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ON NON-EXISTENCENESS OF EQUIFOCAL SUBMANIFOLDS WITH NON-FLAT SECTION*

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Abstract. We first prove a certain kind of splitting theorem for an equifocal submanifold with non-flat section in a simply connected symmetric space of compact type, where an equifocal submanifold means a submanifold with parallel focal structure. By using the splitting theorem, we prove that there exists no equifocal submanifold with non-flat section in an irreducible simply connected symmetric space of compact type whose codimension is greater than the maximum of the multiplicities of roots of the symmetric space or the maximum added one. In particular, it follows that there exists no equifocal submanifold with non-flat section in some irreducible simply connected symmetric spaces of compact type and that there exists no equifocal submanifold with non-flat section in simply connected compact simple Lie group whose codimension is greater than two.

Key words. Equifocal submanifold, polar action.

AMS subject classifications. Primary 53C40, 53C35

1. Introduction. A properly immersed complete submanifold M in a simply connected symmetric space G/K is called a *submanifold with parallel focal structure* if the following conditions hold:

(PF-i) the restricted normal holonomy group of M is trivial,

(PF-ii) if v is a parallel normal vector field on M such that v_{x_0} is a multiplicity k focal normal of M for some $x_0 \in M$, then v_x is a multiplicity k focal normal of M for all $x \in M$,

(PF-iii) for each $x \in M$, there exists a properly embedded complete connected submanifold through x meeting all parallel submanifolds of M orthogonally.

This notion was introduced by Ewert ([E2]). In [A], [AG] and [AT], this submanifold is simply called an *equifocal submanifold*. In this paper, we also shall use this name and assume that all equifocal submanifolds have trivial normal holonomy group. The submanifold as in (PF-iii) ia called a *section of M through x*, which is automatically totally geodesic. Note that Terng-Thorbergsson [TeTh] originally introduced the notion of an equifocal submanifold under the assumption that the sections is flat. The condition (PF-ii) is equivalent to the following condition:

(PF-ii′) for each parallel unit normal vector field v of M, the set of all focal radii along the geodesic γ_{v_x} with $\dot{\gamma}_{v_x}(0) = v_x$ is independent of the choice of $x \in M$.

Note that, under the condition (PF-i), the condition (PF-iii) is equivalent to the following condition:

(PF-iii′) M has Lie triple systematic normal bundle (in the sense of [Koi1]).

In fact, (PF-iii)⇒(PF-iii′) is trivial and (PF-iii′)⇒(PF-iii) is shown as follows. If (PF-iii′) holds, then it is shown by Proposition 2.2 of [HLO] that $\exp^\perp(T^\perp_x M)$ meets all parallel submanifolds of M orthogonally for each $x \in M$, where $\exp^\perp$ is the normal exponential map of M. Also, it is clear that $\exp^\perp(T^\perp_x M)$ is properly embedded.

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Thus (PF-iii) follows. An isometric action of a compact Lie group H on a Riemannian manifold is said to be *polar* if there exists a properly embedded complete connected submanifold Σ meeting every principal orbits of the H-action orthogonally. The submanifold Σ is called a *section* of the action. If Σ is flat, then the action is said to be *hyperpolar*. Principal orbits of polar actions are equifocal submanifolds and those of hyperpolar actions are equifocal ones with flat section. Conversely, homogeneous equifocal submanifolds (resp. homogeneous equifocal ones with flat section) in the symmetric spaces are catched as principal orbits of polar (resp. hyperpolar) actions on the spaces. U. Christ [Ch] showed that complete connected equifocal submanifolds with flat section of codimension greater than one in irreducible simply connected symmetric spaces of compact type are homogeneous. Kollross [Kol1] classified hyperpolar actions on irreducible symmetric spaces of compact type up to orbit equivalence. According to the classification, all hyperpolar actions of cohomogeneity greater than one on the spaces are Hermann actions. By imitating the proof of Theorem B of [Koi3], it is shown that the principal orbits of Hermann actions on the spaces are curvature-adapted except for three exceptional actions ((2), (4) and (7) in P256 of [Co]).

In 1997, Heintze and Liu [HL] showed that an isoparametric submanifold in a Hilbert space is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if the associated Coxeter group is decomposable. In 1998, by using this splitting theorem of Heintze-Liu, Ewert [E1] showed that an equifocal submanifold with flat section in a simply connected symmetric space of compact type is decomposed into a non-trivial (extrinsic) product of two such submanifolds if and only if the associated Coxeter group is decomposable.

In this paper, we first prove the following splitting theorem for an equifocal submanifold with non-flat section in a simply connected symmetric space of compact type.

THEOREM A. *Let M be an equifocal submanifold with non-flat section in a simply connected symmetric space G/K of compact type and Σ be a section of M. Then M is decomposed into a non-trivial extrinsic product of two equifocal submanifolds if and only if the restricted holonomy group of* (*the induced metric on*) *Σ is reducible.*

Next we prove the following fact in terms of Theorem A.

THEOREM B. *Let M be an equifocal submanifold with non-flat section in an irreducible simply connected symmetric space G/K of compact type. Then each section of M is isometric to a sphere or a real projective space.*

For equifocal submanifolds with non-flat section, some open problems remain, for example the following.

OPEN PROBLEM 1. *Does there exist an equifocal submanifold with non-flat section in an irreducible symmetric space of compact type and rank greater than one?*

This includes the following open problem.

OPEN PROBLEM 2. *Are all polar actions on irreducible symmetric spaces of compact type and rank greater than one hyperpolar?*

L. Biliotti [B] gave the following partial answer for this problem.

All polar actions on irreducible Hermitian symmetric spaces of compact type and rank greater than one are hyperpolar.

In 1985, Dadok [D] classified polar actions on spheres up to orbit equivalence. According to the classification, those actions are orbit equivalent to the restrictions to hyperspheres of the linear isotropy actions of symmetric spaces. In 1999, Podestà and Thorbergsson [PoTh1] classified (non-hyperpolar) polar actions on simply connected rank one symmetric spaces of compact type other than spheres up to orbit equivalence. Kollross [Kol2] has recently showed that there exists no (non-hyperpolar) polar action on irreducible symmetric spaces of type I and rank greater than one. See [H] about symmetric spaces of type I. Thus homogeneous equifocal submanifolds in irreducible symmetric space of type I are classified completely. All isoparametric submanifolds of codimension greater than one in a sphere are (curvature-adapted) equifocal submanifolds with non-flat section. According to the homogeneity theorem by Thorbergsson ([Th]), they are homogeneous and hence they are catched as principal orbits of the linear isotropy actions of symmetric spaces of rank greater than two.

By using Theorem B, we can show the following fact for Open Problem 1.

THEOREM C. (i) *There exists no equifocal submanifold with non-flat section in an irreducible simply connected symmetric space G/K of compact type other than spheres whose codimension is greater than*

$$r_0 := \begin{cases} m_{G/K} & (\triangle\,:\,\text{reduced}) \\ m_{G/K}+1 & (\triangle\,:\,\text{non}-\text{reduced}) \end{cases}$$

as $m_{G/K} := \max\{m_\alpha \,|\, \alpha \in \triangle\}$, *where* $\triangle$ *is the root system of* G/K *and* m_α *is the multiplicity of* α.

(ii) *There exists no curvature-adapted equifocal submanifold with non-flat section in an irreducible simply connected symmetric space* G/K *of compact type other than spheres whose codimension is greater than*

$$m'_{G/K} := \max\{m_\alpha \,|\, \alpha \in \triangle \text{ s.t. } |\langle \alpha,\beta\rangle| \le \langle \alpha,\alpha\rangle \text{ for all } \beta \in \triangle\},$$

where $\triangle$ *and* m_α *are as above and* $\langle\ ,\ \rangle$ *is the inner product of the dual space of a fixed maximal abelian subspace of* $T_{eK}(G/K)\,(\subset \mathrm{Lie}\,G)$.

According to the statement (i) of Theorem C and the following table for $m_{G/K}$ and $m'_{G/K}$, we can give the following partial answer for Open problem 1.

THEOREM D. *There exists no equifocal submanifold with non-flat section in irreducible simply connected symmetric spaces of compact type belonging to the classes (AI), (CI), (EI), (EV), (EVIII), (FI) and (G).*

G/K	$m_{G/K}$	$m'_{G/K}$
$SU(m)/SO(m)$	1	1
$SU(2m)/Sp(m)$	4	4
$SU(m)/S(U(l)\times U(m-l))$ $(l\leq\frac{m}{2})$	$2(m-2l)$ $(l<\frac{m}{2})$ 2 $(l=\frac{m}{2})$	2 $(l\geq 2)$ 1 $(l=1)$
$SO(m)/SO(l)\times SO(m-l)$ $(l\leq\frac{m}{2})$	$m-2l$	$m-2l$
$SO(2m)/U(m)$	4	4
$Sp(m)/U(m)$	1	1
$Sp(m)/Sp(l)\times Sp(m-l)$ $(l\leq\frac{m}{2})$	$4(m-2l)$ $(l\leq\frac{m-1}{2})$ 4 $(l=\frac{m}{2})$	4
$E_6/Sp(4)$	1	1
$E_6/SU(6)\cdot SU(2)$	2	2
$E_6/Spin(10)\cdot U(1)$	9	6
E_6/F_4	8	8
$E_7/(SU(8)/\{\pm 1\})$	1	1
$E_7/SO'(12)\cdot SU(2)$	4	4
$E_7/E_6\cdot U(1)$	8	8
$E_8/SO'(16)$	1	1
$E_8/E_7\cdot Sp(1)$	8	8
$F_4/Sp(3)\cdot Sp(1)$	1	1
$F_4/Spin(9)$	8	7
$G_2/SO(4)$	1	1
$(G\times G)/\triangle(G)$	2	2

(G : a simply connected compact simple Lie group)

TABLE.

Also, we have the following fact.

THEOREM E. *There exists no equifocal submanifold with non-flat section in a simply connected compact simple Lie group (equipped with a bi-invariant metric) whose codimension is greater than two.*

REMARK 1.1. The root systems of symmetric spaces belonging to the seven classes in Theorem D and simply connected compact simple Lie groups are reduced.

According to Theorem E, we have the following facts for Open problem 2.

COROLLARY F. *All polar actions of cohomogeneity greater than two on simply connected compact simple Lie groups (equipped with a bi-invariant metric) are hyperpolar.*

2. Proof of Theorem A. In this section, we shall prove Theorem A. Without loss of generality, we may assume that G is simply connected and K is connected. Let $\pi : G \to G/K$ be the natural projection and $\phi : H^0([0,1],\mathfrak{g}) \to G$ be the parallel transport map for G, where $\mathfrak{g}$ is the Lie algebra of G and $H^0([0,1],\mathfrak{g})$ is the space of all L^2-integrable paths having $[0,1]$ as the domain. Let $M^* := \pi^{-1}(M)$ and

