ASIAN J. MATH.
Vol. 12, No. 4, pp. 527–544, December 2008

TOPOLOGY OF CO-SYMPLECTIC/CO-KÄHLER MANIFOLDS*

HONGJUN LI†

Abstract. Co-symplectic/co-Kähler manifolds are odd dimensional analog of symplectic/Kähler manifolds, defined early by Libermann in 1959/Blair in 1967 respectively. In this paper, we reveal their topology construction via symplectic/Kähler mapping tori. Namely,

THEOREM. *Co-symplectic manifold = Symplectic mapping torus;*
Co-Kähler manifold = Kähler mapping torus.

Key words. Co-symplectic manifold, symplectic mapping torus, co-Kähler manifold, Kähler mapping torus.

AMS subject classifications. Primary 53C15; Secondary 53C55, 53D35, 57R17

1. Introduction. By a **co-symplectic manifold** in this paper, we mean a (2n+1)-manifold M together with a closed 1-form η and a closed 2-form Φ such that $\eta \wedge \Phi^n$ is a volume form. This was P. Libermann's definition in 1959 [1], under the name of *cosymplectic manifold*. The pair (η, Φ) is called a **co-symplectic structure** on M. We may view co-symplectic manifolds as odd-dimensional analog of symplectic manifolds.

To a co-symplectic manifold $(M; \eta, \Phi)$, there always associates a so called *almost contact metric structure* (ϕ, ξ, η, g). Where ξ is the Reeb vector field (defined by $\iota_\xi \Phi = 0$ and $\eta(\xi) = 1$) and (ϕ, g) may be created simultaneously by polarizing Φ on the hyperplane distribution $\ker \eta$. It satisfies the following identities

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y).$$

When the structure tensor ϕ is parallel with respect to the Levi-Civita connection of g (i.e., $\nabla^g \phi = 0$), we obtain D.E.Blair's cosymplectic manifolds defined in 1967 [2]. In literature, the terminology "cosymplectic manifold" was more referred to Blair's definition, see [3][4][5][6]. In this paper, we would like to call Blair's cosymplectic manifolds simply as **co-Kähler manifolds**, since our result shows that they are really odd-dimensional analog of Kähler manifolds.

By a **mapping torus**, we mean a topological construction described as follows. Let $\varphi \in \mathrm{Diff}(S)$ be a self-diffeomorphism on a closed,connected manifold S. The mapping torus S_φ is obtained from $S \times [0, 1]$ by identifying the two ends via φ, namely,

$$S_\varphi = S \times [0, 1]/(x, 0) \sim (\varphi(x), 1).$$

It is known that S_φ is exactly the total space of a fiber bundle: $S \hookrightarrow S_\varphi \xrightarrow{\pi} S^1$. Let θ be the coordinate on S^1, it induces naturally a **characteristic 1-form** $\eta_\theta = \pi^*(d\theta)$ and a **distinguished vector field** ξ_θ on S_φ. The duality of $(\xi_\theta, \eta_\theta)$ can be expressed by the identity $\eta_\theta(\xi_\theta) = d\theta(\frac{d}{d\theta}) = 1$.

If (S, ω) is a symplectic manifold and φ is a symplectomorphism which satisfies $\varphi^*\omega = \omega$, we call S_φ a **symplectic mapping torus**. In addition, if (S, ω) is a Kähler

*Received October 21, 2007; accepted for publication June 26, 2008.
†Science College, Xi'an Jiaotong University, Xi'an 710049, P.R. China (hjlimath@mail.xjtu.edu.cn).

manifold with associated Hermitian structure (J, h) and φ is a Hermitian isometry which satisfies $\varphi_* \circ J = J \circ \varphi_*$ and $\varphi^* h = h$ (hence also $\varphi^*\omega = \omega$), we call S_φ a **Kähler mapping torus**.

Co-symplectic/co-Kähler manifolds have been investigated extensively, with more attention being focused on their geometric contents. However, up to author's knowledge, their topological construction has not explicitly been revealed before. The aim of this paper is to show the following

Theorem 1. *Co-symplectic manifold = Symplectic mapping torus.*
Theorem 2. *Co-Kähler manifold = Kähler mapping torus.*

Theorem 1 may not be surprised to experts. We sketch it as follows.

On a symplectic mapping torus S_φ, there is a *canonical* co-symplectic structure (η, Φ). Where: $\eta = \eta_\theta$ is taken as the characteristic 1-form, Φ is obtained by first pull-back ω (via $S \times [0,1] \xrightarrow{p} S$) to get a closed 2-form $p^*\omega$ on $S \times [0,1]$ and then glue up $p^*\omega$ at the two ends(since $\varphi^*\omega = \omega$).

For the converse part of Theorem 1, we have a good starting point due to D. Tischler in 1970 [7].

Theorem 0 (Tischler). *A compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form.*

It follows that the half entry η in the co-symplectic structure (η, Φ) suffices to make M being a mapping torus S_φ. The other half Φ will be restricted to fibers of S_φ to get a family of symplectic 2-form $\{\omega_t\}|_{t\in[0,1]}$. Since $[\omega_0] = [\omega_1]$ is a same cohomology class and $\omega_0 = \varphi^*\omega_1$, φ is isotopic to a symplectomorphism ψ. This makes $S_\varphi \cong S_\psi$ becoming a symplectic mapping torus.

Theorem 2 is more interesting. On a Kähler mapping torus S_φ, there is a canonical co-Kähler structure similar to that described above, and this construction had been used for instances in [3][4].

The converse part of Theorem 2 may be too delicate to be sketched here, but we would like to indicate the key point here. On a co-Kähler manifold $(M; \phi, \xi, \eta, g)$, Theorem 1 has made M being a symplectic mapping torus, $M = S_\varphi$. By hypothesis, ϕ is parallel (i.e., $\nabla^g \phi = 0$ or $N_\phi = 0$) on the *foliation* defined by $\ker \eta$, but this does not imply the corresponding parallel-city on the *fibration* $S_\varphi \xrightarrow{\pi} S^1$ defined by the approximated integral 1-form η_θ. The difficulty is how to transfer the parallel ϕ on the "screwy" foliation $\ker \eta$ to a new parallel one on the "regular" fibration $\ker \eta_\theta$. Only from a parallel "regular" structure $(\phi_0, \partial_\theta, \eta_\theta, g_0)$ can we recover a Kähler structure (J, h) on S.

In 1993 in [4], Chinea, de Leon, and Marrero discussed the topology of co-Kähler manifolds. Amongst other things, they obtained the monotone property of Betti numbers of co-Kähler manifolds:

$$1 = b_0 \leq b_1 \leq \cdots \leq b_n = b_{n+1} \geq \cdots \geq b_{2n} \geq b_{2n+1} = 1.$$

By Theorem 2, a co-Kähler manifold is just a Kähler mapping torus S_φ. This enables us to give a purely topological proof of this monotone property.

The paper is organized as follows.

In section 2, we give a brief argument on Tischler's Theorem 0 and then prove Theorem 1. We also show that M admits a co-symplectic structure if and only if $S^1 \times M$ admits a S^1-invariant symplectic structure.

In section 3, two related topics on symplectic mapping tori are discussed. We give an example of type $(S^2 \times S^2)_\varphi$ to show that a mapping torus may admit no co-symplectic structure even though $\varphi \in \mathrm{Diff}^+(S)$ is of orientation-preserving. However, such phenomena can not occur in dimension three.

Section 4 is devoted to Theorem 2. In section 5, as an application of Theorem 2, we give a new proof of monotone property of Betti numbers for co-Kähler manifolds.

2. Co-symplectic manifold = Symplectic mapping torus.

2.1. Mapping torus and Tischler's Theorem. The mapping torus of a self-homeomorphism $\varphi : S \to S$, denoted by S_φ, is the manifold obtained from $S \times [0,1]$ by identifying the ends through φ,

$$S_\varphi = S \times [0,1]/\{(x,0) \sim (\varphi(x),1) | x \in S\}.$$

Equipped with a canonical map $\pi : S_\varphi \to S^1$ defined by $\pi(x,t) = e^{2\pi t\sqrt{-1}}$, S_φ is a fibre bundle over S^1 with fiber S and monodromy φ. Usually, the base circle S^1 is given the coordinate $\theta = 2\pi t$.

