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PARAMETRIZATION OF SING Θ FOR A FANO 3-FOLD OF GENUS 7 BY MODULI OF VECTOR BUNDLES*

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Abstract. According to Mukai, any prime Fano threefold X of genus 7 is a linear section of the spinor tenfold in the projectivized half-spinor space of Spin(10). The orthogonal linear section of the spinor tenfold is a canonical genus-7 curve Γ, and the intermediate Jacobian $J(X)$ is isomorphic to the Jacobian of Γ. It is proven that, for a generic X, the Abel-Jacobi map of the family of elliptic sextics on X factors through the moduli space of rank-2 vector bundles with $c_1 = -K_X$ and $\deg c_2 = 6$ and that the latter is birational to the singular locus of the theta divisor of $J(X)$.

Key words. Spinors, spinor variety, Fano variety, moduli of vector bundles, intermediate Jacobian, Brill–Noether locus, orthogonal Grassmannian, theta divisor, elliptic sextic, symmetric powers of a curve

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0. Introduction. This work is a sequel to the series of papers on moduli spaces $M_X(2;k,n)$ of stable rank-2 vector bundles on Fano 3-folds X with Picard group $\mathbb{Z}$ for small Chern classes $c_1 = k,\ c_2 = n$. The nature of the results depends strongly on the index of X, which is defined as the largest integer that divides the canonical class K_X in $\operatorname{Pic} X$. Historically, the first Fano 3-fold for which the geometry of such moduli spaces was studied was the projective space $\mathbb{P}^3$, the unique Fano 3-fold of index 4. The most part of results for $\mathbb{P}^3$ concerns the problems of rationality, irreducibility or smoothness of the moduli space, see [Barth-1], [Barth-2], [Ha], [HS], [LP], [ES], [HN], [M], [BanM], [GS], [K], [KO], [CTT] and references therein.

The next case is the 3-dimensional quadric Q^3, which is Fano of index 3. Much less is known here, see [OS]. Further, the authors of [SW] identified the moduli spaces $M_X(2;-1,2)$ on all the Fano 3-folds X of index 2 except for the double Veronese cone V_1', which are (in the notation of Iskovskikh) the quartic double solid V_2, a 3-dimensional cubic V_3, a complete intersection of two quadrics V_4, and a smooth 3-dimensional section of the Grassmannian $G(2,5)$ by three hyperplanes V_5. It turns out that all the vector bundles in $M_X(2;-1,2)$ for these threefolds are obtained by Serre's construction from conics. Remark that for $\mathbb{P}^3$ and Q^3 all the known moduli spaces are either rational or supposed to be rational, whilst [SW] provides first nonrational examples.

We will also mention the paper [KT] on the moduli of stable vector bundles on the flag variety $\mathbb{F}(1,2)$, though it is somewhat apart, for $\mathbb{F}(1,2)$ has Picard group $2\mathbb{Z}$. This is practically all what was known on the subject until the year 2000, when a new tool was brought into the study of the moduli spaces: the Abel–Jacobi map to the intermediate Jacobian $J(X)$. For the 3-dimensional cubic $X = V_3$, it was proved in [MT-1], [IM-1] that the open part of $M_X(2;0,2)$ parametrizing the vector bundles obtained by Serre's construction from elliptic quintics is sent by the Abel–Jacobi map isomorphically onto an open subset of $J(X)$. Druel [D] proved the irreducibility of

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$M_X(2;0,2)$ and described its compactification by semistable sheaves; see also the survey [Beau-1]. The other index-2 case, that of the double solid V_2, was considered in [Ti], [MT-2], where it was proved that the vector bundles coming from the elliptic quintics on V_2 form an irreducible component of $M_{V_2}(2;0,3)$ on which the Abel–Jacobi map is quasi-finite of degree 84 over an open subset of the theta-divisor $\Theta \subset J(V_2)$.

In the index-1 case, several descriptions of the moduli spaces $M_X(2;k,n)$ were obtained for the following threefolds: the 3-dimensional quartic [IM-2], the Fano threefold of degree 12 [IM-3] and the one of degree 16 [IR]. The vector bundles studied in these three papers are related respectively to the half-canonical curves of degree 15, elliptic quintics and elliptic sextics. Kuznetsov in [Ku-1], [Ku-2] used the moduli spaces associated to elliptic quintics on the 3-dimensional cubic V_3 and the Fano threefold $X = X_{12}$ of degree 12 to construct semiorthogonal decompositions of the derived categories of sheaves on these threefolds.

