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CALIBRATED ASSOCIATIVE AND CAYLEY EMBEDDINGS*

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Abstract. Using the Cartan-Kähler theory, and results on real algebraic structures, we prove two embedding theorems. First, the interior of a smooth, compact 3-manifold may be isometrically embedded into a G_2-manifold as an associative submanifold. Second, the interior of a smooth, compact 4-manifold K, whose double $doub(K)$ has a trivial bundle of self-dual 2-forms, may be isometrically embedded into a $Spin(7)$-manifold as a Cayley submanifold. Along the way, we also show that Bochner's Theorem on real analytic approximation of smooth differential forms, can be obtained using real algebraic tools developed by Akbulut and King.

Key words. Associative calibration, Cayley calibration.

AMS subject classifications. 53C25, 58A15

1. Introduction.

Let (M^7, g) be a Riemannian 7-manifold whose holonomy group $\mathrm{Hol}(g)$ is a subgroup of the exceptional group G_2. Then M is naturally equipped with a covariantly constant 3-form φ and 4-form $*\varphi$. We call (M, φ, g) a *G_2-manifold*. It is well known that φ and $*\varphi$ are *calibrations* on M, in the sense of Harvey and Lawson [12]. The corresponding calibrated submanifolds in M are called *associative 3-folds* and *coassociative 4-folds*, respectively.

Similarly, if (M^8, g) has $\mathrm{Hol}(g) \subseteq Spin(7)$, then M admits a covariantly constant, self-dual 4-form Ψ, and we call (M, Ψ, g) a *$Spin(7)$-manifold*. The 4-form Ψ is the *Cayley calibration*, and the calibrated submanifolds are *Cayley 4-folds*.

Constructing examples of manifolds with G_2 and $Spin(7)$ holonomy and their calibrated submanifolds is of interest because of their importance in string theory. Also, they provide new examples of volume minimizing submanifolds in a given homology class [12]. In [8], R. Bryant applied the Cartan-Kähler theory to show that: (1) every closed, real analytic, oriented Riemannian 3-fold can be isometrically embedded in a Calabi-Yau 3-fold as a special Lagrangian submanifold; and (2) every closed, real analytic, oriented Riemannian 4-fold with a trivial bundle of self-dual 2-forms can be isometrically embedded in a G_2-manifold as an coassociative submanifold. Moreover, the submanifolds above may be embedded as the fixed locus of a real structure (in the special Lagrangian case), or an anti G_2-involution (in the coassociative case).

In this paper, we will first show that Bryant's constructions can be repeated for the associative and Cayley submanifolds.

THEOREM 1.1. *Assume (K^3, g) is a closed, oriented, real analytic Riemannian 3-manifold. Then there exists a G_2-manifold (N^7, φ) and an isometric embedding $i : K \hookrightarrow N$ such that the image $i(K)$ is an associative submanifold of N. Moreover, (N, φ) can be chosen so that $i(K)$ is the fixed point set of a nontrivial G_2-involution $r : N \to N$.*

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Remark. Bryant showed that K isometrically embeds as a special Lagrangian submanifold of a Calabi-Yau 3-fold CY. This immediately yields an elementary version of Theorem 1.1 as $N = CY \times \mathbb{R}$ naturally carries a G_2 structure such that A isometrically embeds as an associative submanifold. However, $CY \times \mathbb{R}$ has holonomy a subgroup of SU(3). It may be checked that, as long as A is not flat, the N of Theorem 1.1 has holonomy exactly G_2. In particular, these N are not of the form $CY \times \mathbb{R}$. See the remark at the end of §6.

THEOREM 1.2. *Assume (K^4, g) is a closed, oriented, real analytic Riemannian 4-manifold with a trivial bundle of self-dual 2-forms. Then there exists a Spin(7)-manifold (N^8, Ψ) and an isometric embedding $i : K \hookrightarrow N$ whose image is a Cayley submanifold in N. Moreover, (N, Ψ) can be chosen so that $i(K)$ is the fixed locus of a nontrivial Spin(7)-involution $r : N \to N$.*

We refer the reader to [8, §0.4] for a discussion of Cartan-Kähler theory that will be used in the constructions.

Making use of the real analytic implicit function theorem and a theorem of Nash-Tognoli we are able to show that Theorems 1.1 and 1.2 extend to interiors of compact, smooth manifolds. In particular, assume that K is a compact, oriented, smooth manifold, possibly with boundary. Let *doub*(K) denote the *doubling of* K: glue two copies of K together along the boundary with the identity map. If K is closed $(\partial M = \emptyset)$ then *doub*$(K) = K$. The manifold *doub*(K) is closed and orientable, and admits the structure of a real analytic Riemannian manifold; see Lemma 5.5. Then we have the following two corollaries.

THEOREM 1.3. *Let A be the interior of a smooth, orientable, compact 3-manifold K with nonempty boundary. Then A admits a compatible real analytic Riemannian structure. There exists a G_2-manifold (N^7, φ) and an isometric embedding $i : A \hookrightarrow N$ such that $i(A)$ is an associative submanifold in N. Moreover (N, φ) may be chosen so that $i(A)$ is the fixed locus of a nontrivial G_2-involution $r : N \to N$.*

THEOREM 1.4. *Let A be the interior of a smooth, orientable, compact 4-manifold K with nonempty boundary. Then A admits a compatible real analytic Riemannian structure. Assume also that the bundle of self-dual 2-forms over doub(K) is trivial. There exists a Spin(7)-manifold (N^8, Ψ) and an isometric embedding $i : A \hookrightarrow N$ whose image $i(A)$ is a Cayley submanifold in N. Moreover, (N, Ψ) may be chosen so that $i(A)$ is the fixed point set of a nontrivial Spin(7)-involution $r : N \to N$.*

Theorems 1.1 & 1.3 and Theorems 1.2 & 1.4 are proven in §6 and 7, respectively. Also note that in all these theorems N does not have to be a (locally) product manifold.

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2. Associative submanifolds of G_2-manifolds.

2.1. G_2-manifolds and the associative calibration . On the imaginary octonians $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$ let $x = (x^j)$ denote the standard linear coordinates, set $\mathrm{d}x^{jk} := \mathrm{d}x^j \wedge \mathrm{d}x^k$, and define the 3-forms $\mathrm{d}x^{jk\ell} := \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^\ell$ and

$$\varphi_0 := \mathrm{d}x^{123} + \mathrm{d}x^1 \wedge \left(\mathrm{d}x^{45} + \mathrm{d}x^{67}\right) + \mathrm{d}x^2 \wedge \left(\mathrm{d}x^{46} - \mathrm{d}x^{57}\right) + \mathrm{d}x^3 \wedge \left(-\mathrm{d}x^{47} - \mathrm{d}x^{56}\right) .$$

The simple Lie group G_2 is the subgroup of GL(7) preserving φ_0 [7].

A G_2*-structure* on M^7 is a principle right G_2-bundle $\pi : P \to M$. The elements of $P_x = \pi^{-1}(x)$ are linear isomorphisms $u : T_xM \to \mathbb{R}^7$, and the right action is given by $u \cdot a = a^{-1} \circ u$. The G_2-structure induces a well-defined 3-form φ on M via $\varphi_x = u^*\varphi_0$. Additionally, M admits a unique metric g and volume form $*1$ (also obtained by pull-back) for which $u : T_xM \to \mathbb{R}^7$ is an oriented isometry. In particular, $(*\varphi)_x = u^*(*\varphi_0)$.

We say that (M, φ) is a G_2*-manifold* when φ and $*\varphi$ are closed. Equivalently, the G_2-structure is torsion-free [11]. In this case, φ is parallel, M is Ricci-flat [4, 10.64], and the metric is real analytic in harmonic coordinates [10, Th. 5.2]. Since φ is harmonic it follows that φ is real analytic as well.

Assume (M, φ) is a G_2-manifold. Then φ is the *associative calibration*. The 3-dimensional submanifolds $i : X^3 \hookrightarrow M$ calibrated by φ are the *associative submanifolds*. Associative submanifolds are plentiful in G_2-manifolds: it is a consequence of the Cartan-Kähler theorem [8, §0.4] that every associative $E^3 \subset T_zM$ is tangent to an associative $X^3 \subset M$. Moreover,

Lemma 2.1. *Every real-analytic 2-dimensional submanifold* Y^2 *of a* G_2*-manifold* (M^7, φ) *lies in a unique associative* X^3.

The flat case $(M, \varphi) = (\mathbb{R}^7, \varphi_0)$ was proven by Harvey and Lawson [12, Th.4.1]. Given Lemma 2.2 below, the proof (at the end of this section) is a simple application of the Cartan-Kähler theorem [8, §0.4].

The fundamental identity [12, Th.1.6]

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2 \;=\; |u \wedge v \wedge w|^2$$

implies that $i^*\varphi = \mathrm{d}vol$ precisely when $i^*\chi = 0$. Here χ is the vector-valued 3-form defined by

$$\langle \chi(u, v, w), z\rangle = *\varphi(u, v, w, z)\,.$$

In particular, the associative submanifolds are the 3-dimensional integral manifolds of $\{\chi = 0\}$. In the flat case $(\mathbb{R}^7, \varphi_0)$,

$$*\varphi_0 = \mathrm{d}x^{4567} + \mathrm{d}x^{23} \wedge \left(\mathrm{d}x^{45} + \mathrm{d}x^{67}\right) + \mathrm{d}x^{31} \wedge \left(\mathrm{d}x^{46} - \mathrm{d}x^{57}\right) + \mathrm{d}x^{12} \wedge \left(-\mathrm{d}x^{47} - \mathrm{d}x^{56}\right)\,,$$

and

$$\begin{aligned}
\chi_0 = &- \left(\mathrm{d}x^{357} - \mathrm{d}x^{346} - \mathrm{d}x^{256} - \mathrm{d}x^{247}\right) \partial_{x^1}\\
&- \left(\mathrm{d}x^{367} + \mathrm{d}x^{345} + \mathrm{d}x^{156} + \mathrm{d}x^{147}\right) \partial_{x^2}\\
&+ \left(\mathrm{d}x^{267} + \mathrm{d}x^{245} + \mathrm{d}x^{157} - \mathrm{d}x^{146}\right) \partial_{x^3}\\
&- \left(\mathrm{d}x^{567} + \mathrm{d}x^{235} - \mathrm{d}x^{136} - \mathrm{d}x^{127}\right) \partial_{x^4}\\
&+ \left(\mathrm{d}x^{467} + \mathrm{d}x^{234} - \mathrm{d}x^{137} + \mathrm{d}x^{126}\right) \partial_{x^5}\\
&- \left(\mathrm{d}x^{457} + \mathrm{d}x^{237} + \mathrm{d}x^{134} + \mathrm{d}x^{125}\right) \partial_{x^6}\\
&+ \left(\mathrm{d}x^{456} + \mathrm{d}x^{236} + \mathrm{d}x^{135} - \mathrm{d}x^{124}\right) \partial_{x^7}\,.
\end{aligned}$$

Notice that the coefficient 3-forms are $\chi_{0,j} := -\partial_{x^j} \lrcorner (*\varphi_0)$.

