Given a differential Rees algebra $\mathcal{G}$, in [9, Part 5] it is shown that, using functions derived from those from Theorem 11.1, it is possible to construct a finite sequence of permissible transformations so that the weighted transform of $\mathcal{G}$ is within the monomial case. This would mean, that if $\mathcal{G}$ is of codimensional type $\geq e \geq 1$ in a neighborhood of a point, then its elimination algebra in some e-codimensional smooth scheme is monomial. More precisely, the following result can be proven:

Corollary 11.3. *[9, Part 5] Let $\mathcal{G}^{(n)}$ be a Differential Rees algebra on a smooth n-dimensional scheme $V^{(n)}$ over a perfect field k, and let $x \in$ Sing $\mathcal{G}^{(n)}$ be a simple point. Consider a locally $\mathcal{G}^{(n)}$-admissible projection in a neighborhood of x,*

$$\begin{array}{cccc} \beta_{n,n-\tau}: & V^{(n)} & \rightarrow & V^{(n-\tau_x)} \\ & x & \rightarrow & x'. \end{array} \tag{25}$$

Then there is a finite sequence of permissible transformations of d-dimensional basic objects,

$$(V^{(n)}, \mathcal{G}^{(n)}, E = \{\emptyset\}) \leftarrow (V_1^{(n)}, \mathcal{G}_1^{(n)}, E_1) \leftarrow \ldots \leftarrow (V_r^{(n)}, \mathcal{G}_r^{(n)}, E_r) \tag{26}$$

which induces a finite sequence of permissible transformations of $(d-\tau_x)$-dimensional basic objects,

$$(V^{(n-\tau_x)}, \mathcal{G}^{(n-\tau_x)}, E^{(n-\tau_x)}) \leftarrow (V_1^{(n-\tau_x)}, \mathcal{G}_1^{(n-\tau_x)}, E_1^{(n-\tau_x)}) \leftarrow \ldots \leftarrow (V_r^{(n-\tau_x)}, \mathcal{G}_r^{(n-\tau_x)}, E_r^{(n-\tau_x)}),$$

and commutative diagrams of local admissible projections, elimination algebras y permissible transformations,

$$\begin{array}{ccccccc} (V^{(n)}, x) & \leftarrow & (V_1^{(n)}, x_1) & \leftarrow & \ldots & \leftarrow & (V_r^{(n)}, x_r) \\ \mathcal{G}^{(n)} & & \mathcal{G}_1^{(n)} & & & & \mathcal{G}_r^{(n)} \\ \downarrow & & \downarrow & & & & \downarrow \\ (V^{(n-\tau_x)}, x') & \leftarrow & (V_1^{(n-\tau_x)}, x_1') & \leftarrow & \ldots & \leftarrow & (V_r^{(n-\tau_x)}, x_r') \\ \mathcal{G}^{(n-\tau_x)} & & \mathcal{G}_1^{(n-\tau_x)} & & & & \mathcal{G}_r^{(n-\tau_x)}, \end{array}$$

so that locally, in a neighborhood of $x_r \in$ Sing $\mathcal{G}_r^{(n)}$, $\mathcal{G}_r^{(n-\tau_x)}$ is within the monomial case (i.e., up to integral closure, it can be assumed that $\mathcal{G}_r^{(n-\tau_x)} = \mathcal{O}_{V^{(n-\tau_x)},x_r'}[\mathcal{M}W^s]$, where $\mathcal{M}$ is a monomial ideal supported on $E_r^{(n-\tau_x)}$).

Moreover, the sequence of permissible monoidal transformations (26), and the pull-back of the monomial algebra $\mathcal{O}_{V^{(n-\tau_x)},x_r'}[\mathcal{M}W^s]$ in $\mathcal{O}_{V_r^{(n)},x_r}$ are independent of the local admissible projection fixed in (25).

A detailed study of the monomial case is made in [5], where in addition it is given a short proof of embedded desingularization of surfaces in arbitrary characteristic following this approach (cf. [5, Part III]).

