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ON THE CREPANCY OF THE GIESEKER-UHLENBECK MORPHISM*

ZHENBO QIN† AND QI ZHANG‡

Abstract. The Gieseker-Uhlenbeck morphism from the moduli space of Gieseker semistable rank-2 sheaves over an algebraic surface to the Uhlenbeck compactification was constructed by Jun Li [Li1] (see also [Uhl, Mor]). We prove that if the anti-canonical divisor of the surface is effective and the first Chern class of the semistable sheaves is odd, then the Gieseker-Uhlenbeck morphism is crepant.

Key words. Gieseker stability, Uhlenbeck compactification, crepant

AMS subject classifications. Primary 14D20; Secondary 14D21, 14E05

1. Introduction. A well-known result of Donaldson [Don1] says that slope-stable rank-2 vector bundles over a complex algebraic surface are in one-to-one correspondence with irreducible anti-self-dual connections on certain principal bundles over the underlying smooth 4-manifold. The moduli space of these slope-stable rank-2 bundles has a natural compactification in algebraic geometry, namely, the moduli space of Gieseker semistable rank-2 sheaves. On the other hand, the moduli space of irreducible anti-self-dual connections has a natural compactification in gauge theory, namely, the Uhlenbeck compactification [Uhl]. J. Li [Li1] (see also [Mor]) showed that the Uhlenbeck compactification is a reduced projective scheme, and constructed a morphism from the Gieseker moduli space to the Uhlenbeck compactification. We define this morphism to be the *Gieseker-Uhlenbeck morphism.* The goal of this paper is to study the crepancy of this morphism. For our purpose, a birational morphism $f : Y_1 \to Y_2$ is *crepant* if Y_1 is normal, Y_2 is regular in codimension-1 and $\mathbb{Q}$-Gorenstein [KMM], and $K_{Y_1} = f^*K_{Y_2}$.

To state our result, let X be a surface with canonical class K_X. Fix a divisor c_1 and an ample divisor H on X, and fix an integer c_2. Let $\overline{\mathfrak{M}}_H(c_1, c_2)$ be the moduli space of Gieseker H-semistable rank-2 sheaves on X with Chern classes c_1 and c_2, and let $\overline{\mathfrak{U}}_H(c_1, c_2)$ be the corresponding Uhlenbeck compactification.

THEOREM 1.1. *Let X be a simply connected surface with $-K_X \geq 0$, and let H be an ample divisor with odd $(c_1 \cdot H)$. Assume that $\overline{\mathfrak{M}}_H(c_1, c_2)$ is non-empty. Then the Gieseker-Uhlenbeck morphism $\Psi_H : \overline{\mathfrak{M}}_H(c_1, c_2) \to \overline{\mathfrak{U}}_H(c_1, c_2)$ is crepant.*

Note that X is necessarily a rational surface or a $K3$ surface. The basic properties of the moduli space $\overline{\mathfrak{M}}_H(c_1, c_2)$ are summarized in Lemma 3.1. Our main idea to prove Theorem 1.1 is to show that when $\overline{\mathfrak{M}}_H(c_1, c_2 - 1)$ is non-empty, Ψ_H drops the Picard numbers by one, i.e., the Picard number of $\overline{\mathfrak{M}}_H(c_1, c_2)$ is one more than that of $\overline{\mathfrak{U}}_H(c_1, c_2)$. First of all, we see from [Li2] that the Picard number of $\overline{\mathfrak{M}}_H(c_1, c_2)$ is $1 + \rho$ where ρ denotes the Picard number of X. Next, notice that the birational morphism Ψ_H contracts the irreducible *boundary divisor* in $\overline{\mathfrak{M}}_H(c_1, c_2)$ which consists of non-locally free semistable sheaves. So the Picard number of $\overline{\mathfrak{U}}_H(c_1, c_2)$ is at most

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†Department of Mathematics, University of Missouri, Columbia, MO 65211, USA (zq@math.missouri.edu). Partially supported by an NSF grant.
‡Department of Mathematics, University of Missouri, Columbia, MO 65211, USA (qi@math.missouri.edu).

ρ. On the other hand, the results in [Li1] implies that the determinant line bundle constructed there are contained in

$$(\Psi_H)^*\text{Pic}\big(\overline{\mathfrak{U}}_H(c_1,c_2)\big) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Replacing H by other ample divisors sufficiently close to H, we obtain ρ linearly independent determinant line bundles. Hence the Picard number of $\overline{\mathfrak{U}}_H(c_1,c_2)$ is at least ρ. It follows that the Picard number of $\overline{\mathfrak{U}}_H(c_1,c_2)$ is precisely ρ.

We remark that Theorem 1.1 also follows directly from the Proposition 4.6 in [Li3]. Our alternative approach may be viewed as an elementary proof in the case that $-K_X \ge 0$. Results in this paper will be used in an upcoming joint work of Wei-Ping Li and the first named author, where all the extremal (with respect to the Gieseker-Uhlenbeck morphism Ψ_H) 1-point Gromov-Witten invariants of the moduli space $\overline{\mathfrak{M}}_H(c_1,c_2)$ with $X = \mathbb{P}^2$ have been computed.

Finally, we point out that the Gieseker-Uhlenbeck morphism is a natural generalization of the Hilbert-Chow morphism from the Hilbert scheme of points on a surface to the symmetric product of the surface. The Hilbert-Chow morphism and the Hilbert scheme have been studied intensively in recent years due to their elegant connections with string theory, representation theory and Ruan's Cohomological Resolution Conjecture (see [Nak, Gro, LQW, LL, Ruan] and the references there). It would be interesting to see whether these results could be extended to the Gieseker-Uhlenbeck morphism and the Gieseker moduli space. Indeed, a relation between the Gieseker moduli space and representation theory has been established in [Bar]. We plan to investigate the Gieseker-Uhlenbeck morphism and the Gieseker moduli space in more details in our future work.