Given an arbitrary G_2 manifold (M^7, φ), let $\{\omega^1, \ldots, \omega^7\}$ be a local G_2 coframing. That is, $\omega^j_x = u^*_x \mathrm{d}x^j$ for smoothly varying isometries $u_x : T_xM \to \mathbb{R}^7$ in P_x. Let

$\{e_j\}$ denote the dual framing. Then local expressions for $\varphi = u^*\varphi_0$, $*\varphi = u^*(*\varphi_0)$ and $\chi = u^*\chi_0$ are given by replacing the terms $\mathrm{d}x$ and ∂_x in φ_0, $*\varphi_0$ and χ_0 with ω and e, respectively. The associative submanifolds of (M,φ) are the 3-dimensional integral manifolds of $\{\chi_j := -e_j \lrcorner (*\varphi)\}$.

LEMMA 2.2. *Let $\mathcal{I}$ be the ideal algebraically generated by the coefficient 3-forms χ_j. Then $\mathcal{I}$ is well-defined and closed under exterior differentiation ($\mathrm{d}\mathcal{I} \subset \mathcal{I}$).*

Proof. That $\mathcal{I}$ is well-defined (i.e. does not depend on choice of local G_2-coframing u_x) is immediate from the G_2-invariance of the forms φ, $*\varphi$ and χ. To see that $\mathcal{I}$ is differentially closed recollect that the Lie derivative of any form α by a vector field X is $\mathcal{L}_X\alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$, so that

$$\begin{aligned} d\chi_j &= -d\,(e_j \lrcorner *\varphi) \\ &= e_j \lrcorner d(*\varphi) \;-\; \mathcal{L}_{e_j} *\varphi \\ &= -\mathcal{L}_{e_j} *\varphi\,, \qquad\qquad (*\varphi \text{ is closed})\,, \\ &= \big(e_j \lrcorner d\omega^k\big) \wedge \chi_k\,. \end{aligned}$$

The last line follows from an application of [6, §V.8, Ex.8]. □

Proof of Lemma 2.1. Since Y is 2-dimensional and $\mathcal{I}$ is generated by 3-forms, Y is a priori an integral manifold. In order to apply the Cartan-Kähler theorem we must show that: (i) Y is regular; and (ii) the polar space $H(T_yY)$ is of dimension 3 for every $y \in Y$. (See [8, §0.4] for a review of polar spaces, the variety of p-dimensional integral elements $V_p(\mathcal{I})_y$ in the Grassmannian $\mathrm{Gr}(p, T_yM)$ and the Cartan-Kähler theorem in this context.)

Regularity is easily confirmed, and in the course of doing so we will see that the polar space is of dimension three. Fix $y \in Y$. Since $\mathcal{I}$ is generated by 3-forms, $V_2(\mathcal{I})_y = \mathrm{Gr}(2, T_yM)$, and $V_2(\mathcal{I})$ is (trivially) a smooth submanifold of $\mathrm{Gr}(2, TM)$ near T_yY. Whence, T_yY is ordinary.

Because G_2 acts transitively on 2-planes, there is no loss of generality in assuming that T_yY is spanned by $\{e_1, e_2\}$. The polar space of T_yY is $H(T_yY) = \{v \in T_yM \mid \psi(v, e_1, e_2) = 0 \;\forall\; \psi \in \mathcal{I}^3\}$. In our case it is straightforward to see that $H(T_yY) = \{v \in T_yM \mid \chi_j(v, e_1, e_2) = 0 \;\forall\; j\}$ is spanned by $\{e_1, e_2, e_3\}$. Whence the extension rank $r(T_yY) = dim H(T_yY) - (2+1) = 0$ is constant function $V_2(\mathcal{I}) = \mathrm{Gr}(2, TM)$, and Y is regular. The result follows from the Cartan-Kähler theory. □

2.2. G_2-involutions. One way of finding examples of associative submanifolds is to investigate the fixed point sets of G_2 involutions, [14, Prop.10.8.1].

Let $\sigma : M \to M$ be a nontrivial isometric involution of a G_2-manifold (M, g). This means that $\sigma : M \to M$ is a diffeomorphism satisfying $\sigma^*(g) = g$ and $\sigma^2 = id$, but $\sigma \neq 1$.

LEMMA 2.3. *Let (M, φ, g) be a G_2-manifold and let $\sigma : M \to M$ be a nontrivial isometric involution preserving φ, i.e. $\sigma^*(\varphi) = \varphi$. Then the fixed point set $A = \{p \in M \mid \sigma(p) = p\}$ is an associative 3-fold in M.*

For the details of the proof, see [14]. Note that the fixed point set A is a closed submanifold of M. This is because A can be represented as a preimage of 0 under a continuous map, $\sigma - Id$. This does not contradict our assumption that A can be open. The reason for this is when we thicken an open manifold A to obtain the G_2-manifold M, then A (even if it is open in K) will be always a closed submanifold of M.

Note that there is a similar construction for the coassociative case [14, Prop.10.8.5].

3. Cayley submanifolds of *Spin*(7)-manifolds.

3.1. *Spin*(7)-manifolds and the Cayley calibration. The octonians $\mathbb{O} = \mathbb{R}^8$ are equipped with a triple (and quadruple) cross product. This cross product defines a 4-form $\Psi_0(u,v,w,z) = \langle u \times v \times w, z\rangle$. Given linear coordinates $x = (x^0, x^1, \dots, x^7)$ on $\mathbb{R}^8$,

$$
\begin{aligned}
\widehat{\Psi_0} &= \mathrm{d}x^0 \wedge \phi_0 + *\phi_0 \\
&= \mathrm{d}x^{0123} + \mathrm{d}x^{4567} + \left(\mathrm{d}x^{01} + \mathrm{d}x^{23}\right) \wedge \left(\mathrm{d}x^{45} + \mathrm{d}x^{67}\right) \\
&\quad + \left(\mathrm{d}x^{02} + \mathrm{d}x^{31}\right) \wedge \left(\mathrm{d}x^{46} + \mathrm{d}x^{75}\right) + \left(\mathrm{d}x^{03} + \mathrm{d}x^{12}\right) \wedge \left(\mathrm{d}x^{74} + \mathrm{d}x^{65}\right) .
\end{aligned}
$$

The exceptional $Spin(7) \subset \mathrm{SO}(8)$ is the subgroup preserving Ψ_0 [7, 12].

A $Spin(7)$-*structure* on M^8 is a principle right $Spin(7)$-bundle $\pi : P \to M$. The elements of $P_x = \pi^{-1}(x)$ are linear isomorphisms $u : T_xM \to \mathbb{R}^8$, and the right action is given by $u \cdot a = a^{-1} \circ u$. The $Spin(7)$-structure induces a well-defined 4-form Ψ on M via $\Psi_x = u^*\Psi_0$. As in the G_2 case, M also admits a unique metric g and volume form $*1$ for which $u : T_xM \to R^8$ is an oriented isometry.

We say M is a $Spin(7)$-*manifold* when Ψ is closed. (Equivalently, Ψ is parallel and the $Spin(7)$-structure is torsion-free [7].) In this case, M is Ricci-flat [4, 10.65], and the metric is real analytic in harmonic coordinates [10, Th. 5.2]. Since Ψ is harmonic it follows that Ψ is real analytic as well.

Given a $Spin(7)$-manifold, the 4-form Ψ is a calibration, known as the *Cayley calibration*. The 4-dimensional submanifolds $i : X^4 \to M$ calibrated by Ψ are the *Cayley submanifolds*. Cayley submanifolds are plentiful in $Spin(7)$-manifolds: The Cartan-Kähler theory implies that given a Cayley plane $E^4 \subset T_zM$, there exists a Cayley submanifold X^4 tangent to E. Moreover,

LEMMA 3.1. *Every real-analytic 3-dimensional submanifold Y^3 of a $Spin(7)$-manifold (M^8, Ψ) lies in a unique Cayley X^4.*

The flat case $(M, \Psi) = (\mathbb{R}^8, \Psi_0)$ was proven by Harvey and Lawson [12, Th.4.3]. The proof follows from Lemma 3.3 below and the Cartan-Kähler Theorem [8, §0.4], and is given at the end of this section.

The 4-form satisfies the relation [12, Ch.5, Th.1.28]

$$\Psi_0(u,v,w,z)^2 + |\mathrm{Im}(u \times v \times w \times z)|^2 = |u \wedge v \wedge w \wedge z|^2 . \tag{3.2}$$

Let τ_0 denote the vector-valued 4-form $\tau_0(u,v,w,z) = \mathrm{Im}(u,v,w,z)$. Given a $Spin(7)$-manifold (M^8, Ψ) and a submanifold $i : X^4 \hookrightarrow M$, notice that $i^*\Psi = \mathrm{d}vol_X$ if and only if $i^*\tau = 0$, where τ is the vector valued 4-form $\tau_x = u^*\tau_0$. Consequently the Cayley submanifolds are the integral submanifolds of $\tau = 0$. Let $\{\omega^0, \dots, \omega^7\}$ be a local $Spin(7)$-coframing. That is, $\omega^j_x = u^*_x \mathrm{d}x^j$, $j = 0, \dots, 7$, for smoothly varying isometries

$u_x : T_xM \to \mathbb{R}^8$ in P_x. Let $\{e_j\}$ denote the dual framing. Write $\tau = \sum_1^7 \tau^j e_j$. Then X is Cayley if and only if $i^*\tau^j = 0$, $j = 1, \ldots, 7$.