Part 4. Local presentations and elimination of variables. The purpose of this part is to give a short proof of the following proposition:

Proposition 11.4. *Let $V^{(n)}$ be a smooth n-dimensional scheme over a perfect field k, and let $X(\subset V^{(n)})$ be a reduced non-smooth d-dimensional subscheme. Then,* *the stratum corresponding to the maximum value of the Hilbert-Samuel of X can be described locally, in an étale neighborhood of each closed point, as a closed set in an N-dimensional smooth scheme, with $N \leq d$.*

Motivation. In a neighborhood of a closed point $x \in X$, the Hilbert-Samuel stratum can be seen as the singular locus of an algebra over an n-dimensional smooth scheme, say $(V^n, \mathcal{G}, E = \{\emptyset\})$. Moreover $\mathcal{G}$ can be taken to be a differential Rees algebra,

$$\mathcal{G} = \mathcal{O}_{V^{(n)}} \oplus I_1 W \oplus I_2 W^2 \oplus \dots.$$

When the characteristic is zero, the $\tau_{\mathcal{G},x}$ invariant at x is closely related to the previous N: $n - \tau_{\mathcal{G},x} = N$ is the smallest choice so the proposition holds locally at x. Or, in other words, $n - \tau_{\mathcal{G},x} = N$ is the smallest integer so that, locally in a neighborhood of x there is a smooth $n - \tau_{\mathcal{G},x}$-dimensional smooth variety $\overline{V}$ so that $\mathcal{I}(\overline{V}) \subset I_1$.

In the case of characteristic zero, where there is a theory of maximal contact, this means that the lowering of the maximum of the Hilbert-Samuel function in a neighborhood of $x \in X$ is equivalent to finding a resolution of an N-dimensional basic object $(\overline{V}, \overline{\mathcal{G}}, \{\emptyset\})$ with $N \leq d$, which can be defined by restriction. In particular, and using the notation as in [34] and [16], one can attach to the Hilbert-Samuel stratum a d-dimensional general basic object, where d denotes the dimension of X along any closed point of this stratum.

Here we show, with a different approach, that with the same starting point $(V^n, \mathcal{G}, E = \{\emptyset\})$, a new algebra $\overline{\mathcal{G}}$ can be defined over $\overline{V}$. $\overline{V}$ is not a smooth subscheme in the case of positive characteristic, but $\overline{V}$ is smooth and a generic projections $V^n \to \overline{V}$ is defined by successive elimination of $\tau_{\mathcal{G},x}$-variables.

We prove Proposition 11.4 using the so called Hironaka's trick. The argument obviously makes use of the notion of the τ-invariant of a singularity, and also of the existence of local presentations of a Rees algebra in a neighborhood of a simple point. Local presentations will be treated in Section 12, and the proof of the proposition will be addressed in Section 13.

12. Local presentations, τ-sequences and nested sequences. Proposition 12.1 below gives an interesting and useful presentation of a Rees algebra in a neighborhood of a simple closed point. Let $\mathcal{H} \odot \mathcal{L}$ denote the smallest Rees algebra containing the Rees algebras $\mathcal{H}$ and $\mathcal{L}$.

Proposition 12.1. **Local relative presentation.** [5, Proposition 2.11] *Let $x \in Sing\, \mathcal{G}$ be a simple closed point, and let*

$$V^{(d)} \overset{\beta_{d,d-1}}{\longrightarrow} V^{(d-1)}$$

be a $\mathcal{G}$-admissible projection in a neighborhood of x. Let $f_n W^n \in \mathcal{G}$ be an element of order n in $\mathcal{O}_{V^{(d)},x}$, transversal to $\beta_{d,d-1}$. Then, in a suitable neighborhood U of x, $\mathcal{G}$ is, up to integral closure, equal to

$$\mathcal{O}_{V^{(d)}}(U)\left[f_n W^n W^n, D_r(f_n) W^{n-r}; 1 \leq r \leq n-1\right] \odot \beta^*_{d,d-1}(\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}),$$

where D_r runs through all differential operators in $Diff^{\,r}(V^{(d)}/V^{(d-1)})$, with $1 \leq r \leq n-1$. If $\mathcal{G}$ is a differential Rees algebra, then f_n can be chosen so that its initial form defines a linear subspace of codimension one in $\mathbb{T}_{V^{(d)},x}$.

The previous result also has a formulation when $\tau_{\mathcal{G},x} \geq e > 1$, and for a local $\mathcal{G}$-admissible projection $V^{(d)} \overset{\beta_{d,d-e}}{\longrightarrow} V^{(d-e)}$. Here the existence of *τ-sequences* plays a role.