According to Mukai, any Fano threefold $X = X_{12}$ is a linear section of the spinor tenfold in the projectivized half-spinor space of Spin(10). The orthogonal linear section of the spinor tenfold is a canonical genus-7 curve Γ, and the intermediate Jacobian $J(X)$ is isomorphic to the Jacobian of Γ. It is proved in [IM-3] that $M_X(2;1,5)$ is isomorphic to Γ. Kuznetsov remarks that the last moduli space is fine and provides a natural universal bundle on it.

Here we work on the same variety $X = X_{12}$, but consider the moduli space $M_X(2;1,6)$. We prove that all the vector bundles represented by points of $M_X(2;1,6)$ are obtained by Serre's construction from reduced sextics which deform to elliptic sextics (Proposition 7.4). The main result (Theorem 6.4 and Corollary 7.5) is the following: for generic X, $M_X(2;1,6)$ is irreducible and the Abel–Jacobi map sends it birationally onto the singular locus $\operatorname{Sing}\Theta$ of the theta-divisor of $J(X)$. Our construction provides no universal bundle on $M_X(2;1,6)$, and it seems very likely that this moduli space is not fine.

Throughout the paper, we extensively use the Iskovskikh–Prokhorov–Takeuchi birational transformations that can be obtained by a blowup with center in a point p, a conic q or a twisted rational cubic C_3^0 followed by a flop and a contraction of one divisor (Section 1). The existence of such transformations is proved in [Tak], [Isk-P] by techniques from Mori theory. The principal idea is the following. The anticanonical class $-K_{\tilde{X}}$ of the blowup $\tilde{X}$ of X along one of the above centers is nef and big and defines a small contraction of $\tilde{X} \longrightarrow W$ onto some Fano 3-fold W with terminal singularities. By a result of Kollár [Kol-1], there exists a flop $\tilde{X} \dashrightarrow \tilde{Y}$ over W. The flop is a birational map, biregular on the complement of finitely many flopping curves which are exactly the curves contracted to the singular points of W. The thus obtained variety $\tilde{Y}$ admits a birational contraction $\tilde{Y} \longrightarrow Y$ onto another Fano threefold Y with Picard group $\mathbb{Z}$. The composition $X \dashrightarrow \tilde{X} \dashrightarrow \tilde{Y} \longrightarrow Y$ is what we call an Iskovskikh–Prokhorov–Takeuchi transformation.

If one applies this construction to a conic q in X, then the resulting birational map Ψ_q (see Diagram 2) ends up in the 3-dimensional quadric Q^3, and the last blowdown in its decomposition is the contraction of a divisor onto a curve $\Gamma_{10}^7 \subset Q^3$ of genus 7 and degree 10. The curve Γ_{10}^7 is identified with the projection of Γ, the orthogonal linear section associated to X, from two points $u, v \in \Gamma$. This allows us to parameterize the family of conics in X by the symmetric square $\Gamma^{(2)}$. Further, the rational normal quartics C_4^0 in X meeting q at 2 points are transformed by Ψ_q into conics in Q^3 meeting Γ_{10}^7 in 4 points. If we denote the 4 points u_1, u_2, u_3, u_4, then the divisor $u + v + \sum u_i$ on Γ belongs to W_6^1. The Brill–Noether locus W_6^1 is nothing else but the singular locus of the theta-divisor in $J(\Gamma)$, and the Abel–Jacobi image of the degenerate elliptic sextic C_4^0+q is minus the class of $u+v+\sum u_i$. Any elliptic sextic in X defines a rank-2 vector bundle $\mathcal{E}$ via Serre's construction $\mathcal{S}$. We show that the fibers of $\mathcal{S}$ are the projective spaces $\mathbb{P}^3=\mathbb{P}H^0(X,\mathcal{E})$ and those of the Abel–Jacobi map on elliptic sextics are finite unions of these $\mathbb{P}^3$'s. Further, we verify that the reducible sextics of type C_4^0+q in a generic fiber of the Abel–Jacobi map form an irreducible curve. Hence the fiber of the Abel–Jacobi map is just one copy of $\mathbb{P}^3$, which implies the birationality part of the main result.