LEMMA 3.3. *Let $\mathcal{I}$ denote the ideal generated algebraically by the τ^j. Then $\mathcal{I}$ is well-defined and closed under exterior differentiation.*

Proof. That $\mathcal{I}$ is well-defined is a consequence of the *Spin*(7) invariance of τ.

To see that $\mathcal{I}$ is closed, note that (3.2) and the fact that Ψ is parallel imply that τ is also parallel. Hence

$$\begin{aligned} & 0 = \nabla\tau \ = \ \nabla\left(\tau^j \otimes e_j\right) \ = \ \nabla\tau^j \otimes e_j + \tau^j \otimes \nabla e_j \ = \ \nabla\tau^j \otimes e_j - \tau^j \otimes \theta^k_j e_k \\ \implies \quad & \nabla\tau^j = \tau^k \otimes \theta^j_k \,. \end{aligned}$$

Above, θ is the $\mathfrak{spin}(7)$–valued connection form. Since the exterior derivative $\mathrm{d}\tau^j$ is the skew-symmetrization of the covariant derivative $\nabla\tau^j$, we have $\mathrm{d}\tau^j = \tau^k \wedge \theta^j_k$. □

Proof of Lemma 3.1. As in the case of Lemma 2.1, the proof is a straightforward application of the Cartan-Kähler Theorem. See [8, §0.4] for a review of integral elements, polar spaces and the Cartan-Kähler theory in the context. As $\mathcal{I}$ is generated by 4-forms, and Y is of dimension three, Y is trivially an integral manifold. Similarly, $V_3(\mathcal{I}) = \mathrm{Gr}_3(TM)$, and T_yY is ordinary, for all $y \in Y$.

In a *Spin*(7) coframing the τ^j are given by

$$\begin{aligned}
\tau^1 &= (\omega^{03} - \omega^{12}) \wedge (\omega^{46} + \omega^{57}) - (\omega^{02} + \omega^{13}) \wedge (\omega^{47} - \omega^{56}) \\
\tau^2 &= (\omega^{01} - \omega^{23}) \wedge (\omega^{47} - \omega^{56}) - (\omega^{03} - \omega^{12}) \wedge (\omega^{45} - \omega^{67}) \\
\tau^3 &= (\omega^{02} + \omega^{13}) \wedge (\omega^{45} - \omega^{67}) - (\omega^{01} - \omega^{23}) \wedge (\omega^{46} + \omega^{57}) \\
\tau^4 &= \omega^{1234} - \omega^{0235} + \omega^{0136} - \omega^{0127} + \omega^{0567} - \omega^{1467} + \omega^{2457} - \omega^{3456} \\
\tau^5 &= \omega^{1235} + \omega^{0234} + \omega^{0137} + \omega^{0126} - \omega^{1567} - \omega^{0467} - \omega^{3457} - \omega^{2456} \\
\tau^6 &= \omega^{1236} + \omega^{0237} - \omega^{0134} - \omega^{0125} - \omega^{2567} - \omega^{3467} + \omega^{0457} + \omega^{1456} \\
\tau^7 &= \omega^{1237} - \omega^{0236} - \omega^{0135} + \omega^{0124} - \omega^{3567} + \omega^{2467} + \omega^{1457} - \omega^{0456} \,.
\end{aligned}$$

Cf. [18, (6.10)].

Fix y. Since *Spin*(7) acts transitively on 3-planes [7, Th.4], we may assume that $T_yY = \mathrm{span}\{e_0, e_1, e_2\}$. Then $H(T_yY) = \mathrm{span}\{e_0, e_1, e_2, e_3\}$, and the extension rank is zero. It follows from the Cartan-Kähler theorem that Y lies in a unique Cayley 4-manifold. □

3.2. *Spin*(7)-involutions. There are examples of Cayley submanifolds which are the fixed point sets of *Spin*(7) involutions, [14, Prop.10.8.6].

As in G_2 case, let $\sigma : M \to M$ be a nontrivial isometric involution of a *Spin*(7)-manifold (M, Ψ, g) satisfying $\sigma^*(g) = g$ and $\sigma^2 = id$, but $\sigma \neq 1$.

LEMMA 3.4. *Let (M, Ψ, g) be a Spin(7)-manifold and let $\sigma : M \to M$ be a nontrivial isometric involution preserving Ψ, i.e. $\sigma^*(\Psi) = \Psi$. Then each connected component of the fixed point set $A = \{p \in M |\ \sigma(p) = p\}$ is either a Cayley 4-fold in M or a single point.*

For the details of the proof, see [14].

4. G-structures and ideals. The primary purpose of this section is to introduce the principle objects of interest, and establish notation. Detailed discussions may be found in [15, Ch.II] and [7, §1].

4.1. G-structures. Given a smooth manifold M of dimension n, let $\pi : \mathcal{F} \to M$ denote its bundle of $\mathbb{R}^n$–valued coframes. The fibre $\pi^{-1}(x) =: \mathcal{F}_x$ over $x \in M$ is the collection of linear isomorphisms $u : T_xM \to \mathbb{R}^n$. This is a principle right $\mathrm{GL}(n)$–bundle with action of $a \in \mathrm{GL}(n)$ given by $u \cdot a := a^{-1} \circ u$. Given a subgroup $G \subset \mathrm{GL}(n)$, a *G-structure* is a principle sub-bundle $\mathcal{P} \subset \mathcal{F}$ with structure group G.

Example. When $G = \mathrm{SO}_n$, there is a unique Riemannian metric on M for which $\mathcal{P}$ is the bundle of (oriented) orthonormal coframes.

In the case that $G \subset \mathrm{SO}(n)$, let $\overline{\mathcal{P}} := \mathcal{P} \cdot \mathrm{SO}(n)$ be the $\mathrm{SO}(n)$-bundle of orthonormal coframes. The corresponding Riemannian metric on M is the *underlying metric of the G-structure.*

4.2. Flat structures. Given a coordinate neighborhood $x : U \to \mathbb{R}^n$ on M, notice that $\mathrm{d}x$ is a local section $U \to \mathcal{F}$. We say the G-structure $\mathcal{P}$ is *flat* when M every $p \in M$ admits a coordinate chart such that $\mathrm{d}x$ is a local section of $\mathcal{P}$.

Clearly, $\mathcal{F}$ is a flat $\mathrm{GL}(n)$-structure. Every orientable M admits a $\mathrm{SL}(n)$-structure, given by the volume form. (Alternatively, every $\mathrm{SL}(n)$-structure on M uniquely determines a volume form.) Because the volume form may always be expressed locally as $\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$ in some local coordinate system, every $\mathrm{SL}(n)$-structure is flat.

4.3. Connections. Given a G-structure $\pi : \mathcal{P} \to M$, a tangent vector $v \in T_u\mathcal{P}$ is *vertical* if $\pi_*(v) = 0$. A differential p-form Ω on $\mathcal{P}$ is *semi-basic* if $v \lrcorner \Omega = 0$ for all vertical v. There is a canonically defined, $\mathbb{R}^n$–valued semi-basic 1-form η on $\mathcal{P}$: given $v \in T_u\mathcal{P}$,

$$\eta(v) \ := \ u \circ \pi_*(v)\,.$$

The components of $\eta = (\eta^1, \ldots, \eta^n)$ give a basis of the semi-basic 1-forms on $\mathcal{P}$.

Let $V_u := \mathit{ker}\,\pi_* \subset T_u\mathcal{P}$ denote the vertical subspace at $u \in \mathcal{P}$. A *connection* on $\mathcal{P}$ is a smooth distribution $H_u \subset T_u\mathcal{P}$ that is complimentary to V_u and invariant under the right action of G. Equivalently, $\pi_* : H_u \to T_xM$ is an isomorphism, $x = \pi(u)$; and $(R_a)_*H_u = H_{u\cdot a}$, where $R_a : \mathcal{P} \to \mathcal{P}$ is the map $u \mapsto u \cdot a$.

The connection H determines a $\mathfrak{g}$–valued *connection 1-form* θ satisfying $(R_a)^*\theta = \mathrm{ad}(a^{-1})\theta$ as follows. Setting $\Theta_{|H_u} \equiv 0$, it remains to specify θ on V_u. Every $X \in \mathfrak{g}$ determines a vertical vector field X^* on $\mathcal{P}$: given $a(t) \in G$ with $a(0) = \mathrm{Id}$ and $a'(0) = X$, define $X^*_u = \frac{\mathrm{d}}{\mathrm{d}t} u \cdot a(t)|_{t=0}$. These X^*_u span V_u, and defining $\theta(X^*) = X$ determines θ. Clearly the connection form is $\mathfrak{g}$–valued. We leave it to the reader to confirm that $(R_a)^*\theta = \mathrm{ad}(a^{-1})\theta$. Conversely, any $\mathfrak{g}$–valued 1-form satisfying this condition determines a connection H via the assignment $H_u := \{\theta_u = 0\}$.

4.4. Torsion. It can be shown that a $\mathfrak{g}$–valued 1-form θ is a connection form if and only if $\mathrm{d}\eta^j = -\theta^j_k \wedge \eta^k + T^j_{k\ell}\eta^k \wedge \eta^\ell$, with $T^j_{k\ell} + T^j_{\ell k} = 0$, [13, Prop. 8.3.3]. The functions $T^j_{k\ell}$ define a map $T := T^j_{k\ell}\frac{\partial}{\partial x^j} \otimes \mathrm{d}x^k \wedge \mathrm{d}x^\ell : \mathcal{P} \to \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*$, called the *torsion of* θ. Any other connection 1-form $\tilde\theta$ differs from θ by a $\mathfrak{g}$-valued semi-basic

1-form, $\tilde\theta^j_k = \theta^j_k + c^j_{k\ell}\eta^\ell$. The corresponding change in torsion is $\tilde T^j_{k\ell} - T^j_{k\ell} = c^j_{k\ell} - c^j_{\ell k}$. In particular, $\tilde T - T$ takes values in the image of the skew-symmetrizing map $\delta : \mathfrak{g}\otimes(\mathbb{R}^n)^* \subset \mathbb{R}^n \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \to \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*$. This leads to the definition of the *torsion of the G-structure $\mathcal{P}$* as $[T] : \mathcal{P} \to h^0(\mathfrak{g}) := (\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*)/\delta(\mathfrak{g}\otimes(\mathbb{R}^n)^*)$.