Conventions. Throughout the paper, unless otherwise specified, (semi)stability means Gieseker (semi)stability. For a smooth variety, we make no distinctions between its divisors and the corresponding line bundles, and between its group of divisors modulo linear equivalence relation and its Picard group.

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2. Preliminaries.

2.1. The moduli space of Gieseker semistable sheaves. Let X be a smooth complex projective surface, and let H be an ample divisor on X. For a sheaf V on X, denote the Hilbert polynomial of V by

$$\chi_H(V;n) = \sum_i (-1)^i h^i(X, V \otimes \mathcal{O}_X(nH)). \tag{2.1}$$

A torsion free sheaf V on X is H-stable (resp. H-semistable) if

$$\frac{\chi_H(W;n)}{\text{rank}(W)} < \frac{\chi_H(V;n)}{\text{rank}(V)} \qquad (\text{resp. } \le)$$

for every proper subsheaf $W \subset V$ and $n \gg 0$. Fix a divisor c_1 on X and an integer c_2. Let $\overline{\mathfrak{M}}_H(c_1,c_2)$ be the moduli space of H-semistable rank-2 torsion free sheaves V with Chern classes c_1 and c_2, and let $\mathfrak{M}_H(c_1,c_2) \subset \overline{\mathfrak{M}}_H(c_1,c_2)$ be the open subset consisting of locally free sheaves in $\overline{\mathfrak{M}}_H(c_1,c_2)$. It is well-known that the moduli space $\overline{\mathfrak{M}}_H(c_1,c_2)$ is a projective scheme with the expected dimension

$$\mathfrak{d} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_X). \tag{2.2}$$

For fixed c_1 and H, it is proved in [Don2, Fri2, GL, Li3, O'G, Zuo] that if $c_2 \gg 0$, then $\overline{\mathfrak{M}}_H(c_1, c_2)$ is normal and irreducible with the expected dimension.

Let $\mathrm{Num}(X)$ be the group of divisors on X modulo numerical equivalence relation. Let $C_X \subset \mathrm{Num}(X) \otimes \mathbb{R}$ be the nef cone of X.

Definition 2.1. (see Definition 1.1.1 in [Qin]) Fix c_1 and c_2 as above.

(i) For $\alpha \in \mathrm{Num}(X) \otimes \mathbb{R}$, we define W^α to be the subset

$$C_X \cap \{\beta \in \mathrm{Num}(X) \otimes \mathbb{R} | \, \alpha \cdot \beta = 0\};$$

(ii) Define $W(c_1, c_2)$ to be the set of all the subsets W^α where α is the numerical equivalence class of a divisor of the form $(2F - c_1)$ such that

$$-(4c_2 - c_1^2) \leq \alpha^2 < 0;$$

(iii) A *chamber* $\mathcal{C}$ *of type* (c_1, c_2) is a connected component of the nef cone C_X cut out by all the elements $W^\alpha \in W(c_1, c_2)$.

Lemma 2.2. *Fix a divisor* c_1 *on* X *and an integer* c_2*.*

(i) If two ample divisors H *and* H' *are contained in the same chamber of type* (c_1, c_2)*, then* H*-(semi)stability coincides with* H'*-(semi)stability;*

(ii) If $(c_1 \cdot H)$ *is odd, then* H *is contained in certain chamber of type* (c_1, c_2)*; moreover, every sheaf* $V \in \overline{\mathfrak{M}}_H(c_1, c_2)$ *is* H*-stable (in fact,* H*-slope-stable), and the moduli space* $\overline{\mathfrak{M}}_H(c_1, c_2)$ *is a fine moduli space.*

Proof. (i) is the Theorem 1.3.3 in [Qin]. If $(c_1 \cdot H)$ is odd, then a standard argument shows that H is contained in certain chamber of type (c_1, c_2) and that every sheaf $V \in \overline{\mathfrak{M}}_H(c_1, c_2)$ is H-stable (in fact, H-slope-stable). Furthermore, by the Remark A.7 in [Muk], a universal sheaf over $\overline{\mathfrak{M}}_H(c_1, c_2) \times X$ exists. □

2.2. The Uhlenbeck compactification. Let $(c_1 \cdot H)$ be odd, and assume that the open subset $\mathfrak{M}_H(c_1, c_2)$ is dense in $\overline{\mathfrak{M}}_H(c_1, c_2)$. By Lemma 2.2 (ii), H-semistability implies H-stability. So the quasi-projective variety $\mathfrak{M}_H(c_1, c_2)$ has a Uhlenbeck compactification

$$\overline{\mathfrak{U}}_H(c_1, c_2) = \coprod_{i \geq 0} \mathfrak{M}_H(c_1, c_2 - i) \times \mathrm{Sym}^i(X) \tag{2.3}$$

according to [Uhl, Li1, Mor]. Moreover, J. Li constructed a birational morphism

$$\Psi_H : \overline{\mathfrak{M}}_H(c_1, c_2) \to \overline{\mathfrak{U}}_H(c_1, c_2) \tag{2.4}$$

sending $V \in \overline{\mathfrak{M}}_H(c_1, c_2)$ to the pair (V^{**}, η) where V^{**} is the double dual of V and

$$\eta = \sum_{x \in X} h^0\big(X, (V^{**}/V)_x\big) \, x.$$

It follows that the restriction $\Psi_H|_{\mathfrak{M}_H(c_1,c_2)}$ is the identity map on $\mathfrak{M}_H(c_1, c_2)$, and that the *boundary divisor* $\overline{\mathfrak{M}}_H(c_1, c_2) - \mathfrak{M}_H(c_1, c_2)$ is contracted by Ψ_H to the subscheme $\coprod_{i \geq 1} \mathfrak{M}_H(c_1, c_2 - i) \times \mathrm{Sym}^i(X)$ in $\overline{\mathfrak{U}}_H(c_1, c_2)$.

