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DESINGULARIZATION AND SINGULARITIES OF SOME MODULI SCHEME OF SHEAVES ON A SURFACE*

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Abstract. Let X be a nonsingular projective surface over $\mathbb{C}$, and H_- and H_+ be ample line bundles on X in adjacent chamber of type (c_1, c_2). Let $0 < a_- < a_+ < 1$ be adjacent minichambers, which are defined from H_- and H_+, such that the moduli scheme $M(H_-)$ of rank-two a_--stable sheaves with Chern classes (c_1, c_2) is non-singular. We shall construct a desingularization of $M(a_+)$ by using $M(a_-)$. As an application, we study whether singularities of $M(a_+)$ are terminal or not in some cases where X is ruled or elliptic.

Key words. Moduli scheme of stable sheaves on a surface, singularities, desingularization.

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1. Introduction. Let X be a projective non-singular surface over $\mathbb{C}$, H an ample line bundle on X. Denote by $M(H)$ the coarse moduli scheme of rank-two H-stable sheaves with fixed Chern class $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbb{Z}$. In this paper we think about singularities and desingularization of $M(H)$ from the view of wall-crossing problem of H and $M(H)$.

Let H_- and H_+ be ample line bundles on X separated by only one wall of type (c_1, c_2). For a parameter $a \in (0, 1)$, one can define the a-stability of sheaves in such a way that a-stability of sheaves with fixed Chern class equals H_--stability (resp. H_+-stability) if a is sufficiently close to 0 (resp. 1) , and there is a coarse moduli scheme $M(a)$ of rank-two a-stable sheaves with Chern classes (c_1, c_2). Let a_- and $a_+ \in (0, 1)$ be parameters which are separated by only one miniwall. Assume $M_- = M(a_-)$ is non-singular. One can find such a_- when X is ruled or elliptic. We construct a desingularization $\tilde{\pi}_+ : \tilde{M} \to M_+$ of $M_+ = M(a_+)$ by using M_- and wall-crossing methods, and apply it to consider whether singularities of M_+ are terminal or not when X is ruled or elliptic.

Let $\overline{M}(H)$ denote the Gieseker-Maruyama compactification of $M(H)$. By [10], when X is minimal and its Kodaira dimension is positive, $\overline{M}(H)$ has the nef canonical divisor if $\dim \overline{M}(H)$ equals its expected dimension and if H is sufficiently close to K_X. Thus, to understand minimal models of a moduli scheme of stable sheaves, it can be meaningful to study singularities on $M(H)$. As a problem to be solved, it is desirable to extend results in this article to the case where M_- is not necessarily non-singular but its singularities are terminal (Remark 2.5).

NOTATION. *For a k-scheme S, X_S is $X \times S$ and* $\mathrm{Coh}(X_S)$ *is the set of coherent sheaves on X_S. For $s \in S$ and $E_S \in \mathrm{Coh}(X_S)$, E_s means $E \otimes k(s)$. For E and $F \in \mathrm{Coh}(X)$,* $\mathrm{ext}^i(E, F) := \dim \mathrm{Ext}^i_X(E, F)$ *and* $\hom(E, F) = \dim \mathrm{Hom}_X(E, F)$. $\mathrm{Ext}^i_X(E, E)^0$ *indicates* $\mathrm{Ker}(\mathrm{tr} : \mathrm{Ext}^i(E, E) \to H^0(\mathcal{O}_X))$. *For $\eta \in \mathrm{NS}(X)$, we define $W^\eta \subset \mathrm{Amp}(X)$ by* $\{H \in \mathrm{Amp}(X) \mid H \cdot \eta = 0\}$.