In topology, it is important to know when a manifold M fibers over S^1. In 1970 [7], D.Tischler gave a good criterion on this topic. Since it is a starting point of this paper, we would like to repeat his proof here.

Theorem 0 (D. Tischler). *A compact manifold admits a non-vanishing closed 1-form if and only if the manifold fibres over a circle.*

Proof. If M is a fibration over S^1 with bundle map $\pi : M \to S^1$, then the pull-back $\pi^*(d\theta)$ will be a non-vanishing closed 1-form on M.

Let M admit a non-vanishing closed 1-form, saying η . The compactness of M implies that $[\eta]$ represents a non-zero de Rham cohomology class in $H^1(M, \mathbf{R})$. There is an obvious homomorphism $H^1(S^1, \mathbf{R}) \to H^1(M, \mathbf{R})$ sending $[d\theta]$ to $[\eta]$, where $[d\theta]$ denotes the generator of $H^1(S^1, \mathbf{R})$.

In case $[\eta] \in H^1(M; \mathbf{Z})$ represents an integral cohomology class, the above homomorphism can be realized by a differentiable map $\pi : M \to S^1$ such that $\pi^*(d\theta) = \eta$, since $H^1(M; \mathbf{Z})$ can be identified with the group of homotopy classes $[M, S^1]$.

Generally, $H^1(M; \mathbf{R})$ admits an integral basis $\{[\pi_1^*(d\theta)], \cdots, [\pi_k^*(d\theta)]\}$, where $k = \dim H^1(M; \mathbf{R})$. This gives an expression $\eta = \sum_{i=1}^k r_i \pi_i^*(d\theta) + dh$ for real numbers $\{r_i\}$ and real valued function h. The last term dh can be absorbed in the previous summand so that we may assume $\eta = \sum_{i=1}^k r_i \pi_i^*(d\theta)$. This absorbing process follows since $\pi_1^*(d\theta) + dh = (\pi_1 + \Pi \circ h)^*(d\theta)$, where $\Pi : \mathbf{R} \to S^1$ is the covering map and the right hand addition is induced by group multiplication on S^1. Now replacing $\{r_i\}$ by appropriate rational numbers $\{\frac{n_i}{d}\}$ with common divisor d, we can make $\|\eta - \sum_{i=1}^k \frac{n_i}{d} \pi_i^*(d\theta)\| = \|\eta - \frac{1}{d}\pi^*(d\theta)\|$ arbitrary small, where the norm comes from a Riemannian metric on M and $\pi = \sum_{i=1}^k n_i \pi_i$ is well-defined. It follows that $\pi^*(d\theta)$ is non-vanishing and so π is a submersion. This shows that $\pi : M \to S^1$ is a fiber map. □

The key point in this argument is that one can always approximate an arbitrary non-vanishing closed 1-form η by an *integral* closed 1-form η_θ. η_θ defines a fibration

$\pi : M \to S^1$ on M such that $\eta_\theta = \pi^*(d\theta)$, but η defines only a co-dimensional one foliation $\ker \eta$ on M which may be of "screwy".

On a mapping torus S_φ with fibre map $\pi : S_\varphi \to S^1$, the non-vanishing closed 1-form $\eta_\theta = \pi^*(d\theta)$ is called a *characteristic* 1*-form.* Dual to η_θ, there is a distinguished vector field ξ_θ such that $\pi_*(\partial_\theta) = \frac{d}{d\theta}$. In fact, S_φ is locally a product $S \times I$, where I is an interval. Hence the coordinate θ in S^1 gives a local coordinate in S_φ, and the projection $\pi : S_\varphi \to S^1$ is a submersion on these coordinates. This defines a vector field ξ_θ which corresponds to $\frac{\partial}{\partial\theta}$ in S^1. ξ_θ is called a *distinguished vector field* on the mapping torus S_φ.

It may be well-known that ξ_θ is always a volume-preserving vector field on S_φ, see Proposition 4. But in dimension three, ξ_θ can never be a Reeb field of any contact form on S_φ unless $S = T^2$.

2.2. Proof of Theorem 1. In a mapping torus S_φ, when (S, ω) is a closed symplectic manifold with $\varphi : S \to S$ a symplectomorphism such that $\varphi^*\omega = \omega$, we say that S_φ is a symplectic mapping torus. A co-symplectic manifold M, in the sense of Libermann, is a $(2n+1)$-manifold together with a co-symplectic structure (η, Φ) consisting of a closed 1-form η and a closed 2-form Φ such that $\eta \wedge \Phi^n$ is a volume-form.

A co-symplectic pair (η, Φ) can be seen as a geometrical structure on an odd-manifold, analogous to a symplectic structure on an even-manifold. On the other hand, a symplectic mapping torus S_φ is purely a topological construction. The first main result of this paper is the following

THEOREM 1. *A closed manifold M admits a co-symplectic structure if and only if it is a symplectic mapping torus $M = S_\varphi$. In short:*

Co-symplectic manifold = Symplectic mapping torus.

A mapping torus is characterized by a non-vanishing closed 1-form η, by Tischler's Theorem 0. A symplectic mapping torus is characterized by a co-symplectic structure (η, Φ), by Theorem 1. Hence, Theorem 1 can be regarded as an extension of Tischler's Theorem 0 in the best way to cover "symplectic" information.

The proof of Theorem 1 will be finished after three lemmas. The sufficient part of Theorem 1 is just the following lemma, and this result may be known to authors in [3][4].

LEMMA 1. *If $M = S_\varphi$ is a symplectic mapping torus, then M admits a canonical co-symplectic structure $(\eta_\theta, \Phi_\theta)$.*

Proof. Let $M = S_\varphi$ be a symplectic mapping torus. Topologically, S_φ is obtained from $S \times [0, 1]$ by identifying two ends through the diffeomorphism φ, namely, $S_\varphi = (S \times [0, 1])/\varphi$. This construction can well be seen by the following diagram:

$$\begin{array}{ccccc} S & \xleftarrow{p_1} & S \times [0,1] & \xrightarrow{\rho} & S_\varphi \\ & & p_2 \downarrow & & \downarrow \pi \\ & & [0,1] & \xrightarrow{\Pi} & S^1. \end{array}$$

where $\{p_1, p_2\}$ are projections and $\rho : S \times [0, 1] \to S_\varphi$ is the identification map. The symplectic 2-form ω on S is pulled-back to a non-vanishing closed 2-form $p_1^*\omega$ on

$S \times [0,1]$. Then $p_1^*\omega$ can be glued up, since $\varphi^*\omega = \omega$, to give a non-vanishing closed 2-form Φ_θ on S_φ. Let $\eta_\theta = \pi^*(d\theta)$ be the characreristic 1-form as before. Since p_1 and p_2 are independent projections from $S \times [0,1]$, one checks easily that $\eta_\theta \wedge \Phi_\theta^n \neq 0$. Hence, $(\eta_\theta, \Phi_\theta)$ defines a co-symplectic structure on $M = S_\varphi$. □

Now, let's consider the necessary part of Theorem 1.

Let M be a co-symplectic manifold with co-symplectic structure (η, Φ). Since η is a non-vanishing closed 1-form, it follows from Tischler's Theorem 0 that M is a mapping torus $M = S_\varphi$ for some diffeomorphism $\varphi : S \to S$. It remains only to show that $M = S_\varphi$ is indeed a symplectic mapping torus.

As in Lemma 1, let $\rho : S \times [0,1] \to S_\varphi$ be the identification map. Let $i_\tau(x) = (x, \tau) : S \to S \times [0,1]$ denote the inclusion for all $\tau \in [0,1]$.

Lemma 2. *S is a symplectic manifold. More precisely,*

(i) $\forall\ \tau \in [0,1], \quad \omega_\tau = (\rho \circ i_\tau)^*\Phi$ *is a symplectic 2-form on S.*

(ii) $\omega_0 = \varphi^*\omega_1$ *and* $[\omega_0] = [\omega_1] \in H^2(S, \mathbf{R})$.