Definition 2.3. We define $\Psi := \Psi_H$ to be the *Gieseker-Uhlenbeck morphism.*

We outline the construction of Ψ and refer to [Li1] for details. Let k be an *even* integer sufficiently large, and let $C \in |kH|$ be an irreducible and smooth curve with genus g_C. Choose a line bundle $\tilde{\theta}_C$ on the curve C such that

$$\deg(\tilde{\theta}_C) = g_C - 1 - \frac{(c_1 \cdot C)}{2}. \tag{2.5}$$

By Lemma 2.2 (ii), a universal sheaf $\mathcal{V}$ over $\overline{\mathfrak{M}}_H(c_1, c_2) \times X$ exists. Let

$$\mathcal{L}(C, \tilde{\theta}_C) = \mathrm{Det}\big(R\tilde{\pi}_{1*}(\mathcal{V}|_{\overline{\mathfrak{M}}_H(c_1,c_2)\times C} \otimes \tilde{\pi}_2^*\tilde{\theta}_C)\big)^{-1} \tag{2.6}$$

where $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are the projections on $\overline{\mathfrak{M}}_H(c_1, c_2) \times C$. For $m \gg 0$, there exists a base-point-free linear series in $H^0\big(\overline{\mathfrak{M}}_H(c_1, c_2), \mathcal{L}(C, \tilde{\theta}_C)^{\otimes m}\big)$ which induces a morphism $\Psi : \overline{\mathfrak{M}}_H(c_1, c_2) \to \mathbb{P}^N$ for a suitable integer N. The image $\Psi(\overline{\mathfrak{M}}_H(c_1, c_2))$ is precisely the Uhlenbeck compactification $\overline{\mathfrak{U}}_H(c_1, c_2)$. It follows that

$$\mathcal{L}(C, \tilde{\theta}_C) \in \Psi^*\mathrm{Pic}\big(\overline{\mathfrak{U}}_H(c_1, c_2)\big) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.7}$$

3. The Gieseker-Uhlenbeck morphism is crepant. Throughout this section, we assume that X is a simply connected surface with effective anti-canonical divisor $-K_X$ and that $(c_1 \cdot H)$ is odd. So X is either a rational surface or a $K3$ surface. Our goal is to prove that the Gieseker-Uhlenbeck morphism $\Psi = \Psi_H : \overline{\mathfrak{M}}_H(c_1, c_2) \to \overline{\mathfrak{U}}_H(c_1, c_2)$ is crepant.

3.1. The Gieseker moduli space $\overline{\mathfrak{M}}_H(c_1, c_2)$. The moduli space $\overline{\mathfrak{M}}_H(c_1, c_2)$ has been studied extensively by various authors. We refer to the three books [OSS, Fri2, HL] for further references. The following summarizes some properties of $\overline{\mathfrak{M}}_H(c_1, c_2)$ relevant to us.

LEMMA 3.1. *Let X be simply connected with $-K_X \geq 0$, and let H be an ample divisor with odd $(c_1 \cdot H)$. Assume that $\overline{\mathfrak{M}}_H(c_1, c_2) \neq \emptyset$.*

(i) The moduli space $\overline{\mathfrak{M}}_H(c_1, c_2)$ is smooth, fine and irreducible with dimension

$$\mathfrak{d} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_X).$$

Moreover, the open subset $\mathfrak{M}_H(c_1, c_2)$ is dense in $\overline{\mathfrak{M}}_H(c_1, c_2)$;

(ii) If we further assume that $\overline{\mathfrak{M}}_H(c_1, c_2 - 1) \neq \emptyset$, then the Picard number of $\overline{\mathfrak{M}}_H(c_1, c_2)$ is one more than the Picard number of X.

Proof. (i) By Lemma 2.2 (ii), $\overline{\mathfrak{M}}_H(c_1, c_2)$ is a fine moduli space. The smoothness and dimension of $\overline{\mathfrak{M}}_H(c_1, c_2)$ can be found in [MaM]. The irreducibility follows from the Corollary 10 in [MaE]. To show that $\mathfrak{M}_H(c_1, c_2)$ is dense in the irreducible variety $\overline{\mathfrak{M}}_H(c_1, c_2)$, it suffices to prove that $\mathfrak{M}_H(c_1, c_2)$ is not empty. Let $V \in \overline{\mathfrak{M}}_H(c_1, c_2)$. Then the double dual V^{**} is stable. By the Corollary 1.5 in [Art], the sheaf V is smoothable. Hence $\mathfrak{M}_H(c_1, c_2)$ is not empty.

(ii) By the Theorem 3.8 [1] of [Li2], there exists a homomorphism

$$\overline{\Phi}: \ \mathrm{Pic}(X \times X)^\sigma \oplus \mathbb{Z} \to \mathrm{Pic}(\overline{\mathfrak{M}}_H(c_1, c_2)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{12}\right]$$

[1] More precisely, the proof of the Theorem 3.8 of [Li2] needs to be modified slightly since its statement is for $c_2 \gg 0$. For instance, we need to replace the bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ in the proof of the Proposition 2.1 of [Li2] by two bundles $\mathcal{E}_1 \in \mathfrak{M}_H(c_1, c_2)$ and $\mathcal{E}_2 \in \mathfrak{M}_H(c_1, c_2 - 1)$. Similar remarks apply when we use the results of [Li4] in (3.9).

which has finite kernel and co-kernel. Here $\sigma : X \times X \to X \times X$ is the automorphism exchanging the factors. Since X is simply connected, we have

$$\mathrm{Pic}(X \times X) = \pi_1^*\mathrm{Pic}(X) \oplus \pi_2^*\mathrm{Pic}(X) \tag{3.1}$$

where π_1, π_2 are the two projections on $X \times X$. It follows that

$$\mathrm{Pic}(X \times X)^\sigma \cong \mathrm{Pic}(X).$$

Therefore, the Picard number of $\overline{\mathfrak{M}}_H(c_1, c_2)$ is one more than that of X. □