2. Desingularization of M_+ by using M_- . We begin with background materials. Let H_- and H_+ be ample divisors lying in neighboring chambers of type $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbb{Z}$, and H_0 an ample divisor in the wall W of type (c_1, c_2) which lies in the closure of chambers containing H_- and H_+ respectively. (Refer to [8] about

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the definition of wall and chamber.) Assume that $M = H_+ - H_-$ is effective. For a number $a \in [0,1]$ one can define the a-stability of a torsion-free sheaf E using

$$P_a(E(n)) = \{(1-a)\chi(E(H_-)(nH_0)) + a\chi(E(H_+(nH_0)))\}/\operatorname{rk}(E).$$

There is the coarse moduli scheme $\bar{M}(a)$ of rank-two a-semistable sheaves on X with Chern classes (c_1, c_2). Denote by $M(a)$ its open subscheme of a-stable sheaves. When one replace $H_\pm$ by $NH_\pm$ if necessary, $M(0)$ (resp. $M(1)$) equals the moduli scheme of H_--semistable (resp. H_+-semistable) sheaves. There exist finite numbers $a_1 \dots a_l \in (0,1)$ called minichambers such that $\bar{M}(a)$ and $M(a)$ changes only when a passes a miniwall. Refer to [2, Section 3] for details. Fix numbers a_- and a_+ separated by the only one miniwall, and indicate $\bar{M}_\pm = \bar{M}(a_\pm)$ and $M_\pm = M(a_\pm)$ for short. From [9, Section 2], the subset

$$\bar{M}_- \supset P_- = \big\{[E] \bigm| E \text{ is not } a_+\text{-semistable}\big\}$$
$$\big(\text{resp.} \quad \bar{M}_+ \supset P_+ = \big\{[E] \bigm| E \text{ is not } a_-\text{-semistable}\big\}\big)$$

is contained in M_- (resp. M_+) and endowed with a natural closed subscheme structure of M_- (resp. M_+). Let η be a element of

$$A^+(W) = \big\{\eta \in \operatorname{NS}(X) \bigm| \eta \text{ defines } W,\ 4c_2 - c_1^2 + \eta^2 \geq 0 \text{ and } \eta \cdot H_+ > 0)\big\}.$$

After [2, Definition 4.2] we define

$$T_\eta = M(1, (c_1+\eta)/2, n) \times M(1, (c_1-\eta)/2, m),$$

where n and m are numbers defined by

$$n + m = c_2 - (c_1 - \eta^2)/4 \quad \text{and} \quad n - m = \eta \cdot (c_1 - K_X)/2 + (2a_0 - 1)\eta \cdot (H_+ - H_-),$$

and $M(1, (c_1+\eta)/2)$ is the moduli scheme of rank-one torsion-free sheaves on X with Chern classes $((c_1+\eta)/2, n)$. If F_{T_η} (resp. G_{T_η}) is the pull-back of a universal sheaf of $M(1, (c_1+\eta)/2, n)$ (resp. $M(1, (c_1-\eta)/2, m)$) to X_{T_η}, then we have an isomorphism

$$P_- \simeq \coprod_{\eta \in A^+(W)} \mathbf{P}_{T_\eta}\left(Ext^1_{X_{T_\eta}/T_\eta}(F_{T_\eta}, G_{T_\eta}(K_X))\right) \tag{1}$$

from [9, Section 5].

Proposition 2.1 ([9] Proposition 4.9). *The blowing-up of M_- along P_- agrees with the blowing-up of M_+ along P_+. So we have blowing-ups*

$$M_- \xleftarrow{\pi_-} B_{P_-}(M_-) = B_{P_+}(M_+) \xrightarrow{\pi_+} M_+.$$

By taking $4c_2 - c_1^2$ to be sufficiently large with respect to H_- and H_+, we can assume from [6] and [7] that $M_\pm \supset \operatorname{Sing}(M_\pm) := \{E \mid \operatorname{ext}^2(E,E)^0 \neq 0\}$ satisfies $\operatorname{codim}(M_\pm, \operatorname{Sing}(M_\pm)) \geq 2$ and that $P_\pm \subset M_\pm$ is nowhere dense, and hence both M_- and M_+ are normal l.c.i. schemes and birationally equivalent. Suppose that $A^+(W) = \{\eta\}$ for simplicity and denote $T_\eta = T$. From Hironaka's desingularization theorem, there is a sequence of blowing-ups

$$M_N \longrightarrow M_{N-1} \dots \longrightarrow M_- \tag{2}$$

along non-singular centers $Z_i \subset M^i$ such that the ideal sheaf of $\mathcal{O}_{M_N}$ generated by pull-back of the ideal sheaf of $P_- \subset M_-$ is invertible.