In order to handle degree-6 curves, we start with lines, conics, then continue by rational normal quartics, each time constructing higher degree curves as smoothings of the reducible one. Thus we prove auxiliary results on the families of low degree curves which may be of interest themselves. For example, we identify the curve $\tau(X)$ of lines in X with tne Brill–Noether locus $W_5^1(\Gamma)$ and determine its genus $g_{\tau(X)}=43$ (Proposition 2.1). We prove that the surface of conics $\mathcal{F}(X)$ is isomorphic to $\Gamma^{(2)}$ (Proposition 2.2). This result was also obtained by [Ku-2] via a different approach using the Fourier–Mukai transform $D^b(X)\longrightarrow D^b(\Gamma)$. It is curious to note that $\mathcal{F}(X)$ remains nonsingular for *all* nonsingular X.

Proceeding to curves of higher degree, we show that the families of rational normal cubics and quartics in X are irreducible (Lemmas 4.1, 4.3). We prove that the family of degenerate elliptic sextics of the form C_4^0+q in X is irreducible (Lemma 5.1). A standard monodromy argument together with the result of N. Perrin [P-2] on the irreducibility of the family of elliptic curves of given degree on the spinor tenfold Σ allow us to deduce the irreducibility of the family of elliptic sextics in X and that of the moduli space $M_X(2;1,6)$.

On several occasions, we use the rigidity of the symmetric square of Γ in the following sense: $\Gamma^{(2)}$ has neither nontrivial self-maps, nor maps to a curve. Though the subject seems to be classical, we did not find appropriate references and included the proof of the rigidity of $\Gamma^{(2)}$ for a generic curve of genus $g\geq 5$ in the last section (Proposition 8.1).

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1. Preliminaries. Let $\Sigma=\Sigma_{12}^{10}$ be the spinor tenfold in $\mathbb{P}^{15}$. It is a homogeneous space of the complex spin group Spin(10), the unique closed orbit of Spin(10) in the projectivized half-spinor representation $\mathrm{Spin}(10):\mathbb{P}^{15}\circlearrowright$. It can be also interpreted as one of the two components of the orthogonal Grassmannian $G(4;Q)=\Sigma^+\sqcup\Sigma^-$ parametrizing the linear subspaces $\mathbb{P}^4$ of $\mathbb{P}^9$ contained in a given smooth 8-dimensional quadric $Q=Q^8\subset\mathbb{P}^9$. See [Mu-1], [RS] or Section 1 of [IM-3] for more details and for expicit equations of Σ.

The Fano threefold X_{12} is a smooth 3-dimensional linear section of Σ by a subspace $\mathbb{P}^8\subset\mathbb{P}^{15}$. We will also consider smooth linear sections of Σ by linear subspaces $\mathbb{P}^7$ and $\mathbb{P}^6$, which are K3 surfaces, resp. canonical curves of degree 12. The Gauss dual $\Sigma^\vee\subset\mathbb{P}^{15\vee}$ of Σ is naturally identified with Σ via the so called fundamental form on $\mathbb{P}^{15}$, and to a linear section $V=\mathbb{P}^{7+k}\cap\Sigma$ for $k=-1,0$, resp. 1 we can associate the orthogonal linear section $\check{V}=(\mathbb{P}^{7+k})^\perp\cap\Sigma^\vee$. The orthogonal linear section of a Fano 3-fold X_{12} is a canonical genus-7 curve $\Gamma=\Gamma_{12}^7$, and that of a K3 surface $(k=0)$ is another K3 surface. By [Mu-1], $\Gamma=\check{X}$ is not an arbitrary smooth curve of genus 7, but a sufficiently generic one: it has no g_4^1, neither g_6^2.

If we identify Σ with $\Sigma^+\subset G(4;Q)$, then $\Sigma^\vee$ is naturally identified with the other component Σ^- of $G(4;Q)$. Denote by $\mathbb{P}^{15\pm}$ the half-spinor space spanned by $\Sigma^\pm$, so that $\mathbb{P}^{15+} = \mathbb{P}^{15}$ and $\mathbb{P}^{15-} = \mathbb{P}^{15\vee}$. For $c \in \Sigma^\pm$, introduce the following notation:

$\mathbb{P}^4_c$, the linear subspace of Q represented by c;

$\mathbb{P}^{14}_c$, the tangent hyperplane to $\Sigma^\mp$ in $\mathbb{P}^{15\mp}$ represented by c;

H_c, the corresponding hyperplane section $\mathbb{P}^{14}_c \cap \Sigma^\mp$;

$\varepsilon(c)$, the sign of c, that is $\varepsilon(c) \in \{+,-\}$ and $c \in \Sigma^{\varepsilon(c)}$.

The following proposition lists some useful properties of $\Sigma^\pm$.