3.2. The boundary of the moduli space $\overline{\mathfrak{M}}_H(c_1, c_2)$. In this subsection, we study the boundary of the moduli space $\overline{\mathfrak{M}}_H(c_1, c_2)$, i.e., the subset consisting of all the non-locally free sheaves in $\overline{\mathfrak{M}}_H(c_1, c_2)$. Note that the non-emptiness of the moduli space $\overline{\mathfrak{M}}_H(c_1, c_2 - 1)$ implies the non-emptiness of $\overline{\mathfrak{M}}_H(c_1, c_2)$. Assume that $\overline{\mathfrak{M}}_H(c_1, c_2 - 1)$ is nonempty. Let

$$\mathfrak{B} = \overline{\mathfrak{M}}_H(c_1, c_2) - \mathfrak{M}_H(c_1, c_2) \tag{3.2}$$

be the boundary. Recall the Gieseker-Uhlenbeck morphism Ψ from (2.4). Put

$$\mathfrak{B}_* = \Psi^{-1}(\mathfrak{M}_H(c_1, c_2 - 1) \times X) \subset \overline{\mathfrak{M}}_H(c_1, c_2).$$

Then, $\mathfrak{B}_*$ is an open and dense subset of the boundary divisor $\mathfrak{B}$. Also, $\mathfrak{B}_*$ parametrizes all the sheaves $V \in \overline{\mathfrak{M}}_H(c_1, c_2)$ sitting in exact sequences of the form:

$$0 \to V \to V_1 \to \mathcal{O}_x \to 0$$

for some bundle $V_1 \in \mathfrak{M}_H(c_1, c_2 - 1)$ and some point $x \in X$. To give a global description of $\mathfrak{B}_*$, take a universal sheaf $\mathcal{V}_1^0$ over $\mathfrak{M}_H(c_1, c_2 - 1) \times X$. Let

$$\mathbb{P}_* = \mathbb{P}(\mathcal{V}_1^0),$$

and let $\mathcal{O}_{\mathbb{P}_*}(1)$ be the tautological line bundle over $\mathbb{P}_*$. Then there is a surjection:

$$\pi^*\mathcal{V}_1^0 \to \mathcal{O}_{\mathbb{P}_*}(1) \to 0$$

where $\pi : \mathbb{P}_* = \mathbb{P}(\mathcal{V}_1^0) \to \mathfrak{M}_H(c_1, c_2 - 1) \times X$ is the natural projection. Consider

$$\pi \times \mathrm{Id}_X : \ \mathbb{P}_* \times X \ \to \ \mathfrak{M}_H(c_1, c_2 - 1) \times X \times X.$$

Let Δ_X be the diagonal of $X \times X$, and $\alpha : \mathfrak{M}_H(c_1, c_2 - 1) \times \Delta_X \to \mathfrak{M}_H(c_1, c_2 - 1) \times X$ be the obvious isomorphism. Then, we have the isomorphisms:

$$\begin{aligned} (\pi \times \mathrm{Id}_X)^{-1}(\mathfrak{M}_H(c_1, c_2 - 1) \times \Delta_X) &\cong \mathbb{P}(\alpha^*\mathcal{V}_1^0), \\ \tilde{\pi}^*\mathcal{V}_1^0|_{(\pi \times \mathrm{Id}_X)^{-1}(\mathfrak{M}_H(c_1, c_2 - 1) \times \Delta_X)} &\cong \tilde{\alpha}^*\mathcal{V}_1^0 \end{aligned} \tag{3.3}$$

where $\tilde{\pi} : \mathbb{P}_* \times X \to \mathfrak{M}_H(c_1, c_2 - 1) \times X$ is the composition of $\pi \times \mathrm{Id}_X$ and

$$\mathfrak{M}_H(c_1, c_2 - 1) \times X \times X \to \mathfrak{M}_H(c_1, c_2 - 1) \times X$$

which denotes the projection to the product of the first and third factors in $\mathfrak{M}_H(c_1, c_2 - 1) \times X \times X$, and $\tilde{\alpha} : \mathbb{P}(\alpha^*\mathcal{V}_1^0) \to \mathfrak{M}_H(c_1, c_2 - 1) \times X$ is the composition of the natural projection $\mathbb{P}(\alpha^*\mathcal{V}_1^0) \to \mathfrak{M}_H(c_1, c_2 - 1) \times \Delta_X$ and α.

Combining the isomorphism (3.3) with the canonical surjection

$$\tilde{\alpha}^*\mathcal{V}_1^0 \to \mathcal{O}_{(\pi\times \mathrm{Id}_X)^{-1}(\mathfrak{M}_H(c_1,c_2-1)\times\Delta_X)}(1) \to 0$$

over $(\pi\times\mathrm{Id}_X)^{-1}(\mathfrak{M}_H(c_1,c_2-1)\times\Delta_X)$, we obtain a surjection over $\mathbb{P}_*\times X$:

$$\tilde{\pi}^*\mathcal{V}_1^0 \to \mathcal{O}_{(\pi\times \mathrm{Id}_X)^{-1}(\mathfrak{M}_H(c_1,c_2-1)\times\Delta_X)}(1) \to 0.$$

Let $\mathcal{V}'$ be the kernel. Then $\mathcal{V}'$ is flat over $\mathbb{P}_*$, and we have an exact sequence

$$0 \to \mathcal{V}' \to \tilde{\pi}^*\mathcal{V}_1^0 \to \mathcal{O}_{(\pi\times \mathrm{Id}_X)^{-1}(\mathfrak{M}_H(c_1,c_2-1)\times\Delta_X)}(1) \to 0. \tag{3.4}$$