Definition 12.2. Let $\mathcal{G} = \oplus_n I_n W^n$ be a Rees algebra in a d-dimensional smooth scheme V over a field k, let $x \in \operatorname{Sing} \mathcal{G}$ be a simple point, and let k' be the residue field at x. We will say that a set of homogeneous elements $f_1 W^{n_1}, \dots, f_s W^{n_s} \in \mathcal{G}$ is a *$\tau_{\mathcal{G},x}$-sequence of length s* if for $j = 1, \dots, s$:

i. $n_j = p^{e_j}$;
ii. $\mathrm{In}_x f_j, \in Gr_{\mathcal{O}_{V,x}} \simeq k'[Z_1, \dots, Z_d]$ is a k'-linear combination of $Z_1^{p^{e_j}}, \dots, Z_d^{p^{e_j}}$ for some $e_j \in \mathbb{N}$;
iii. The class of $\mathrm{In}_x f_j$ is a regular element at the graded ring $Gr_{\mathcal{O}_{V,x}} / \langle \mathrm{In}_x f_i : i \neq j \rangle$.

By definition, if $f_1 W^{n_1}, \dots, f_s W^{n_s} \in \mathcal{G}$ is a $\tau_{\mathcal{G},x}$-sequence of length s, then $s \leq \tau_{\mathcal{G},x}$. A $\tau_{\mathcal{G},x}$-sequence $f_1 W^{n_1}, \dots, f_s W^{n_s} \in \mathcal{G}$ is said to be a *maximal-$\tau_{\mathcal{G},x}$-sequence* if $\tau_{\mathcal{G},x} = s$.

Remark 12.3. Let $f_1 W^{n_1}, \dots, f_s W^{n_s} \in \mathcal{G}$ be a $\tau_{\mathcal{G},x}$-sequence. If char $k = 0$ then condition (ii) says that $\mathrm{In}_x f_1, \dots, \mathrm{In}_x f_s \in Gr_{\mathcal{O}_{V,x}}$ are linear forms, while condition (iii) means that they are linearly independent. If char $k = p > 0$, then, up to a change of the base field, it can be assumed that $\mathrm{In}_x f_j \in Gr_{\mathcal{O}_{V,x}}$ is some p^{e_j}-th power of a linear form for $j = 1, \dots, s$. Condition (iii) indicates that these linear forms are independent (see 7.2 and 7.3). Notice that if $f_1 W^{n_1}, f_2 W^{n_2} \dots, f_s W^{n_s}$ is a $\tau_{\mathcal{G},x}$-sequence, then so is $(f_1)^p (W^{n_1})^p, f_2 W^{n_2} \dots, f_s W^{n_s}$. In particular it can always be assumed that $n_1 = \dots = n_s$.

When $\mathcal{G}$ is a differential Rees algebra, then there is a maximal $\tau_{\mathcal{G},x}$-sequence at each simple point $x \in \operatorname{Sing} \mathcal{G}$ (see 7.3). However given a permissible monoidal transformation,

$$\begin{array}{ccc} (V^{(d)}, x) & \leftarrow & (V^{(d)'}, x') \\ \mathcal{G} & & \mathcal{G}', \end{array}$$

in general, it is no longer true that the strict transforms of a $\tau_{\mathcal{G},x}$-sequence form a $\tau_{\mathcal{G}',x'}$-sequence. This motivates the introduction of another type of sequences: *nested sequences.*

Definition 12.4. Let $\mathcal{G}^{(d)}$ be a Rees algebra, and let $x \in \operatorname{Sing} \mathcal{G}^{(d)}$ be a simple point with $\tau_{\mathcal{G}^{(d)},x} \geq s$. Suppose that there is a $\mathcal{G}^{(d)}$-admissible projection to some $(d-s)$-dimensional smooth scheme in a neighborhood of x,

$$(V^{(d)}, x) \to (V^{(d-s)}, x_s), \tag{27}$$

and a factorization into admissible projections

(28)
$$\begin{array}{ccccccccc} (V^{(d)}, x) & \overset{\beta_{d,d-1}}{\longrightarrow} & (V^{(d_1)}, x_1) & \to & \dots & \to & (V^{(d-(s-1))}, x_{s-1}) & \overset{\beta_{d-(s-1),d-s}}{\longrightarrow} & (V^{(d-s)}, x_s) \\ \mathcal{G}^{(d)} & & \mathcal{G}^{(d-1)} & & \dots & & \mathcal{G}^{(d-(s-1))} & & \mathcal{G}^{(d-s)}. \end{array}$$

A set of homogeneous elements $f_1 W^{n_1}, f_2 W^{n_2}, \dots, f_s W^{n_s} \in \mathcal{G}^{(d)}$ is said to be a *$\mathcal{G}^{(d)}$-nested sequence relative to the sequence (28)* if

$$f_1 W^{n_1} \in \mathcal{G}^{(d)},\ f_2 W^{n_2} \in \mathcal{G}^{(d-1)}, \dots,\ f_s W^{n_s} \in \mathcal{G}^{(d-(s-1))},$$

and f_i is transversal to $\beta_{d-(i-1),d-i}$ for $i=1,\dots,s$.