Example. When $\mathfrak{g} = \mathfrak{so}(n)$, it is easy to show that $h^0(\mathfrak{so}(n)) = \{0\}$. This is equivalent to the existence of a torsion-free g–compatible connection on a Riemannian manifold (M, g). (This is the existence-half of the fundamental theorem of Riemannian geometry. The uniqueness-half is equivalent to $\mathfrak{so}(n)^{(1)} := \mathit{ker}\,\delta = (\mathfrak{so}(n)\otimes(\mathbb{R}^n)^*) \cap (\mathbb{R}^n \otimes S^2(\mathbb{R}^n)^*) = \{0\}$. In general, $\mathfrak{g}^{(1)}$ records the changes in connection that preserve torsion.) When $\mathfrak{g} \subset \mathfrak{so}(n)$, we have $h^0(\mathfrak{g}) = (\mathfrak{so}(n)/\mathfrak{g}) \otimes (\mathbb{R}^n)^*$.

4.5. Torsion-free *G*-structures. We say that $\mathcal{P}$ is *torsion-free* when $[T] \equiv 0$, and M is a *G-manifold*. In the case that $G \subset \mathrm{SO}(n)$, let $i : \mathcal{P} \to \overline{\mathcal{P}}$ denote the inclusion, and $\overline{H}$ the Levi-Civita connection on $\overline{\mathcal{P}}$. It is not difficult to see that $\mathcal{P}$ is torsion-free if and only if $\overline{H} \subset i_* T\mathcal{P}$. Equivalently, $\mathcal{P}$ is preserved under parallel transport in $\overline{\mathcal{P}}$.

Torsion may be viewed as a first-order obstruction to flatness. Here is one way to see this. Suppose M carries a G-structure $\mathcal{P}$. Let $x : U \to \mathbb{R}^n$ be a coordinate system about $z \in M$. We may assume that the local section $\mathrm{d}x : U \to \mathcal{F}$ satisfies $\mathrm{d}x_z \in \mathcal{P}_z$. The coordinates define a local, flat G-structure $\mathcal{P}_0 := \mathrm{d}x \cdot G$ over U. The following lemma is well-known.

Lemma 4.1. *The G-structure $\mathcal{P}$ is torsion-free if and only if for all $z \in M$, there exist local coordinates (x, U) such that $\mathcal{P}$ and $\mathcal{P}_0$ are tangent at $\mathrm{d}x_z$.*

4.6. *G*-structures as sections and 1-flatness. Let $S = \mathcal{F}/G$, and consider the bundle $\overline{\pi} : S \to M$,

$$\begin{tikzcd} \mathcal{F} \arrow[dd, "\pi"'] \arrow[dr, "\rho"] & \\ & S \arrow[dl, "\overline{\pi}"] \\ M & \end{tikzcd}$$

Notice that G-structures on M are in one-to-one correspondence with $\overline{\pi}$-sections $\sigma : M \to S$.

We say $\sigma : M \to S$ is *flat* when the corresponding G-structure is flat. The section σ is *1-flat* if it is flat to first-order at every point. That is, if every $x \in M$ admits an open neighborhood U carrying a flat G-structure with corresponding section $\sigma_0 : U \to S$ such that $\sigma(x) = \sigma_0(x)$, and $\sigma(M)$ and $\sigma_0(U)$ are tangent at $\sigma(x)$. By Lemma 4.1, *$\mathcal{P}$ is torsion-free if and only if σ is 1-flat.*

4.7. Admissible groups. Given $G \subset \mathrm{SO}(n)$, let $\Lambda^*(\mathbb{R}^n)^G$ denote the G-invariant constant coefficient differential forms on $\mathbb{R}^n$. We say G is *admissible* if it is the subgroup of $\mathrm{GL}(n)$ fixing all the forms in $\Lambda^*(\mathbb{R}^n)^G$. (A priori, this subgroup contains G.) In [7] Bryant showed that G_2 and $\mathit{Spin}(7)$ are admissible. The ring $\Lambda^*(\mathbb{R}^n)^{G_2}$ is generated by φ_0 and $*\varphi_0$ (cf. §2.1), and $\Lambda^*(\mathbb{R}^n)^{\mathit{Spin}(7)}$ by Ψ_0 (cf. §3.1).

4.8. A differential ideal on S . This subsection and the following borrow heavily from [8].

Every p-form α on $\mathbb{R}^n$ defines a semi-basic p-form $\hat\alpha$ on $\mathcal{F}$ via

$$\hat\alpha_u(v_1,\ldots,v_p) \ := \ \alpha(\eta(v_1),\ldots,\eta(v_p))\,.$$

When α is G-invariant, $\hat\alpha$ is invariant under the right-action of G on $\mathcal{F}$ and therefore descends to a well-defined p-form on S, also denoted by $\hat\alpha$. Given a G-structure $\mathcal{P}$ with corresponding section $\sigma : M \to S$, the pull-back $\sigma^*\hat\alpha$ defines a p-form α_σ on M. Recall that σ is torsion-free if and only if the G-structure is preserved under parallel transport by the underlying Levi-Civita connection. In particular, α_σ is parallel, and therefore closed, if σ is torsion-free. Whence $\sigma^*(\mathrm{d}\hat\alpha)=0$.

We denote by $\mathcal{I}$ both the ideal on $\mathcal{F}$ and the ideal on S that is generated algebraically by $\mathrm{d}\hat\alpha$, $\alpha\in\Lambda^*\mathbb{R}^n$. Graphs of torsion-free $\sigma : M\to S$ are necessarily integral manifolds of $\mathcal{I}$. The converse need not hold; see [8, §0.5.5] for an example.

4.9. Strong admissibility. Given $k\le n$, let $V(\mathcal{I},\overline\pi)\subset \mathrm{Gr}_k(TS)$ denote the k-dimensional integral elements $E\subset T_sS$ that are $\overline\pi$-transverse; that is, the projection $\overline\pi_* : E\to T_{\overline\pi(s)}M$ is injective. As noted above, $V_n(\mathcal{I},\overline\pi)$ contains the set of n-planes tangent to the graph of a torsion-free section σ. When G is admissible, and $V_n(\mathcal{I},\overline\pi)$ consists of exactly these tangent planes, then we say G is *strongly admissible*. As a result any section $\sigma : M\to S$ whose image in S is an integral manifold of $\mathcal{I}$ is necessarily torsion-free. Both G_2 and $Spin(7)$ are strongly admissible [7].

Recall from §4.4 that the torsion of a G-structure $\sigma : M\to S$ lives in $\mathfrak{h}^0(\mathfrak{g})$. Since $V_n(\mathcal{I},\overline\pi)$ contains the tangent planes to torsion-free $\sigma(M)$, and torsion is a first-order invariant, we must have

$$\mathit{codim}(V_n(\mathcal{I},\pi),\mathrm{Gr}_n(T\mathcal{F}))\le \mathit{dim}\,\mathfrak{h}^0(\mathfrak{g})\,,$$

with equality precisely when G is strongly admissible.

4.10. Integral elements on S and $\mathcal{F}$. Define $V_k(\mathcal{I},\pi)\subset\mathrm{Gr}_k(T_u\mathcal{F})$ to be the k-dimensional integral elements of $\mathcal{I}$ that are π-transverse. Observe that $\rho_* : V_k(\mathcal{I},\pi)\to V_k(\mathcal{I},\overline\pi)$ is a surjection. Given $E\,,\ E'\in V_k(\mathcal{I},\pi)$, we have $\rho_*(E)=\rho_*(E')$ if and only if $E\equiv E'$ mod $\mathfrak{g}_u$. Here $\mathfrak{g}_u := \mathit{ker}(\rho_{*|T_u\mathcal{F}})$. (Alternatively, $\mathfrak{g}_u\subset V_u$ is the vertical subspace of $T_u\mathcal{F}$ identified with $\mathfrak{g}$ under the right action of G at $u\in\mathcal{F}$.) In particular, for fixed E the set of all such E' is naturally identified with $\mathrm{Hom}(E,\mathfrak{g}_u)$. This set is of dimension $k\,\mathit{dim}(G)$. As $\mathit{dim}\,T\mathcal{F}-\mathit{dim}\,TS = n\,\mathit{dim}(G)$, we have

$$\mathit{codim}(V_n(\mathcal{I},\overline\pi)\,,\,\mathrm{Gr}_n(TS)) \ = \ \mathit{codim}(V_n(\mathcal{I},\pi)\,,\,\mathrm{Gr}_n(T\mathcal{F})) \tag{4.2}$$

in the case that $k=n$.