By the universal property of $\overline{\mathfrak{M}}_H(c_1,c_2)$, the sheaf $\mathcal{V}'$ induces a morphism

$$\mathbb{P}_* \to \overline{\mathfrak{M}}_H(c_1,c_2)$$

which is injective with image $\mathfrak{B}_*$. Since both $\mathfrak{B}_*$ and $\mathbb{P}_*$ are smooth, $\mathfrak{B}_*\cong\mathbb{P}_*$ by the Zariski's Main Theorem. For simplicity, we just write $\mathfrak{B}_*=\mathbb{P}_*$. Hence,

$$\mathfrak{B}_* = \mathbb{P}_* = \mathbb{P}(\mathcal{V}_1^0). \tag{3.5}$$

Lemma 3.2. *Assume that the moduli space $\overline{\mathfrak{M}}_H(c_1,c_2-1)$ is non-empty. Let $\mathfrak{d}=4c_2-c_1^2-3\chi(\mathcal{O}_X)$. Let $\mathfrak{f}\cong\mathbb{P}^1$ be a fiber of the natural projection*

$$\pi : \mathfrak{B}_* = \mathbb{P}_* = \mathbb{P}(\mathcal{V}_1^0) \to \mathfrak{M}_H(c_1,c_2-1)\times X,$$

and let $N := N_{\mathfrak{f}\subset\overline{\mathfrak{M}}_H(c_1,c_2)}$ be the normal bundle of $\mathfrak{f}$ in $\overline{\mathfrak{M}}_H(c_1,c_2)$. Then,

(i) $\mathfrak{f}\cdot K_{\overline{\mathfrak{M}}_H(c_1,c_2)}=0$;

(ii) $\mathfrak{f}\cdot\mathfrak{B}=-2$ and $N\cong\mathcal{O}_\mathfrak{f}^{\oplus(\mathfrak{d}-2)}\oplus\mathcal{O}_\mathfrak{f}(-2)$;

(iii) $T_{\overline{\mathfrak{M}}_H(c_1,c_2)}|_\mathfrak{f}\cong\mathcal{O}_\mathfrak{f}^{\oplus(\mathfrak{d}-2)}\oplus\mathcal{O}_\mathfrak{f}(-2)\oplus\mathcal{O}_\mathfrak{f}(2)$.

Proof. (i) Assume that $\mathfrak{f}=\pi^{-1}(V_1,x)$ where $(V_1,x)\in\mathfrak{M}_H(c_1,c_2-1)\times X$. Let $\mathcal{V}_\mathfrak{f}=\mathcal{V}'|_{\mathfrak{f}\times X}$. Restricting (3.4) to $\mathfrak{f}\times X$ yields the exact sequence

$$0\to\mathcal{V}_\mathfrak{f}\to\pi_2^*V_1\to\pi_1^*\mathcal{O}_\mathfrak{f}(1)|_{\mathfrak{f}\times\{x\}}\to 0 \tag{3.6}$$

where π_1 and π_2 are the projections on $\mathfrak{f}\times X$. Since H-semistability coincides with H-stability and $H^1(X,\mathcal{O}_X)=0$, the tangent sheaf of $\overline{\mathfrak{M}}_H(c_1,c_2)$ is isomorphic to $\mathcal{E}xt^1_{\pi_1}(\mathcal{V},\mathcal{V})$ where $\mathcal{V}$ denotes a universal sheaf over $\overline{\mathfrak{M}}_H(c_1,c_2)\times X$. Hence

$$\mathfrak{f}\cdot K_{\overline{\mathfrak{M}}_H(c_1,c_2)} = -c_1\left(\mathcal{E}xt^1_{\pi_1}(\mathcal{V}_\mathfrak{f},\mathcal{V}_\mathfrak{f})\right)\in A^1(\mathfrak{f})\cong\mathbb{Z}.$$

Note also that $\mathcal{E}xt^0_{\pi_1}(\mathcal{V}_\mathfrak{f},\mathcal{V}_\mathfrak{f})\cong\mathcal{O}_\mathfrak{f}$, and $\mathcal{E}xt^2_{\pi_1}(\mathcal{V}_\mathfrak{f},\mathcal{V}_\mathfrak{f})=0$ or $\mathcal{O}_\mathfrak{f}$. Therefore,

$$\begin{aligned}\mathfrak{f}\cdot K_{\overline{\mathfrak{M}}_H(c_1,c_2)} &= \sum_{i=0}^{2}(-1)^i\mathrm{ch}_1\big(\mathcal{E}xt^i_{\pi_1}(\mathcal{V}_\mathfrak{f},\mathcal{V}_\mathfrak{f})\big)\\ &= \{\pi_{1*}\big(\mathrm{ch}(\mathcal{V}_\mathfrak{f})^\tau\cdot\mathrm{ch}(\mathcal{V}_\mathfrak{f})\cdot\pi_2^*\mathrm{td}(X)\big)\}_1 \qquad (3.7)\end{aligned}$$

where $\{\,\}_1$ denotes the component in $A^1(\mathfrak{f})\cong\mathbb{Z}$, and τ is the action on the Chow group $A^*(\cdot)$ sending an element $\alpha\in A^i(\cdot)$ to $(-1)^i\alpha$. By (3.6),

$$\mathrm{ch}(\mathcal{V}_\mathfrak{f}) = \pi_2^*\mathrm{ch}(V_1) - \pi_1^*\mathrm{ch}(\mathcal{O}_\mathfrak{f}(1))\cdot\pi_2^*\mathrm{ch}(\mathcal{O}_x).$$

A straight-forward computation shows that $\mathfrak{f} \cdot K_{\overline{\mathfrak{M}}_H(c_1,c_2)} = 0$.