$\widetilde{M} := (\pi \circ \phi)^{-1}(M)$. Since G is simply connected and K is connected, M^* and $\widetilde{M}$ are connected. Denote by A (resp. $\widetilde{A}$) the shape tensor of M (resp. $\widetilde{M}$) and by $\nabla^\perp$ (resp. $\widetilde{\nabla}^\perp$) the normal connection of M (resp. $\widetilde{M}$). Let Σ_x be the section of M through $x\,(\in M)$. Assume that the restricted holonomy group of Σ_x is reducible. Fix $x_0 \in M$. We have the non-trivial orthogonal decomposition $T_{x_0}\Sigma_{x_0} = W_1 \oplus W_2$, which is invariant with respect to the restricted holonomy group of Σ_{x_0} at x_0. Since M has trivial normal holonomy group, there exists the $\nabla^\perp$-parallel subbundle D_i^N of the normal bundle $T^\perp M$ of M with $(D_i^N)_{x_0} = W_i$ $(i = 1, 2)$. For each $x \in M$, it is easy to show that there exists an isometry f of a neighborhood of x_0 in Σ_{x_0} onto a neighborhood of x in Σ_x such that f_{*x_0} coincides with the parallel translation (with respect to $\nabla^\perp$) along any curve in M from x_0 to x. From this fact, it follows that, for each $x \in M$, the orthogonal decomposition $T_x\Sigma_x = (D_1^N)_x \oplus (D_2^N)_x$ is invariant with respect to the restricted holonomy group of Σ_x at x. Let $\widetilde{D}_i^N$ $(i = 1, 2)$ be the subbundles of the normal bundle $T^\perp \widetilde{M}$ of $\widetilde{M}$ with $(\pi \circ \phi)_{*u}((\widetilde{D}_i^N)_u) = (D_i^N)_{(\pi\circ\phi)(u)}$ $(u \in \widetilde{M})$ and D_i^{N*} $(i = 1, 2)$ be those of $T^\perp(M^*)$ with $\pi_*((D_i^{N*})_g) = (D_i^N)_{\pi(g)}$ $(g \in G)$. According to Lemma 1A.4 of [PoTh1], the focal set of (M, x) consists of finitely many totally geodesic hypersurfaces in Σ_x. Denote by $\mathfrak{L}_x$ the set of all focal hypersurfaces of (M, x). Let $\psi_x : \widehat{\Sigma}_x \to \Sigma_x$ be the universal covering of Σ_x. According to the de Rham's decomposition theorem, $\widehat{\Sigma}_x$ is isometric to the (non-trivial) Riemannian product $\widehat{\Sigma}_x^1 \times \widehat{\Sigma}_x^2$, where $\widehat{\Sigma}_x^i$ $(i = 1, 2)$ is the complete totally geodesic submanifold of $\widehat{\Sigma}_x$ through $\hat{x} \in \psi_x^{-1}(x)$ such that $(\psi_x)_{*\hat{x}}(T_{\hat{x}}\widehat{\Sigma}_x^i) = (D_i^N)_x$. By retaking the decomposition $T_{x_0}^\perp M = W_1 \oplus W_2$ if necessary, we may assume that $\widehat{\Sigma}_x^1$ has no Euclidean part in the de Rham's decomposition for each $x \in M$. Let $\widehat{\mathfrak{L}}_x := \{\psi_x^{-1}(L) \,|\, L \in \mathfrak{L}_x\}$. According to Corollary 3.6 of [Kol2], elements of $\widehat{\mathfrak{L}}_x$ are either $L_1 \times \widehat{\Sigma}_x^2$-type ($L_1$: a totally geodesic hypersurface of $\widehat{\Sigma}_x^1$) or $\widehat{\Sigma}_x^1 \times L_2$-type ($L_2$: a totally geodesic hypersurface of $\widehat{\Sigma}_x^2$), where we need the fact that $\widehat{\Sigma}_x^1$ has no Euclidean part. Denote by $\widehat{\mathfrak{L}}_x^1$ (resp. $\widehat{\mathfrak{L}}_x^2$) the set of all elements of $\widehat{\mathfrak{L}}_x$ of $L_1 \times \widehat{\Sigma}_x^2$-type (resp. of $\widehat{\Sigma}_x^1 \times L_2$-type) and set $\mathfrak{L}_x^i := \{L \in \mathfrak{L}_x \,|\, \psi_x^{-1}(L) \in \widehat{\mathfrak{L}}_x^i\}$ $(i = 1, 2)$. Let $V' := \mathrm{Span}(\bigcup_{u\in\widetilde{M}} T_u^\perp \widetilde{M})$, $V_i := \mathrm{Span}(\bigcup_{u\in\widetilde{M}} (\widetilde{D}_i^N)_u)$ $(i = 1, 2)$ and $V_0 := (V')^\perp$. Also, let $(\widetilde{D}_0^T)_u := \bigcap_{v\in T_u^\perp\widetilde{M}} \mathrm{Ker}\,\widetilde{A}_v$, $(\widetilde{D}_1^T)_u := \left(\bigcap_{v\in(\widetilde{D}_2^N)_u} \mathrm{Ker}\,\widetilde{A}_v\right) \ominus (\widetilde{D}_0^T)_u$ and $(\widetilde{D}_2^T)_u := \left(\bigcap_{v\in(\widetilde{D}_1^N)_u} \mathrm{Ker}\,\widetilde{A}_v\right) \ominus (\widetilde{D}_0^T)_u$, where $u \in \widetilde{M}$. Without loss of generality, we may assume that $\widetilde{M}$ includes the zero element $\hat{0}$ of $H^0([0,1],\mathfrak{g})$, where we note that $\hat{0}$ is the constant path at the zero element 0 of $\mathfrak{g}$. Let $\widetilde{M}' := \widetilde{M} \cap V'$. First we prepare the following fact.

PROPOSITION 2.1. *We have* $\widetilde{M} = \widetilde{M}' \times V_0 \subset V' \times V_0 = H^0([0,1],\mathfrak{g})$.

Proof. First we shall show $V_0 \subset (\widetilde{D}_0^T)_u$ for each $u \in \widetilde{M}$, where we regard $(\widetilde{D}_0^T)_u (\subset T_u H^0([0,1],\mathfrak{g}))$ as a subspace of $H^0([0,1],\mathfrak{g})$ under the identification of $T_u H^0([0,1],\mathfrak{g})$ with $H^0([0,1],\mathfrak{g})$. From the definition of V_0, we have $V_0 \subset T_u\widetilde{M}$ for each $u \in \widetilde{M}$. Let $(\widetilde{D}_0^T)_u^\perp$ be the orthogonal complement of $(\widetilde{D}_0^T)_u$ in $T_u\widetilde{M}$. Clearly we have $(\widetilde{D}_0^T)_u^\perp = \sum_{v\in T_u^\perp\widetilde{M}} \left(\bigoplus_{\lambda\in\mathrm{Spec}\,\widetilde{A}_v\setminus\{0\}} \mathrm{Ker}(\widetilde{A}_v - \lambda\,\mathrm{id})\right)$, where $\mathrm{Spec}\,\widetilde{A}_v$ is the spectrum of $\widetilde{A}_v$. Let

$X \in \operatorname{Ker}(\widetilde{A}_v - \lambda\,\mathrm{id})$ ($v \in T^{\perp}_u \widetilde{M}$, $\lambda \in \operatorname{Spec} \widetilde{A}_v \setminus \{0\}$). Let J_X be the strongly Jacobi field along the normal geodesic γ_v with $\gamma'_v(0) = v$ satisfying $J_X(0) = X$ (hence $J'_X(0) = -A_v X$). Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve in M with $\alpha'(0) = X$ and $\widetilde{v}$ be the parallel normal vector field along α with $\widetilde{v}_0 = v$. Define a map $\delta : (-\varepsilon, \varepsilon) \times [0, \infty) \to H^0([0,1], \mathfrak{g})$ by $\delta(t,s) := \gamma_{\widetilde{v}_t}(s)$, where $\gamma_{\widetilde{v}_t}$ is the normal geodesic in $H^0([0,1], \mathfrak{g})$ with $\gamma'_{\widetilde{v}_t}(0) = \widetilde{v}_t$. Then we have $\delta_*(\frac{\partial}{\partial t}|_{t=0}) = J_X$. Since $\delta(t,0) - \delta(t, \frac{1}{\lambda}) \in T^{\perp}_{\alpha(t)} \widetilde{M} \subset V'$ for each $t \in (-\varepsilon, \varepsilon)$, we have $\delta_*(\frac{\partial}{\partial t}|_{t=s=0}) - \delta_*(\frac{\partial}{\partial t}|_{t=0, s=\frac{1}{\lambda}}) \in V'$. On the other hand, we have $\delta_*(\frac{\partial}{\partial t}|_{t=s=0}) = X$ and $\delta_*(\frac{\partial}{\partial t}|_{t=0,s=\frac{1}{\lambda}}) = 0$. Hence we have $X \in V'$. From the arbitrariness of X, it follows that $\operatorname{Ker}(\widetilde{A}_v - \lambda\,\mathrm{id}) \subset V'$. Furthermore, it follows from the arbitrarinesses of λ and v that $(\widetilde{D}^T_0)^{\perp}_u \subset V'$, that is, $V_0 \subset (\widetilde{D}^T_0)_u$. Since $V_0 \subset (\widetilde{D}^T_0)_u \subset T_u \widetilde{M}$ for any $u \in \widetilde{M}$, we have $\widetilde{M} = \bigcup_{u \in \widetilde{M}'} (u + V_0) = \widetilde{M}' \times V_0 \subset V' \times V_0$. ∎

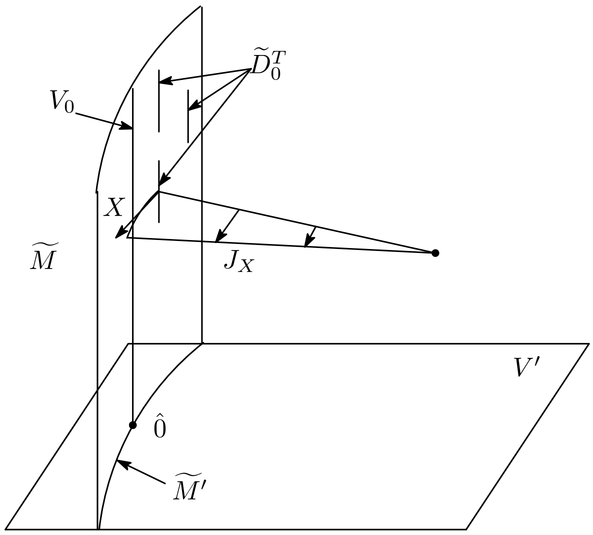

FIG. 1.

Define distributions D^T_0, D^T_1 and D^T_2 on M by

$$
\begin{aligned}
(D^T_0)_x &:= \left(\bigcap_{v \in T^{\perp}_x M} \operatorname{Ker} A_v \right) \cap g_* \left(\mathfrak{c}_{g_*^{-1} T_x M} (g_*^{-1} T^{\perp}_x M) \right), \\
(D^T_1)_x &:= \left(\left(\bigcap_{v \in (D^N_2)_x} \operatorname{Ker} A_v \right) \cap g_* \left(\mathfrak{c}_{g_*^{-1} T_x M} (g_*^{-1} (D^N_2)_x) \right) \right) \ominus (D^T_0)_x, \\
(D^T_2)_x &:= \left(\left(\bigcap_{v \in (D^N_1)_x} \operatorname{Ker} A_v \right) \cap g_* \left(\mathfrak{c}_{g_*^{-1} T_x M} (g_*^{-1} (D^N_1)_x) \right) \right) \ominus (D^T_0)_x,
\end{aligned}
$$

for each $x = gK \in M$, where $\mathfrak{c}_*(\sharp)$ is the centralizer of $\sharp$ in $*$. Take an arbitrary $v \in T^{\perp}_{eK} M$. Let $\mathfrak{a}_v$ be a maximal abelian subspace of $\mathfrak{p} := T_{eK}(G/K)$ containing v and $\mathfrak{p} = \mathfrak{a}_v + \sum_{\alpha \in \triangle^v_+} \mathfrak{p}^v_\alpha$ be the root space decomposition with respect to $\mathfrak{a}_v$. Note that

$$
T_{eK} M = \mathfrak{a}_v \cap T_{eK} M + \sum_{\alpha \in \triangle^v_+} (\mathfrak{p}^v_\alpha \cap T_{eK} M) \tag{2.2}
$$

and

$$T^{\perp}_{eK}M = \mathfrak{a}_v \cap T^{\perp}_{eK}M + \sum_{\alpha\in\triangle^v_+}(\mathfrak{p}^v_\alpha \cap T^{\perp}_{eK}M) \tag{2.3}$$

because M is equifocal and hence it has Lie triple systematic (hence root decomposable) normal bundle. Let $\widetilde{\mathfrak{a}}_v$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_v$ and $\mathfrak{a}^v_{\mathfrak{f}} := \widetilde{\mathfrak{a}}_v \cap \mathfrak{f}$, where $\mathfrak{f}$ is the Lie algebra of K. For $X \in \mathfrak{p}^v_\alpha$ ($\alpha \in \triangle^v_+$), we define $X_{\mathfrak{f}}$ as the element of $\mathfrak{f}$ such that $\mathrm{ad}(a)(X) = \alpha(a)X_{\mathfrak{f}}$ and $\mathrm{ad}(a)(X_{\mathfrak{f}}) = -\alpha(a)X$ for all $a \in \mathfrak{a}_v$. For $X \in \mathfrak{p}^v_\alpha$, $Y \in \widetilde{\mathfrak{a}}_v$ and $k \in \mathbf{Z}$, we define loop vectors $l^i_{X,k}$, $l^i_{X_{\mathfrak{f}},k}$ and $l^i_{Y,k} \in H^0([0,1],\mathfrak{g})$ ($i = 1,2$) by

$$\begin{aligned} l^1_{X,k}(t) &= l^1_{X_{\mathfrak{f}},k}(t) = X\cos(2k\pi t) - X_{\mathfrak{f}}\sin(2k\pi t),\\ l^2_{X,k}(t) &= l^2_{X_{\mathfrak{f}},k}(t) = X\sin(2k\pi t) + X_{\mathfrak{f}}\cos(2k\pi t),\\ l^1_{Y,k}(t) &= Y\cos(2k\pi t),\ \ l^2_{Y,k}(t) = Y\sin(2k\pi t). \end{aligned}$$

For a general $Z \in \mathfrak{g}$, we define loop vectors $l^i_{Z,k} \in H^0([0,1],\mathfrak{g})$ ($i = 1,2,\ k \in \mathbf{Z}$) by

$$l^i_{Z,k} := l^i_{Z_0,k} + \sum_{\alpha\in\triangle^v_+}\left(l^i_{Z_{\mathfrak{p},\alpha},k} + l^i_{Z_{\mathfrak{f},\alpha},k}\right),$$

where $Z = Z_0 + \sum_{\alpha\in\triangle^v_+}(Z_{\mathfrak{p},\alpha} + Z_{\mathfrak{f},\alpha})$ ($Z_0 \in \widetilde{\mathfrak{a}}_v$, $Z_{\mathfrak{p},\alpha} \in \mathfrak{p}^v_\alpha$, $Z_{\mathfrak{f},\alpha} \in \mathfrak{f}^v_\alpha := \{X_{\mathfrak{f}} \mid X \in \mathfrak{p}^v_\alpha\}$). Denote by $\widehat{*}$ the constant path at $* \in \mathfrak{g}$. Note that $\widehat{*}$ is the horizontal lift of $*\,(\in \mathfrak{g} = T_eG)$ to $\widehat{0}$. Then, according to Propositions 3.1 and 3.2 of [Koi2] and those proofs, we have the following relations.