Proof. Let η_θ be the characteristic 1-form on $M = S_\varphi$. By proof of Tischler's Theorem 0, η_θ is an approximation of η. Hence, being (η, Φ) a co-symplectic structure implies that (η_θ, Φ) is also a co-symplectic structure on M.

$\forall\ \tau \in [0,1]$, the pull-back $\omega_\tau = (\rho \circ i_\tau)^*\Phi$ is clearly a closed 2-form on S. We need to show its non-degeneracy.

Locally, on a trivialization $U \times J$ of S_φ, Φ can be written as

$$\Phi|_{U\times J} = \sum_{i<j} a_{ij}(x,t) dx_i \wedge dx_j + \sum_{k=1}^{2n} b_k(x,t) dx_k \wedge dt,$$

where $(x,t) = (x_1, \cdots, x_{2n}, t)$ are local coordinates on $U \times J \subset S_\varphi$ and $\{a_{ij}(x,t), b_k(x,t)\}$ smooth functions on $U \times J$. If $\tau \in J \subset [0,1]$, we have

$$\omega_\tau = (\rho \circ i_\tau)^*\Phi = \sum_{i<j} a_{ij}(x,\tau) dx_i \wedge dx_j.$$

Since $\eta_\theta|_{U\times J} = 2\pi dt$, the following identity holds:

$$(\eta_\theta \wedge \Phi^n)|_{U\times J} = 2\pi \cdot dt \wedge \omega_t^n.$$

This shows that ω_τ^n is nowhere zero and so ω_τ is a symplectic 2-form.

By the construction $S_\varphi = S \times [0,1]/\{(x,0) \sim (\varphi(x),1)\}$, we know that $\rho \circ i_0 = \rho \circ i_1 \circ \varphi$. Hence, $\omega_0 = \varphi^*\omega_1$ by definition. Note also that the two maps $\rho \circ i_0$ and $\rho \circ i_1$ are clearly homotopic, so $[\omega_0] = [\omega_1] \in H^2(S, \mathbf{R})$.

This shows the lemma. □

Lemma 3. *If S_φ admits a co-symplectic structure, then the gluing map φ is isotopic to a symplectomrphism ψ so that $S_\varphi \cong S_\psi$ is indeed a symplectic mapping torus.*

Proof. By Lemma 2, S admits a 1-parameter family of symplectic 2-forms $\{\omega_\tau\}_{\tau\in[0,1]}$ and $[\omega_0] = [\omega_1]$ is a same cohomology class. By Moser's theorem [8] on the stability of symplectic 2-forms, there is a diffeomorphism $F: S \to S$ which is isotopic to the identity and satisfies $\omega_1 = F^*\omega_0$. Again by Lemma 2, $\varphi^*\omega_1 = \omega_0$. It follows that $(F \circ \varphi)^*\omega_0 = \omega_0$. This means that $\psi = F \circ \varphi$ is a symplectomophism on S which is clearly isotopic to φ.

Finally, it is not difficulty to show that if $\varphi, \psi \in \mathrm{Diff}^+(S)$ are isotopic then the two mapping tori S_φ, S_ψ are fiber-preservingly diffeomorphic. □

Summing up, Lemma 1+Lemma 2+Lemma 3, we have completed the whole proof of Theorem 1. □

2.3. S^1-invariant symplectic structure on $S^1 \times M$. In this subsection, we turn to the product $S^1 \times M$ to study co-symplectic manifold M. From this extrinsic point of view, in addition to Theorem 1, we show again that co-symplectic manifolds are really odd dimensional analog of symplectic manifolds.

PROPOSITION 1. *M admits a co-symplectic structure (η, Φ) if and only if the product $S^1 \times M$ admits a S^1-invariant symplectic 2-form Ω.*

Proof. Let's first introduce some symbols on $S^1 \times M$. The coordinate in the S^1-factor of a product $S^1 \times M$ will be denoted by t, and ∂_t and dt are the induced vector field and 1-form on $S^1 \times M$, respectively. We denote by $\pi : S^1 \times M \to M$ the canonical projection and by $s(x) = (0, x) : M \to S^1 \times M$ a fixed section.

On $S^1 \times M$ there is an obvious S^1-action. A r-form Ω on $S^1 \times M$ is called *S^1-invariant* if $\mathcal{L}_{\partial_t}\Omega = 0$.

The necessary part " $\Longrightarrow$": Starting from a co-symplectic structure (η, Φ) on M, we define a closed 2-form Ω on $S^1 \times M$ by

$$\Omega = \pi^*\Phi + \pi^*\eta \wedge dt.$$

It is easy to compute that

$$\begin{aligned} \Omega^{n+1} &= (n+1)\pi^*(\eta \wedge \Phi^n) \wedge dt \\ \mathcal{L}_{\partial_t}\Omega &= 0. \end{aligned}$$

The first equality shows that Ω is a symplectic 2-form on $S^1 \times M$, since $\eta \wedge \Phi^n$ is a volume-form on M. The second equality means that Ω is S^1-invariant.

The sufficient part" $\Longleftarrow$": Let Ω be a S^1-invariant symplectic 2-form on $S^1 \times M$, satisfying $\mathcal{L}_{\partial_t}\Omega = 0$. We define a pair (η, Φ) on M by

$$\eta = s^*\iota_{\partial_t}\Omega, \qquad \Phi = s^*\Omega.$$

Obviously, η and Φ are closed forms on M, since $d\eta = s^*\mathcal{L}_{\partial_t}\Omega = 0$ and $d\Phi = s^*d\Omega = 0$. We claim that the expression

$$\Omega = \pi^*\Phi + \pi^*\eta \wedge dt$$

holds on $S^1 \times M$. Then the identity $\Omega^{n+1} = (n+1)\pi^*(\eta \wedge \Phi^n) \wedge dt$ and the non-degeneracy of Ω implies that $\eta \wedge \Phi^n$ is a volume-form on M, and consequently the pair (η, Φ) is a co-symplectic structure on M.

Locally, on a trivialization $S^1 \times U$ with $U \subset M$, Ω can be expressed as

$$\Omega|_{S^1 \times U} = \sum_{i<j} a_{ij}(t,x)dx_i \wedge dx_j + \sum_{k=1}^{2n+1} b_k(t,x)dx_k \wedge dt,$$

where $x = (x_1, \cdots, x_{2n+1})$ are local coordinates on U and $\{a_{ij}(t,x), b_k(t,x)\}$ are smooth functions on $S^1 \times U$. We compute directly that

$$\Phi|_U = s^*\Omega = \sum_{i<j} a_{ij}(0,x)dx_i \wedge dx_j, \qquad \eta|_U = s^*\iota_{\partial_t}\Omega = \sum_{k=1}^{2n+1} b_k(0,x)dx_k.$$

Now, the condition $\mathcal{L}_{\partial_t}\Omega = 0$ implies that $\{a_{ij}(t,x), b_k(t,x)\}$ are independent to the variable t. Hence, we have

$$\begin{aligned}\Omega|_{S^1\times U} &= \sum_{i<j} a_{ij}(0,x)dx_i \wedge dx_j + \sum_{k=1}^{2n+1} b_k(0,x)dx_k \wedge dt \\ &= \pi^*\Phi|_U + \pi^*\eta|_U \wedge dt.\end{aligned}$$

This establish the disired expression $\Omega = \pi^*\Phi + \pi^*\eta \wedge dt$ on $S^1 \times M$. □

The product $S^1 \times M$ can be seen as a S^1-bundle with trivial free S^1-action. Generally, Fernández, Gray and Morgan discussed when a S^1-bundles with a free S^1-action admits a S^1-invariant symplectic structure in [11]. For our aim of comparison, let's present their main result briefly as the following

PROPOSITION 2. (Theorem 18, [11]) *Let E be a closed symplectic manifold with a free S^1-action leaving the symplectic form invariant. Let $\pi : E \to M$ be the S^1-fibration induced by the S^1-action. Then there exist a closed manifold S and a diffeomorphism $\varphi : S \to S$ such that*

(i) *M is a mapping torus of φ.*

(ii) *There is a 1-parameter family $\{\omega_t\}_{t\in[0,1]}$ of symplectic forms on S with*

$$\omega_0 = \varphi^*\omega_1 \quad \text{and} \quad [\omega_t] = [\omega_0] + t\xi$$

for some φ-invariant class $\xi \in H^2(S, \mathbf{Z})$.