(ii) By (i), $c_1(N) = -2$. Note that $N_{\mathfrak{f} \subset \mathfrak{B}_*} \cong \mathcal{O}_{\mathfrak{f}}^{\oplus(\mathfrak{d}-2)}$. By the exact sequence

$$0 \to N_{\mathfrak{f} \subset \mathfrak{B}_*} \to N \to N_{\mathfrak{B}_* \subset \overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} \to 0, \tag{3.8}$$

$c_1\big(N_{\mathfrak{B}_* \subset \overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}}\big) = -2$. Since $\mathfrak{B}_*$ is an open subset of the boundary divisor $\mathfrak{B}$,

$$N_{\mathfrak{B}_* \subset \overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} = N_{\mathfrak{B} \subset \overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} \cong \mathcal{O}_{\overline{\mathfrak{M}}_H(c_1,c_2)}(\mathfrak{B})|_{\mathfrak{f}}.$$

Hence $\mathfrak{f} \cdot \mathfrak{B} = -2$, $N_{\mathfrak{B}_* \subset \overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} \cong \mathcal{O}_{\mathfrak{f}}(-2)$, and (3.8) is simplified to

$$0 \to \mathcal{O}_{\mathfrak{f}}^{\oplus(\mathfrak{d}-2)} \to N \to \mathcal{O}_{\mathfrak{f}}(-2) \to 0$$

which must split. Therefore, we obtain $N \cong \mathcal{O}_{\mathfrak{f}}^{\oplus(\mathfrak{d}-2)} \oplus \mathcal{O}_{\mathfrak{f}}(-2)$.

(iii) The exact sequence $0 \to T_{\mathfrak{f}} \to T_{\overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} \to N \to 0$ gives rise to

$$0 \to \mathcal{O}_{\mathfrak{f}}(2) \to T_{\overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} \to \mathcal{O}_{\mathfrak{f}}^{\oplus(\mathfrak{d}-2)} \oplus \mathcal{O}_{\mathfrak{f}}(-2) \to 0$$

which again splits. Thus $T_{\overline{\mathfrak{M}}_H(c_1,c_2)}|_{\mathfrak{f}} \cong \mathcal{O}_{\mathfrak{f}}^{\oplus(\mathfrak{d}-2)} \oplus \mathcal{O}_{\mathfrak{f}}(-2) \oplus \mathcal{O}_{\mathfrak{f}}(2)$. □

3.3. The μ map. Assume that the moduli space $\overline{\mathfrak{M}}_H(c_1,c_2)$ is non-empty. By Lemma 3.1 (i), $\mathfrak{M}_H(c_1,c_2)$ is non-empty. Let g be the Kahler metric on the underlying smooth 4-manifold X associated to the ample divisor H. Let P be the $SO(3)$-bundle on X associated to a rank-2 bundle with Chern classes c_1 and c_2. Let $\mathfrak{B}(P)^*$ be the space of gauge equivalence classes of irreducible connections on P. By [Don1], $\mathfrak{M}_H(c_1,c_2)$ can be identified with the subset of $\mathfrak{B}(P)^*$ consisting of anti-self-dual irreducible connections. For simplicity of notations, we regard that

$$\mathfrak{M}_H(c_1,c_2) \subset \mathfrak{B}(P)^*.$$

By the Theorem 0.1 in [Li4], the restriction map is an isomorphism:

$$\text{res}: \ H^2(\mathfrak{B}(P)^*;\mathbb{Q}) \overset{\cong}{\to} H^2(\mathfrak{M}_H(c_1,c_2);\mathbb{Q}). \tag{3.9}$$

A universal $SO(3)$-bundle $\widetilde{\mathcal{V}}$ exists over $B(P)^* \times X$. Let

$$p_1(\widetilde{\mathcal{V}}) \in H^4(B(P)^* \times X;\mathbb{Z})$$

be its first Pontrjagin class. It is known from gauge theory that the map

$$\tilde{\mu}: H_2(X;\mathbb{Q}) \to H^2(\mathfrak{B}(P)^*;\mathbb{Q}),$$

defined by the slant product $\tilde{\mu}(\alpha) = -1/4 \cdot p_1(\widetilde{\mathcal{V}})/\alpha$, is an isomorphism. Put

$$\mu = (\text{res} \circ \tilde{\mu}): \ H_2(X;\mathbb{Q}) \to H^2(\mathfrak{M}_H(c_1,c_2);\mathbb{Q}). \tag{3.10}$$

Note that if $\mathcal{V}$ is a universal sheaf over $\mathfrak{M}_H(c_1,c_2) \times X$, then

$$\mu(\alpha) = -\frac{1}{4} \cdot \left[c_1(\mathcal{V})^2 - 4c_2(\mathcal{V})\right]/\alpha. \tag{3.11}$$

Lemma 3.3. *Assume that the moduli space $\overline{\mathfrak{M}}_H(c_1,c_2)$ is non-empty.*

(i) The map μ is an isomorphism;
(ii) Let $\mathcal{L}(C, \tilde{\theta}_C)$ be the determinant line bundle defined in (2.6). Then,

$$\mathcal{L}(C, \tilde{\theta}_C)|_{\mathfrak{M}_H(c_1,c_2)} = \mu(C) \in H^2(\mathfrak{M}_H(c_1, c_2); \mathbb{Q}). \tag{3.12}$$

Proof. (i) follows from (3.9) since the map $\tilde{\mu}$ is an isomorphism. To prove (ii), note from (2.6) and the Proposition 3.8 (iii) in Chapter V of [FM] that

$$\mathcal{L}(C, \tilde{\theta}_C) = -\mathrm{ch}_1\big(\tilde{\pi}_{1!}(\mathcal{V}|_{\overline{\mathfrak{M}}_H(c_1,c_2)\times C} \otimes \tilde{\pi}_2^*\tilde{\theta}_C)\big). \tag{3.13}$$