It is straightforward to check that, given $E\in V_k(\mathcal{I},\pi)$, the polar spaces satisfy $H(E) = (\rho_*)^{-1}H(\rho_*E)$. In particular, given an integral flag $F=\{E_j\}_{j=0}^n$ in $T_u\mathcal{F}$ and the and the corresponding flag $\overline F=\{\overline E_j=\rho_*E_j\}_{j=0}^n$ in $T_{\rho(u)}S$, we have

$$c_j(F) \ = \ c_j(\overline F) \qquad \text{and} \qquad c(F) \ = \ c(\overline F)\,. \tag{4.3}$$

Finally, it is not difficult to describe the set $V_n(\mathcal{I},\pi)\subset\mathrm{Gr}_n(T\mathcal{F})$. Given a π-transverse $E\in\mathrm{Gr}_n(T\mathcal{F})$ the canonical 1-forms η^j span E^*. In particular, when restricted to E the connection 1-forms θ^j_k may be expressed as linear combinations of the η^j, $\theta^j_k = p^j_{k\ell}\eta^\ell$. The $p^j_{k\ell}$ are functions on $\mathcal{F}$, and $p^j_{k\ell}(u)$ parameterizes the open set of π-transverse n-planes $E \in \mathrm{Gr}_n(T_u\mathcal{F})$. Now, given $\alpha = a_I \mathrm{d}x^I \in \Lambda^p(\mathbb{R}^n)^*$, $I = \{i_1\,,\ \cdots\,,\ i_p\}$ a multi-index, we have $\hat\alpha = a_I\eta^I$. When restricted to $E \in \mathrm{Gr}_n(T\mathcal{F},\pi)$, $\mathrm{d}\eta^j = -\theta^j_k \wedge \eta^k = p^j_{k\ell}\eta^k \wedge \eta^\ell$. We see that the equation $\mathrm{d}\hat\alpha = 0$ is a set of linear conditions on the $p^j_{k\ell}$, and that $V_n(\mathcal{I},\pi)$ is a submanifold of $\mathrm{Gr}_n(T\mathcal{F})$. (The exterior differential system $\mathcal{I}$ with independence condition $\eta^1 \wedge \cdots \wedge \eta^n \neq 0$ is *in linear form*.) In particular, each $E \in V_n(\mathcal{I},\pi)$ is ordinary.

4.11. Canonical flags and regular presentations. To each n-dimensional integral element $E_n \in V_n(\mathcal{I},\pi)$ at $u \in \mathcal{F}$ we may canonically associate a flag $F = \{E_0 \subset E_1 \subset \cdots \subset E_n\}$ by

$$E_k \ := \ \{v \in E_n \mid \eta^j(v) = 0 \ \forall\, j > k\}\,.$$

The polar spaces are $H(E_k) = E_n + (\mathfrak{h}_k)_u$, where the $\mathfrak{h}_k$ are defined as follows. Let $i_k : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ denote the natural inclusion, and set

$$\mathfrak{h}_k \ := \ \{x \in \mathfrak{gl}(n) \mid {i_k}^*(x.\alpha) = 0\,,\ \forall \alpha \in \Lambda^*(\mathbb{R}^n)^G\}\,.$$

Note that $\mathfrak{h}_k$ contains $\mathfrak{g}$, $\mathfrak{h}_{k+1}$ and the space $M_{n,k}\mathbb{R}$ of n-by-n matrices whose first k columns are zero. When G is admissible, $\mathfrak{h}_n = \mathfrak{g}$.

Cartan's Test implies that

$$\sum_{j=0}^{n-1} c_j \ = \ \sum_{j=0}^{n-1} \mathit{codim}(\mathfrak{h}_j, \mathfrak{gl}(n)) \ \leq \ \mathit{codim}(V_n(\mathcal{I},\pi), \mathrm{Gr}_n(T\mathcal{F}))\,.$$

A strongly admissible group G is *regularly presented* when equality holds. We will see that both G_2 and $\mathit{Spin}(7)$ are regularly presented. When G is regularly presented every $E \in V_n(\mathcal{I},\pi)$ is the terminus of a regular flag F. It follows from (4.2, 4.3) and Cartan's Test that every $\overline{E}_n \in V_n(\mathcal{I},\overline{\pi})$ is also the terminus of a regular flag, $\{\overline{E}_j := \rho_*(E_j)\}$.

5. Real algebraic and real analytic structures. In this section we briefly review the basics of real algebraic sets. For more information see [1, 2, 3].

A real algebraic set is the set of solutions of polynomial equations in real variables, a set V of the form $V(I) = \{x \in \mathbb{R}^n \mid p(x) = 0,$ for all $p \in I\}$ where I is a set of polynomial functions $p : \mathbb{R}^n \to \mathbb{R}$.

A point x in an algebraic set $V \subset \mathbb{R}^n$ is called *nonsingular of codimension k in V* if there are polynomials p_i, $i \in \{1,\ldots,k\}$, and a neighborhood $U \subset \mathbb{R}^n$ of x so that $p_i(V) = 0$ and

(i) $V \cap U = U \cap \bigcap_{i=1}^k p_i^{-1}(0)$

(ii) the gradients ∇p_i, are linearly independent on U.

Define $\mathit{dim}V$ to be the maximum of $n-k$ over all nonsingular $x \in V$. Then for an algebraic set V,

$$\mathit{Nonsing}(V) := \{x \in V \mid x \text{ is nonsingular of dimension } \mathit{dim}V\}$$

and $Sing(V) := V \setminus Nonsing(V)$. We say that an algebraic set V is *nonsingular* if $Sing(V) = \emptyset$.

Nash [20] proved that every smooth, closed manifold is a topological component of a nonsingular algebraic set, and conjectured that every smooth, closed manifold is a nonsingular algebraic set. Tognoli verified the Nash's conjecture.

THEOREM 5.1 (Nash-Tognoli [20, 21, 3]). *Let M be a smooth, closed manifold. Then there exists a nonsigular algebraic set V and a diffeomorphism $\phi : M \to V$.*

In [1, 2, 3], Akbulut and King generalized Nash's theorem to interiors of compact manifolds and proved that the interior of a smooth, compact manifold M is diffeomorphic to a nonsingular real algebraic set V which is properly imbedded in $\mathbb{R}^n$ for some n. This established a one-to-one correspondence between interiors of compact smooth manifolds and nonsingular real algebraic sets.

Note that, one can use Akbulut-King's result and the Real Analytic Implicit Function Theorem, Theorem 5.4, to find real analytic metrics on interiors of compact manifolds. In this paper, we won't use this fact as we first double our manifold to obtain a closed manifold, construct the real analytic metric and then take its restriction.

In [5], Bochner proved that on a closed, real analytic manifold the real analytic differential forms are dense in the smooth forms in the uniform topology. Next, we show that one can obtain Bochner's result using real algebraic theory developed by Akbulut and King.

THEOREM 5.2. *Every smooth, closed manifold X can be made a nonsingular real algebraic variety V. Every smooth differential form on V can be approximated by a real analytic differential form.*

Proof. Let X be a smooth, closed manifold. By Nash-Tognoli, it can be made a nonsingular real algebraic variety V. Now, we show that every smooth differential form on V can be approximated by a real analytic differential form. In [3], it was shown that for a nonsingular algebraic set V with dimension k, the classifying Gauss map $\rho : V \to G(k, n)$ of the tangent bundle $TV \to V$ is an entire rational map. Now, let $E(k, n)$ be the universal bundle over the Grassmannian variety of k planes in $\mathbb{R}^n$:

$$\begin{aligned}
&E(k,n) = \{(A, v) \in \mathcal{M}_{\mathbb{R}}(n) \times \mathbb{R}^n |\ A \in G(k,n),\ Av = v\} \\
&\quad \downarrow \pi \\
&G(k,n) = \{A \in \mathcal{M}_{\mathbb{R}}(n) |\ \ A^t = A,\ \ A^2 = A,\ \ trace(A) = k\}
\end{aligned}$$

where $\mathcal{M}_{\mathbb{R}}(n)$ denotes $n \times n$ real matrices. The tangent bundle TV can be identified with the pullback bundle $\rho^* E = \{(x, v) \in V \times E |\ \ \rho(x) = \pi(v)\}$ where ρ and π are both algebraic.

Then we have the following diagram:

$$\begin{array}{ccc}
TV & & E(k,n) \\
\downarrow & & \downarrow \pi \\
V & \longrightarrow & G(k,n)
\end{array}$$

where V, TV are nonsingular algebraic sets. Sections $s : V \to TV \cong \rho^* E$ are given by $s(x) = (x, f(x))$, for some function $f : V \to E$. This means that for real analytic approximation of the sections s, it is sufficient to find real analytic approximations of f.

Now, $V \subset \mathbb{R}^\ell$ and $E(k,n) \subset \mathbb{R}^m$ are both nonsingular algebraic sets for some ℓ, m. By the Weierstrass Approximation Theorem, a real valued smooth map from an open subset of $\mathbb{R}^\ell$ can always be approximated by polynomials. Denote this polynomial approximating f as F. Even though F may not map V to E, we can always take the projection π from the image of F to E. This map, which is from a tubular neighborhood of an algebraic variety E to E, is real analytic. So the composition of F and $\pi : tubular(E) \subset \mathbb{R}^N \to E$ yields a real analytic approximation of f, and thus the section $s = (x, f(x))$.

Since the cotangent and tangent bundles are the dual bundles, the same real analytic approximation holds for differential forms. □

An important property of real analytic functions is that the inverse of a real analytic function is also real analytic.

Theorem 5.3 (Real analytic inverse function theorem). *Let F be real analytic in a neighborhood of $a = (a_1, a_2, ..., a_n)$ and suppose that its derivative at a, $DF(a)$, is nonsingular. Then F^{-1} is defined and real analytic in a neighborhood of $F(a)$.*

The proof of this theorem follows from a special case of the Cauchy-Kowalewsky Theorem [16, 17].

As an important corollary, we obtain the implicit function theorem in the analytic setting.

Theorem 5.4 (Real analytic implicit function theorem). *Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is real analytic in a neighborhood of (x_0, y_0), for some $x_0 \in \mathbb{R}^n$ and some $y_0 \in \mathbb{R}^m$. If $F(x_0, y_0) = 0$ and the $m \times m$ matrix with entries $\frac{\partial F_i}{\partial y_j}(x_0, y_0)$ is nonsingular, then there exists a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is real analytic in a neighborhood of x_0 and is such that $F(x, f(x)) = 0$ holds in a neighborhood of x_0.*

Assume that $V \subset \mathbb{R}^n$ is a algebraic set, and define $Z = Nonsing(V)$. Let g_0 denote the canonical Euclidean metric on $\mathbb{R}^n$. The Implicit Function Theorem for real analytic maps [16, 17] implies that the restriction of g_0 to Z is a real analytic Riemannian metric.

Let K be a smooth, compact manifold. Then the double $doub(K)$ is smooth and closed. From Theorem 5.1 and Theorem 5.4, we deduce the following lemma.