(iii) *The Euler class of $\pi : E \to M$ restrict to $S \times \{0\}$ to give ξ.*

It follows from Proposition 2 that M is always a mapping torus S_φ on a symplectic manifold S but may not be a symplectic mapping torus in general. In case $E = S^1 \times M$(i.e., $\xi = 0$), the conclusion of Proposition 2 coincides with that of Lemma 2, so that M turns out to be a symplectic mapping torus by Lemma 3.

In light of Proposition 1 and Theorem 1, here arises a question:

QUESTION. *If $S^1 \times M$ admits a symplectic structure(not necessarily S^1-invariant), is M a symplectic mapping torus(or, a mapping torus)?*

The question is closely related to a well-known question asked by Taubes in 1994[10]:

CONJECTURE. (Taubes 1994) *If M^3 is a 3-manifold such that $S^1 \times M^3$ admits a symplectic structure, then M^3 fibers over S^1.*

This is a major conjecture with extensive current interest and is becoming one of focus problems in 4-dimensional symplectic topology. Obviously, Proposition 1 and Theorem 1 together gives a trivial observation on Taubes' conjecture. So, our Question can be regarded as a generalization of Taubes' conjecture in high dimensions.

Recently, S. Friedl and S. Vidussi announced that they have proved Taubes's conjecture, see [20].

3. Two related topics.

3.1. 3-dimensional case study. It is natural and reasonable to think that *a mapping torus may not be a symplectic mapping torus* in general. But, surprisingly, this is not the case in dimension 3. Assume S is an orientable closed 2-surface and $\varphi \in \mathrm{Diff}^+(S)$ is an orientation-preserving diffeomorphism. In this case, a symplectic form on S is just an area-form and so a symplectomophism is the same as an area-preserving diffeomorphism.

PROPOSITION 3. *Every $\varphi \in \mathrm{Diff}^+(S)$ is isotopic to an area-preserving diffeomorphism. So, every 3-dimensional mapping torus is also a 3-dimensional symplectic mapping torus.*

Proof. Let σ be an area-form on S. $\forall \varphi \in \mathrm{Diff}^+(S)$, $\varphi^*\sigma$ is also an area-form. Generally, the two area-forms differ by a nowhere-zero function, that is, $\varphi^*\sigma = c(x)\sigma$. Again by Moser's theorem in [8], there exists a $\psi \in \mathrm{Diff}^+(S)$ such that $\psi^*(\varphi^*\sigma) = c\sigma$, where ψ is isotopic to the identity and c a constant. Obviously, $c = \int_S \varphi^*\sigma / \int_S \sigma = 1$.

This shows that $(\varphi\circ\psi)^*\sigma = \sigma$. Hence, $\varphi\circ\psi$ is an area-preserving diffeomorphism which is isotopic to φ. □

Another proof. Surface-diffeomorphisms have been well understood since Thurston's work [12]. For readers familiar with Dehn twists(cf. [13]), let's sketch another proof of Proposition 3, which may be well-known.

Let S be a compact 2-surface (with or without boundary). Let γ be a simple closed curve in the interior of S and $\mathcal{N}(\gamma) \cong S^1 \times [-1, 1]$ be a tubular neighborhood of γ. A *Dehn twist* about γ is a surface diffeomorphism which is the identity outside $\mathcal{N}(\gamma)$ and sends (θ, t) to $(\theta + \pi(t+1), t)$ inside $\mathcal{N}(\gamma)$. If γ bounds a disc or is parallel to a boundary component, then the Dehn twist about γ is isotopic to the identity. Generally, it is elementary to observe that any Dehn twist can be represented by an area-preserving one in its isotopy class. Also, a theorem of Lickorish says that any orientation-preserving homeomorphism on a compact orientable surface is isotopic to a composition of finite number of Dehn twists. Hence, every $\varphi \in \mathrm{Diff}^+(S)$ is isotopic to an area-preserving diffeomorphism. □

PROPOSITION 4. *On a 3-dimensional mapping torus S_φ, no matter φ is area-preserving or not, the distinguished vector field ξ_θ is always volume-preserving in the sense that $\mathcal{L}_{\xi_\theta} vol = 0$ for some volume vol on S_φ.*

Proof. In case φ is an area-preserving diffeomorphism such that $\varphi^*\sigma = \sigma$, ξ_θ is easily seen to be a volume-preserving field. We can lift σ to a 2-form Φ_θ on S_φ so that $\eta_\theta \wedge \Phi_\theta$ is a volume-form on S_φ, as we have done in Lemma 1. Then, $\mathcal{L}_{\xi_\theta}(\eta_\theta \wedge \Phi_\theta) = 0$ shows that ξ_θ is a volume-preserving field on S_φ.

Generally, let σ be an arbitrary area-form on S. On each local trivialization $\pi^{-1}(I) \cong S \times I$ ($I \subset S^1$ an interval), define a local highest form $\sigma \wedge d\theta$ on S_φ. Such local highest forms can be glued together to give a global volume-form on S_φ. Clearly, $\mathcal{L}_{\xi_\theta}(\sigma \wedge d\theta) = 0$, which shows that ξ_θ is a volume-preserving field on S_φ.□

Interesting enough, Proposition 4 can be used to give a new proof of Proposition 3. This is why we present Proposition 4 here.

In fact, let vol be a volume-form on S_φ such that $\mathcal{L}_{\xi_\theta} vol = 0$, and define a closed 2-form $\Phi = \iota_{\xi_\theta} vol$ on S_φ. Then

$$0 = \iota_{\xi_\theta}(\eta_\theta \wedge vol) = vol - \eta_\theta \wedge \Phi$$

shows that (η_θ, Φ) is a co-symplectic structure on S_φ. By Theorem 1, the mapping torus S_φ is also a symplectic mapping torus.

3.2. Mapping torus with no co-symplectic structure. In this subsection, we give an example of a mapping tori with no co-symplectic structure. By Theorem 1 and Proposition 3, such example can only be found in higher dimensions.

For this aim, we need to search a diffeomorphism which can not be isotopic to any symplectomorphism. A symplectomorphism $\varphi \in \mathrm{Symp}(S, \omega)$ must satisfy $\varphi^*\omega = \omega$. When seeing φ^* as a linear isomorphism on cohomology space $H^2(S, \mathbf{R})$, $\varphi^*\omega = \omega$ means that φ^* must have an eigenvalue 1. This fact implies that there may be plenty of diffeomorphisms which are not isotopic to symplectomorphisms. Any diffeomorphism admitting no eigenvalue 1 on the de Rham cohomology $H^2(S)$ provides such an example.

Example. Let $S = S^2 \times S^2$, with area-forms in the first and second 2-spheres being denoted by σ_1 and σ_2 respectively. Clearly, $([\sigma_1^*], [\sigma_2^*])$ is a basis for de Rham cohomology $H^2(S^2 \times S^2)$, where σ_1^*, σ_2^* denote the canonical pull-back 2-forms. Any $\omega = c_1\sigma_1^* + c_2\sigma_2^*$, with c_1, c_2 constants and $c_1c_2 \neq 0$, is a symplectic 2-form on $S^2 \times S^2$.

Define a diffeomorphism φ on $S^2 \times S^2$ by letting φ be the orientation-reversing diffeomorphisms on each S^2-factors. Globally, $\varphi \in \mathrm{Diff}^+(S)$ is an orientation-preserving diffeomorphism. φ induces an isomorphism φ^* on $H^2(S^2 \times S^2)$, expressed explicitly as

$$\varphi^*([\sigma_1^*], [\sigma_2^*]) = ([\sigma_1^*], [\sigma_2^*]) \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}.$$

Hence, φ^* admits no eigenvalue 1 on $H^2(S^2 \times S^2)$ and so φ is not isotopic to any symplectomorphism.