Claim 2.2. *If we set*

$$l_1 = \max\{\mathrm{ext}^1(F_t, G_t(K_X)) \bigm| t \in T\},$$

then we can take the center Z_i in (2) *so that the dimension of Z_i is not greater than $l_1 - 1 + \dim T$.*

Proof. Since one can readily show $\mathrm{ext}^2(F_t, G_t(K_X)) = \hom(G_t, F_t) = 0$ for all $t \in T$, (1) implies that P_- is embedded in a $\mathbf{P}^{l_1}$-bundle over T. Thus for $s \in P_-$, the rank of $\Omega_{P_-} \otimes k(s)$ is not greater than $\dim T + l_1 - 1$. From the exact sequence

$$CN_{P_-/M_-} \longrightarrow \Omega_{M_-}|_{P_-} \longrightarrow \Omega_{P_-} \longrightarrow 0,$$

we can choose local coordinates $g_i \in \mathcal{O}_{M_-,s}$ so that g_i lies in $I_{P_-,s}$ for $i \le \dim M_- - (\dim T + l_1 - 1)$. From [1, Thm. 1.10], one can choose the center Z_i in such a way that the ideal sheaf of Z_i contains the weak transform of I_{P_-} by $M_i \to M_-$, say I_i. If y is a local generator of the exceptional divisor of $M_1 \to M_-$, then g_i/y ($i \le \dim M_- - (\dim T + l_1 - 1)$) are partial coordinating parameters of M_1 and belong to I_1. Since I_{Z_1} contains I_1, the claim holds for $i = 1$. For general i, one can verify the claim in the same way. □

From Proposition 2.1, we obtain a morphism

$$M_N \longrightarrow B(M) := B_{P_-}(M_-) = B_{P_+}(M_+) \longrightarrow M_+$$

and a diagram

(3)
$$\begin{array}{ccccc} & & \tilde{M} := M_N & & \\ & \swarrow^{\tilde{\pi}_-} & \downarrow{\scriptstyle \pi} & \searrow^{\tilde{\pi}_+} & \\ M_- & \xleftarrow[\pi_-]{} & B(M) & \xrightarrow[\pi_+]{} & M_+ \end{array}$$

Therefore we can regard $\tilde{M}$ as a desingularization of M_+.

Next let us calculate $K_{\tilde{M}} - \tilde{\pi}_+^* K_{M_+}$. If we denote by $D_i \subset \tilde{M}$ the pull-back of the exceptional divisor of $M^i \to M^{i-1}$, then

(4)
$$K_{\tilde{M}} - \tilde{\pi}_-^* K_{M_-} = \sum_i [\dim M_- - \dim Z_i - 1]\, D_i.$$

Next consider $\tilde{\pi}_-^*(K_{M_-}) - \tilde{\pi}_+^*(K_{M_+})$. By the proof of Proposition 2.1, which uses elementary transform, we have the following.

Proposition 2.3. *Denote the exceptional divisor $\pi_-^{-1}(P_-) = \pi_+^{-1}(P_+) \subset B(M)$ by D. Suppose we have a universal family $E^-_{M_-} \in \mathrm{Coh}(X_{M_-})$ of M_- and a universal family $E^+_{M_+} \in \mathrm{Coh}(X_{M_+})$ of M_+. If $p : D \to P_+ \to T$ is a natural map, then there are line bundles $L_\pm$ on $P_\pm$ and a line bundle L_0 on $B(M)$ such that we have exact sequences*

(5)
$$0 \longrightarrow \pi_+^* E^+_{M_+} \otimes L_0 \longrightarrow \pi_-^* E^-_{M_-} \xrightarrow{f} p^* G_T \otimes \pi_+^* L_+ \longrightarrow 0$$

in $\mathrm{Coh}(X_{B(M)})$ *and*

$$0 \longrightarrow \pi^* F_T \otimes \pi_-^* L_- \longrightarrow \pi_-^*(E_{M_-}^-)|_{X_D} \stackrel{f|D}{\longrightarrow} p^* G_T \otimes \pi_+^* L_+ \longrightarrow 0 \tag{6}$$

in $\mathrm{Coh}(X_D)$.