LEMMA 2.2. *Let $X \in T_{eK}M \cap \mathfrak{p}^v_\alpha$. Then we have*

$$\begin{aligned} \tilde{A}_{\hat{v}} l^1_{X,k} &= \frac{\alpha(v)}{2k\pi}(\hat{X} - l^1_{X,k}),\\ \tilde{A}_{\hat{v}} l^2_{X,k} &= \frac{\alpha(v)}{2k\pi}(\hat{X}_{\mathfrak{f}} - l^2_{X,k}),\\ \tilde{A}_{\hat{v}}\hat{X} &= \widehat{A_vX} - \frac{\alpha(v)}{2}\hat{X}_{\mathfrak{f}} + \frac{\alpha(v)}{2\pi}\sum_{k\in\mathbf{Z}\setminus\{0\}}\frac{1}{k}l^1_{X,k},\\ \tilde{A}_{\hat{v}}\hat{X}_{\mathfrak{f}} &= -\frac{\alpha(v)}{2}\hat{X} + \frac{\alpha(v)}{2\pi}\sum_{k\in\mathbf{Z}\setminus\{0\}}\frac{1}{k}l^2_{X,k} \end{aligned}$$

and

$$\widetilde{\nabla}^{\perp}_{l^1_{X,k}}\widetilde{v}^L = \widetilde{\nabla}^{\perp}_{l^2_{X,k}}\widetilde{v}^L = \widetilde{\nabla}^{\perp}_{\hat{X}}\widetilde{v}^L = \widetilde{\nabla}^{\perp}_{\hat{X}_{\mathfrak{f}}}\widetilde{v}^L = 0,$$

where $k \in \mathbf{Z}\setminus\{0\}$ and $\widetilde{v}^L$ is the horizontal lift of a parallel normal vector field $\widetilde{v}$ with $\widetilde{v}_0 = v$ along an arbitrary curve α in M with $\dot{\alpha}(0) = X$.

LEMMA 2.3. *Let $w \in T^{\perp}_{eK}M \cap \mathfrak{p}^v_\alpha$. Then we have*

$$\begin{aligned} \tilde{A}_{\hat{v}} l^1_{w,k} &= -\frac{\alpha(v)}{2k\pi}l^1_{w,k},\\ \tilde{A}_{\hat{v}} l^2_{w,k} &= \frac{\alpha(v)}{2k\pi}(\hat{w}_{\mathfrak{f}} - l^2_{w,k}),\\ \tilde{A}_{\hat{v}}\hat{w}_{\mathfrak{f}} &= \frac{\alpha(v)}{2\pi}\sum_{k\in\mathbf{Z}\setminus\{0\}}\frac{1}{k}l^2_{w,k}, \end{aligned}$$

and

$$\widetilde{\nabla}^{\perp}_{l^1_{w,k}}\widetilde{v}^L=-\frac{\alpha(v)}{2k\pi}\hat{w},\ \widetilde{\nabla}^{\perp}_{l^2_{w,k}}\widetilde{v}^L=0,\ \widetilde{\nabla}^{\perp}_{\hat{w}_{\mathfrak{f}}}\widetilde{v}^L=\frac{\alpha(v)}{2}\hat{w},$$

where $k\in\mathbf{Z}\setminus\{0\}$ *and* $\widetilde{v}^L$ *is as in Lemma 2.2.*

LEMMA 2.4. *Let* $X\in\mathfrak{a}_v$ *and* $Y\in\mathfrak{a}^v_{\mathfrak{f}}$. *Then we have*

$$\tilde{A}_{\hat{v}}l^i_{X,k}=\tilde{A}_{\hat{v}}l^i_{Y,k}=\tilde{A}_{\hat{v}}\hat{Y}=0,$$

$$\widetilde{\nabla}^{\perp}_{l^i_{X,k}}\widetilde{v}^L=\widetilde{\nabla}^{\perp}_{l^i_{Y,k}}\widetilde{v}^L=\widetilde{\nabla}^{\perp}_{\hat{Y}}\widetilde{v}^L=0$$

and

$$\tilde{A}_{\hat{v}}\hat{X}=\widehat{A_vX},\quad \widetilde{\nabla}^{\perp}_{\widehat{X}}\widetilde{v}^L=0\quad(\text{when } X\in\mathfrak{a}_v\cap T_{eK}M),$$

where $i=1,2,\ k\in\mathbf{N}$ *and* $\widetilde{v}^L$ *is as in Lemma 2.2.*

From Lemmas 2.2 $\sim$ 2.4, we can show the following relations.

LEMMA 2.5. *At* $\hat{0}\in\widetilde{M}$, $(\widetilde{D}^T_0)_{\hat{0}}$ *is equal to*

$$\begin{aligned}&\mathrm{Span}\{\widehat{X}\,|\,X\in(D^T_0)_{eK}\}\oplus\mathrm{Span}\{\widehat{\eta}\,|\,\eta\in\mathfrak{c}_{\mathfrak{f}}(T^{\perp}_{eK}M)\}\\&\quad\oplus\mathrm{Span}\{l^i_{Z,k}\,|\,Z\in\mathfrak{c}_{\mathfrak{g}}(T^{\perp}_{eK}M),\ i=1,2,\ k\in\mathbf{Z}\setminus\{0\}\}\end{aligned}$$

and $(\widetilde{D}^T_{j_1})_{\hat{0}}$ *is equal to*

$$\begin{aligned}&\mathrm{Span}\{\widehat{X}\,|\,X\in(D^T_{j_1})_{eK}\}\oplus\mathrm{Span}\{\widehat{\eta}\,|\,\eta\in\mathfrak{c}_{\mathfrak{f}}((D^N_{j_2})_{eK})\ominus\mathfrak{c}_{\mathfrak{f}}(T^{\perp}_{eK}M)\}\\&\quad\oplus\mathrm{Span}\{l^i_{Z,k}\,|\,Z\in\mathfrak{c}_{\mathfrak{g}}((D^N_{j_2})_{eK})\ominus\mathfrak{c}_{\mathfrak{g}}(T^{\perp}_{eK}M),\ i=1,2,\ k\in\mathbf{Z}\setminus\{0\}\},\end{aligned}$$

where $(j_1,j_2)=(1,2)$ *or* $(2,1)$.

Proof. According to Lemmas 2.2 $\sim$ 2.4, we have

$$\begin{aligned}\mathrm{Ker}\,\widetilde{A}_{\hat{v}}=&\ \mathrm{Span}\{\widehat{X}\,|\,X\in\mathrm{Ker}\,A_v\cap\mathfrak{c}_{T_{eK}M}(v)\}\\&\oplus\mathrm{Span}\{\widehat{X}_{\mathfrak{f}}\,|,X\in\mathfrak{c}_{T_{eK}M}(v)\ominus\mathfrak{a}_v\}\oplus\mathrm{Span}\{\widehat{\xi}\,|\,\xi\in\mathfrak{a}^v_{\mathfrak{f}}\}\\&\oplus\mathrm{Span}\{l^i_{X,k}\,|\,X\in\mathfrak{c}_{T_{eK}M}(v)\ominus\mathfrak{a}_v,\ i=1,2,\ k\in\mathbf{Z}\setminus\{0\}\}\\&\oplus\mathrm{Span}\{\widehat{w}_{\mathfrak{f}}\,|\,w\in\mathfrak{c}_{T^{\perp}_{eK}M}(v)\ominus\mathfrak{a}_v\}\\&\oplus\mathrm{Span}\{l^2_{w,k}\,|\,w\in\mathfrak{c}_{T^{\perp}_{eK}M}(v)\ominus\mathfrak{a}_v,\ k\in\mathbf{Z}\setminus\{0\}\}\\&\oplus\mathrm{Span}\{l^i_{\xi,k}\,|\,\xi\in\widetilde{\mathfrak{a}}_v,\ i=1,2,\ k\in\mathbf{N}\}\\&\oplus\mathrm{Span}\{l^1_{w,k}\,|\,w\in\mathfrak{c}_{T^{\perp}_{eK}M}(v)\ominus\mathfrak{a}_v,\ k\in\mathbf{Z}\setminus\{0\}\}\\=&\ \mathrm{Span}\{\widehat{X}\,|\,X\in\mathrm{Ker}\,A_v\cap\mathfrak{c}_{T_{eK}M}(v)\}\oplus\mathrm{Span}\{\widehat{\eta}\,|\,\eta\in\mathfrak{c}_{\mathfrak{f}}(v)\}\\&\oplus\mathrm{Span}\{l^i_{Z,k}\,|\,Z\in\mathfrak{c}_{\mathfrak{g}}(v),\ i=1,2,\ k\in\mathbf{Z}\setminus\{0\}\}.\end{aligned}$$

Hence we have the desired relations. □

From Lemmas 2.2 $\sim$ 2.4, we have the following lemma.

LEMMA 2.6. *Assume that* $v \in D_i^N$. *Let* $\widetilde{v}^L$ *be as in Lemma 2.2. Then the statements* (i) *and* (ii) *hold.*

(i) *For each* $X \in T\widetilde{M}$, *we have* $\widetilde{\nabla}^{\perp}_X \widetilde{v}^L \in \widetilde{D}_i^N$.

(ii) *For each* $Y \in \widetilde{D}_j^T \oplus \widetilde{D}_0^T$ $(j \neq i)$, *we have* $\widetilde{\nabla}_Y \widetilde{v}^L = 0$.

Proof. Without loss of generality, we may assume that the base point of X is $\hat{0}$. First we shall show the statement (i). According to (2.2), (2.3) and Lemmas 2.2 $\sim$ 2.4, we have only to show $\widetilde{\nabla}^{\perp}_X \widetilde{v}^L \in \widetilde{D}_i^N$ in case of $X = l^1_{w,k}$ or $\widehat{w}_{\mathfrak{f}}$ $(w \in T^{\perp}_{eK} M \cap \mathfrak{p}^v_\alpha)$. If $w \in D_i^N$, then it follows from Lemma 2.3 that $\widetilde{\nabla}^{\perp}_X \widetilde{v}^L \in \mathrm{Span}\{\widehat{w}\} \subset \widetilde{D}_i^N$. If $w \in D_j^N$ $(j \neq i)$, then we have $\alpha(v) = 0$ because the sectional curvature of $\mathrm{Span}\{v, w\}$ is equal to 0. Hence it follows from Lemma 2.3 that $\widetilde{\nabla}^{\perp}_X \widetilde{v}^L = 0$. Thus the statement (i) is shown. Next we shall show the statement (ii). From (i), we have $\widetilde{\nabla}^{\perp}_Y \widetilde{v}^L = 0$. Also, from the definitions of $\widetilde{D}_j^T$ and $\widetilde{D}_0^T$, we have $\widetilde{A}_{\hat{v}} Y = 0$. Hence, we obtain $\widetilde{\nabla}_Y \widetilde{v}^L = 0$. □

By using (ii) of Lemma 2.6, we prove the following lemma.

LEMMA 2.7. *For each* $u \in \widetilde{M}$, *the tangent space* $T_u \widetilde{M}$ *is orthogonally decomposed as* $T_u \widetilde{M} = (\widetilde{D}_1^T)_u \oplus (\widetilde{D}_2^T)_u \oplus (\widetilde{D}_0^T)_u$.