Lemma 5.5. *Let $doub(K)$ be the double of a smooth, compact manifold K. Then $doub(K)$ admits a compatible real analytic structure and a real analytic metric g.*

6. The associative embedding: proof of Theorems 1.1 & 1.3. Bryant has shown that the group G_2 is admissible [7, Prop.1]. Now consider the differential system $\mathcal{I}$ of §4.8. Here $n = 7$ and the indices j, k range over $1, \ldots, 7$. On $\mathcal{F}$, the ideal is generated by the 4-form $\mathrm{d}\widehat{\varphi_0}$ and the 5-form $\mathrm{d}(\widehat{*\varphi_0})$ where

$$\begin{aligned}
\widehat{\varphi_0} &= \eta^{123} + \eta^1 \wedge \left(\eta^{45} + \eta^{67}\right) + \eta^2 \wedge \left(\eta^{46} - \eta^{57}\right) + \eta^3 \wedge \left(-\eta^{47} - \eta^{56}\right) \\
\widehat{*\varphi_0} &= \eta^{4567} + \eta^{23} \wedge \left(\eta^{45} + \eta^{67}\right) + \eta^{31} \wedge \left(\eta^{46} - \eta^{57}\right) + \eta^{12} \wedge \left(-\eta^{47} - \eta^{56}\right) .
\end{aligned}$$

Given a $\mathfrak{gl}(7)$-valued connection form θ^j_k on $\mathcal{F}$, we have $\mathrm{d}\eta^j = -\theta^j_k \wedge \eta^k$. Let $p^j_{k\ell}(u)$ be the functions parameterizing $\mathrm{Gr}_7(T_u\mathcal{F}, \pi)$ introduced in §4.10: on any $E \in \mathrm{Gr}_7(T_u\mathcal{F}, \pi)$ we have $\theta^j_k = p^j_{k\ell}\eta^\ell$. Then the equations $\mathrm{d}\widehat{\varphi}_0 = 0 = \mathrm{d}\widehat{*\varphi}_0$ defining $V_7(\mathcal{I}, \pi)$ in $\mathrm{Gr}_7(T\mathcal{F})$ are linear conditions on the parameters $p^j_{k\ell}$. The first equation is equivalent to 35 (independent) linear constraints on the $p^j_{k\ell}$, and the second equation imposes an additional 14. Hence

$$\mathit{codim}(V_7(\mathcal{I}, \pi), \mathrm{Gr}_7(\mathcal{F})) = 49\,.$$

Moreover, by work of Fernandez and Gray [11] we know that $\mathit{dim}\, h^0(\mathfrak{g}_2) = 49$. Whence G_2 is strongly admissible (§4.9).

Fix $E \in V_7(\mathcal{I}, \pi)$, and let F denote the canonical flag of §4.11. Next we compute the sequence of polar spaces. Since $(\Lambda^*\mathbb{R}^7)^{G_2}$ contains no 1- or 2-forms, we have $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}_2 = M_7\mathbb{R} \simeq \mathbb{R}^{49}$. Next, $i_3^*(x.\varphi_0) = (x^1_1 + x^2_2 + x^3_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, so that $\mathfrak{h}_3 = \{x \in M_7\mathbb{R} \mid x^1_1 + x^2_2 + x^3_3 = 0\}$. Similarly, $i_4^*(x.\varphi_0) = 0 = i_4^*(x.*\varphi_0)$ implies that $\mathfrak{h}_4 \subset \mathfrak{h}_3$ is given by the four additional equations

$$0 \;=\; x^1_4 - x^6_3 - x^7_2 \;=\; x^2_4 + x^5_3 + x^7_1 \;=\; x^3_4 - x^5_2 + x^6_1 \;=\; x^5_1 + x^6_2 - x^7_3\,.$$

Continuing in this fashion we find that $\mathit{codim}(\mathfrak{h}_5) = 15$, $\mathit{codim}(\mathfrak{h}_6) = 13$, and $\mathfrak{h}_7 = \mathfrak{g}_2$ so that $\mathit{codim}(\mathfrak{h}_7) = 35$.

Whence the polar space codimensions are $(c_0, c_1, \ldots, c_6) = (0, 0, 0, 1, 5, 15, 28)$ and $\sum c_j = 49$. In particular, $G_2 \subset \mathrm{SO}(7)$ is regularly presented (§4.11). Additionally, Cartan's Test implies that $V_7(\mathcal{I}, \pi)$ is a codimension 49 submanifold of $\mathrm{Gr}(\mathcal{F}, \pi)$, and each $E \in V_7(\mathcal{I}, \pi)$ is the terminus of a (canonical) regular flag F.

This completes the necessary preliminaries for G_2. For Theorem 1.1 we assume that (K^3, g) is a closed, oriented real analytic Riemannian 3-manifold. As an oriented 3-manifold K is smoothly parallelizable by a result of Wu [19]. Using Bochner's result, [5], or Theorem 5.2, we can conclude that K admits a real analytic parallelization.

In the case of Theorem 1.3, invoke Lemma 5.5 to endow $\mathit{doub}(K)$ with a compatible real analytic Riemannian structure $(\mathit{doub}(K), g)$. (If K is closed, then $\mathit{doub}(K) = K$.) As above, $\mathit{doub}(K)$ admits a real analytic parallelization.

The rest of the argument applies to both theorems. Let $A = \mathit{int}(K)$. (In Theorem 1.1, $A = K$.) Assume A is connected, else apply the theorem to each connected component individually. Restrict the Riemannian metric g and real analytic parallelization to A. The Gramm-Schmidt process yields an orthonormal parallelization, and 1-forms ω_1, ω_2 and ω_3 such that

$$g \;=\; {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2\,,$$

and $\mathrm{d}\mathit{vol}_g = \omega_1 \wedge \omega_2 \wedge \omega_3$.

Let $M = A \times \mathbb{R}^4$, and let $y = (y^4, y^5, y^6, y^7)$ be linear coordinates on $\mathbb{R}^4$. Regard the y^j as functions on M and identify A with the 0-section $A \times \{0\}$. The 1-forms $\{\omega, \mathrm{d}y\}$ form a coframing of M and define a global section $s : M \to \mathcal{F}$. The corresponding trivialization of $\pi : \mathcal{F} \to M$ is given by associating to each $u \in \mathcal{F}_z$ the unique $g = g(u) \in \mathrm{GL}(7)$ such that $u = g^{-1} \circ s(z)$. With respect to the trivialization, the canonical 1-forms $\eta = (\eta^j)$ are given by

$$\eta_{(z,g)}(v) \;=\; g^{-1}\begin{pmatrix} \omega(\pi_* v) \\ \mathrm{d}y(\pi_* v) \end{pmatrix}\,, \quad v \in T_{z,g}(M \times \mathrm{GL}(7))\,.$$

It will be convenient notationally to identify $\mathcal{F}$ with the trivialization $M \times \mathrm{GL}(7)$.

Define an involution $r : M \to M$ by $r(p,y) = (p,-y)$, $(p,y) \in A \times \mathbb{R}^4 = M$. Lift r to an involution, also denoted by r, of $\mathcal{F}$ by defining $r(u) = r^*(u)$. That is, given $u : T_zM \to \mathbb{R}^7$ in $\mathcal{F}_z$ we define $r(u) : T_{r(z)}M \to \mathbb{R}^7$ in $\mathcal{F}_{r(z)}$ to be the map sending $v \in T_{r(z)}M$ to $u(r_*v)$. With respect to the trivialization $\mathcal{F} \simeq A \times \mathbb{R}^4 \times \mathrm{GL}(7)$, we have $r(p,y,g) = (p,-y,Rg)$, where

$$R \;=\; \begin{pmatrix} I_3 & 0 \\ 0 & -I_4 \end{pmatrix}.$$

Notice that $R \in G_2$, so that $r : \mathcal{F} \to \mathcal{F}$ preserves the ρ-fibres, and r descends to a well-defined involution of S. Also, $\pi \circ r = r \circ \pi$ implies that $r^*\eta = \eta$, so that

$$r^*\left(\widehat{\varphi_0}\right) \;=\; \widehat{\varphi_0} \quad \text{and} \quad r^*\left(\widehat{*\varphi_0}\right) \;=\; \widehat{*\varphi_0}\,.$$

Whence $r^*\mathcal{I} = \mathcal{I}$, and r carries integral manifolds of $\mathcal{I}$ to integral manifolds of $\mathcal{I}$.

We are now ready to apply the Cartan-Kähler theorem to prove Theorem 1.1. Define a lift $f_3 : A \to S$ by $f_3(p,0) = \rho\,(p,0,I_7)$, and let X_3 denote the image. Since X_3 is three-dimensional, and $\mathcal{I}$ is generated by a 4- and 5-form, X_3 is trivially an integral manifold. Because $R \in G_2$, X_3 lies in the fixed locus of r. We will use the Cartan-Kähler theorem to thicken (in four steps) X_3 to a seven-dimensional r-invariant integral manifold that projects diffeomorphically onto a neighborhood N of $A \subset M$. Moreover, the induced G_2-structure on N will have the properties that (i) $A \subset N$ is associative, and (ii) the metric induced on N by the G_2-structure agrees with g when restricted to A. The construction is repetitive and very similar to that of [8], so after detailing Steps 1 and 2 below, we will sketch the remaining steps.

Step 1. Thicken X_3 to a 4-dimensional integral manifold X_4 of $\mathcal{I}$. Let $z = f_3(p)$. To compute the polar space $H(T_zX_3)$, note that $T_zX_3 = \rho_*T_p$, where T_p is the 3-plane tangent to the lift $\{(p,0,I_7) \mid p \in A\}$ in $\mathcal{F}$. There exists an $E_7 \in V_7(\mathcal{I},\pi)$ containing T_p. Given this, it is clear that $T_p = E_3$ in the canonical regular flag F terminating in E_7. Thus T_zX_3 is $\overline{E}_3$ in the canonical regular flag $\overline{F}$ (cf. §4.11).