The exact sequence (6) is the relative a_+-Harder Narashimhan filtration of $E_{M_-}^-$. Here we remark that generally a universal family of M_- exists only ètale-locally, but one can generalize this proposition to general case with straightforward labor. Suppose $L_\pm$ and L_0 in this proposition are trivial for simplicity. From (5)

$$\begin{aligned}
&\tilde{\pi}_-^* K_{M_-} - \tilde{\pi}_+^* K_{M_+}\\
&\quad= \pi_-^* \det \mathbf{R}\mathcal{H}om_{X_{M_-}/M_-}(E_{M_-}^-, E_{M_-}^-) - \pi_+^* \det \mathbf{R}\mathcal{H}om_{X_{M_+}/M_+}(E_{M_+}^+, E_{M_+}^+)\\
&\quad= \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(\pi_-^* E_{M_-}^-, \pi_-^* E_{M_-}^-)\\
&\qquad - \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(\pi_+^* E_{M_+}^+, \pi_+^* E_{M_+}^+)\\
&\quad= \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^-, E_{B(M)}^+) + \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^+, \pi^* G_T)\\
&\qquad - \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^-, E_{B(M)}^+)\\
&\qquad + \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(\pi_+^* G_T, E_{B(M)}^+)\\
&\quad= \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^-, G_D) + \det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(G_D, E_{B(M)}^+).
\end{aligned}$$

If $i : D \hookrightarrow B(M)$ is inclusion, then by (6)

$$\begin{aligned}
\det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^-, G_D) &= \det i_* \mathbf{R}\mathcal{H}om_{X_D/D}(E_{B(M)}^-|_D, G_D) =\\
&\det i_* \mathbf{R}\mathcal{H}om_{X_D/D}(F_D, G_D) + \det i_* \mathbf{R}\mathcal{H}om_{X_D/D}(G_D, G_D).
\end{aligned} \tag{7}$$

Since $\det \mathcal{O}_D = D$, (7) equals $[\chi(F_t, G_t) + \chi(G_t, G_t)]\, D$ for any $t \in D$. By the Serre duality

$$\begin{aligned}
&\det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(G_D, E_{B(M)}^+)\\
&\quad= \det \mathbf{R}\mathcal{H}om_{B(M)}(\mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^+, G_D(K_X)), \mathcal{O}_{B(M)})\\
&\quad= -\det \mathbf{R}\mathcal{H}om_{X_{B(M)}/B(M)}(E_{B(M)}^+, G_D(K_X))\\
&\quad= -\det i_* \mathbf{R}\mathcal{H}om_{X_D/D}(E_{B(M)}^+|_D, G_D(K_X))\\
&\quad= -[\chi(F_t, G_t(K_X)) + \chi(G_t, G_t(K_X))]\, D = -[\chi(G_t, F_t) + \chi(G_t, G_t)]\, D.
\end{aligned}$$

Therefore

$$\pi_-^* K_{M_-} - \pi_+^* K_{M_+} = [\chi(F_t, G_t) - \chi(G_t, F_t)] D = 2(c_1(F_t) - c_1(G_t)) \cdot K_X. \tag{8}$$

Moreover, we put

$$\tilde{\pi}^* D = \sum_{i=0}^{N-1} \lambda_i\, D_i. \tag{9}$$

When $\dim M_- - (l_1 - 1 + \dim T) > 0$, all λ_i are 1. Indeed, the proof of Claim 2.2 says that some element $g \in I_{P_-}$ satisfies that if y is a local generator of the exceptional divisor of $M_1 \to M_-$, then g/y is a partial coordinating parameter of M_1. Thus the pull-back of I_{P_-} by $M_1 \to M_-$ is divided by y, but cannot be divided by y^2, which implies $\lambda_1 = 1$. One can show $\lambda_i = 1$ similarly. Consequently, from (4), (8) and (9), we have shown the following.

Proposition 2.4. *In the diagram* (3) *it holds that*

$$K_{\tilde{M}} - \tilde{\pi}_+^* K_{M_+} = \sum_{i=0}^{N-1} [\dim M_- - \dim Z_i - 1 + \lambda_i 2(c_1(F_t) - c_1(G_t)) \cdot K_X]\, D_i. \tag{10}$$

with $\lambda_i \geq 1$. *If* $\dim M_- > l_1 - 1 + \dim T$ *then* $\lambda_i = 1$ *and*

$$\dim M_- - \dim Z_i - 1 + 2\lambda_i (c_1(F_t) - c_1(G_t)) \cdot K_X \geq \dim M_- - (l_1 - 1 + \dim T) - 1 + 2(c_1(F_t) - c_1(G_t))K_X.$$

One can use this proposition to verify whether singularities in M_+ is terminal or not.