It is a standard result, by Mayer-Vietoris sequence, that the second Betti number b_2 of a mapping torus S_φ is given by

$$b_2(S_\varphi) = \text{dim of coker}(\mathbf{1} - \varphi^*)|_{H^1(S)} + \text{dim of } \ker(\mathbf{1} - \varphi^*)|_{H^2(S)}$$

In our example, $S = S^2 \times S^2$ and $\varphi^* = -\mathbf{1}$. Hence, $H^2((S^2 \times S^2)_\varphi) = 0$. This shows that the mapping torus $(S^2 \times S^2)_\varphi$ can never become a symplectic mapping torus. □

It is valuable to make a final remark. If $M = S_\varphi$ is a mapping torus on a non-symplectic base manifold S, then there is no sense to talk about "symplectic mapping torus" for S_φ. But we can ask if $S^1 \times S_\varphi$ admits a symplectic structure, in light of Proposition 1. Clearly, this question is closely related to the previous Question. A positive answer to the former will imply a negative answer to the latter. As an example, let's examine $M = S^4_{id} = S^4 \times S^1$. Then, $S^1 \times S^4_{id} = S^1 \times S^4 \times S^1$ can never admit any symplectic structure since each class in $H^2(S^1 \times S^4_{id})$ has vanishing cup square.

4. Co-Kähler manifold = Kähler mapping torus.

4.1. Co-Kähler manifolds. Besides cosymplectic manifolds in Libermann's sense, Blair [2] gave another definition of cosymplectic manifolds which were more referred in literature(cf. [3][4][5][6]). In this paper, we name cosymplectic manifolds in Blair's sense as *co-Kähler manifolds*. This subsection is devoted to a brief introduction to related concepts. All the materials come from Blair's book [14].

Analogous to almost complex structures on even dimensional manifolds, we have a concept of *almost contact structures* on odd dimensional manifolds M^{2n+1}. It consists of a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \text{and} \quad \eta(\xi) = 1.$$

It follows that $\phi\xi = 0$ and $\eta \circ \phi = 0$. The endomorphism ϕ has maximal rank $2n$ on the hyperplane distribution $\ker \eta$. In other words, the tangent sub-bundle $\ker \eta$ is invariant under ϕ.

A Riemannian metric g is said to be *compatible* with the almost contact structure (ϕ, ξ, η) on M, if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Analogous to almost Hermitian structures, such a compatible metric always exists, and we call (ϕ, ξ, η, g) an *almost contact metric structure*.

On an almost contact metric manifold $(M^{2n+1}; \phi, \xi, \eta, g)$, we can define a *fundamental 2-form* Φ by

$$\Phi(X, Y) = g(X, \phi Y).$$

Locally, there is a g-orthonormal ϕ-basis $\{X_1, \cdots, X_n, \phi X_1, \cdots, \phi X_n, \xi\}$ on M, with $\eta(X_k) = 0$, $1 \leq k \leq n$. Let $\{\alpha^1, \cdots, \alpha^n, \beta^1, \cdots, \beta^n, \eta\}$ be the dual of the local ϕ-basis, then the fundamental 2-form Φ can be expressed as $\Phi = \sum_{k=1}^{n} \alpha^k \wedge \beta^k$. It follows that $\eta \wedge \Phi^n$ is a volume-form on M. But, not necessarily either η or Φ is closed.

The almost contact metric structure (ϕ, ξ, η, g) is called:

integrable iff the Nijenhuis torsion of ϕ vanishes, $N_\phi = 0$;

normal iff $N_\phi + 2d\eta \otimes \xi = 0$;

parallel iff $\nabla^g \phi = 0$, where ∇^g denotes the Levi-Civita connection of g.

Here, the Nijenhuis torsion N_ϕ(a little bit different from N_J) is defined by

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y].$$

It is well-known that the normal condition implies both $\mathcal{L}_\xi \phi = 0$ and $\mathcal{L}_\xi \eta = 0$. Note also that ϕ being parallel is a stronger condition than normal. It implies both ξ and η are parallel(by differenting $\phi^2 = -I + \eta \otimes \xi$). It also implies both η and Φ are closed, by the following important

PROPOSITION 5. (Theorem 6.7, [14]) *An almost contact metric structure (ϕ, ξ, η, g) is normal with both η and Φ closed if and only if ϕ is parallel.* □

In 1967, Blair defined a cosymplectic structure to be a normal almost contact metric structure with both η and Φ closed. It follows that cosymplectic manifolds in Blair's sense are always co-symplectic manifolds in Liberman's sense.

Conversely, a co-symplectic structure (η, Φ) in Libermann's sense also determines an almost contact metric structure (ϕ, ξ, η, g) which is compatible with the co-symplectic structure (η, Φ)("compatible" means that $g(X, \phi Y) = \Phi(X, Y)$). This can be seen as follows. ξ is taken as the Reeb field of (η, Φ) defined by $\eta(\xi) = 1$ and $\iota_\xi \Phi = 0$. (ϕ, g) may be created simultaneously by polarizing Φ on $\ker \eta$ and then extending in the ξ direction (but no canonical way to obtain (ϕ, g)).

Notice that if an almost contact metric structure (ϕ, ξ, ϕ, g) is compatible with a co-symplectic structure (η, Φ) then the three conditions of ϕ being integrable, normal, and parallel are all equivalent.

We define a "co-Kähler structure" to be a parallel co-symplectic structure (η, Φ) in the sense that (η, Φ) admits a compatible almost contact metric structure (ϕ, ξ, η, g) such that ϕ is parallel. Obviously, co-Kähler manifolds coincide with cosymplectic manifolds in Blair's sense by Proposition 5.

In this way, we may roughly say that a co-Kähler manifold is a co-symplectic manifold endowed with a parallel almost contact metric structure, very analogous to the fact that a Kähler manifold is a symplectic manifold endowed with a parallel almost complex structure.

4.2. Proof of Theorem 2. In a symplectic mapping torus S_φ, when (S, ω) is a Kähler manifold with associated Hermitian structure (J, h) and φ is a Hermitian isometry satisfying $\varphi_* \circ J = J \circ \varphi_*$ and $\varphi^* h = h$(hence also $\varphi^* \omega = \omega$), S_φ is said to be a Kähler mapping torus. A co-Kähler manifold is a $(2n+1)$-manifold M together with a parallel co-symplectic structure (η, Φ). The second main result of this paper is the following

THEOREM 2. *A closed manifold M is a co-Kähler manifold if and only if it is a Kähler mapping torus S_φ. In short:*
Co-Kähler manifold = Kähler mapping torus.

Here, we see an interesting development from Theorem 0 to Theorem 1 and to Theorem 2. At each step, we are only adding geometric structures on the underlying topology.

The sufficient part of Theorem 2 is just the following lemma. We remark that a similar construction was used in [3][4].

LEMMA 4. *If S_φ is a Kähler mapping torus, then it admits a parallel co-symplectic structure.*

Proof. We will construct an almost co-Hermitian structure (ϕ, ξ, η, g) on S_φ as follows. Obviously, (ξ, η) can be taken as the distinguished pair $(\xi_\theta, \eta_\theta)$ on S_φ. To define the pair (ϕ, g), recall the construction of S_φ and maps $S \xrightarrow{i_\tau} S \times [0,1] \xrightarrow{\rho} S_\varphi$, where i_τ is an embedding and ρ a local diffeomorphism. Locally, any vector field on S_φ is a finite sum of two types of vector fields:

$$\text{(I).} \quad c_1 \cdot (\rho \circ i_\tau)_* Z, \qquad \text{(II).} \quad c_2 \cdot \xi_\theta.$$

Here Z is a local vector field on S and $\{c_1, c_2\}$ local smooth functions on S_φ. The Kähler structure (J, h) on S can be extended and pulled-back to $S \times [0,1]$, respectively, and then descend to S_φ to yield a pair $(\phi, \overline{h})$. The two descending processes follow since $\varphi_* \circ J = J \circ \varphi_*$ and $\varphi^* h = h$. The desired Riemannian metric g on S_φ is defined by

$$g(X,Y) = \overline{h}(X,Y) + \eta_\theta(X)\eta_\theta(Y).$$

More precisely, we define the pair (ϕ, g) on S_φ by

$$\phi(\xi_\theta) = 0, \quad \phi((\rho \circ i)_* Z) = (\rho \circ i)_*(JZ);$$

$$g(\xi_\theta, \xi_\theta) = 1, \quad g(\xi_\theta, (\rho \circ i)_* Z) = 0, \quad g((\rho \circ i)_* Z_1), (\rho \circ i)_* Z_2) = h(Z_1, Z_2).$$

Where $i = i_\tau$ represents a typical embedding.