PROPOSITION 1.1. *The following assertions hold:*

(i) For $c, d \in G(4;Q)$, $\varepsilon(c) = \varepsilon(d)$, that is c, d lie in the same component of $G(4;Q)$, if and only if $\dim(\mathbb{P}^4_c \cap \mathbb{P}^4_d) \in \{0,2,4\}$.

(ii) For $c, d \in G(4;Q)$, $\varepsilon(c) = -\varepsilon(d)$, that is c, d belong to different components of $G(4;Q)$, if and only if $\dim(\mathbb{P}^4_c \cap \mathbb{P}^4_d) \in \{-1,1,3\}$*, where the negative dimension corresponds to the empty set.*

(iii) Let $c \in G(4;Q)$. Then $H_c = \{a \in G(4;Q) \mid \dim(\mathbb{P}^4_c \cap \mathbb{P}^4_d) \in \{1,3\}\} = \{d \in \Sigma^{-\varepsilon(c)} \mid \mathbb{P}^4_c \cap \mathbb{P}^4_d \neq \varnothing\}$.

(iv) The hyperplane $\mathbb{P}^{14}_c$ is tangent to $\Sigma^{-\varepsilon(c)}$ along a linear 4-dimensional subspace $\mathbb{P}^4 \subset \Sigma^{-\varepsilon(c)}$, which we will denote by Π^4_c, and $\Pi^4_c = \{d \in G(4;Q) \mid \dim(\mathbb{P}^4_c \cap \mathbb{P}^4_d) = 3\}$. Any 3-space $\mathbb{P}^3 \subset Q$ determines in a unique way a pair $\mathbb{P}^4_c$, $\mathbb{P}^4_d$ of 4-subspaces of Q containing $\mathbb{P}^3$, so Π^4_c is naturally identified with the dual of $\mathbb{P}^4_d$.

(v) H_c is a cone whose vertex (= ridge) is Π^4_c and whose base is the Grassmannian $G(2,5)$, embedded in a standard way into $\mathbb{P}^9 \simeq (\Pi^4_c)^\perp$. The linear projection with center Π^4_c identifies the open set $U_c = H_c \smallsetminus \Pi^4_c$ with the universal vector subbundle of $\mathbb{C}^5 \times G(2,5)$ of rank 3.

Proof. The assertions (i), (ii) are classical, see for example [Mu-1]. For a proof of (iii)–(v) see [IM-3], Lemma 3.4. ∎

The families of lines and conics on the spinor tenfold are easy to describe:

PROPOSITION 1.2. *(i) Fix a plane $\mathbb{P}^2$ contained in $Q = Q^8$. Then*

$$\ell^\pm_{\mathbb{P}^2} = \{c \in \Sigma^\pm \mid \mathbb{P}^2 \subset \mathbb{P}^4_c\}$$

is a line in $\Sigma^\pm$. Every line in $\Sigma^\pm$ is of this form. The variety $\tau(\Sigma^\pm)$ is thus identified with the Grassmannian $G(2;Q)$ parametrizing the planes $\mathbb{P}^2$ contained in Q.

(ii) Fix a point $p \in Q$. Then

$$Q^{6\pm}_p = \{c \in \Sigma^\pm \mid p \in \mathbb{P}^4_c\}$$

is a nonsingular 6-dimensional quadric contained in $\Sigma^\pm$. Any conic q in $\Sigma^\pm$ belongs to one of the following two types: either q lies in a plane $\mathbb{P}^2$ contained in Q, or there exist a unique point $p \in Q$ depending on q, and a plane $\mathbb{P}^2$ in the linear span $\mathbb{P}^{7\pm}_p$ of $Q^{6\pm}_p$ such that $q = Q^{6\pm}_p \cap \mathbb{P}^2$.

More generally, for any quadric q^k of dimension $k = 0,1,\dots,6$ contained in $\Sigma^\pm$, either its span $\mathbb{P}^{k+1}$ is contained in $\Sigma^\pm$, or there exists a unique point $p \in Q$ such that $\mathbb{P}^{k+1} \subset \mathbb{P}^{7\pm}_p$ and $q^k = \mathbb{P}^{k+1} \cap Q^{6\pm}_p$.

Proof. Assertion (i) is proved in [RS], Section 3. For the part (ii), see [Mu-1], 1.14–1.15. ∎

We will often use the following property of the plane linear sections of Σ, whose proof is obtained by a refinement of the proof of Proposition 1.16 in [Mu-1]:

Lemma 1.3. *Let* $\mathbb{P}^2$ *be a plane in* $\mathbb{P}^{15}$. *If* $\mathbb{P}^2 \cap \Sigma$ *is finite, then* $\mathrm{length}(\mathbb{P}^2 \cap \Sigma) \leq 3$.