Proof. Take unit vectors v_i belonging to $(\widetilde{D}_i^N)_u$ $(i = 1, 2)$. According to (i) of Lemma 2.6, we have $\widetilde{R}^{\perp}(X, Y) v_1 \in (\widetilde{D}_1^N)_u$ for any $X, Y \in T_u \widetilde{M}$, where $\widetilde{R}^{\perp}$ is the curvature tensor of the normal connection of $\widetilde{M}$. Hence, it follows from the Ricci equation that $[\widetilde{A}_{v_1}, \widetilde{A}_{v_2}] = 0$. Therefore, we have

$$T_u \widetilde{M} = \bigoplus_{\lambda \in \mathrm{Spec}\, \widetilde{A}_{v_1}} \bigoplus_{\mu \in \mathrm{Spec}\, \widetilde{A}_{v_2}} \left(\mathrm{Ker}(\widetilde{A}_{v_1} - \lambda\, \mathrm{id}) \cap \mathrm{Ker}(\widetilde{A}_{v_2} - \mu\, \mathrm{id}) \right), \tag{2.1}$$

where $\mathrm{Spec}\, \widetilde{A}_{v_i}$ $(i = 1, 2)$ is the spectrum of $\widetilde{A}_{v_i}$. Set $\widetilde{\mathfrak{L}}^i_u := \{ ((\pi \circ \phi)|_{T^{\perp}_u \widetilde{M}})^{-1}(L) \mid L \in \mathfrak{L}^i_{\pi(u)} \}$ $(i = 1, 2)$. The family $\widetilde{\mathfrak{L}}^1_u \cup \widetilde{\mathfrak{L}}^2_u$ gives the family of all focal hypersurfaces of $\widetilde{M}$ at u. Let $\lambda \in \mathrm{Spec}\, \widetilde{A}_{v_1} \setminus \{0\}$ and $\mu \in \mathrm{Spec}\, \widetilde{A}_{v_2} \setminus \{0\}$. We shall show $\mathrm{Ker}(\widetilde{A}_{v_1} - \lambda\, \mathrm{id}) \cap \mathrm{Ker}(\widetilde{A}_{v_2} - \mu\, \mathrm{id}) = \{0\}$. Suppose that $\mathrm{Ker}(\widetilde{A}_{v_1} - \lambda\, \mathrm{id}) \cap \mathrm{Ker}(\widetilde{A}_{v_2} - \mu\, \mathrm{id}) \neq \{0\}$. Take $X (\neq 0) \in \mathrm{Ker}(\widetilde{A}_{v_1} - \lambda\, \mathrm{id}) \cap \mathrm{Ker}(\widetilde{A}_{v_2} - \mu\, \mathrm{id})$. The point $u + \frac{1}{\lambda} v_1$ and $u + \frac{1}{\mu} v_2$ are focal points along the normal geodesics γ_{v_1} and γ_{v_2}, respectively. Hence there exist $L_1 \in \widetilde{\mathfrak{L}}^1_u$ with $u + \frac{1}{\lambda} v_1 \in L_1$ and $L_2 \in \widetilde{\mathfrak{L}}^2_u$ with $u + \frac{1}{\mu} v_2 \in L_2$. Let $w_\theta := \cos\theta \cdot v_1 + \frac{\lambda}{\mu} \sin\theta \cdot v_2$ $(0 \leq \theta \leq \frac{\pi}{2})$. Since $A_{w_\theta} X = \lambda(\sin\theta + \cos\theta) X$, the point $u + \frac{1}{\lambda(\sin\theta + \cos\theta)} w_\theta$ is a focal point along γ_{w_θ} for each $\theta \in [0, \frac{\pi}{2}]$. Define a curve $c : [0, \frac{\pi}{2}] \to H^0([0,1], \mathfrak{g})$ by $c(\theta) := u + \frac{1}{\lambda(\sin\theta + \cos\theta)} w_\theta$ $(\theta \in I)$, which is smooth and regular. For each $\theta \in [0, \frac{\pi}{2}]$, we have $c(\theta) \in \bigcup_{L \in \widetilde{\mathfrak{L}}^1_u \cup \widetilde{\mathfrak{L}}^2_u} (L \cap \mathrm{Span}\{v_1, v_2\})$. For simplicity, we set $F := \bigcup_{L \in \widetilde{\mathfrak{L}}^1_u \cup \widetilde{\mathfrak{L}}^2_u} (L \cap \mathrm{Span}\{v_1, v_2\})$. Since F is a family of affine lines in $\mathrm{Span}\{v_1, v_2\}$ which are parallel to $\mathrm{Span}\{v_1\}$ or $\mathrm{Span}\{v_2\}$ and c is a regular curve in F, c lies in the only affine line belonging to F. It is clear that the affine lines $L_1 \cap \mathrm{Span}\{v_1, v_2\}$ and $L_2 \cap \mathrm{Span}\{v_1, v_2\}$ are mutually distinct. These facts contradict $c(0) \in L_1$ and $c(\frac{\pi}{2}) \in L_2$ (see Fig. 2). Therefore we have $\mathrm{Ker}(\widetilde{A}_{v_1} - \lambda\, \mathrm{id}) \cap \mathrm{Ker}(\widetilde{A}_{v_2} - \mu\, \mathrm{id}) = \{0\}$. This fact together with (2.1) deduces $\bigoplus_{\lambda\in\mathrm{Spec}\,\widetilde{A}_{v_1}\setminus\{0\}}\mathrm{Ker}(\widetilde{A}_{v_1}-\lambda\,\mathrm{id})\subset\mathrm{Ker}\,\widetilde{A}_{v_2}$. From the arbitrariness of v_2, we have $\bigoplus_{\lambda\in\mathrm{Spec}\,\widetilde{A}_{v_1}\setminus\{0\}}\mathrm{Ker}(\widetilde{A}_{v_1}-\lambda\,\mathrm{id})\subset(\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u$. That is, the orthogonal complement $((\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u)^\perp$ of $(\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u$ is contained in $\mathrm{Ker}\,\widetilde{A}_{v_1}$. From the arbitrariness of v_1, we have $((\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u)^\perp\subset(\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_2)_u$, which implies $((\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u)^\perp\subset(\widetilde{D}^T_2)_u$. On the other hand, we have $((\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u)\cap(\widetilde{D}^T_2)_u=\{0\}$. Hence we have $T_u\widetilde{M}=(\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u\oplus(\widetilde{D}^T_2)_u$ and $((\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u)^\perp=(\widetilde{D}^T_2)_u$. After all we have $T_u\widetilde{M}=(\widetilde{D}^T_0)_u\oplus(\widetilde{D}^T_1)_u\oplus(\widetilde{D}^T_2)_u$ (orthogonal direct sum). □

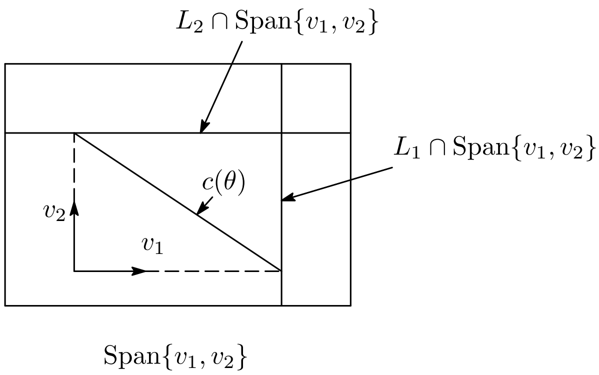

FIG. 2.

Next we prepare the following lemma.

LEMMA 2.8. (i) *The distributions* $\widetilde{D}^T_i\oplus\widetilde{D}^T_0$ $(i=1,2)$ *are totally geodesic.*
(ii) *The distributions* $\widetilde{D}^T_i$ $(i=1,2)$ *are totally geodesic.*

Proof. For simplicity, set $\widetilde{D}^T_{i0}:=\widetilde{D}^T_i\oplus\widetilde{D}^T_0$ $(i=1,2)$. Denote by $\widetilde{h}$ (resp. $\widetilde{h}_{10}$) the second fundamental form of $\widetilde{M}$ (resp. $\widetilde{D}^T_{10}$), by $\widetilde{A}^{10}$ the shape tensor of $\widetilde{D}^T_{10}$, by $\widetilde{\nabla}$ (resp. $\nabla^{\widetilde{M}}$) the Levi-Civita connection of $H^0([0,1],\mathfrak{g})$ (resp. $\widetilde{M}$) and by $\nabla^{\perp_2}$ the normal connection of $\widetilde{D}^T_2$. Also, denote by $\overline{\nabla}$ the connection of the bundle $T^*\widetilde{M}\otimes T^*\widetilde{M}\otimes T^\perp\widetilde{M}$ induced from $\nabla^{\widetilde{M}}$ and $\widetilde{\nabla}^\perp$. Let $X,Y\in(\widetilde{D}^T_{10})_u$ and $Z\in(\widetilde{D}^T_2)_u$. Let $\widetilde{X}$ (resp. $\widetilde{Y}$) be a section of $\widetilde{D}^T_{10}$ with $\widetilde{X}_u=X$ (resp. $\widetilde{Y}_u=Y$) and $\widetilde{Z}$ be a section of $\widetilde{D}^T_2$ with $\widetilde{Z}_u=Z$. For any $v_1\in(\widetilde{D}^N_1)_u$, we have $\langle\widetilde{h}(Y,Z),v_1\rangle=\langle\widetilde{A}_{v_1}Z,Y\rangle=0$ because of $(\widetilde{D}^T_2)_u\subset\mathrm{Ker}\,\widetilde{A}_{v_1}$. Also, for any $v_2\in(\widetilde{D}^N_2)_u$, we have $\langle\widetilde{h}(Y,Z),v_2\rangle=\langle\widetilde{A}_{v_2}Y,Z\rangle=0$ because of $(\widetilde{D}^T_{10})_u\subset\mathrm{Ker}\,\widetilde{A}_{v_2}$. Hence we have $\widetilde{h}(Y,Z)=0$. From the arbitrarinesses of Y,Z and u, we have $\widetilde{h}(\widetilde{D}^T_{10},\widetilde{D}^T_2)=0$. Also, we can show $\widetilde{h}(\widetilde{D}^T_{10},\widetilde{D}^T_{10})\subset\widetilde{D}^N_1$ and $\widetilde{h}(\widetilde{D}^T_2,\widetilde{D}^T_2)\subset\widetilde{D}^N_2$. Let $X,Y,Z,\widetilde{Y}$ and $\widetilde{Z}$ be as above. It follows from $\widetilde{h}(\widetilde{D}^T_{10},\widetilde{D}^T_2)=0$ that

$$
\begin{aligned}
(\overline{\nabla}_X\widetilde{h})(Z,Y)&=\widetilde{\nabla}^\perp_X(\widetilde{h}(\widetilde{Z},\widetilde{Y}))-\widetilde{h}(\nabla^{\widetilde{M}}_X\widetilde{Z},Y)-\widetilde{h}(Z,\nabla^{\widetilde{M}}_X\widetilde{Y})\\
&=\widetilde{h}(A^{10}_ZX,Y)-\widetilde{h}(Z,h_{10}(X,Y))\\
&\equiv-\widetilde{h}(Z,h_{10}(X,Y))\qquad(\mathrm{mod}\ (\widetilde{D}^N_1)_u).
\end{aligned}\tag{2.2}
$$

Also, it follows from $\widetilde{h}(\widetilde{D}^T_{10},\widetilde{D}^T_{10})\subset\widetilde{D}^N_1$ and Lemma 2.6 that

$$(\overline{\nabla}_Z\widetilde{h})(X,Y)=\widetilde{\nabla}^{\perp}_Z(\widetilde{h}(\widetilde{X},\widetilde{Y}))-\widetilde{h}(\nabla^{\perp_2}_Z\widetilde{X},Y)-\widetilde{h}(X,\nabla^{\perp_2}_Z\widetilde{Y}) \equiv 0 \ (\mathrm{mod}\,(\widetilde{D}^N_1)_u). \tag{2.3}$$

By (2.2), (2.3) and the Codazzi equation, we have $\widetilde{h}(Z,h_{10}(X,Y))\in(\widetilde{D}^N_1)_u$. On the other hand, it follows from $\widetilde{h}(\widetilde{D}^T_2,\widetilde{D}^T_2)\subset\widetilde{D}^N_2$ that $\widetilde{h}(Z,h_{10}(X,Y))\in(\widetilde{D}^N_2)_u$. Hence we have $\widetilde{h}(Z,h_{10}(X,Y))=0$. According to the proof of Lemma 2.7, we have

$$(\widetilde{D}^T_2)_u=\bigoplus_{v_2\in(\widetilde{D}^N_2)_u}\ \bigoplus_{\mu\in\mathrm{Spec}\widetilde{A}_{v_2}\setminus\{0\}}\mathrm{Ker}(\widetilde{A}_{v_2}-\mu\,\mathrm{id}).$$

If $Z\in\mathrm{Ker}(\widetilde{A}_{v_2}-\mu\,\mathrm{id})$ ($\mu\in\mathrm{Spec}\widetilde{A}_{v_2}\setminus\{0\}$), then we have

$$\langle\widetilde{h}(Z,h_{10}(X,Y)),v_2\rangle=\langle\widetilde{A}_{v_2}Z,h_{10}(X,Y)\rangle=\mu\langle h_{10}(X,Y),Z\rangle=0,$$

that is, $\langle h_{10}(X,Y),Z\rangle=0$. From the arbitrariness of $Z\in(\widetilde{D}^T_2)_u$, it follows that $h_{10}(X,Y)=0$. From the arbitrarinesses of X and Y, it follows that $h_{10}=0$, that is, $\widetilde{D}^T_{10}$ is totally geodesic. Similarly, we can show that $\widetilde{D}^T_{20}$ is totally geodesic. By the similar discussion, we can show the statement (ii). □

By using Lemmas 2.6∼2.8, we show the following fact.