It remains to check the following three things:
(i). $(\phi, \xi_\theta, \eta_\theta, g)$ is an almost contact metric structure on S_φ, that is

$$\phi^2 = -I + \eta_\theta \otimes \xi_\theta, \quad \eta_\theta(\xi_\theta) = 1, \quad g(X,Y) = g(\phi X, \phi Y) + \eta_\theta(X)\eta_\theta(Y).$$

(ii). The fundamental 2-form of $(\phi, \xi_\theta, \eta_\theta, g)$ is precisely that of Φ in the co-symplectic structure (η_θ, Φ) as constructed in Lemma 1.
(iii). The Nijenhuis torsion N_ϕ vanishes, or saying that ϕ is parallel.

We check only (iii) here. Recall that N_ϕ is defined by

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y].$$

Firstly, since $\phi(\xi_\theta) = 0$, we have obviously that $N_\phi(\xi_\theta, \xi_\theta) = 0$.
Secondly, by property of bracket $[\xi_\theta, (\rho \circ i)_* Z] = \rho_*[\frac{d}{d\theta}, i_* Z] = 0$, we have

$$N_\phi(\xi_\theta, (\rho \circ i)_* Z) = \phi^2[\xi_\theta, (\rho \circ i)_* Z] - \phi[\xi_\theta, \phi((\rho \circ i)_* Z)] = 0.$$

Finally, iterated using two identities $\phi((\rho \circ i)_* Z) = (\rho \circ i)_*(JZ)$ and $[(\rho \circ i)_*(Z_1), (\rho \circ i)_*(Z_2)] = (\rho \circ i)_*[Z_1, Z_2]$, and also note $\eta_\theta((\rho \circ i)_* Z) = 0$, we compute that

$$N_\phi((\rho \circ i)_* Z_1, (\rho \circ i)_* Z_2) = (\rho \circ i)_* N_J(Z_1, Z_2) = 0.$$

In this way, we obtain a parallel co-symplectic structure on S_φ. □

For the necessary of Theorem 2, let's observe our question closely.

Let M be a co-Kähler manifold with co-symplectic structure (η, Φ). From Theorem 1, we know that M is a symplectic mapping torus S_φ for a symplectomorphism $\varphi \in \mathrm{Symp}(S, \omega)$. We need to show that S_φ is indeed a Kähler mapping torus. On S_φ, there is a canonical co-symplectic structure $(\eta_\theta, \Omega_\theta)$ as in Lemma 1, where the integral closed 1-form $\eta_\theta = \pi^*(d\theta)$ may come from an approximating process on η as in Tischler's Theorem 0. By assumption (η, Φ) is parallel, that is, it admits an associated almost contact metric structure (ϕ, ξ, η, g) such that $\nabla^g \phi = 0$ or $N_\phi = 0$. But we don't yet know whether $(\eta_\theta, \Omega_\theta)$ is parallel. Only from a "parallel + regular" structure $(\phi_0, \partial_\theta, \eta_\theta, g_0)$ can we recover a Kähler structure (J, h) on S. So, our first task is to associate to $(\eta_\theta, \Omega_\theta)$ with an almost contact metric structure $(\phi_0, \xi_\theta, \eta_\theta, g_0)$ such that $\nabla^{g_0} \phi_0 = 0$ or $N_{\phi_0} = 0$.

This situation can be evaluated as follows. η defines a co-dimensional one foliation $\ker \eta$ on M whose leaf is integrable (since $N_\phi = 0$) but may be "screwy". η_θ

defines a fibration $\pi : S_\varphi \to S^1$ on S_φ which behaves "regular" in topology. η_θ is an approximation of η. Thus, it is reasonable to expect a transfer from a parallel (ϕ, g) on the "screwy" $\ker \eta$ to a new parallel (ϕ_0, g_0) on the "regular" $\ker \eta_\theta$.

Such structure transfer will be carried out in a way of "along the flow of ξ". Let's first define local difeomorphisms using the flow of ξ, aiming to transfer $\ker \eta_\theta$ to $\ker \eta$.

Let $\{\varphi_t\}_{t\in\mathbf{R}}$ be the flow of the vector field ξ, they consist of isometries of g since ξ is parallel. $\forall p \in S_\varphi$, there passes through a fiber of the fibration $\pi : S_\varphi \to S^1$. Such a fiber $(= (\rho \circ i_\tau)S)$ is denoted by S_{p,η_θ}. Let $q = \varphi_s(p)$ be any point on the flow line of ξ through p, there passes through q a leaf of the foliation defined by η. Such a leaf is denoted by $S_{q,\eta}$. Near p, the flow lines of ξ define a local diffeomorphism from the fiber to the leaf, which will be denoted by $\widetilde{\varphi}_p : S_{p,\eta_\theta} \supset U_p \to V_q \subset S_{q,\eta}$. Obviously, the tangent map $(\widetilde{\varphi}_p)_*$ is an isomorphism which transfers $\ker \eta_\theta$ to $\ker \eta$.

Note the following fact. If $\widetilde{\varphi}_{p_1} : U_{p_1} \to V_{q_1}$ and $\widetilde{\varphi}_{p_2} : U_{p_2} \to V_{q_2}$ are two such local diffeomorphisms with $\{p_1, p_2\}$ in a same fiber and $U_{p_1} \bigcap U_{p_2} \neq \emptyset$, then on the overlap they differ only by an isometry of g. That is to say, $\widetilde{\varphi}_{p_2} = \varphi_\tau \circ \widetilde{\varphi}_{p_1}$ for some $\tau \in \mathbf{R}$. This follows since ξ is parallel.

For this reason, when such local diffeomorphisms are used to do a construction, we prefer to choose $q = p$ and use $\widetilde{\varphi}_p : U_p \to V_p$. We will see that such a simplify causes no essential difference in our construction.

Now, let's transfer the pair (ϕ, g) mainly defined on $\ker \eta$ to a new pair (ϕ_0, g_0) mainly defined on $\ker \eta_\theta$ along the flow lines of ξ.

- ϕ_0 is defined mainly as a conjugation of ϕ:

$$\begin{cases} \phi_0 = \widetilde{\varphi}_{p*}^{-1} \circ \phi \circ \widetilde{\varphi}_{p*} : \ker \eta_\theta \to \ker \eta_\theta, \\ \qquad \text{and setting}: \ \phi_0(\xi_\theta) = 0. \end{cases}$$

- g_0 is defined mainly as the pull-back of g:

$$\begin{cases} g_0(X, Y) = g(\widetilde{\varphi}_{p*}X, \widetilde{\varphi}_{p*}Y), \ \ \forall \ \ X, Y \in \ker \eta_\theta, \\ \qquad \text{and setting}: \ \xi_\theta \text{ orthonormal to } \ker \eta_\theta. \end{cases}$$

We need to show that the two definitions are independent of the choice of local diffeomorphisms $\{\widetilde{\varphi}_p\}$, so that (ϕ_0, g_0) are globally well-defined. If $\{\widetilde{\varphi}_{p_1}, \widetilde{\varphi}_{p_2}\}$ are two such choices, they differ only by an isometry φ_τ of g: $\widetilde{\varphi}_{p_2} = \varphi_\tau \circ \widetilde{\varphi}_{p_1}$. It follows immediately that g_0 is well-defined. To see ϕ_0 is well-defined, noticing the fact that $\mathcal{L}_\xi \phi = 0$. This gives us a formula $\varphi_{\tau*} \circ \phi = \phi \circ \varphi_{\tau*}$ which implies that $\widetilde{\varphi}_{p_2*}^{-1} \circ \phi \circ \widetilde{\varphi}_{p_2*} = \widetilde{\varphi}_{p_1*}^{-1} \circ \phi \circ \widetilde{\varphi}_{p_1*}$.