Informally speaking, this means that Σ has no 4-secant 2-planes. As Σ is an intersection of quadrics, any intersection $\mathbb{P}^2 \cap \Sigma$ that contains a subscheme of length 4 is either a line, or a line plus a point, or a conic, or the whole plane $\mathbb{P}^2$.

Let now $X = X_{12}$ be a smooth Fano threefold of degree 12. We will describe the Iskovskikh–Prokhorov–Takeuchi ([Isk-P], [Tak]) birational maps Φ_x, Ψ_q, resp. $\Psi_{C_3^0}$ associated to a point $x \in X$, a conic $q \subset X$, resp. a rational normal cubic $C_3^0 \subset X$ (Theorems 4.5.8, 4.4.11, 4.6.3 in [Isk-P]; see also Theorems 6.3 and 6.5 of [IM-3] for the first two). For the reader's convenience, we will briefly remind their structure. Each of these maps is a composition of three birational modifications: blowup of a point or a curve in X, flop and blowdown of some divisor onto a curve. The blowup gives a 3-fold $\tilde{X}$ with nef and big anticanonical class and 2 exceptional divisors. The first one is that of the blowdown $\tilde{X} \longrightarrow X$. The contraction of the second one provides a new 3-fold Y, but before the contraction, one has to make a flop in finitely many irreducible curves $C \subset \tilde{X}$ characterized by the condition $C \cdot K_{\tilde{X}} = 0$.

Start by Φ_x, the birational map associated to a generic point $x \in X$. It is a birational isomorphism of X onto $Y = Y_5$, the Del Pezzo variety of degree 5, that is a nonsingular 3-dimensional linear section $\mathbb{P}^6 \cap G(2,5)$ of the Grassmannian in $\mathbb{P}^9$. Its structure is described by Diagram 1:

$$
\begin{tikzcd}
E_X \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[rr, dashed, "\text{flop}"] \arrow[d, "\sigma_X"'] \arrow[ddr, "\tilde{\pi}"] & & \tilde{Y} \arrow[d, "\sigma_Y"] \arrow[ddl, "\tilde{\eta}"'] & E_Y \arrow[l, hook'] \arrow[d] \\
x \arrow[r, hook] \arrow[u, dashed, bend left, "\text{blowup of } x"] & X \arrow[rr, dashed, "\Phi_x"] \arrow[dr, dashed, "\pi"'] & & Y \arrow[dl, dashed, "\eta"] & \Gamma \arrow[l, hook'] \arrow[u, dashed, bend right, "\text{blowup of } \Gamma"'] \\
& & W & &
\end{tikzcd}
$$

Diagram 1. *The birational isomorphism* $\Phi_x : X \dashrightarrow Y_5 = G(2,5) \cap \mathbf{P}^6$.

In the diagram, $\pi = \pi_{2x}$ is the double projection from x, that is the rational map $X \dashrightarrow \mathbb{P}^4$ defined by the linear system of hyperplanes in $\mathbb{P}^8$ tangent to X at x, $\Gamma = \Gamma_{12}^7$ is a canonical genus-7 curve contained in Y, and η the projection by the linear system of quadrics containing Γ. The map Φ_x is given by the incomplete linear system $|\mathcal{O}_X(3 - 7x)|$ and the opposite map Φ_x^{-1} by the linear system $|\mathcal{O}_Y(12 - 7\Gamma)|$. The curve Γ is isomorphic to the orthogonal linear section $\Gamma = \check{X}$ of Σ^-, denoted by the same symbol. Both projections π, η are birational and end up in the same singular quartic 3-fold $W \subset \mathbb{P}^4$. When lifted to $\tilde{X}$ and $\tilde{Y}$, they become regular morphisms defined by the anticanonical linear system: $\tilde{\pi} = \varphi_{|-K_{\tilde{X}}|}$, $\tilde{\eta} = \varphi_{|-K_{\tilde{Y}}|}$. The essential point in this diagram is that the flop $\tilde{X} \dashrightarrow \tilde{Y}$ is *a flop over* W, that is $\tilde{\pi}, \tilde{\eta}$ are small morphisms contracting the flopping curves to isolated singular points of W, and these flopping curves are the only indeterminacies of the flop. We showed in [IM-3] that for generic X, x, the flopping curves in X are the 24 conics passing through x, and those in Y are the 24 bisecant lines to Γ contained in Y.