LEMMA 2.9. *We have* $V'=V_1\oplus V_2$ (*orthogonal direct sum*).

Proof. Clearly we have $V'=V_1+V_2$. We have only to show $V_1\perp V_2$. Take arbitrary $u_1,u_2\in\widetilde{M}$ and arbitrary $v_j\in(\widetilde{D}^N_j)_{u_i}$ ($(i,j)=(1,2),(2,1)$). Define a subset $U(u_1)$ of $H^0([0,1],\mathfrak{g})$ by $U(u_1):=\bigcup_{u\in L^{\widetilde{D}^T_{10}}_{u_1}}L^{\widetilde{D}^T_{20}}_u$, where $L^{\widetilde{D}^T_{10}}_{u_1}$ (resp. $L^{\widetilde{D}^T_{20}}_u$) is the leaf of $\widetilde{D}^T_{10}$ (resp. $\widetilde{D}^T_{20}$) through u_1 (resp. u). Since $\widetilde{M}$ is complete, $\widetilde{D}^T_{10}$ is totally geodesic by Lemma 2.8 and $\widetilde{D}^T_2$ is the orthogonal complementary distribution of $\widetilde{D}^T_{10}$ by Lemma 2.7, $\widetilde{D}^T_2$ is an Ehresmann connection for the foliation consisting of integral manifolds of $\widetilde{D}^T_{10}$ (see [BH]). Note that the discussions in [BH] are valid in the infinite dimensional case. From the infinite dimensional version of the discussion in [BH], it follows that $U(u_1)=\widetilde{M}$. Therefore we have $L^{\widetilde{D}^T_{10}}_{u_1}\cap L^{\widetilde{D}^T_{20}}_{u_2}\neq\emptyset$. Take $u_3\in L^{\widetilde{D}^T_{10}}_{u_1}\cap L^{\widetilde{D}^T_{20}}_{u_2}$ and curves $\alpha_i:[0,1]\to L^{\widetilde{D}^T_{i0}}_{u_i}$ ($i=1,2$) with $\alpha_i(0)=u_i$ and $\alpha_i(1)=u_3$. According to (ii) of Lemma 2.6, we have $P^{\widetilde{\nabla}}_{\alpha_i}(v_j)\in(\widetilde{D}^N_j)_{u_3}$ ($i=1,2$), where $P^{\widetilde{\nabla}}_{\alpha_i}$ is the parallel translation along α_i with respect to $\widetilde{\nabla}$. Hence we obtain $\langle v_1,v_2\rangle=0$. Therefore, it follows from the arbitrarinesses of v_1 and v_2 that $V_1\perp V_2$. □

Fix $x_0\in M$. According to Lemma 2.15 and Proposition 2.16 of [E2], the focal set of (M,x_0) consists of finitely many totally geodesic hypersurfaces in the section Σ_{x_0} through x_0. Let $\mathfrak{L}_{x_0}$ be the family of all the focal hypersurfaces. The focal hypersurfaces divide Σ_{x_0} into some open cells. Denote by $\triangle$ the component containing $0\in T^{\perp}_{x_0}M$ of the inverse image by $\exp^{\perp}_{x_0}$ of the open cell containing x_0. Define a map $f:M\times\triangle\to G/K$ by $f(x,v):=\exp^{\perp}_x(\widetilde{v}_x)$ ($(x,v)\in M\times\triangle$), where $\widetilde{v}$ is the parallel normal vector field of M with $\widetilde{v}_{x_0}=v$. Let $U:=f(M\times\triangle)$, which is an open dense subset of G/K consisting of non-focal points of M. For each $v \in \triangle$, denote by M_v the parallel submanifold $\eta_{\widetilde{v}}(M)$ of M, where $\eta_{\widetilde{v}}$ is the end-point map for $\widetilde{v}$, that is, $\eta_{\widetilde{v}}(x) = f(x,v)$ $(x \in M)$. Let E_i^N $(i=1,2)$ be the distribution on U such that $E_i^N|_M = D_i^N$, $E_i^N|_{\Sigma_x}$ is a parallel distribution on Σ_x for each $x \in M$ and that $E_i^N|_{M_v}$ is a normal parallel subbundle of $T^\perp M_v$ for each $v \in \triangle$. Denote by $(D_i^T)^v$ $(i=0,1,2)$ the distributions on M_v corresponding to the distributions D_i^T $(i=0,1,2)$ on M. It is shown that $(D_i^T)^v = (\eta_{\widetilde{v}})_*(D_i^T)$. For each $i \in \{0,1,2\}$, the distributions $(D_i^T)^v$'s $(v \in \triangle)$ give a distribution on U. Denote by E_i^T $(i=0,1,2)$ this distribution on U. Set $E_i := E_i^T \oplus E_i^N$ and $E_{i0} := E_i^T \oplus E_i^N \oplus E_0^T$ $(i=1,2)$. Let $\widetilde{U} := (\pi\circ\phi)^{-1}(U)$, which is an open dense subset of $H^0([0,1],\mathfrak{g})$. For each $v \in \triangle$, denote by $\widetilde{M_v}$ the submanifold $\eta_{\widetilde{v}^L}(\widetilde{M})$, where $\eta_{\widetilde{v}^L}$ is the end-point map for the horizontal lift $\widetilde{v}^L$ of $\widetilde{v}$. Note that $\eta_{\widetilde{v}^L}(\widetilde{M})$ is not a parallel submanifold of $\widetilde{M}$ because $\widetilde{v}^L$ is not parallel with respect to the normal connection of $\widetilde{M}$. Let $\widetilde{E}_i^N$ $(i=1,2)$ be the horizontal lift of E_i^N to $\widetilde{U}$. Denote by $(\widetilde{D}_i^T)^v$ the distributions on $\widetilde{M_v}$ corresponding to the distributions $\widetilde{D}_i^T$ $(i=0,1,2)$ on $\widetilde{M}$. For each $i \in \{0,1,2\}$, the distributions $(\widetilde{D}_i^T)^v$'s $(v\in\triangle)$ give a distribution on $\widetilde{U}$. Denote by $\widetilde{E}_i^T$ $(i=0,1,2)$ this distribution. Set $\widetilde{E}_i := \widetilde{E}_i^T \oplus \widetilde{E}_i^N$ and $\widetilde{E}_{i0} := \widetilde{E}_i^T \oplus \widetilde{E}_i^N \oplus \widetilde{E}_0^T$ $(i=1,2)$. By using Lemmas 2.5 and 2.8, we show the following lemma.

LEMMA 2.10. (i) *The distributions* $\widetilde{E}_{i0}$ $(i=1,2)$ *are totally geodesic.*
(ii) *The distributions* $\widetilde{E}_i$ $(i=1,2)$ *are totally geodesic.*

Proof. For each $X \in \Gamma(TM)$, we define $\overline{X} \in \Gamma(TU)$ by $\overline{X}_{f(x,v)} := (\eta_{\widetilde{v}})_{*x}(X_x)$ $((x,v) \in M\times\triangle)$, where $\eta_{\widetilde{v}}$ is as above. Also, for each $w \in \triangle$, we define $\overline{w} \in \Gamma(TU)$ by $\overline{w}_{f(x,v)} := P^{\Sigma_x}_{\gamma_{\widetilde{v}_x}}(\widetilde{w}_x)$ $((x,v)\in M\times\triangle)$, where $\widetilde{w}$ is the parallel normal vector field of M with $\widetilde{w}_{x_0} = w$ and $P^{\Sigma_x}_{\gamma_{\widetilde{v}_x}}$ is the parallel translation along the geodesic $\gamma_{\widetilde{v}_x} : [0,1] \to \Sigma_x$ with $\gamma'_{\widetilde{v}_x}(0) = \widetilde{v}_x$ with respect to the Levi-Civita connection of Σ_x. Note that $P^{\Sigma_x}_{\gamma_{\widetilde{v}_x}}$ coincides with the parallel translation along $\gamma_{\widetilde{v}_x}$ with respect to the Levi-Civita connection of G/K because Σ_x is totally geodesic. Without loss of generality, we may assume $x_0 = eK$. We suffice to show that $\widetilde{E}_{i0}$ $(i=1,2)$ and $\widetilde{E}_i$ $(i=1,2)$ have $\hat{0}$ as a geodesic point. Easily we can show that if $X \in \Gamma(D_i^T)$ (resp. $w \in \triangle \cap (D_j^N)_{eK}$), then $\overline{X} \in \Gamma(E_i^T)$ (resp. $\overline{w} \in \Gamma(E_j^N)$), where $i=0,1,2$ and $j=1,2$. We shall show that $\widetilde{E}_{10}$ has $\hat{0}$ as a geodesic point. From Lemma 2.5, we have

$$
\begin{aligned}
(\widetilde{E}_{10})_{\hat{0}} = {} & \mathrm{Span}\{\widehat{X} \mid X \in (D_{10}^T)_{eK}\} \oplus \mathrm{Span}\{\widehat{\eta} \mid \eta \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK})\} \\
& \oplus \mathrm{Span}\{l^i_{Z,k} \mid Z \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK}),\ i=1,2,\ k \in \mathbf{Z}\setminus\{0\}\} \\
& \oplus \mathrm{Span}\{\widehat{w} \mid w \in (D_1^N)_{eK}\}.
\end{aligned} \tag{2.4}
$$

Denote by $\widetilde{h}_{10}$ the second fundamental form of $\widetilde{E}_{10}$. First we show $\widetilde{h}_{10}((\widetilde{D}_1^N)_{\hat{0}}, (\widetilde{D}_1^N)_{\hat{0}}) = 0$. Let $w_1, w_2 \in (D_1^N)_{eK}$. Denote by ∇, ∇^* and $\widetilde{\nabla}$ the Levi-Civita connection of $G/K, G$ and $H^0([0,1],\mathfrak{g})$. Denote by $(\cdot)^L$ (resp. $(\cdot)^*$) the horizontal lift of $(\cdot)$ to $H^0([0,1],\mathfrak{g})$ (resp. G). According to Lemmas 2.2 and 2.3 in [Koi2], we have

$$
\widetilde{\nabla}_{\widehat{w}_1}\overline{w}_2^L = (\nabla^*_{w_1}\overline{w}_2^*)^L_{\hat{0}} - t[w_1,w_2] + \frac{1}{2}[w_1,w_2]^L_{\hat{0}} = (\nabla_{w_1}\overline{w}_2)^L_{\hat{0}} - t[w_1,w_2],
$$

where $t[w_1,w_2]$ is the H^0-path in $\mathfrak{g}$ assigning $t[w_1,w_2]$ to each $t \in [0,1]$. Since E_1^N is totally geodesic, we have $\nabla_{w_1}\overline{w}_2 \in (D_1^N)_{eK}$ and hence $(\nabla_{w_1}\overline{w}_2)^L_{\hat{0}} \in (\widetilde{E}_{10})_{\hat{0}}$ by (2.4).