To see that such an E_7 exists, recall that $(p^j_{k\ell}) \in \mathbb{R}^{7^3}$ parameterizes the open set of π-transverse $E \in \mathrm{Gr}_7(T_u\mathcal{F},\pi)$, and $V_7(\mathcal{I},\pi)$ is a linear subspace of $\mathbb{R}^{7^3}$ of codimension 49. The condition that T^3_p lie in some E_7 holds if we are free to specify the values of $p^j_{k1}, p^j_{k2}, p^j_{k3}$ in $V_7(\mathcal{I},\pi)$. It can be checked that these variables are independent in $V_7(\mathcal{I},\pi) \subset \mathbb{R}^{7^3}$, so such a specification is always possible.

We may now compute $\mathit{dim}\, H(T_{f_3(p)}X_3) = \mathit{dim}\, H(\overline{E}_3) = 41$. Hence X_3 is a regular integral manifold of extension rank 37. Note that

$$\mathit{dim}\, S \;=\; 42\,;$$

so to apply the Cartan-Kähler Theorem we need to construct a 5-dimensional manifold Z_3 that contains X_3 with tangent space at $z \in X_3$ transverse to $H(T_zX_3)$.

Let $W_1 \subset M_7\mathbb{R}$ be the 1-dimensional subspace of matrices of the form

$$\begin{pmatrix} x_1 I_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_1 \in \mathbb{R}\,.$$

Notice that $W_1 \cap \mathfrak{h}_3 = \{0\}$ and $RW_1 = W_1$. Since $\mathfrak{g}_2 \subset \mathfrak{h}_3$, the affine space $I_7 + W_1$ intersects G_2 transversely at $I_7 \in \mathrm{GL}(7)$. Hence, there is a neighborhood U_1 of 0 in W_1 such that the map $U_1 \to \mathrm{GL}(7)/G_2$ sending $x \mapsto (I + x)G_2$ is an embedding.

Define a 5-dimensional manifold $Z_3 \subset S$ by

$$Z_3 \ := \ \left\{ \rho\left(p, (y^4, 0, 0, 0), I_7 + x\right) \ | \ p \in A\,,\ y^4 \in \mathbb{R}\,,\ x \in U_1 \right\} .$$

As constructed Z_3 contains X_3, is r-invariant and $H(T_z X_3) \cap T_z Z_3$, $z \in X_3$, is of dimension 4. The Cartan-Kähler theorem concludes that there exists a real analytic, 4-dimensional integral manifold $Y_4 \subset Z_3$ containing X_3. Since $r^*\mathcal{I} = \mathcal{I}$, $r(Y_4)$ is also an integral manifold. And since X_3 and Z_3 are r-invariant, we have $X_3 \subset r(Y_4) \subset Z_3$. By the uniqueness part of the Cartan-Kähler theorem the r-invariant $X_4 := Y_4 \cap r(Y_4)$ is also a 4-dimensional integral manifold of $\mathcal{I}$.

Given $z \in X_3$ note that the 4-plane $T_z X_4 = H(T_z X_3) \cap T_z Z_3$ is (i) $\overline{\pi}$-transverse, and (ii) the $\overline{E}_4$ of a regular flag $\overline{F}$. Transversality implies that a neighborhood of X_3 in X_4 projects diffeomorphically onto a neighborhood N_4 of A in $A \times \mathbb{R} \subset M$. Shrinking X_4 if necessary, we may assume that X_4 is image of a section $N_4 \to S$. Item (ii) implies that $T_z X_4$ is regular. Since regularity is an open condition, again shrinking X_4 if necessary, we may assume that X_4 is regular. Finally, we may suppose (shrinking again if necessary) that X_4 is connected. Whence the extension rank of X_4 is 32.

Step 2. Thicken X_4 to a 5-dimensional integral manifold X_5. To apply the Cartan-Kähler theorem we must construct a 10-dimensional manifold Z_4 containing X_4 so that $T_z Z_4$ and $H(T_z X_4)$ are transverse along X_4. Define $W_5 \subset M_7\mathbb{R}$ to be the 5-dimensional subspace

$$\begin{pmatrix} x_1 & 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_5 & x_3 & x_3 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_3 & 0 & 0 & 0 & 0 \\ x_4 & x_4 & x_5 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 .$$

Notice that $W_1 \subset W_5$, $W_5 \cap \mathfrak{h}_4 = \{0\}$, and $RW_5 = W_5$. Since $\mathfrak{g}_2 \subset \mathfrak{h}_4$, the affine space $I_7 + W_5$ intersects G_2 transversely at $I_7 \in \mathrm{GL}(7)$. Hence there is a neighborhood U_5 of $0 \in W_5$ such that the map $U_5 \to \mathrm{GL}(7)/G_2$ sending $x \to (I + x)G_2$ is an embedding.

Define a 10-dimensional manifold $Z_4 \subset S$ by

$$Z_4 \ := \ \left\{ \rho\left(p, (y^4, y^5, 0, 0), I_7 + x\right) \ | \ p \in A\,,\ (y^4, y^5) \in \mathbb{R}^2\,,\ x \in U_5 \right\} .$$

By construction Z_4 contains X_4, is r-invariant, and the intersection $H(T_z X_4) \cap T_z Z_4$, $z \in X_4$, is of dimension five. Thus, the Cartan-Kähler theorem yields a 5-dimensional, real analytic integral manifold Y^5 such that $X_4 \subset Y_5 \subset Z_4$. Since $\mathcal{I}$ is preserved under r, $r(Y_5)$ is also an integral manifold. The r-invariance of X_4 and

Z_4 implies $X_4 \subset r(Y_5) \subset Z_4$. The uniqueness portion of the Cartan-Kähler theorem assures us that $X_5 := Y_5 \cap r(Y_5)$ is also a 5-dimensional, real analytic integral manifold of $\mathcal{I}$.

Given $z \in X_3$, $T_z X_5 = H(T_z X_4) \cap T_z Z_4$ is (i) $\overline{\pi}$-transverse, and (ii) the $\overline{E}_5$ of a canonical regular flag. Transversality implies that a neighborhood of X_3 in X_5 projects diffeomorphically onto a neighborhood N_5 of A in $A \times \mathbb{R}^2 \subset M$. So, shrinking X_5 if necessary, we may assume that it is the (connected) image of a section $N_5 \to S$. Moreover, since regularity is an open condition, a neighborhood of X_3 in X_5 will be regular. Hence, again shrinking X_5 if necessary, we may take X_5 to be a regular integral manifold. The extension rank of X_5 is 21.

Steps 3 & 4. As in Steps 1 & 2 we may thicken X_5 to a 6-dimensional integral manifold X_6 of extension rank 7. Then X_6 is thickened to a 7-dimensional integral manifold X_7 that is an r-invariant connected image of a section $\sigma : N \to S$ over an open neighborhood N of A in M.

The finish. As a section $N \to S$, σ represents a G_2-structure on the 7-dimensional N. The corresponding 3-form on N is $\varphi := \sigma^*(\widehat{\varphi_0})$. By construction $\sigma(N) \subset S$ is an integral manifold of $\mathcal{I}$. Equivalently, the G_2-structure is torsion-free, and (N, φ) is a G_2-manifold.

The relation $r \circ \sigma = \sigma \circ r$ implies that $r : M \to M$ restricts to an involution on N, and that $r^*\varphi = \varphi$. Whence r is a G_2-involution (§2.2), Lemma 2.3. It follows immediately that A, as the fixed point locus of r in N, is associative.

Let h denote the metric induced on N by the G_2-structure. At $z \in A$, $\{\omega_1, \omega_2, \omega_3, \mathrm{d}y^4, \mathrm{d}y^5, \mathrm{d}y^6, \mathrm{d}y^7\}$ is a G_2 coframing of $T_z^* N$. In particular,

$$\begin{aligned} \varphi_z &= \omega_{123} + \omega_1 \wedge \left(\mathrm{d}y^{45} + \mathrm{d}y^{67}\right) + \omega_2 \wedge \left(\mathrm{d}y^{46} - \mathrm{d}y^{57}\right) \\ &\quad + \omega_3 \wedge \left(-\mathrm{d}y^{47} - \mathrm{d}y^{56}\right) \,, \\ h_z &= {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2 + (\mathrm{d}y^4)^2 + (\mathrm{d}y^5)^2 + (\mathrm{d}y^6)^2 + (\mathrm{d}y^7)^2 \,. \end{aligned}$$

Whence $h_{|A} = g$, and the inclusion $i : A \hookrightarrow N$ is an isometry. (Also, $i^*\varphi = \omega_{123} = \mathrm{d}vol_A$, proving again that A is associative.) This completes the proof of Theorem 1.1.

Remark. We remarked in §1 after the statement of Theorem 1.1 that, so long as A is not flat, $\mathrm{Hol}(N) = G_2$. In particular, $N \neq CY \times \mathbb{R}$. This may be seen as follows. Suppose that $\mathrm{Hol}(N) \neq G_2$. Then $\mathrm{Hol}(N) \subseteq \mathrm{SU}(3)$ and N is an $\mathrm{SU}(3)$-manifold.

Set $\mathbf{i} = \sqrt{-1}$ and take the SU(3) action on $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$ that fixes the forms $\mathrm{d}x^7$,

$$\begin{aligned} \omega_0 &= \mathrm{d}x^{16} - \mathrm{d}x^{25} - \mathrm{d}x^{34} \,, \\ \Upsilon_0 &= (\mathrm{d}x^1 + \mathbf{i}\,\mathrm{d}x^6) \wedge (\mathrm{d}x^2 - \mathbf{i}\,\mathrm{d}x^5) \wedge (\mathrm{d}x^3 - \mathbf{i}\,\mathrm{d}x^4) \,. \end{aligned}$$

Then SU(3) acts trivially on the second factor and by the standard representation on the first.

Let $R = \mathcal{F}/\mathrm{SU}(3)$ and consider

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mu"] \arrow[dd, "\pi"'] & R \arrow[d, "\nu"] \\ & S \arrow[dl, "\overline{\pi}"] \\ N & \end{tikzcd}$$

Above, $\nu \circ \mu = \rho$. Let $\tilde{\pi} = \overline{\pi} \circ \nu : R \to N$.