The map $\Psi_q : X \dashrightarrow Q^3$ of the second type is a birational isomorphism from X to a 3-dimensional quadric $Q^3 \subset \mathbb{P}^4$, associated to a generic conic $q \subset X$. It is given by the linear system $|\mathcal{O}_X(2-3q)|$, and its inverse Ψ_q^{-1} by $|\mathcal{O}_{Q^3}(8-3\Gamma_q)|$. Its structure is described by Diagram 2:

$$\begin{tikzcd}
E_X \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[r, dashed, "\text{flop}", "\varphi"'] \arrow[d, "\sigma_X"] & \tilde{Q}^3 \arrow[d, "\sigma_Q"] & E_Q \arrow[l, hook'] \arrow[d] \\
q \arrow[r, hook] \arrow[u, dashed, bend left, "\text{blowup of } q"] & X \arrow[r, dashed, "\Psi_q"] & Q^3 & \Gamma_q \arrow[l, hook'] \arrow[u, dashed, bend right, "\text{blowup of } \Gamma_q"']
\end{tikzcd}$$

DIAGRAM 2. *The birational isomorphism* $\Psi_q : X \dashrightarrow Q^3$.

In this diagram, $\Gamma_q \subset Q^3$ is a curve of degree 10 and genus 7, isomorphic to the orthogonal linear section Γ associated to X (see Corollary 5.12 in [IM-3]). It is not canonically embedded, for it has genus 7 and lies in $\mathbb{P}^4$. By the geometric Riemann–Roch Theorem, there is a unique unordered pair of points $u, v \in \Gamma$ such that $\mathcal{O}_{Q^3}(1)|_{\Gamma_q} \simeq \mathcal{O}_\Gamma(K-u-v)$, where K denotes the canonical class, and $\Gamma_q \subset \mathbb{P}^4$ is the image of Γ under projection from the line $\overline{uv}$. We will denote it sometimes $\Gamma_{u,v}$ in place of Γ_q. The flopping curves in X are the 14 lines meeting q, and those in Q^3 are the 14 trisecants of Γ_q contained in Q^3.

$$\begin{tikzcd}
E_X \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[r, dashed, "\text{flop}"] \arrow[d, "\sigma_X"] & \widetilde{\mathbb{P}^3} \arrow[d, "\sigma_{\mathbb{P}^3}"] & E_{\mathbb{P}^3} \arrow[l, hook'] \arrow[d] \\
C_3^0 \arrow[r, hook] \arrow[u, dashed, bend left, "\text{blowup of } C_3^0"] & X \arrow[r, dashed, "\Psi_{C_3^0}"] & \mathbb{P}^3 & \Gamma_9^7 \arrow[l, hook'] \arrow[u, dashed, bend right, "\text{blowup of } \Gamma_9^7"']
\end{tikzcd}$$

DIAGRAM 3. *The birational isomorphism* $\Psi_{C_3^0} : X \dashrightarrow \mathbb{P}^3$.

The map $\Psi_{C_3^0}$ of the third type is a birational isomorphism of X onto $\mathbb{P}^3$ and is described by Diagram 3. In this diagram, C_3^0 is a sufficiently generic rational cubic curve in X, and $\Gamma_9^7 \subset \mathbb{P}^3$ is a nonsingular curve of degree 9 and genus 7 which is a projection of the canonical curve $\Gamma = \check{X}$ from three points $u, v, w \in \Gamma$. The direct map $\Psi_{C_3^0}$ is given by the linear system $|\mathcal{O}_X(3-4C_3^0)|$ and its inverse by $|\mathcal{O}_{\mathbb{P}^3}(15-4\Gamma_9^7)|$. The flopping curves in X are the 21 lines meeting C_3^0, and those in $\mathbb{P}^3$ are the 21 quadrisecants to Γ_9^7.

2. Lines and conics in X_{12}. We will start the study of curves on X with a description of the families of lines and conics in terms of the orthogonal curve $\Gamma = \check{X}$.

PROPOSITION 2.1. *Let $X = X_{12}$ be any transversal linear section $\mathbb{P}^8 \cap \Sigma$, $\Gamma = \check{X}$ its orthogonal curve and $\tau(X) = \mathrm{Hilb}_X^{t+1}$ the Hilbert scheme of lines in X, where a* *"line" is a subscheme of X with Hilbert polynomial $P(t) = t+1$. Let $R(X)$ be the surface swept by the lines in X: $R(X) = \bigcup_{v\in\tau(X)} \ell_v$. Then the following statements hold.*

(i) $\tau(X)$ is a connected locally complete intersection curve of arithmetic genus 43, isomorphic to the Brill–Noether locus $W^1_5(\Gamma)$.