Let π_1 and π_2 be the two projections on $\overline{\mathfrak{M}}_H(c_1, c_2) \times X$. Then,

$$c_1(\mathcal{V}) = \pi_1^*\mathcal{D} + \pi_2^*c_1$$

for some divisor $\mathcal{D}$ on $\overline{\mathfrak{M}}_H(c_1, c_2)$. Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be the two projections on $\overline{\mathfrak{M}}_H(c_1, c_2) \times C$. By the Grothendieck-Riemann-Roch Theorem,

$$\begin{aligned}\mathrm{ch}\big(\tilde{\pi}_{1!}(\mathcal{V}|_{\overline{\mathfrak{M}}_H(c_1,c_2)\times C} \otimes \tilde{\pi}_2^*\tilde{\theta}_C)\big) &= \tilde{\pi}_{1*}\big(\mathrm{ch}(\mathcal{V}|_{\overline{\mathfrak{M}}_H(c_1,c_2)\times C}) \cdot \tilde{\pi}_2^*\mathrm{ch}(\tilde{\theta}_C) \cdot \tilde{\pi}_2^*\mathrm{td}(C)\big)\\ &= \frac{(c_1 \cdot C)}{2}\,\mathcal{D} - \tilde{\pi}_{1*}c_2\big(\mathcal{V}|_{\overline{\mathfrak{M}}_H(c_1,c_2)\times C}\big)\\ &= \frac{(c_1 \cdot C)}{2}\,\mathcal{D} - c_2(\mathcal{V})/C\end{aligned}$$

where we have used (2.5) in the second equality. A direct computation yields

$$\begin{aligned}\mu(C) &= -\frac{1}{4}\left[c_1(\mathcal{V})^2 - 4c_2(\mathcal{V})\right]/C\\ &= -\frac{(c_1 \cdot C)}{2}\,\mathcal{D} + c_2(\mathcal{V})/C.\end{aligned}$$

Therefore, $\mathcal{L}(C, \tilde{\theta}_C) = \mu(C) \in H^2(\mathfrak{M}_H(c_1, c_2); \mathbb{Q})$ in view of (3.13). ∎

When $c_1 = 0$, Lemma 3.3 (ii) is the Proposition 1.1 in Chapter V of [FM].

3.4. The Gieseker-Uhlenbeck morphism is crepant.

PROPOSITION 3.4. *Assume that the moduli space $\overline{\mathfrak{M}}_H(c_1, c_2 - 1)$ is non-empty. Recall the boundary divisor $\mathfrak{B}$ from definition (3.2). Then,*
(i) the Picard number of $\overline{\mathfrak{U}}_H(c_1, c_2)$ is equal to that of X;
(ii) $\mathrm{Pic}\big(\overline{\mathfrak{M}}_H(c_1, c_2)\big) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}\mathfrak{B} \oplus \Psi^*\mathrm{Pic}\big(\overline{\mathfrak{U}}_H(c_1, c_2)\big) \otimes_{\mathbb{Z}} \mathbb{Q}$*;*
(iii) $C \in \mathbb{Q}\mathfrak{f} \subset \mathrm{Num}\big(\overline{\mathfrak{M}}_H(c_1, c_2)\big) \otimes_{\mathbb{Z}} \mathbb{Q}$ if C is a curve contracted by Ψ.

Proof. (i) Let ρ denote the Picard number of X. For simplicity, denote the spaces $\overline{\mathfrak{M}}_H(c_1, c_2)$ and $\overline{\mathfrak{U}}_H(c_1, c_2)$ by $\overline{\mathfrak{M}}_H$ and $\overline{\mathfrak{U}}_H$ respectively.

First of all, since the boundary divisor $\mathfrak{B}$ is contracted by Ψ, we see from Lemma 3.1 (ii) that the Picard number of $\overline{\mathfrak{U}}_H$ is at most ρ. To see the other direction, note that H is contained in certain chamber $\mathcal{C}$ of type (c_1, c_2) since $(c_1 \cdot H)$ is odd. Choose ample divisors $H_2, \ldots, H_\rho \in \mathcal{C}$ such that

$$H_1 := H,\ H_2, \ldots,\ H_\rho$$

form a basis of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For each i, we have the Gieseker-Uhlenbeck morphism

$$\Psi_{H_i} :\ \overline{\mathfrak{M}}_{H_i} \to \overline{\mathfrak{U}}_{H_i},$$

and for a suitable choose $C_i \in |k_i H_i|$ with $k_i > 0$, the determinant line bundle

$$\mathcal{L}(C_i, \tilde{\theta}_{C_i}) \in (\Psi_{H_i})^* \mathrm{Pic}\big(\overline{\mathfrak{U}}_{H_i}\big) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By Lemma 2.2 (i) and (2.3), the spaces $\overline{\mathfrak{M}}_{H_i}$, $\mathfrak{M}_{H_i}$ and $\overline{\mathfrak{U}}_{H_i}$ are independent of i, and thus can be identified with $\overline{\mathfrak{M}}_H$, $\mathfrak{M}_H$ and $\overline{\mathfrak{U}}_H$ respectively. Moreover, we see from the definition of Ψ_H in (2.4) that $\Psi_{H_i} = \Psi_{H_1} = \Psi_H$ for all the i. So we have

$$\mathcal{L}(C_1, \tilde{\theta}_{C_1}), \ldots, \mathcal{L}(C_\rho, \tilde{\theta}_{C_\rho}) \ \in \ (\Psi_H)^* \mathrm{Pic}\big(\overline{\mathfrak{U}}_H\big) \otimes_{\mathbb{Z}} \mathbb{Q} \ \subset \ \mathrm{Pic}\big(\overline{\mathfrak{M}}_H\big) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (3.14)$$

We claim that the ρ line bundles in (3.14) are linearly independent. If they were linearly dependent, then their restrictions to $\mathfrak{M}_H(c_1, c_2) \subset \overline{\mathfrak{M}}_H(c_1, c_2)$ would be linearly dependent. By (3.12), the cohomology classes

$$\mu(C_1) = \mu(k_1 H_1), \ldots, \ \mu(C_\rho) = \mu(k_\rho H_\rho)$$

in $H^2(\mathfrak{M}_H(c_1, c_2); \mathbb{Q})$ would be linearly dependent. By Lemma 3.3 (i), the classes

$$k_1 H_1, \ldots, k_\rho H_\rho \in H_2(X; \mathbb{Q})$$

would be linearly dependent. This is impossible since $H_1, \ H_2, \ldots, \ H_\rho$ form a basis of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and since $c_1 : \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^2(X; \mathbb{Q}) \cong H_2(X; \mathbb{Q})$ is injective.