If $f_1W^{n_1}, f_2W^{n_2},\dots,f_sW^{n_s}\in\mathcal{G}$ is nested, then

$$\mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_2^{(d-1)},\dots,f_s^{(d-(s-1))}\rangle$$

is a complete intersection (cf. [9, 11.7]). Moreover, if $V^{(d)}\leftarrow V^{(d)'}$ is a permissible monoidal transformation, $\mathcal{G}^{(d)'}$ is the weighted transform of $\mathcal{G}^{(d)}$ in $V^{(d)'}$, and $x'\in\operatorname{Sing}\mathcal{G}^{(d)'}$ is a closed point dominating x, then the strict transforms of $f_1,f_2,\dots,f_s$ in $V^{(d)'}$, $f_1',f_2',\dots,f_s'$, form a $\mathcal{G}^{(d)'}$-nested sequence relative to the transform of sequence (28) (see Theorem 10.8). In [9, Proposition 11.8 and Corollary 11.9] it is shown that if $\mathcal{G}$ is a differential Rees algebra, then for any simple point $x\in\operatorname{Sing}\mathcal{G}$ there is a maximal τ_x-sequence that is also nested relative to some sequence of admissible projections.

Remark 12.5. Observe that if $x\in\operatorname{Sing}\mathcal{G}^{(d)}$ is a closed point with $\tau_{\mathcal{G}^{(d)}x}=s\geq 1$, then it follows that there is local admissible projection as in (27), a factorization as in (28) and a $\mathcal{G}^{(d)}$-nested sequence $f_1W^{t_1},\dots,f_sW^{t_s}$ such that, locally in a neighborhood U of x, up to integral closure, $\mathcal{G}^{(d)}$ can be assumed to be equal to:

$$\mathcal{O}_{V^{(n)}}(U)\left[f_1W^{t_1}, D_1^{r_1}(f_1)W^{t_1-r_1}\dots,f_sW^{t_s}, D_s^{r_s}(f_s)W^{t_s-r_s}\right]\odot\mathcal{G}^{(d-s)}$$

where $D_i^{r_i}$ runs through all differential operators in $\operatorname{Diff}^{\,r_i}(V^{(d-(i-1))}/V^{(d-i)})$ for $i=1,\dots,s$ and $r_i=1,\dots,t_i-1$.

13. Proof of Proposition 11.4. Assume the hypotheses of Proposition 11.4. Let $U\subset V^{(n)}$ be an étale neighborhood of a closed point x in the highest Hilbert-Samuel stratum of X, denoted here by $HS_{X,x}$ (this is a closed set in U). Then there is a standard basis $f_1,\dots,f_m\in\mathcal{O}_{V^{(n)}}(U)$ of $\mathcal{I}(X)$ such that, if F_{b_i} denotes the maximum multiplicity locus of $V(\langle f_i\rangle)$ for $i=1,\dots,m$, then

$$HS_{X,x}=\cap F_{b_i}.$$

And moreover, if $V^{(n)}\leftarrow V^{(n)'}$ is a monoidal transformation with center $Y\subset HS_{X,x}$, and if $x'\in X'\subset V^{(d)'}$ is a closed point in the strict transform of X, X', dominating x where the Hilbert-Samuel Function has the same value as in x, then in a neighborhood of x',

$$HS_{X',x'}=\cap F'_{b_i}$$

where F'_{b_i} denotes the maximum multiplicity locus of the strict transform of $V(\langle f_i\rangle)$, say $V(\langle f_i'\rangle)$. This statement is a Rees algebra version of a well known result of Hironaka, usually referred to as the idealistic exponent associated to the Hilbert-Samuel stratum (cf. [26]).

Let $\mathcal{G}^{(n)}$ be the differential Rees algebra generated by $f_1W^{b_1},\dots,f_nW^{b_n}$. Then

$$\operatorname{Sing}\mathcal{G}^{(n)}=HS_{X,x}.$$

Note that $\operatorname{Sing}\mathcal{G}^{(n)}$ has dimension strictly smaller than d because X is reduced and non-smooth.