(ii) If X is generic, then $\tau(X)$ is nonsingular and every line $\ell \subset X$ has normal bundle $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.

(iii) If X is generic, then the generic line on X meets eight other lines and $R(X) \in |\mathcal{O}_X(7)|$.

Proposition 2.2. *Under the hypotheses of the previous proposition, let $\mathcal{F}(X)$ denote the Hilbert scheme Hilb^{2t+1}_X of conics on X (the "Fano surface" of X), where a "conic" is a subscheme of X with Hilbert polynomial $P(t) = 2t+1$. Then the following statements hold:*

(i) A generic conic q is nonsingular and $\mathcal{N}_{q/X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

(ii) $\mathcal{F}(X)$ is isomorphic to $\Gamma^{(2)}$, where $\Gamma^{(2)}$ denotes the symmetric square of Γ.

(iii) There are 24 conics passing through a generic point of X.

We will start by conics.

Proof of Proposition 2.2. For part (i), see [Isk-P], Proposition 4.2.5, Remark 4.2.8 and Theorem 4.5.10. Part (iii) was proved in [IM-3], Theorem 6.3 (f). We will now prove (ii).

We are going to construct an isomorphism $\lambda : \mathcal{F}(X) \xrightarrow{\sim} \Gamma^{(2)}$. We will describe the construction of $\lambda(q)$ for a closed point $q \in \mathcal{F}(X)$; it is clear how one can extend it to T-points of $\mathcal{F}(X)$ for any scheme T.

Since X is a linear section of $\Sigma = \Sigma^+$ and does not contain planes, it does not contain conics of the first type in the sense of Proposition 1.2. Hence to any conic $q \subset X$ we can associate a unique point $p = p(q) = \bigcap_{x\in q} \mathbb{P}^{4+}_x \in Q^8$, so that $q = \mathbb{P}^{2+}(q) \cap Q^{6+}_p$, where $\mathbb{P}^{2+}(q)$ denotes the linear span $\langle q\rangle$ of q. We can rewrite it as $q = \mathbb{P}^{8+}(X) \cap \mathbb{P}^{7+}_p \cap \Sigma^+ \subset \mathbb{P}^{15+}$, where $\mathbb{P}^{8+}(X) = \langle X\rangle$, $\mathbb{P}^{7+}_p = \langle Q^{6+}_p\rangle$ and $\mathbb{P}^{2+}(q) = \mathbb{P}^{8+}(X) \cap \mathbb{P}^{7+}_p$. If we now pass to the orthogonal complements in $(\mathbb{P}^{15+})^\vee = \mathbb{P}^{15-}$, we obtain:

$$\mathbb{P}^{12-}(q) := \mathbb{P}^{2+}(q)^\perp = \langle \mathbb{P}^{8+}(X)^\perp, (\mathbb{P}^{7+}_p)^\perp\rangle = \langle \mathbb{P}^{6-}(\Gamma), \mathbb{P}^{7-}_p\rangle,$$

where $\mathbb{P}^{6-}(\Gamma) = \langle\Gamma\rangle$. Thus $\mathbb{P}^{6-}(\Gamma), \mathbb{P}^{7-}_p$ are not in general position, but intersect in a line $\mathbb{P}^1$. The triple intersection $\mathbb{P}^{6-}(\Gamma)\cap\mathbb{P}^{7-}_p\cap\Sigma^-$ can be seen as $(\mathbb{P}^{6-}(\Gamma)\cap\mathbb{P}^{7-}_p)\cap\Sigma^- = \mathbb{P}^1 \cap \Sigma^-$, or $\mathbb{P}^{6-}(\Gamma) \cap (\mathbb{P}^{7-}_p \cap \Sigma^-) = \mathbb{P}^{6-}(\Gamma) \cap Q^{6-}_p$, or else as $\mathbb{P}^{7-}_p \cap (\mathbb{P}^{6-}(\Gamma) \cap \Sigma^-) = \mathbb{P}^{7-}_p \cap \Gamma$. Hence it is a subscheme of length 2 contained in Γ, that is an element of $\Gamma^{(2)}$. We define:

$$\lambda(q) := \mathbb{P}^{6-}(\Gamma) \cap \mathbb{P}^{7-}_p \cap \Sigma^- \in \Gamma^{(2)}.$$