(ii) Our result follows from (i), Lemma 3.1 (ii) and the fact that

$$\mathfrak{B} \notin \Psi^* \mathrm{Pic}(\overline{\mathfrak{U}}_H(c_1, c_2)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(iii) Let $a = (C \cdot \mathfrak{B})/2 \in \mathbb{Q}$. Then, $(C + a\mathfrak{f}) \cdot \mathfrak{D} = 0$ for every divisor

$$\mathfrak{D} \in \mathrm{Pic}(\overline{\mathfrak{M}}_H(c_1, c_2)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

in view of (ii). Hence, $C + a\mathfrak{f} = 0$. It follows that $C \in \mathbb{Q}\mathfrak{f}$. ☐

Remark 3.5. Proposition 3.4 (iii) can be sharpened as follows. If there exist two divisors D_1 and D_2 on X such that $(D_1 - D_2) \cdot (D_1 + D_2 + c_1 - K_X) = \pm 1$, then

$$C \in \mathbb{Z}\mathfrak{f} \subset \mathrm{Num}\big(\overline{\mathfrak{M}}_H(c_1, c_2)\big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

whenever C is a curve contracted by Ψ. For instance, this condition holds when $X = \mathbb{P}^2$, $c_1 = -\ell$, $D_1 = -\ell$ and $D_2 = 0$, where ℓ denotes a line in X.

Theorem 3.6. *Let X be a simply connected surface with $-K_X \geq 0$, and let H be an ample divisor with odd $(c_1 \cdot H)$. Assume that $\overline{\mathfrak{M}}_H(c_1, c_2)$ is non-empty. Then the Gieseker-Uhlenbeck morphism $\Psi_H : \overline{\mathfrak{M}}_H(c_1, c_2) \to \overline{\mathfrak{U}}_H(c_1, c_2)$ is crepant.*

Proof. First of all, if $\overline{\mathfrak{M}}_H(c_1, c_2 - 1)$ is empty, then $\overline{\mathfrak{M}}_H(c_1, c_2) = \overline{\mathfrak{U}}_H(c_1, c_2)$ and Ψ_H is the identity map. Hence our statement is trivially true.

Next, assume that $\overline{\mathfrak{M}}_H(c_1, c_2 - 1)$ is non-empty. By Proposition 3.4 (ii),

$$K_{\overline{\mathfrak{M}}_H(c_1, c_2)} = a\mathfrak{B} + \Psi^* \mathfrak{D}$$

for some $a \in \mathbb{Q}$ and some $\mathbb{Q}$-Cartier divisor $\mathfrak{D}$ on $\overline{\mathfrak{U}}_H(c_1, c_2)$. Intersecting both sides with $\mathfrak{f}$ and applying Lemma 3.2 (i) and (ii) force $a = 0$. So $K_{\overline{\mathfrak{M}}_H(c_1, c_2)} = \Psi^* \mathfrak{D}$. Note that the canonical class $K_{\overline{\mathfrak{U}}_H(c_1,c_2)}$ exists as a Weil divisor since $\overline{\mathfrak{U}}_H(c_1,c_2)$ is regular in codimension-1. Since $\Psi|_{\mathfrak{M}_H(c_1,c_2)}$ is the identity map on $\mathfrak{M}_H(c_1,c_2)$ and

$$K_{\overline{\mathfrak{M}}_H(c_1,c_2)} = \Psi^*\mathfrak{D},$$

$K_{\overline{\mathfrak{U}}_H(c_1,c_2)}$ coincides with the $\mathbb{Q}$-Cartier divisor $\mathfrak{D}$ on the open subset

$$\mathfrak{M}_H(c_1,c_2) \subset \overline{\mathfrak{U}}_H(c_1,c_2).$$

Since $\overline{\mathfrak{U}}_H(c_1,c_2) - \mathfrak{M}_H(c_1,c_2)$ is codimension-2 in $\overline{\mathfrak{U}}_H(c_1,c_2)$, we obtain $K_{\overline{\mathfrak{U}}_H(c_1,c_2)} = \mathfrak{D}$. Hence, $K_{\overline{\mathfrak{U}}_H(c_1,c_2)}$ is $\mathbb{Q}$-Cartier and $K_{\overline{\mathfrak{M}}_H(c_1,c_2)} = \Psi^* K_{\overline{\mathfrak{U}}_H(c_1,c_2)}$. □

REMARK 3.7. It is unclear whether the Uhlenbeck compactification $\overline{\mathfrak{U}}_H(c_1,c_2)$ is normal or not. However, applying the Stein Factorization Theorem (see the Theorem 2.26 in [Iit]) to the Gieseker-Uhlenbeck morphism Ψ_H, we can prove that the natural morphism from the normalization of $\overline{\mathfrak{U}}_H(c_1,c_2)$ to $\overline{\mathfrak{U}}_H(c_1,c_2)$ is bijective.

REMARK 3.8. Under the conditions of Theorem 3.6, it has been proved in [Bar] that the Gieseker-Uhlenbeck morphism $\Psi_H : \overline{\mathfrak{M}}_H(c_1,c_2) \to \overline{\mathfrak{U}}_H(c_1,c_2)$ is strictly semi-small with respect to certain natural stratification of $\overline{\mathfrak{U}}_H(c_1,c_2)$.

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