Since $\mathcal{G}^{(n)}$ is a differential Rees algebra, there is a regular system of parameters $z_1, \dots, z_n \in \mathcal{O}_{V^{(n)},x}$ and elements $g_1 W^{p^{e_1}}, \dots, g_{\tau_x} W^{p^{e_{\tau_x}}} \in \mathcal{G}^{(n)}$ such that for $i = 1, \dots, \tau_x$,

$$\mathrm{In}_x g_i = \mathrm{In}_x z_i^{p^{e_i}} \in \mathrm{Gr}_{V^{(n)},x}, \tag{29}$$

i.e., the $g_1, \dots, g_{\tau_x}$ form a maximal $\tau_{\mathcal{G}^{(n)},x}$-sequence. We assume this sequence to be nested and attached to some sequence of local admissible projections

(30)

$$\begin{array}{ccccccc} (V^{(n)}, x_0 = x) & \overset{\beta_{n,n-1}}{\longrightarrow} & \dots & \to & (V^{(n-(s-1))}, x_{s-1}) & \overset{\beta_{n-(\tau_x-1),n-\tau_x}}{\longrightarrow} & (V^{(n-\tau_x)}, x_{\tau_x}) \\ g_1 W^{p^{e_1}} \in \mathcal{G}^{(n)} & & \dots & & g_{\tau_x} W^{p^{e_{\tau_x}}} \in \mathcal{G}^{(n-(\tau_x-1))} & & \mathcal{G}^{(n-\tau_x)}. \end{array}$$

This defines, by composition defines a local admissible projection,

$$\beta_{n,n-\tau_x} : V^{(n)} \to V^{(n-\tau_x)}.$$

Up to integral closure, we can assume that the equality

$$\mathcal{G} = \mathcal{O}_{V^{(n)}}[g_1 W^{p^{e_1}}, \dots, g_{\tau_x} W^{p^{e_{\tau_x}}}, D^{\alpha_{1,j}} g_1 W^{p^{e_1}-\alpha_{1,j}}, \dots, D^{\alpha_{\tau_x,j}} g_{\tau_x} W^{p^{e_{\tau_x}}-\alpha_{\tau_x,j}}] \odot \mathcal{G}^{(n-\tau_x)}$$

holds in a neighborhood of x, where $D^{\alpha_{i,j}}$ runs over all differentials relative to the morphism $\beta_{n-(i-1),n-i} V^{(n-(i-1))} \to V^{(n-i)}$, and $\mathcal{G}^{(n-\tau_x)}$ is not simple at $\beta_{n,n-\tau_x}(x)$ (see Remark 12.5). Theorem 8.7 ensures that $\mathcal{G}$ is weakly equivalent to

$$\mathcal{G}' = \mathcal{O}_{V^{(n)}}[g_1 W^{c_1}, \dots, g_{\tau_x} W^{\tau_x}] \odot \mathcal{G}^{(n-\tau_x)}. \tag{31}$$

Denote by m_x the maximal ideal in $\mathcal{O}_{V^{(n)},x}$. Then observe that by condition (29), for $i = 1, \dots, \tau_x$,

$$g_i W^{p^{e_i}} \in \langle z_i \rangle W \odot m_x^{a_i} W^{d_i}$$

with $a_i > d_i$ (since $\mathcal{G}^{(n-\tau_x)}$ is not simple at $\beta_{n,n-\tau_x}(x)$), and whence, by (31),

$$\mathcal{G}' \subset \langle z_1, \dots, z_{\tau_x} \rangle W \odot \mathcal{H} = \mathcal{J} \odot \mathcal{H},$$

where $\mathcal{H}$ is not simple at x, and $\mathcal{J} = \langle z_1, \dots, z_{\tau_x} \rangle W$.

Consider the multiplication by an affine line, $V^{(n)} \times \mathbb{A}^1_k$, and the natural extensions of $\mathcal{G}'$, $\mathcal{J}$, and $\mathcal{H}$ by the corresponding projection, say $\mathcal{G}''$, $\mathcal{J}''$, and $\mathcal{H}''$ respectively. Then the line $\mathbb{L} = V(z_1, \dots, z_n) \subset V^{(d)} \times \mathbb{A}^1_k$ is contained in Sing $\mathcal{G}''$. Note here that $X \times \mathbb{A}^1_k$ is is a closed subscheme in $V^{(n)} \times \mathbb{A}^1_k$, of dimension $d+1$, and that Sing $\mathcal{G}''$ is the stratum corresponding to the highest value of its Hilbert-Samuel function, properly included in $X \times \mathbb{A}^1_k$ if X is non-smooth and reduced.