Lemma 5. *Let S_φ be a symplectic mapping torus with distinguished pair $(\xi_\theta, \eta_\theta)$. Every parallel almost contact metric structure (ϕ, ξ, η, g) on S_φ can be transferred to a new parallel almost contact metric structure $(\phi_0, \xi_\theta, \eta_\theta, g_0)$.*

Proof. The identity $\phi_0^2 = -I + \eta_\theta \otimes \xi_\theta$ holds obviously from the definition.

The compatible condition $g_0(\phi_0 X, \phi_0 Y) = g_0(X, Y) - \eta_\theta(X)\eta_\theta(Y)$ holds obviously in both cases of $X = Y = \xi_\theta$ and $X = \xi_\theta, Y \in \ker \eta_\theta$. For the case $X, Y \in \ker \eta_\theta$, we check by definitions of (ϕ_0, g_0) that

$$\begin{aligned} g_0(\phi_0 X, \phi_0 Y) &= g(\phi \circ \widetilde{\varphi}_{p*}X, \phi \circ \widetilde{\varphi}_{p*}Y) \\ &= g(\widetilde{\varphi}_{p*}X, \widetilde{\varphi}_{p*}Y) = g_0(X, Y). \end{aligned}$$

Thus, we obtain an almost contact metric structure $(\phi_0, \xi_\theta, \eta_\theta, g_0)$ on S_φ.

It remains to verify the parallel condition $\nabla^{g_0}\phi_0 = 0$.

We indicate firstly that $(\phi_0, \xi_\theta, \eta_\theta, g_0)$ determines a fundamental 2-form Φ_0 by $\Phi_0(X, Y) = g_0(\phi_0 X, Y)$ which may not coincide with Φ_θ in the canonical co-symplectic structure $(\eta_\theta, \Phi_\theta)$ on S_φ. By Proposition 5, to show the parallel condition $\nabla^{g_0}\phi_0 = 0$ we need only to show that ϕ_0 is integrable ($N_{\phi_0} = 0$) and Φ_0 is closed ($d\Phi_0 = 0$).

- $N_{\phi_0} = 0$: $N_{\phi_0}(\xi_\theta, \xi_\theta) = 0$ is obvious. For $X \in \ker \eta_\theta$, $N_{\phi_0}(X, \xi_\theta) = 0$ follows mainly since $[X, \xi_\theta] = 0$, the same argument was used in Lemma 4. In the case of $X, Y \in \ker \eta_\theta$, the expression $\phi_0 = \widetilde{\varphi}_{p*}^{-1} \circ \phi \circ \widetilde{\varphi}_{p*}$ implies that $N_{\phi_0}(X, Y) = \widetilde{\varphi}_{p*}^{-1} N_\phi(\widetilde{\varphi}_{p*}X, \widetilde{\varphi}_{p*}Y) = 0$.

- $d\Phi_0 = 0$: $\Phi_0(X, Y) = 0$ in case of $X = \xi_\theta$ or $Y = \xi_\theta$. On $\ker \eta_\theta$, using the definitions of (ϕ_0, g_0), we have

$$\Phi_0(X, Y) = g_0(\phi_0 X, Y) = g(\phi\widetilde{\varphi}_{p*}X, \widetilde{\varphi}_{p*}Y) = \Phi(\widetilde{\varphi}_{p*}X, \widetilde{\varphi}_{p*}Y).$$

So, restricting on $\ker \eta_\theta$ and the fibers, Φ_0 seems a pull-back $\Phi_0 = (\widetilde{\varphi}_p)^*(\Phi)$.

In the following, we show that Φ_0 is really a pull-back of Φ, locally.

For this aim, let's first "thicken" the diffeomorphisms $\widetilde{\varphi}_p$. Assume $\widetilde{\varphi}_p : U_p \to V_p$ has its explicit defining domain U_p and image V_p. For a sufficient small $\delta > 0$, we define $\overline{\widetilde{\varphi}}_p : U_p \times (-\delta, \delta) \to V_p \times (-\delta, \delta)$ by $\overline{\widetilde{\varphi}}_p(x, t) = \varphi_t \circ \widetilde{\varphi}_p(x)$. The notation $V_p \times (-\delta, \delta)$ may have some vague here, but it is convenient to use since the leaves are parallel. Then, we have $\Phi_0 = (\overline{\widetilde{\varphi}}_p)^*(\Phi)$, locally. It follows immediately that $d\Phi_0 = 0$.

This completes the proof of the Lemma 5. □

LEMMA 6. *Let S_φ be a symplectic mapping torus with distinguished pair $(\xi_\theta, \eta_\theta)$, then a parallel almost contact metric structure $(\phi, \xi_\theta, \eta_\theta, g)$ on S_φ induces naturally to a Kähler structure $(\omega; J, h)$ on S so that S_φ becomes a Kähler mapping torus.*

Proof. Since (ϕ, g) is compatible with the distinguished pair $(\xi_\theta, \eta_\theta)$ on S_φ, it is clear that (ϕ, g) restricts to an Hermitian structure (J, h) on each fiber of $\pi : S_\varphi \to S^1$. The integrable condition $N_J = 0$ follows directly from $N_\phi = 0$, as in the argument of Lemma 4. If Φ denotes the fundamental closed 2-form of (ϕ, g), then it descends to a symplectic 2-form ω on a fiber S which satisfies $\omega(X, Y) = h(JX, Y)$. Hence, $(\omega; J, h)$ is Kähler structure on S.

If we cut S_φ along a fiber S, we see that the monodromy $\varphi \in \mathrm{Diff}^+(S)$ automatically satisfies $\varphi^*\omega = \omega$, $\varphi^*h = h$ and $J \circ \varphi_* = \varphi_* \circ J$. This means that S_φ is a Kähler mapping torus. □

Summing up, Lemma 4+Lemma 5+Lemma 6, we have finished the whole proof of Theorem 2. □

5. Monotone Betti numbers of co-Kähler manifolds. In 1993, Chinea, de Leon, and Marrero [4] discussed the topology of co-Kähler manifolds. Amongst other things, they obtained the monotone property of Betti numbers of co-Kähler manifolds.

THEOREM 3. (Chinea, de León and Marrero, [4]) *The Betti numbers $\{b_k\}$ of a co-Kähler manifold M^{2n+1} satisfy a monotone property:*

$$1 = b_0 \leq b_1 \leq \cdots \leq b_n = b_{n+1} \geq b_{n+2} \geq \cdots \geq b_{2n+1} = 1.$$

Their method is to compute the dimension of *effective-harmonic forms*, a method similar to that we used on Kähler manifolds. By Theorem 2, co-Kähler manifolds are nothing but Kähler mapping tori S_φ, their Betti numbers are easily computed from the knowledge of S and φ. This enables us to give a purely topological proof of Theorem 3.

5.1. Kähler manifolds. Symplectic structures have certain rigidity. For instance, all even Betti numbers of a closed symplectic manifold are nonzero. Kähler structures impose more strong restriction on the underlying topology. Some classical results on Kähler manifolds $(S;\omega)$ are listed below(cf. [15][16]).

(i). The even dimensional Betti numbers b_{2k} are nonzero;
(ii). The odd dimensional Betti numbers b_{2k-1} are even;
(iii). Betti numbers satisfy a monotone property: $b_{r-2} \le b_r$ for $r \le n$.
(iv). S has the strong Lefchetz property, i.e., $\mathbf{L}^{n-r} : \Omega^r_H(S) \to \Omega^{2n-r}_H(S)$ is an isomorphism for $0 \le r \le n$, where $\mathbf{L}$ is an operator defined below.

We are specially interested in the monotone property of Betti numbers, that is property (iii). In the following, we recall briefly the main steps to establish this property.

Step 1. On any symplectic manifold (S, ω), we define two operators $\mathbf{L}$ and $\mathbf{\Lambda}$ acting on differential forms α by

$$\mathbf{L}\alpha = \alpha \wedge \omega, \qquad \mathbf{\Lambda}\alpha = * \circ \mathbf{L} \circ *\alpha$$

where $*$ denotes the Hodge star isomorphism induced by the compatible Riemannian metric.

A r-form α is said to be *effective* on S if $\mathbf{\Lambda}\alpha = 0$.

A r-form α is said to be *harmonic* if $\Delta\alpha = 0$, where $\Delta = d \circ \delta + \delta \circ d$ is the Laplace-Beltrami operator on S and $\delta = \pm * \circ d \circ *$. On closed manifolds, $\Delta\alpha = 0$ if and only if $d\alpha = 0$ and $\delta\alpha = 0$.