Also, we have $[w_1, w_2] \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK})$ and hence $t[w_1, w_2] \in (\widetilde{E}_{10})_{\hat{0}}$ by (2.4). Therefore, we have $\widetilde{\nabla}_{\widehat{w}_1} \overline{w}_2^L \in (\widetilde{E}_{10})_{\hat{0}}$, that is, $\widetilde{h}_{10}(\widehat{w}_1, \widehat{w}_2) = 0$. Thus we have

$$\widetilde{h}_{10}((\widetilde{D}_1^N)_{\hat{0}}, (\widetilde{D}_1^N)_{\hat{0}}) = 0. \tag{2.5}$$

Set $\widetilde{E}_{10}^T := \widetilde{E}_1^T \oplus \widetilde{E}_0^T$. Next we show that $\widetilde{h}_{10}((\widetilde{E}_{10}^T)_{\hat{0}}, (\widetilde{E}_{10}^T)_{\hat{0}}) = 0$. Let $\widetilde{X}, \widetilde{Y} \in \Gamma(\widetilde{E}_{10}^T)$. For each $w \in (D_2^N)_{eK}$, we have

$$\langle \widetilde{h}(\widetilde{X}_{\hat{0}}, \widetilde{Y}_{\hat{0}}), \widehat{w} \rangle = \langle \widetilde{A}_{\widehat{w}} \widetilde{X}_{\hat{0}}, \widetilde{Y}_{\hat{0}} \rangle = 0$$

from the definition of $\widetilde{E}_{10}^T$. Hence we have $\widetilde{h}(\widetilde{X}_{\hat{0}}, \widetilde{Y}_{\hat{0}}) \in (\widetilde{D}_1^N)_{\hat{0}} \subset (\widetilde{E}_{10})_{\hat{0}}$. Also, since $\widetilde{D}_{10}^T$ is totally geodesic by Lemma 2.8, we have $\nabla^{\widetilde{M}}_{\widetilde{X}_{\hat{0}}} \widetilde{Y} \in (\widetilde{D}_{10}^T)_{\hat{0}} \subset (\widetilde{E}_{10})_{\hat{0}}$. Therefore, we have $\widetilde{h}_{10}(\widetilde{X}_{\hat{0}}, \widetilde{Y}_{\hat{0}}) = 0$. Thus we have

$$\widetilde{h}_{10}((\widetilde{E}_{10}^T)_{\hat{0}}, (\widetilde{E}_{10}^T)_{\hat{0}}) = 0. \tag{2.6}$$

Next we show $\widetilde{h}_{10}((\widetilde{E}_{10}^T)_{\hat{0}}, (\widetilde{D}_1^N)_{\hat{0}}) = 0$. Let $w \in (D_1^N)_{eK}$. According to (2.4), we suffices to show that $\widetilde{h}_{10}(\widehat{X}, \widehat{w})$ $(X \in (D_{10}^T)_{eK})$, $\widetilde{h}_{10}(\widehat{\eta}, \widehat{w})$ $(\eta \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK}))$ and $\widetilde{h}_{10}(l_{Z,k}^i, \widehat{w})$ $(Z \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK}),\ i = 1, 2,\ k \in \mathbf{Z} \setminus \{0\})$ vanish. According to Lemmas 2.2 and 2.3 in [Koi2], we have $\widetilde{\nabla}_{\widehat{X}} \overline{w}^L = (\nabla_X \overline{w})_{\hat{0}}^L - t[X, w]$, $\widetilde{\nabla}_{\widehat{\eta}} \overline{w}^L = -t[\eta, w]$ and $\widetilde{\nabla}_{l_{Z,k}^i} \overline{w}^L = -[\int_0^t l_{Z,k}^i(t)dt,\, w]$. Also, we can show $\nabla_X \overline{w} = -A_w X \in (D_{10}^T)_{eK}$, $[X, w] \in \mathfrak{c}_{\mathfrak{f}}((D_2^N)_{eK})$, $[\eta, w] \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK}) \cap ((D_{10}^T)_{eK} \oplus (D_1^N)_{eK})$ and $[\int_0^t l_{Z,k}^i(t)dt, w] \in \mathfrak{c}_{\mathfrak{g}}((D_2^N)_{eK}) \cap ((D_{10}^T)_{eK} \oplus (D_1^N)_{eK})$ for each fixed $t \in [0, 1]$. Hence it follows from (2.4) that $\widetilde{\nabla}_{\widehat{X}} \overline{w}^L$, $\widetilde{\nabla}_{\widehat{\eta}} \overline{w}^L$ and $\widetilde{\nabla}_{l_{Z,k}^i} \overline{w}^L$ belong to $(\widetilde{E}_{10})_{\hat{0}}$. That is, we have $\widetilde{h}_{10}(\widehat{X}, \widehat{w}) = \widetilde{h}_{10}(\widehat{\eta}, \widehat{w}) = \widetilde{h}_{10}(l_{Z,k}^i, \widehat{w}) = 0$. Thus we have

$$\widetilde{h}_{10}((\widetilde{E}_{10}^T)_{\hat{0}}, (\widetilde{D}_1^N)_{\hat{0}}) = 0. \tag{2.7}$$

Similarly, we can show $\widetilde{h}_{10}((\widetilde{D}_1^N)_{\hat{0}}, (\widetilde{E}_{10}^T)_{\hat{0}}) = 0$, which together with (2.5) $\sim$ (2.7) and $(\widetilde{E}_{10})_{\hat{0}} = (\widetilde{E}_{10}^T)_{\hat{0}} \oplus (\widetilde{D}_1^N)_{\hat{0}}$ implies that $(\widetilde{h}_{10})_{\hat{0}} = 0$, that is, $\hat{0}$ is a geodesic point of $\widetilde{E}_{10}$. This completes the proof of the totally geodesicness of $\widetilde{E}_{10}$. Similarly, we can show that $\widetilde{E}_{20}$ and $\widetilde{E}_i$ $(i = 1, 2)$ are totally geodesic. □

Let $\widetilde{M}_i(u) := \widetilde{M} \cap (u + V_i)$ and $(F_i)_u := T_u \widetilde{M}_i(u)$ $(u \in \widetilde{M},\ i = 1, 2)$.

Lemma 2.11. *The correspondence* $F_i : u \mapsto (F_i)_u$ $(u \in \widetilde{M})$ *gives a totally geodesic distribution on* $\widetilde{M}$ *having* $\widetilde{M}_i(u)$*'s* $(u \in \widetilde{M})$ *as integral manifolds, where* $i = 1, 2$.

Proof. Fix $u_0 \in \widetilde{M}$. From (ii) of Lemma 2.6, it follows that $V_i = \mathrm{Span}\left(\bigcup_{u \in L_{u_0}^{\widetilde{D}_i^T}} (\widetilde{D}_i^N)_u \right)$, where $L_{u_0}^{\widetilde{D}_i^T}$ is the leaf of $\widetilde{D}_i^T$ through u_0. On the other hand, it follows from Lemma 2.10 that $(\widetilde{D}_i^N)_u$'s $(u \in L_{u_0}^{\widetilde{D}_i^T})$ are contained in $T_{u_0} L_{u_0}^{\widetilde{D}_i^T} \oplus (\widetilde{D}_i^N)_{u_0}$. Hence we have $V_i \subset T_{u_0} L_{u_0}^{\widetilde{D}_i^T} \oplus (\widetilde{D}_i^N)_{u_0}$ and hence $\widetilde{M}_i(u_0) \subset L_{u_0}^{\widetilde{D}_i^T}$. It is clear that

$\widetilde{M}_i(u_0)$ is totally geodesic in $L_{u_0}^{\widetilde{D}_i^T}$. Also, according to Lemma 2.8, $L_{u_0}^{\widetilde{D}_i^T}$ is totally geodesic in $\widetilde{M}$. Hence $\widetilde{M}_i(u_0)$ is totally geodesic in $\widetilde{M}$. This completes the proof. $\square$

By using this lemma, we can show the following fact.

LEMMA 2.12. *The submanifold* $\widetilde{M}_i(u)$*'s* ($u \in \widetilde{M}$) *are integral manifolds of* $\widetilde{D}_i^T$ ($i = 1, 2$).

Proof. Let $\widetilde{M}'(u) := \widetilde{M} \cap (u + V')$ ($u \in \widetilde{M}$). Since $V' = V_1 \oplus V_2$ (orthogonal direct sum) by Lemma 2.9, we have $T_u\widetilde{M}'(u) = (F_1)_u \oplus (F_2)_u$ (orthogonal direct sum) for each $u \in \widetilde{M}$. Also, it follows from Lemma 2.7 that $T_u\widetilde{M}'(u) = (\widetilde{D}_1^T)_u \oplus (\widetilde{D}_2^T)_u$ (orthogonal direct sum) for each $u \in \widetilde{M}$. On the other hand, it follows from the proof of Lemma 2.11 that $(F_i)_u \subset (\widetilde{D}_i^T)_u$ ($u \in \widetilde{M}$, $i = 1, 2$). These facts imply $F_i = \widetilde{D}_i^T$ ($i = 1, 2$). Hence the statement of this lemma follows. $\square$

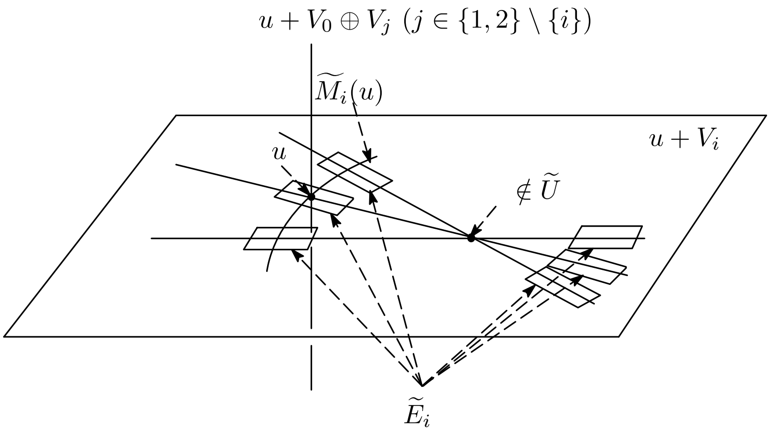

FIG. 3.

By using Lemma 2.11, we can show the following fact.

LEMMA 2.13. *For any two points* u_1 *and* u_2 *of* $\widetilde{M}'$, $\widetilde{M}_1(u_1)$ *intersects with* $\widetilde{M}_2(u_2)$.

Proof. Denote by $\mathfrak{F}_1$ the foliation on $\widetilde{M}'$ consisting of the integral manifolds of $F_1|_{\widetilde{M}'}$. Since $\mathfrak{F}_1$ is totally geodesic by Lemma 2.11 and the induced metric on each leaf of $\mathfrak{F}_1$ is complete, $F_2|_{\widetilde{M}'}$ is an Ehresmann connection for $\mathfrak{F}_1$ in the sense of Blumenthal-Hebda and hence the statement of this lemma follows (see [BH]). $\square$

By using this lemma and imitating the proof of Corollary 3.11 of [HL], we can show the following fact.

LEMMA 2.14. *For any* $u_0 \in \widetilde{M}_i (= \widetilde{M}_i(\hat{0}))$, *the translation map* $f_{u_0} : V' \to V'$ *defined by* $f_{u_0}(u) := u + u_0$ ($u \in V'$) *maps* $\widetilde{M}_j (= \widetilde{M}_j(\hat{0}))$ *isometrically onto* $\widetilde{M}_j(u_0)$, *where* $(i, j) = (1, 2)$ *or* $(2, 1)$.

By using this lemma and imitating the proof of Corollary 3.12 of [HL], we can show the following fact.

PROPOSITION 2.15. *We have* $\widetilde{M}' = \widetilde{M}_1 \times \widetilde{M}_2 \subset V_1 \times V_2 = V'$.

Define ideals $\mathfrak{g}'$ and $\mathfrak{g}_i$ $(i = 1, 2)$ by

$$\mathfrak{g}' := \mathrm{Span} \bigcup_{x^* \in M^*} \{g_{0*} v(x^*)_*^{-1} g_{0*}^{-1} \mid v \in T^\perp_{x^*} M^*,\ g_0 \in G\},$$
$$\mathfrak{g}_i := \mathrm{Span} \bigcup_{x^* \in M^*} \{g_{0*} v(x^*)_*^{-1} g_{0*}^{-1} \mid v \in ((D_i^N)_{\pi(x^*)})^*_{x^*},\ g_0 \in G\}.$$

Also, set $\mathfrak{g}_0 := \mathfrak{g} \ominus \mathfrak{g}'$, which is also an ideal of $\mathfrak{g}$. Let G' and G_i $(i = 0, 1, 2)$ be the connected Lie subgroups of G whose Lie algebras are $\mathfrak{g}'$ and $\mathfrak{g}_i$ $(i = 0, 1, 2)$, respectively. Since G/K is simply connected, we may assume that G is simply connected. So we have $G = G' \times G_0$ and $G' = G_1 \times G_2$. By imitating the proof of Lemma 5.1 of [Koi4], we can show the following fact.

LEMMA 2.16. *We have* $V' \subset H^0([0, 1], \mathfrak{g}')$ *and* $V_i \subset H^0([0, 1], \mathfrak{g}_i)$ $(i = 1, 2)$.

Also, by using Lemma 2.9 and imitating the proof of Lemma 3.7 of [E1], we can show the following fact.

LEMMA 2.17. *We have* $\mathfrak{g}_1 \perp \mathfrak{g}_2$ *and hence* $H^0([0, 1], \mathfrak{g}') = H^0([0, 1], \mathfrak{g}_1) \oplus H^0([0, 1], \mathfrak{g}_2)$ (*orthogonal direct sum*).