Remark 2.5. It is desirable to extend this article to the case where M_- is not necessarily non-singular but its singularities are terminal. It is a problem that we can not use (4) since M_- is not non-singular.

3. Examples: ruled or elliptic surface. We shall give examples of $M_\pm$ with M_- non-singular. If a surjective morphism $X \to C$ to a nonsingular curve C exists, then by [3, p.142] we have a (c_1, c_2)-suitable polarization, that is, an ample line bundle H such that H does not lie on any wall of type (c_1, c_2), and for any wall $W = W^\eta$ of type (c_1, c_2), we have $\eta \cdot f = 0$ or $\mathrm{Sign}(f \cdot \eta) = \mathrm{Sign}(H \cdot \eta)$. From [3, p.159, p.201], if X is a ruled surface or an elliptic surface, then any rank-two sheaf E of type (c_1, c_2) which is stable respect to (c_1, c_2)-suitable polarization is good, i.e. $\mathrm{Ext}^2(E, E)^0 = 0$.

(A) First we suppose that X is a (minimal) ruled surface. When $c_1 \cdot f$ is odd $M(H)$ is empty for (c_1, c_2)-suitable polarization. Thus we assume $c_1 = 0$. If a rank-two sheaf E of type (c_1, c_2) is stable with respect to a polarization H such that $H \cdot K_X < 0$, then E is good and so $M(H)$ is nonsingular. Hence we assume that $W^{K_X} \cap \mathrm{Amp}(X) \neq \emptyset$, so $2 \leq g = g(C)$ and $e(X) \leq 2g - 2$ from the description of $\mathrm{Amp}(X)$ [4, Prop. V.2.21]. Since $\dim \mathrm{NS}(X) = 2$, if we move polarization H from a (c_1, c_2)-suitable one, then $M(H)$ may begin to admit singularities when H passes the wall W^{K_X}. Let H_- and H_+ be ample line bundles separated by only one wall W^{K_X}. $M(H_-)$ is non-singular, and $E^+ \in \mathbf{P}_+$ has a non-trivial exact sequence

$$0 \longrightarrow G = L \otimes I_{Z_l} \longrightarrow E^+ \longrightarrow F = L^{-1} \otimes I_{Z_r} \longrightarrow 0 \tag{11}$$

with $-2L \sim mK_X$. About this filtration we have $\mathrm{Ext}^2_-(E^+, E^+) = 0$ since $p_g(X) = 0$ (See [5, p. 49] for $\mathrm{Ext}_\pm$), and

$$\begin{aligned}\mathrm{ext}^2(E^+, E^+) = \mathrm{ext}^2_+(E^+, E^+) &= \mathrm{ext}^2(L \otimes I_{Z_l}, L^{-1} \otimes I_{Z_r}) \\ &= \hom(I_{Z_r}, \mathcal{O}(K_X + 2L) \otimes I_{Z_l}).\end{aligned}$$

Since W^{K_X} defines a wall, $H^0(\mathcal{O}(K_X + 2L)) = 0$ unless $2L + K_X = 0$. Hence $\mathrm{ext}^2(E^+, E^+)^0 \neq 0$ if and only if $-2L = K_X$ and $Z_l \subset Z_r$. As a result when one defines a-stability using $H_\pm$,

$$\chi^a(E^+) - \chi^a(L \otimes I_{Z_l}) = Aa + B + l(Z_l)$$

for some constant A and B, and so the moduli scheme $M(a)$ of a-stable sheaves begins to admit singularities just when a passes a miniwall a_0 defined by

$$l(Z_l) = \begin{cases} c_2/2 - (g-1) & \text{if } c_2 \text{ is even} \\ (c_2-1)/2 - (g-1) & \text{if } c_2 \text{ is odd.} \end{cases}$$