If $\mathrm{Hol}(N) \subseteq \mathrm{SU}(3)$, then N is an SU(3)-manifold. In particular, N admits a section $\tau : N \to R$ such that $\nu \circ \tau = \sigma$. Rename $\mathcal{I} = \mathcal{I}_{G_2}$, and let $\mathcal{I}_{\mathrm{SU}(3)}$ be the ideal generated by $\mathrm{d}\eta^7$, $\mathrm{d}\omega$ and $\mathrm{d}\Upsilon$, where

$$\begin{aligned} \omega &= \eta^{16} - \eta^{25} - \eta^{34} \\ \Upsilon &= (\eta^1 + \mathbf{i}\, \eta^6) \wedge (\eta^2 - \mathbf{i}\, \eta^5) \wedge (\eta^3 - \mathbf{i}\, \eta^4)\,. \end{aligned}$$

The image $\tau(N)$ is necessarily an integral manifold of $\mathcal{I}_{\mathrm{SU}(3)}$.

Fix $z \in X_3$, and let $E \in V_7(\mathcal{I}_{G_2}, \pi)$ denote the 7-plane constructed in the proof above with $\rho_* E = T_z X_7 = T_z \sigma(N) \in V_7(\mathcal{I}_{G_2}, \overline{\pi})$. If $\tau : N \to R$ exists, then there exists $E' \in V_7(\mathcal{I}_{\mathrm{SU}(3)}, \pi) \subset V_7(\mathcal{I}_{G_2}, \pi)$ such that $\mu_* E' \in V_7(\mathcal{I}_{\mathrm{SU}(3)}, \tilde{\pi})$ is tangent to $\tau(N)$ and $\rho_* E' = \nu_*(\mu_* E') = \rho_* E$. As noted in §4.10, this implies $E' \equiv E \bmod \mathfrak{g}_2$. A lengthy computation confirms that this is possible if and only if A is flat.

7. The Cayley embedding: proof of Theorems 1.2 & 1.4.

Remark. As an application of the Cartan-Kähler Theorem the proof of Theorems 1.2 and 1.4 is very like the proof of Theorems 1.1 and 1.3. However, unlike 3-manifolds, the 4-manifold $A = int(K)$ may not admit a global parallelism. We assume that the bundle of self-dual 2-forms on $doub(K)$ is trivial in order to obtain the structure necessary to apply the Cartan-Kähler Theorem.

The group $Spin(7)$ is admissible [7]; $\Lambda^*(\mathbb{R}^8)^{Spin(7)}$ is generated by the 4-form Ψ_0. The differential system $\mathcal{I}$ of §4.8 is generated by the exterior derivative of

$$\begin{aligned} \widehat{\Psi_0} = \eta^{0123} + \eta^{4567} &+ \left(\eta^{01} + \eta^{23}\right) \wedge \left(\eta^{45} + \eta^{67}\right) \\ &+ \left(\eta^{02} + \eta^{31}\right) \wedge \left(\eta^{46} + \eta^{75}\right) + \left(\eta^{03} + \eta^{12}\right) \wedge \left(\eta^{74} + \eta^{65}\right)\,. \end{aligned}$$

The condition that $\mathrm{d}\widehat{\Psi_0} = 0$ is 56 independent linear equations on the functions $p^j_{k\ell}(u)$ parameterizing $\mathrm{Gr}_8(T_u\mathcal{F}, \pi)$ (cf. §4.10). Thus

$$codim(\mathrm{V}_8(\mathcal{I}, \pi), \mathrm{Gr}_8(T\mathcal{F})) \;=\; 56\,.$$

Since $dim\, \mathfrak{h}^0(\mathfrak{spin}(7)) = 56$ [7, Prop.4], it follows that $Spin(7)$ is strongly admissible, c.f. §4.9.

Fix $E \in V_8(I, \pi)$, and let F denote the canonical flag of §4.11. The subspaces $\mathfrak{h}_k$ of §4.11 are: $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}_3 = M_8\mathbb{R} \simeq \mathbb{R}^{64}$. The subspace $\mathfrak{h}_4 \subset M_8\mathbb{R}$ is defined by the single equation $x^0_0 + x^1_1 + x^2_2 + x^3_3 = 0$. As in §6 calculations of the remaining $\mathfrak{h}_j$ lead us to the polar space codimensions: $(c_0, c_1, \ldots, c_8) = (0,0,0,0,1,5,15,35,43)$ and $\sum c_j = 56$, so that $Spin(7)$ is regularly presented, c.f §4.11. Cartan's Test concludes that $V_8(\mathcal{I}, \pi)$ is a codimension 56 (observed above) submanifold of $\mathrm{Gr}_8(T\mathcal{F}, \pi)$, and each $E \in V_8(\mathcal{I}, \pi)$ is the terminus of a canonical regular flag F.

This completes the necessary preliminaries for *Spin*(7). For Theorem 1.2 we assume that (K^4, g) is a closed oriented, real analytic Riemannian 4-manifold, and that the bundle $\Lambda^2_+(K)$ of self-dual 2-forms over K is smoothly trivial. Bochner's result or Theorem 5.2 implies that $\Lambda^2_+(K)$ is real-analytically trivial. In particular, there exist globally defined real-analytic self-dual 2-forms Ω_1, Ω_2, Ω_3 such that $\Omega_j \wedge \Omega_k = 2\delta_{jk}\, \mathrm{d}vol_g$.

In the case of Theorem 1.4, invoke Lemma 5.5 to endow $doub(K)$ with a compatible real analytic Riemannian structure $(doub(K), g)$. (If K is closed, then $doub(K) = K$.) As above, $doub(K)$ admits globally defined real-analytic self-dual 2-forms Ω_1, Ω_2, Ω_3 such that $\Omega_j \wedge \Omega_k = 2\delta_{jk}\, \mathrm{d}vol_g$.

The rest of the argument applies to both theorems. Let $A = int(K)$. (In Theorem 1.2, $A = K$.) Assume A is connected, else apply the theorem to each connected component individually. Restrict the Riemannian metric g and 2-forms Ω_j to A. While A may not admit a global coframing, it is not difficult to check that there exist local orthonormal coframings $\{\omega^a\}_{a=1}^4$ such that

$$\Omega_1 = \omega^1\wedge\omega^2 + \omega^3\wedge\omega^4\,, \quad \Omega_2 = \omega^1\wedge\omega^3 - \omega^2\wedge\omega^4\,, \quad \Omega_3 = \omega^1\wedge\omega^4 + \omega^2\wedge\omega^3\,.$$

This choice of coframing is unique up to the action of SU(2). (Recall, SU(2) is the subgroup of SO(4) preserving the two forms $\frac{i}{2}(\zeta^1\wedge\overline{\zeta}^1 + \zeta^2\wedge\overline{\zeta}^2)$ and $\zeta^1\wedge\zeta^2$, where $\zeta^1 = \omega^1 + i\omega^2$ and $\zeta^2 = \omega^3 + i\omega^4$.) We may identify SU(2) with the subgroup of *Spin*(7) fixing $\mathrm{d}x^j$, $j = 0, \ldots, 3$, via

$$\left\{ \begin{pmatrix} \mathrm{Id}_4 & 0 \\ 0 & P \end{pmatrix} \middle|\, P \in \mathrm{SU}(2) \right\} \subset \mathit{Spin}(7)\,.$$

Let $M = \mathbb{R}^4 \times A$ with linear coordinates $\{y^j\}_{j=0}^3$ on $\mathbb{R}^4$. Then

$$\Psi \;=\; \mathrm{d}y^{0123} + \tfrac{1}{2}\Omega_1\wedge\Omega_1 + \big(\mathrm{d}y^{01} + \mathrm{d}y^{23}\big)\wedge\Omega_1 + \big(\mathrm{d}y^{02} + \mathrm{d}y^{31}\big)\wedge\Omega_2 - \big(\mathrm{d}y^{03} + \mathrm{d}y^{12}\big)\wedge\Omega_3$$

defines a *Spin*(7)-structure on M. Let $\sigma : M \to S$ denote the corresponding section.

Define an involution on M by $r(y,p) = (-y,p)$. Note that $r^*\Psi = \Psi$. Define a covering involution $r : \mathcal{F} \to \mathcal{F}$ as follows. Given $u : T_zM \to \mathbb{R}^8$, let $r(u)$ be the coframe $r^*(u) : T_{r(z)}M \to \mathbb{R}^8$. Then $r^*(\eta) = \eta$ and $r^*\widehat{\Psi_0} = \widehat{\Psi_0}$. Let $\{\omega^a\}$ be a coframing of an open set $U \subset L$. Then $\{\mathrm{d}y^j, \omega^a\}$ defines a trivialization $\mathcal{F}_{|\mathbb{R}^4\times U} := \pi^{-1}(\mathbb{R}^4\times U) \simeq \mathbb{R}^4 \times U \times \mathrm{GL}_8\mathbb{R}$. Notice that $\mathcal{F}_{|\mathbb{R}^4\times U}$ is invariant under r and, with respect to the trivialization, is given by $r(y,p,g) = (-y,p,Rg)$, where

$$R \;=\; \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \in \mathit{Spin}(7)\,.$$

In particular, r descends to a well-defined involution on S.

From this point on the proof of Theorem 1.2 is very similar to the proof of Theorem 1.1; so we merely sketch the main steps. Define $X_4 = \sigma(A)$. Since $\mathcal{I}$ is generated by a 5-form, and X_4 is 4-dimensional, X_4 is trivially an integral manifold of $\mathcal{I}$. Since $r\circ\sigma = \sigma\circ r$, it follows that X_4 lies in the fixed point locus of $r : S \to S$.

It remains to select the subspaces $W_{d_1} \subset W_{d_2} \subset W_{d_3} \subset W_{d_4} \subset M_8\mathbb{R}$, $(d_4, d_5, d_6, d_7) = (1, 5, 15, 35)$, so that (i) $dim W_{d_j} = d_j$, (ii) $W_{d_s} \cap \mathfrak{h}_s = \{0\}$, and

(iii) $RW_d \subset W_d$. Because the coframing $\omega, \mathrm{d}y$ of M is defined only up to the SU(2) action it is also necessary that we pick the subspaces so that $\mathrm{SU}(2)\,W_d \subset W_d$. We leave this exercise to the reader.

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