Let $V_0' := H^0([0, 1], \mathfrak{g}') \ominus V'$ and $V_{i,0} := H^0([0, 1], \mathfrak{g}_i) \ominus V_i$ $(i = 1, 2)$. Clearly we have $V_0' = V_{1,0} \oplus V_{2,0}$. Set $\widetilde{M}'_{H^0} := \widetilde{M} \cap H^0([0, 1], \mathfrak{g}')$ and $\widetilde{M}_{i,H^0} := \widetilde{M} \cap H^0([0, 1], \mathfrak{g}_i)$ $(i = 1, 2)$. It follows from Proposition 2.1 that $\widetilde{M}'_{H^0} = \widetilde{M}' \times V_0'$ and $\widetilde{M}_{i,H^0} = \widetilde{M}_i \times V_{i,0}$ $(i = 1, 2)$. Furthermore, it follows from Proposition 2.15 that $\widetilde{M} = \widetilde{M}_{1,H^0} \times \widetilde{M}_{2,H^0} \times H^0([0, 1], \mathfrak{g}_0)$. It is clear that the parallel transport map ϕ for G is decomposed as $\phi = \phi_1 \times \phi_2 \times \phi_0$, where ϕ_i $(i = 0, 1, 2)$ is the parallel transport map for G_i. Set $M^*_{i,H^0} := \phi_i(\widetilde{M}_{i,H^0})$ $(i = 1, 2)$. Clearly we have $M^* = M^*_{1,H^0} \times M^*_{2,H^0} \times G_0 \subset G_1 \times G_2 \times G_0 = G$. Let $(\mathfrak{g}, \theta)$ be the orthogonal symmetric Lie algebra of G/K. By imitating the discussion in Section 4 of [E1], we can show the following fact.

LEMMA 2.18. *We have* $\theta(\mathfrak{g}_i) = \mathfrak{g}_i$ $(i = 0, 1, 2)$.

Let $\mathfrak{f}_i := \mathrm{Fix}(\theta|_{\mathfrak{g}_i})$ and $K_i := \exp_{G_i}(\mathfrak{f}_i)$, where $i = 0, 1, 2$. Since G/K is simply connected, we have $G/K = G_1/K_1 \times G_2/K_2 \times G_0/K_0$. Denote by π_i the natural projection of G_i onto G_i/K_i $(i = 0, 1, 2)$. Let $M_{i,H^0} := \pi_i(M^*_{i,H^0})$ $(i = 1, 2)$. Now we prove Theorem A.

Proof of Theorem A. Assume that the holonomy group of the section Σ is reducible. Then, under the above notations, we have $M = M_{1,H^0} \times M_{2,H^0} \times G_0/K_0 \subset G_1/K_1 \times G_2/K_2 \times G_0/K_0 = G/K$. Let $\mathfrak{t} := T^\perp_{eK} M$ and $\mathfrak{t}_i$ $(i = 1, 2)$ be the normal space of M_{i,H^0} in G_i/K_i. Since M is equifocal, $\mathfrak{t}$ is a Lie triple system. Hence it follows that $\mathfrak{t}_i$ $(i = 1, 2)$ are Lie triple systems. This fact implies that M_{i,H^0} $(i = 1, 2)$ have Lie triple systematic normal bundle. On the other hand, it is clear that M_{i,H^0}

$(i=1,2)$ satisfy the conditions (PF-i) and (PF-ii). Thus M_{i,H^0} $(i=1,2)$ is equifocal. The converse is trivial. □

3. Proof of Theorems B and C. In this section, we shall prove Theorems B and C in terms of Theorem A. For its purpose, we first show the following fact.

Proof of Theorem B. Let Σ be the section of M through $x_0 = g_0K\,(\in M)$ and $\pi_\Sigma : \widehat{\Sigma} \to \Sigma$ be the universal covering of Σ. Since G/K is irreducible, it follows from Theorem A that the holonomy group of Σ is irreducible, that is, $\widehat{\Sigma}$ is irreducible. Since Σ is totally geodesic in G/K, it is a symmetric space. Hence $\widehat{\Sigma}$ is an irreducible simply connected symmetric space. On the other hand, according to Lemma 1A.4 of [PoTh1], Σ and hence $\widehat{\Sigma}$ admit a totally geodesic hypersurface. Hence, it follows from the result in [CN] that $\widehat{\Sigma}$ is isometric to a sphere, that is, Σ is isometric to a sphere or a real projective space (of constant curvature). □

We prepare the following lemma to prove Theorem C.

LEMMA 3.1. *Let Σ be a totally geodesic submanifold of positive constant curvature in a symmetric space G/K of compact type. Take a unit tangent vector v of Σ at gK and let $T_{eK}G/K = \mathfrak{a}_v + \sum\limits_{\alpha\in\triangle_+^v} \mathfrak{p}_\alpha^v$ be the root space decomposition with respect to a maximal abelian subspace $\mathfrak{a}_v$ (equipped with a lexicographical ordering) containing $g_*^{-1}v$. Then we can express as $T_{gK}\Sigma = \mathrm{Span}\{g_*w_\alpha\} + T_{gK}\Sigma \cap g_*\mathfrak{p}_\alpha^v$ for some $\alpha \in \triangle_+^v$, where w_α is the vector of $\mathfrak{a}_v$ defined by $\alpha(\cdot) = \langle w_\alpha, \cdot\rangle$.*

Proof. Let $\mathfrak{t} := g_*^{-1}T_{gK}\Sigma$. Since the tangent bundle of Σ is Lie triple systematic and hence root decomposable in the sense of [Koi1], we can express as $\mathfrak{t} = \mathrm{Span}\{g_*^{-1}v\} + \sum\limits_{\alpha\in(\triangle_+^v)'} (\mathfrak{t}\cap\mathfrak{p}_\alpha^v)$ for some $(\triangle_+^v)' \subset \triangle_+^v$, where $\mathfrak{t}\cap\mathfrak{p}_\alpha \neq \{0\}$ $(\alpha \in (\triangle_+^v)')$. Denote by κ the positive constant curvature of Σ. We have $\alpha(g_*^{-1}v)^2 = \kappa$ for any $\alpha \in (\triangle_+^v)'$. Fix $\alpha \in (\triangle_+^v)'$. Since $\mathfrak{t}$ is a Lie triple system, we have $[[g_*^{-1}v, \mathfrak{t}\cap\mathfrak{p}_\alpha^v], \mathfrak{t}\cap\mathfrak{p}_\alpha^v] \subset \mathfrak{t}$. On the other hand, we have $[[g_*^{-1}v, \mathfrak{t}\cap\mathfrak{p}_\alpha^v], \mathfrak{t}\cap\mathfrak{p}_\alpha^v] \subset [[\mathfrak{a}_v, \mathfrak{p}_\alpha^v], \mathfrak{p}_\alpha^v] \subset \mathfrak{p}_{2\alpha}^v + \mathrm{Span}\{w_\alpha\}$. Since $(2\alpha(g_*^{-1}v))^2 \neq \kappa$, we have $2\alpha \notin (\triangle_+^v)'$. Hence we have $[[g_*^{-1}v, \mathfrak{t}\cap\mathfrak{p}_\alpha^v], \mathfrak{t}\cap\mathfrak{p}_\alpha^v] \subset \mathfrak{t}\cap\mathrm{Span}\{w_\alpha\}$, which implies together with $g_*^{-1}w_\alpha \in [[g_*^{-1}v, \mathfrak{t}\cap\mathfrak{p}_\alpha^v], \mathfrak{t}\cap\mathfrak{p}_\alpha^v]$ and $\mathfrak{t}\cap\mathfrak{a}_v = \mathrm{Span}\{g_*^{-1}v\}$ that $g_*^{-1}v = \pm\frac{w_\alpha}{||w_\alpha||}$. Since this relation holds for every $\alpha \in (\triangle_+^v)'$, we see that $(\triangle_+^v)'$ is a one-point set. Let $(\triangle_+^v)' = \{\alpha_0\}$. Then we have $\mathfrak{t} = \mathrm{Span}\{w_{\alpha_0}\} + \mathfrak{t}\cap\mathfrak{p}_{\alpha_0}^v$, that is, $T_{gK}\Sigma = \mathrm{Span}\{g_*w_{\alpha_0}\} + T_{gK}\Sigma\cap g_*\mathfrak{p}_{\alpha_0}^v$.
□

Now we shall prove Theorem C in terms of Theorem B and this lemma.

Proof of Theorem C. Let M be an equifocal submanifold with non-flat section in an irreducible simply connected symmetric space G/K of compact type other than spheres. Let Σ be the section of M through $x = gK\,(\in M)$. According to Theorem B, the section Σ is isometric to a sphere or a real projective space. Take an arbitrary unit normal vector v of M at x, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} := T_{eK}G/K$ $(\subset \mathfrak{g} := \mathrm{Lie}\,G)$ containing $g_*^{-1}v$ and let $\mathfrak{p} = \mathfrak{a}_v + \sum\limits_{\alpha\in\triangle_+^v} \mathfrak{p}_\alpha^v$ be the root space decomposition with respect to $\mathfrak{a}_v$ (equipped with a lexicographical ordering). Since Σ is totally geodesic and of positive constant curvature, it follows from Lemma 3.1 that $v = \pm\frac{g_* w_{\alpha_0}}{||w_{\alpha_0}||}$ and $T_x\Sigma = \mathrm{Span}\{g_* w_{\alpha_0}\} + T_x\Sigma \cap g_*\mathfrak{p}^v_{\alpha_0}$ for some $\alpha_0 \in \triangle^v_+$. Therefore we have $\mathrm{codim}\, M \leq m_{G/K} + 1$. Assume that the root system of G/K is reduced. Let L be the focal submanifold of M through some focal point $p(= \bar{g}K)$ of (M, x). Take arbitrary X and $Y \in T^\perp_p L$. Since the slice action of L at p (which is the action on $T^\perp_p L$) is variationally complete, there exists a sequence $\{\widetilde{\Sigma}_1, \cdots, \widetilde{\Sigma}_k\}$ of the sections of the action such that $X \in \widetilde{\Sigma}_1$, $Y \in \widetilde{\Sigma}_k$ and $\dim(\widetilde{\Sigma}_i \cap \widetilde{\Sigma}_{i+1}) \geq 1$ $(i = 1, \cdots, k-1)$ (see [BS]). Let $\Sigma_i := \exp_p(\widetilde{\Sigma}_i)$ $(i = 1, \cdots, k)$, which are sections of M. Take $v_i(\neq 0) \in T_p(\Sigma_i \cap \Sigma_{i+1})$. According to Lemma 3.1, we have $T_p\Sigma_i \subset \mathrm{Span}\{v_i\} + \bar{g}_*\mathfrak{p}^{v_i}_{\alpha_i}$, $T_p\Sigma_{i+1} \subset \mathrm{Span}\{v_i\} + \bar{g}_*\mathfrak{p}^{v_i}_{\beta_i}$ and $\mathrm{Span}\{\bar{g}_* w_{\alpha_i}\} = \mathrm{Span}\{\bar{g}_* w_{\beta_i}\} = \mathrm{Span}\{v_i\}$ $(\bar{g}K = p)$, where α_i and β_i are roots of the positive root system $\triangle^{v_i}_+$ with respect to a maximal abelian subspace $\mathfrak{a}_{v_i}$ containing $\bar{g}_*^{-1} v_i$, w_{α_i}, w_{β_i}, $\mathfrak{p}^{v_i}_{\alpha_i}$ and $\mathfrak{p}^{v_i}_{\beta_i}$ are as in Lemma 3.1. Since $\triangle^{v_i}$ is reduced by the assumption, it follows from $\mathrm{Span}\{\bar{g}_* w_{\alpha_i}\} = \mathrm{Span}\{\bar{g}_* w_{\beta_i}\}$ that $\alpha_i = \beta_i$. Set $\mathfrak{t}_i := \mathrm{Span}\{w_{\alpha_i}\} + \mathfrak{p}^{v_i}_{\alpha_i}$, which is a Lie triple system. Set $T_i := \exp_p(\bar{g}_*\mathfrak{t}_i)$. Since $T_p\Sigma_i \cup T_p\Sigma_{i+1} \subset \bar{g}_*\mathfrak{t}_i$ by $\alpha_i = \beta_i$, we have $\Sigma_i \cup \Sigma_{i+1} \subset T_i$ and hence $T_1 = \cdots = T_{k-1}$, that is, $\mathfrak{t}_1 = \cdots = \mathfrak{t}_{k-1}$. Hence we have $X, Y \in \bar{g}_*\mathfrak{t}_1$. It follows from the arbitrarinesses of X and Y that $T^\perp_p L \subset \bar{g}_*\mathfrak{t}_1$. Therefore we have $\dim\Sigma < \dim T^\perp_p L \leq \dim \mathfrak{t}_1 \leq m_{G/K} + 1$, that is, $\mathrm{codim}\, M \leq m_{G/K}$. This completes the proof of the statement (i). Next we shall show the statement (ii). Assume that M is curvature-adapted. Let $\{\lambda_i \,|\, i \in I\}$ be the spectrum of A_v. Since M is curvature-adapted, we have

$$T_x M = \left(\sum_{\alpha \in \triangle^v_+ \cup \{0\}} \left(\bigoplus_{i \in I_{\alpha,v}} \left(\left(\bigoplus_{\beta \in \triangle^v_\alpha} g_*\mathfrak{p}^v_\beta \right) \cap \mathrm{Ker}(A_v - \lambda_i \cdot \mathrm{id}) \right) \right) \right),$$

where $\triangle^v_\alpha := \{\beta \in \triangle^v_+ \cup \{0\} \,|\, \beta(g_*^{-1}v) = \alpha(g_*^{-1}v)\}$, $I_{\alpha,v} := \{i \in I \,|\, (\bigoplus_{\beta \in \triangle^v_\alpha} g_*\mathfrak{p}^v_\beta)$ $\cap \mathrm{Ker}(A_v - \lambda_i \cdot \mathrm{id}) \neq \{0\}$.Note that $\mathfrak{p}^v_0 = \mathfrak{a}_v$. Then the set of all focal radii along γ_v is given by

$$\begin{aligned} &\left(\bigcup_{\alpha \in \triangle^v_+ \setminus \triangle^v_0} \{ \frac{1}{\alpha(g_*^{-1}v)} (\arctan \frac{\alpha(g_*^{-1}v)}{\lambda_i} + j\pi) \,|\, i \in I_{\alpha,v},\, j \in \mathbf{Z}\} \right) \\ &\cup \left(\bigcup_{\alpha \in \triangle^v_0} \{ \frac{1}{\lambda_i} \,|\, i \in I_{\alpha,v}\} \right) \end{aligned} \tag{3.1}$$