Let a_- and a_+ be minichambers separated by only one miniwall a_0. $M(a_+) = M_+$ has singularities along $P_+ \times_T T'$, where

$$T' = \left\{ (L \otimes I_{Z_l}, L^{-1} \otimes I_{Z_r}) \;\middle|\; -2L = K_X \right\}_{red} \subset M(1, K_X/2, l(Z_l)) \times M(1, -K_X/2, l(Z_r)).$$

(B) Suppose that X is an elliptic surface with a section σ and $c_1 = \sigma$. In contrast to ruled surfaces, $K_X^2 = 0$ and so $W^{K_X} \cap \mathrm{Amp}(X)$ is always empty, though one can study some singularities appearing in $M(H)$ by Proposition 2.4. Let $\pi : X \to C$ be an elliptic fibration, $f \in \mathrm{NS}(X)$ its fiber class, $d = -\deg R^1\pi_*(\mathcal{O}_X) - \sigma^2 \geq 0$. We have a natural map to a ruled surface $\kappa : X \to \mathbf{P}(\pi_*(\mathcal{O}(2\sigma))) = \mathbf{P}(\mathcal{E}_2)$. Since $\kappa_*(\sigma)$ is a section of $\mathbf{P}(\mathcal{E}_2)$, and since the pull-back of an ample line bundle by a finite map is ample, $L = af$ satisfies $W^{2L-c_1} \cap \mathrm{Amp}(X) \neq \emptyset$ if $a > 0$ from the description of the ample cone of a ruled surface. Let c_1 be σ and $c_2 = (c_1 - L) \cdot L = a$. Then any sheaf E with non-trivial exact sequence

$$0 \longrightarrow F = L \longrightarrow E \longrightarrow G = L^{-1} \otimes c_1 \longrightarrow 0, \tag{12}$$

whose Chern class equals (c_1, c_2), is stable with respect to a (c_1, c_2)-suitable ample line bundle. Indeed, $(2L - c_1) \cdot f < 0$ and so $\pi_*(\mathcal{O}(2L - c_1)) = 0$ and $R^1\pi_*(\mathcal{O}(2L - c_1))$ commutes with base change. Thus the exact sequence

$$0 \longrightarrow H^1(C, \pi_*(\mathcal{O}(2L - c_1))) \longrightarrow H^1(X, \mathcal{O}(2L - c_1)) \longrightarrow H^0(E, R^1\pi_*(\mathcal{O}(2L - c_1)))$$

shows that the restriction of the exact sequence (12) to a general fiber is non-trivial, and so a corollary of Artin's theorem for vector bundles on an elliptic curve [3, p. 89] and a basic property of a suitable polarization [3, p. 144] deduce that E is stable with respect to a suitable polarization. Thereby such E is good. Let H_- and H_+ be ample line bundles which lie in no wall of type (c_1, c_2) with $(2L - c_1) \cdot H_- < 0 < (2L - c_1) \cdot H_+$. One can define a-stability by them. Let a_0 be a miniwall such that $\chi^{a_0}(\mathcal{O}(L)) = \chi^{a_0}(\mathcal{O}(2L - c_1))$, $a_- < a_0 < a_+$ minichambers, and $M_\pm = M(a_\pm)$. Then some connected components of $P_- \subset M_-$ contains any sheaf E with non-trivial exact sequence (12), and some neighborhood of them in M_- is non-singular. It induces a desingularization of some open neighborhood of connected components $\mathcal{K}_+$ of P_+ consisting of sheaves E^+ with a non-trivial exact sequence

$$0 \longrightarrow L^{-1} \otimes c_1 \longrightarrow E^+ \longrightarrow L \longrightarrow 0$$

as in Section 2.

We have in case of (A) $\mathrm{ext}^1(G, F) \leq 1$, and in case of (B) $\mathrm{ext}^1(G, F) = h^0(c_1 - 2L + K_X) - \chi(c_1 - 2L) \leq 2c_2 + C(X)$ with some constant $C(X)$ independent of c_2 because $h^0(c_1 - 2L + K_X) = 0$ if $a = c_2$ is sufficiently large. Thus in both cases one can show that, if c_2 is sufficiently large, then all singularities of M_+ along above-mentioned sheaves are terminal.

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