Now blow-up at a closed point $x_0 = V(\langle z_1, \dots, z_n, t \rangle) \in \mathbb{L}$, $V_1 \to V^{(n)} \times \mathbb{A}^1_k$ (here t is a local parameter in some point of $\mathbb{A}^1_k$). Continue with a finite sequence of blow-ups at the intersection of the successive strict transforms of $\mathbb{L}$, $\mathbb{L}_i$, and the exceptional divisors E_i,

$$\begin{array}{ccccccccc} V^{(n)} \times \mathbb{A}^1_k & \leftarrow & V_1^{(n+1)} & \leftarrow & V_2^{(n+1)} & \leftarrow & \dots & \leftarrow & V_N^{(n+1)} \\ x_0 & & E_1 \cap \mathbb{L}_1 & & E_2 \cap \mathbb{L}_2 & & \dots & & E_N, \mathbb{L}_N. \end{array}$$

Then, for each $i = 1, \dots, N$, the weighted transform of $\mathcal{G}''$ in $V_i^{(n+1)}$, $\mathcal{G}''_i$, is contained in

$$\mathcal{J}''_i \odot \mathcal{I}(E_i)^{s_i} \mathcal{H}''_i,$$

where $\mathcal{J}_i''$ and $\mathcal{H}_i''$ denote the weak transforms of $\mathcal{J}''$ and $\mathcal{H}''$ in $V_i^{(n+1)}$, and

$$1 < s_1 < s_2 < \ldots < s_N,$$

since $\mathcal{H}''$ is not simple at x_0. Thus, for N large enough,

$$\text{Sing } \mathcal{G}_N'' \supset \text{Sing } \mathcal{J}_N \cap E_N,$$

where the right hand side is a $(\tau_x + 1)$-codimensional closed subscheme. Therefore, Sing $\mathcal{G}_N''$ contains an $n + 1 - (\tau_x + 1) = n - \tau_x$-dimensional scheme. But $n - \tau_x < \dim X + 1 = d + 1$ because Sing $\mathcal{G}_N''$ is the closed Hilbert-Samuel stratum of a $d+1$-dimensional non-smooth reduced scheme. $\square$

Part 5. Appendix: Defining smooth morphisms on smooth schemes. Given a closed point closed point $x \in V^{(d)}$, it is fairly easy to define a smooth morphism, say $\beta : V^{(d)} \longrightarrow V^{(d')}$, for any positive integer $d' \leq d$. At least at an étale neighborhood of x. To this end note that a regular system of parameters $\{x_1, \dots, x_d\}$ in $\mathcal{O}_{V^{(d)},x}$ defines an inclusion of a polynomial ring in d variables, say $k'[x_1, \dots, x_d]$, where k' is the residue field at x. This in turn says that $(V^{(d)}, x)$ is an étale neighborhood of $(\mathbb{A}^{(d)}, \mathbb{O})$; and plenty of smooth morphisms (in fact, plenty of surjective linear transformations) $(\mathbb{A}^{(d)}, \mathbb{O}) \longrightarrow (\mathbb{A}^{(d')}, \mathbb{O})$ can be constructed. Finally set $\beta : V^{(d)} \longrightarrow V^{(d')}$ as the composition on $V^{(d')} = \mathbb{A}^{(d')}$.

This simple construction is also useful in showing the existence of local retractions: Suppose now that $x \in X^{(d')}$ and $X^{(d')}$ is a smooth subscheme in $V^{(d)}$ locally defined by $x_{d'+1} = 0, \dots, x_d = 0$. There is a natural restriction of β, say $\delta : X^{(d')} \longrightarrow \mathbb{A}^{(d')} = Spec(k'[x_1, \dots, x_{d'}])$, obtained by taking pull-back of the corresponding closed immersion $\mathbb{A}^{(d')} \subset \mathbb{A}^{(d)}$. This restricted map δ is étale at x and taking the fiber product it defines

(32)
$$\begin{tikzcd}
V^{(d)} \arrow[d, "\beta"] & V_1^{(d)} \arrow[l, "\delta_1"'] \arrow[d, "\beta_1"] \\
\mathbb{A}^{(d')} & X^{(d')} \arrow[l, "\delta"']
\end{tikzcd}$$

Here δ_1 is étale, and the smooth morphism β_1 admits a section with image ${\delta_1}^{-1}(X^{(d')})$. This section corresponds to the diagonal map defined by the inclusion $X^{(d')} \subset V^{(d)}$. Finally β_1, together with the section, define a retraction of $V_1^{(d)}$ on $X^{(d')}$.

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