(see Theorem 3.3 of [Koi2]), where $\arctan \frac{\alpha(g_*^{-1}v)}{\lambda_i}$ implies $\frac{\pi}{2}$ when $\lambda_i = 0$. On the other hand, Σ is isometric to a sphere or a real projective space of constant curvature $\alpha_0(g_*^{-1}v)^2$ and the focal set of (M, x) consists of finitely many totally geodesic hypersurfaces in Σ by Lemma 1A.4 of [PoTh1]. By using these facts, we shall show that $\alpha(g_*^{-1}v) \in \{\frac{\alpha_0(g_*^{-1}v)}{k} \,|\, k \in \mathbf{Z} \setminus \{0\}\}$ for each $\alpha \in \triangle^v_+ \setminus (\triangle^v_0 \cup \{\alpha_0\})$. Suppose that $\beta(g_*^{-1}v) \notin \{\frac{\alpha_0(g_*^{-1}v)}{k} \,|\, k \in \mathbf{Z} \setminus \{0\}\}$ for some $\beta \in \triangle^v_+ \setminus (\triangle^v_0 \cup \{\alpha_0\})$. Fix $i_0 \in I_{\beta,v}$. Set $p_1 := \exp^\perp \left(\frac{1}{\beta(g_*^{-1}v)} \arctan \frac{\beta(g_*^{-1}v)}{\lambda_{i_0}} \cdot v \right)$ and $p_2 := \exp^\perp \left(\frac{1}{\beta(g_*^{-1}v)} (\arctan \frac{\beta(g_*^{-1}v)}{\lambda_{i_0}} + \pi) v \right)$. From $\beta(g_*^{-1}v) \notin \{\frac{\alpha_0(g_*^{-1}v)}{k} \,|\, k \in \mathbf{Z} \setminus \{0\}\}$, these focal points p_1 and p_2 belong to mutually distinct focal totally geodesic hypersurfaces S_1 and S_2, respectively. Take a unit normal vector w of M at x which is orthogonal to v. Let $v_\theta := (\cos\theta)v + (\sin\theta)w$ and $\{\lambda^\theta_i \,|\, i \in I^\theta\}$ be the spectrum of A_{v_θ}. Also, let c_i $(i = 1, 2)$ be the (C^∞-)curve in S_i such that $c_i(0) = p_i$

and that, for each θ, $c_i(\theta)$ is an intersection point of S_i with the geodesic γ_{v_θ} satisfying $\dot\gamma_{v_\theta}(0) = v_\theta$. For each θ, we can express as

$$c_1(\theta) = \exp^\perp\left(\frac{1}{\beta_\theta(g_*^{-1}v_\theta)}\arctan\frac{\beta_\theta(g_*^{-1}v_\theta)}{\lambda^\theta_{i_\theta}}\cdot v_\theta\right)\ (\in S_1)$$

and

$$c_2(\theta) = \exp^\perp\left(\frac{1}{\beta_\theta(g_*^{-1}v_\theta)}\left(\arctan\frac{\beta_\theta(g_*^{-1}v_\theta)}{\lambda^\theta_{i_\theta}}+\pi\right)v_\theta\right)\ (\in S_2)$$

in terms of some positive root β_θ with respect to a maximal abelian subspace $\mathfrak{a}_\theta$ containing $g_*^{-1}v_\theta$ and some $i_\theta \in I^\theta_{\beta_\theta}$, where $I^\theta_{\beta_\theta}$ is defined in similar to I_β.

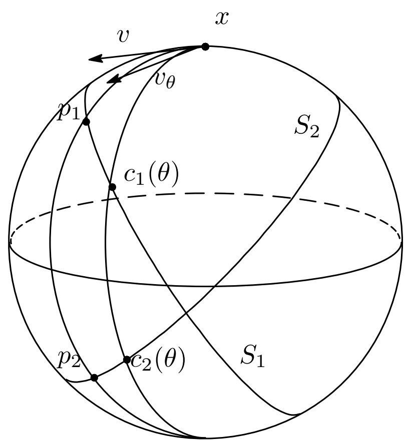

FIG. 4.

The length l_θ of the arc $\widehat{c_1(\theta)c_2(\theta)}$ is equal to $\frac{\pi}{\beta_\theta(g_*^{-1}v_\theta)}$. We may assume that l_θ is not constant by retaking w if necessary. Since l_θ variates continuously with respect to θ, we have $\frac{\beta_{\theta_0}(g_*^{-1}v_{\theta_0})}{\alpha_0(g_*^{-1}v)} \in \mathbf{R}\setminus\mathbf{Q}$ for some θ_0. Then it is shown that the set of all focal points along $\gamma_{v_{\theta_0}}$ is dense in the closed geodesic $\gamma_{v_{\theta_0}}([0, \frac{j\pi}{\alpha_0(g_*^{-1}v)}])$ in Σ, where $j = 1$ when Σ is a real projective space and $j = 2$ when Σ is a sphere. This contradicts the fact that the focal set of (M, x) consists of finitely many totally geodesic hypersurfaces. Therefore, we have $\alpha(g_*^{-1}v) \in \{\frac{\alpha_0(g_*^{-1}v)}{k} \mid k \in \mathbf{Z}\setminus\{0\}\}$ for each $\alpha \in \triangle^v_+ \setminus (\triangle^v_0 \cup \{\alpha_0\})$. Thus we have $\alpha(g_*^{-1}v)^2 \le \alpha_0(g_*^{-1}v)^2$, that is, $|\langle\alpha_0, \alpha\rangle| \le \langle\alpha_0, \alpha_0\rangle$ for any $\alpha \in \triangle^v_+$. Hence we have $2\alpha_0 \notin \triangle^v_+$. By using this fact and imitating the above argument, we can show $T^\perp_p L \subset \mathrm{Span}\{\bar g_* w_{\alpha_1}\} + \bar g_*\mathfrak{p}^{v_1}_{\alpha_1}$ ($\bar gK = p$), where v_1, w_{α_1} and $\mathfrak{p}^{v_1}_{\alpha_1}$ are as above. Since both $\exp_p\left(\mathrm{Span}\{\bar g_* w_{\alpha_1}\} + \bar g_*\mathfrak{p}^{v_1}_{\alpha_1}\right)$ and $\exp_x\left(\mathrm{Span}\{g_* w_{\alpha_0}\} + g_*\mathfrak{p}^{v}_{\alpha_0}\right)$ include Σ and $\dim\Sigma \ge 2$, they have the same constant curvature. Hence it follows from Lemma 3.2 that they coincide with each other. Hence we have $\dim\Sigma < \dim T^\perp_p L \le \dim\mathfrak{p}^v_{\alpha_0} + 1$, that is, $\mathrm{codim}\, M \le \dim\mathfrak{p}^v_{\alpha_0} \le m'_{G/K}$. Thus we obtain the statement (ii). □

Appendix. Polar actions on rank one symmetric spaces other than spheres and real projective spaces are classified by F. Podestà and G. Thorbergsson ([PoTh1]). For example, polar actions on the m-dimensional complex projective space $\mathbf{C}P^m$ are classified as follows.

THEOREM 4.1 ([PoTh1]). *Any (non-hyperpolar) polar action on* $\mathbf{C}P^m$ $(m \geq 2)$ *is orbit equivalent to the action on* $\mathbf{C}P^m$ *induced by the Hopf fibration* $\pi : S^{2m+1} \to \mathbf{C}P^m$ *from the linear isotropy action (which arises the action on a* $(2m+1)$*-dimensional sphere) of a* $2(m+1)$*-dimensional Hermitian symmetric space of rank greater than two.*

REMARK 4.1. In this theorem, if the Hermitian symmetric space is of rank r, then the corresponding polar action on the complex projective space is of cohomogeneity $r-1$.

According to (ii) of Theorem C, the principal orbits of the above polar actions on $\mathbf{C}P^m$ should not be curvature-adapted because $m'_{\mathbf{C}P^m} = 1$. We shall check this fact. Let G/K be a $2(m+1)$-dimensional Hermitian symmetric space of rank $r\,(\geq 3)$ and $\mathfrak{p} := T_{eK}G/K$. Let $\mathfrak{p} = \mathfrak{a} + \sum\limits_{\alpha\in\triangle_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to a maximal abelian subspace $\mathfrak{a}$ (equipped with a lexicographical ordering) of $\mathfrak{p}$ $(\subset \mathfrak{g})$. Let M be the principal orbit of the linear isotropy action of G/K through a W-regular point $a \in \mathfrak{a}$, where W is the Weyl group of G/K. The principal orbit M is contained in the hypersphere $S^{2m+1}(||a||)$ of radius $||a||$ centered at the origin of $\mathfrak{p}$. Let $\pi : S^{2m+1}(||a||) \to \mathbf{C}P^m$ be the Hopf fibration, where $\mathbf{C}P^m$ is of holomorphic sectional curvature $\frac{4}{||a||^2}$. Set $\overline{M} := \pi(M)$, which is a principal orbit of the induced K-action on $\mathbf{C}P^m$. We have $\pi^{-1}(\overline{M}) = M$. Denote by A (resp. $\overline{A}$) the shape tensor of $M\,(\subset S^{2m+1}(||a||))$ (resp. $\overline{M}\,(\subset \mathbf{C}P^m)$). Denote by J (resp. $\overline{J}$) the complex structure of G/K (resp. $\mathbf{C}P^m$). Let $\mathfrak{f}_0$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{f}\,(=\mathrm{Lie}\,K)$ and $\mathfrak{f}_\alpha := \{X \in \mathfrak{f}\,|\,\mathrm{ad}(H)^2X = -\alpha(H)^2X \text{ for all } H \in \mathfrak{a}\}$ $(\alpha \in \triangle_+)$. Let $\triangle'_+ := \{\alpha_1,\cdots,\alpha_r\}\,(\subset \triangle_+)$ be the set of the strongly orthogonal roots. There uniquely exists a central element Z of $\mathfrak{f}$ with $\mathrm{ad}(Z) = J_{eK}$. It is shown that $Z \in \mathfrak{f}_0 + \mathfrak{f}_{\alpha_1} + \cdots + \mathfrak{f}_{\alpha_r}$ and that the $\mathfrak{f}_{\alpha_i}$-component of Z $(i = 1,\cdots,r)$ does not vanish (see Proposition 3.10 of [KW] and the proof). Let $Z = \sum\limits_{i=0}^{r} Z_i$ $(Z_0 \in \mathfrak{f}_0,\ Z_i \in \mathfrak{f}_{\alpha_i})$. Take a unit normal vector v of $M\,(\subset S^{2m+1}(||\mathfrak{a}||))$ at a. We have $A_v|_{\mathfrak{p}_\alpha} = -\frac{\alpha(v)}{\alpha(a)}\mathrm{id}_{\mathfrak{p}_\alpha}$ $(\alpha \in \triangle_+)$ (see Proposition 5 of [TaTa]), where we note $T_aM = \sum\limits_{\alpha\in\triangle_+} \mathfrak{p}_\alpha$. Since $A_vJv = -\sum\limits_{i=1}^{r} \frac{\alpha_i(v)}{\alpha_i(a)}[Z_i, v]$ and $(A_vJv)_H$ is equal to the horizontal lift of $\overline{A}_{\pi_*v}\overline{J}\pi_*v$ to a, we have $\overline{A}_{\pi_*v}\overline{J}\pi_*v = -\sum\limits_{i=1}^{r} \frac{\alpha_i(v)}{\alpha_i(a)}\pi_*[Z_i, v]$. On the other hand, we have $\overline{J}\pi_*v = \sum\limits_{i=1}^{r} \pi_*[Z_i, v]$. From these relations, it follows that $\overline{A}_{\pi_*v}\overline{J}\pi_*v$ and $\overline{J}\pi_*v$ are linearly independent for almost all unit normal vectors v of M at a. This implies that $\overline{M}$ is not curvature-adapted.

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