Fact: $\mathbf{L}$ *is an injective homomorphism, on space of r-forms for $r < n$.*

Step 2. On Kähler manifold (S, ω), the fundamental form ω is parallel with respect to the compatible Rimannian metric and is itself a harmonic 2-form. It follows that if α is a harmonic r-form with $r \le n$ then $\mathbf{L}\alpha = \alpha \wedge \omega$ is also harmonic.

Let $\Omega^r_H(S)$ denotes the space of harmonic r-forms and by $\bar{\Omega}^r_H(S)$ the spaces of effective harmonic r-forms. Then we have a decomposition

Fact: $$\Omega^r_H(S) = \bar{\Omega}^r_H(S) \oplus \mathbf{L}\Omega^{r-2}_H(S), \qquad \text{for} \quad r \le n.$$

Step 3. By Hodge theory, each class in de Rham cohomology $H^r(S)$ has a unique representative by harmonic r-form. Hence, $H^r(S)$ and $\Omega^r_H(S)$ are isomorphic as vector spaces. Then the decomposition in Step 2 and the injectivity of $\mathbf{L}$ in Step 1 imply that the dimension of the space of effective harmonic r-forms $\bar{\Omega}^r_H(S)$ is $b_r - b_{r-2}$ for $r \le n$. So, $b_{r-2} \le b_r$, for $r \le n$. This proves the monotone property (iii) on Kähler manifolds. □

Now, assume $\varphi : S \to S$ is a Hermitian isometry on a Kähler manifold (S, ω) with Hermiatian structure (J, h). Let's consider *φ-invariant forms* on S, namely, forms α satisfy $\varphi^*\alpha = \alpha$. Note that φ^* commutes with the four operators d,δ, $\mathbf{L}$, $\mathbf{\Lambda}$. If we adopt only φ-invariant forms in the above argument, it is not difficulty to check that all Step 1, Step 2 and Step 3 still apply without obstacle.

In this way, we may expect a "φ-invariant topology" of Kähler manifolds. Especially, some kind of monotone property of "φ-invariant Betti numbers" may by established. In the following, however, we will go another way.

On level of cohomology classes, we prefer to consider *φ-invariant classes* in the kernel of the linear map $(\varphi^* - \mathbf{1}) : H^r(S) \to H^r(S)$. This kernel will be denoted by $H^r_\varphi(S)$, namely,

$$H^r_\varphi(S) = \ker\{(\varphi^* - \mathbf{1}) : H^r(S) \to H^r(S)\}.$$

We remark that such subspaces $H^r_\varphi(S)$ are important for computing the cohomology of Kähler mapping torus S_φ. By argument on Mayer-Vietoris sequence of S_φ, the following formula is well-known:

$$b_r(S_\varphi) = \dim H^r_\varphi(S) + \dim H^{r-1}_\varphi(S).$$

The monotone property of "φ-invariant Betti numbers" is now read as:

Proposition 6. *Let φ be a Hermitian isometry on a Kähler manifold (S, ω). Then*

$$\dim H^{r-2}_\varphi(S) \leq \dim H^r_\varphi(S), \qquad for \quad r \leq n.$$

Proof: Let $\{[\alpha_1], \cdots, [\alpha_k]\}$ be a basis of $H^{r-2}_\varphi(S)$. By Hodge theory, we can choose α_i with $1 \leq i \leq r$ to be the unique harmonic representative. Notice that α_i being harmonic if and only if $\varphi^*\alpha_i$ being harmonic , so $\varphi^*[\alpha_i] = [\alpha_i]$ in the level of classes implies $\varphi^*\alpha_i = \alpha_i$ in the level of forms. Since $\varphi^*\omega = \omega$, $\{\alpha_1 \wedge \omega, \cdots, \alpha_r \wedge \omega\}$ consists of φ-invariant harmonic r-forms. It remains only to show $\{[\alpha_1 \wedge \omega], \cdots, [\alpha_r \wedge \omega]\}$ is linearly independent in $H^r_\varphi(S)$. This is easy to see, again by the uniqueness of harmonic representative and the injectivity of operator $\mathbf{L}$. □

5.2. A topological proof of Theorem 3. *A new proof of Theorem 3:* By Theorem 2, a co-Kähler manifold M^{2n+1} is just a Kähler mapping torus S_φ, for a Hermitian isometry φ on Kähler manifold S. The Betti numbers of S_φ satisfy a formula

$$b_r(S_\varphi) = \dim H^{r-1}_\varphi(S) + \dim H^r_\varphi(S)$$

where $H^r_\varphi(S) = \ker\{(\varphi^* - \mathbf{1}) : H^r(S) \to H^r(S)\}$ is the φ-invariant subspaces. By Proposition 6, we have: $\dim H^{r-2}_\varphi(S) \leq \dim H^r_\varphi(S)$, for $r \leq n$. Thus,

$$b_{r-1}(S_\varphi) - b_r(S_\varphi) = \dim H^{r-2}_\varphi(S) - \dim H^r_\varphi(S) \leq 0, \quad \text{for all } r \leq n.$$

This is half of Theorem 3. The other half is the work of Poincaré duality. □

Acknowledgement. This paper was completed during the spring of 2006 when I visited Chern Institute of Mathematics in Nankai University. I wish to express my gratitude to Professor Huitao Feng, who pointed out many errors in previous version and gave me many useful suggestions. Thanks are also due to the director of Chern Institute of Mathematics, Professor Weiping Zhang, for his kind invitation and hospitality.

REFERENCES

[1] P. Libermann, *Sur les automorphismes infinitésimaux des structures symplectiques et de atructures de contact*, Coll. Géom. Diff. Globale, (Bruxelles 1958), Louvain, 1959, pp. 37–59.
[2] D. E. Blair, *The theorey of quasi-Sasakian structures*, J. Diff. Geom., 1 (1967), pp. 331–345.
[3] A. Fujimoto and H. Mutō, *On cosymplectic manifolds*, Tensor, N. S., 28 (1974), pp. 43–52.
[4] D. Chinea, M. de León and J. C. Marrero, *Topology of cosymplectic manifolds*, J. Math. Pures Appl., 72 (1993), pp. 567–591.
[5] D. Chinea, M. de León and J. C. Marrero, *Spectral sequences on Sasakian and cosymplectic manifolds*, Houston J. Math., 23 (1997), pp. 631–649.
[6] J. C. Marrero and E. Padron, *New examples of compact cosymplectic solv-manifolds*, Arch. Math., 34 (1998), pp. 337–345.
[7] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology, 9 (1970), pp. 153–154.
[8] J. Moser, *On volume elements on a manifold*, Trans. Amer. Math. Soc., 120 (1965), pp. 286–294.
[9] M. Farber, *Topology of closed 1-forms*, Mathematical Surveys and Monographs, 108 (2004).
[10] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett., 1 (1994), pp. 809–822.
[11] M. Fernández, A. Gray and J. W. Morgan, *Compact symplectic manifolds with free circle action, and Massey products*, Michigan Math. J., 38 (1991), pp. 271–283.
[12] W. P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc., 19 (1988), pp. 417–431.
[13] D. P. Rolfsen, *Knots and links*, Publish or Perish, 1976.
[14] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math., Vol. 203, Birkhäuser Press, Boston, 2002.
[15] S. I. Goldberg, *Curvature and homology*, Academic Press, New York, 1962.
[16] P. Deligne, P. Griffiths, J. Morgan and D. Sullivin, *Real homotopy theory of Kähler manifolds*, Invent. Math., 29 (1975), pp. 245–274.
[17] B. Hajduk and R. Walczak, *Symplectic forms invariant under free circle actions on 4-manifolds*, Trans. Amer. Math. Soc., 358 (2006), pp. 1953–1970.
[18] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, GTM, 82, Springer-Verlag Press, 1982.
[19] G. W. Whitehead, *Elements of homotopy theory*, GTM, 61, Springer-Verlag Press, 1978.
[20] S. Friedl and S. Vidussi, arXiv:0805.1234v1 [math.GT], 8 May 2008.