The inverse map is defined in exactly the same manner: by Proposition 1.2 (ii) for $k = 0$, a subscheme $\xi \subset \Gamma$ of length 2 is contained in a unique quadric Q^{6-}_p and we define:

$$\lambda^{-1}(\xi) := \mathbb{P}^{8+}(X) \cap \mathbb{P}^{7+}_p \cap \Sigma^+ \in \mathcal{F}(X).$$

$\square$

Remark 2.3. Alexander Kuznetsov [Ku-2] proves the isomorphism $\mathcal{F}(X) \simeq \Gamma^{(2)}$ in a more algebraic way: he shows that the Fourier–Mukai transform associated to an appropriate universal rank-2 vector bundle on $X \times M_X(2;1,5)$ sends the structure sheaf $\mathcal{O}_q$ of a conic $q \subset X$ to the sky-scraper sheaf $\mathcal{O}_\xi$ on $\Gamma = M_X(2;1,5)$ for some $\xi \subset \Gamma$ of length 2.

Remark 2.4. According to Mukai, a generic K3 surface S of degree 12 is a transversal linear section of the spinor tenfold: $S = S+ = \mathbb{P}^7 \cap \Sigma^+$. Applying the same arguments as above with $\mathbb{P}^7$ in place of $\mathbb{P}^{8+}(X)$, we obtain a non-isomorphic K3 surface $S^- = \mathbb{P}^{7\perp} \cap \Sigma^-$ and an isomorphism $\lambda : \mathrm{Hilb}^2(S^+) \xrightarrow{\sim} \mathrm{Hilb}^2(S^-)$ (see also [Mu-3], Example 4).

In our description of the birational transformation Ψ_q (see Diagram 2), we associated a pair of points $u+v$ of Γ to a generic conic q. This gives a rational map

$$\mathcal{F}(X) \longrightarrow \Gamma^{(2)}\ , \quad q \mapsto u+v\ ,$$

which we will temporarily denote by f.

Lemma 2.5. *$\lambda = f$.*

Proof. By Proposiition 8.1, it suffices to prove that f is nonconstant. Then $f \circ \lambda^{-1}$ is a nonconstant rational self-map of $\Gamma^{(2)}$; it is the identity for generic Γ, and by continuity, this is true for any nonsingular Γ.

Let $u+v \in \Gamma^{(2)}$ be a generic degree-2 divisor. Let $\Gamma_{u,v}$ be the curve of degree 10 in $\mathbb{P}^4$ obtained as the image of the canonical curve $\Gamma \subset \mathbb{P}^{6-}(\Gamma)$ under the projection from the line $\overline{uv}$. It is contained in a unique quadric Q^3. By [Mu-2], Theorem 8.1, there is a Fano 3-fold $X' = X'_{12}$, defined as the non-abelian Brill–Noether locus $M_\Gamma(2,K,3)$, and a smooth conic $q \subset X'$, such that $\Gamma_{u,v}$ together with its trisecants is the indeterminacy locus of $\Psi_q^{-1} : Q^3 \dashrightarrow X'$. Since a variety X_{12} is uniquely determined by its orthogonal curve Γ, we have $X \simeq X'$, so f is a dominant map, and this ends the proof. ∎

Proof of Proposition 2.1. By Shokurov's Theorem on the existence of lines, see 4.4.13 in [Isk-P], and by ibid, Proposition 4.4.2, the scheme $\tau(X)$ is of pure dimension 1, and the normal bundle of a line is of type $(0,-1)$ if and only if this line is represented by a nonsingular point of $\tau(X)$. So (ii) is a consequence of (i) together with the smoothness of $W^1_5(\Gamma)$ for a generic curve Γ of genus 7 ([ACGH], IV.4.4 and V.1.6).

Let us prove (i). The easiest way to construct a map from $\tau(X)$ to $W^1_5(\Gamma)$ is by using either one of the birational maps Φ_x or Ψ_q with generic x or q. For example, let us do it for Ψ_q.

Let ℓ be a line in X and q a generic conic. Then ℓ does not meet q and $\tilde{\ell} = \Psi_q(\ell)$ is a conic. Recalculating the degree of ℓ, equal to 1, from the linear system that defines Ψ_q^{-1}, we see that $\tilde{\ell}$ meets Γ_q in a scheme Z of length 5. Denoting by angular brackets the linear span, we have $\langle Z \rangle_{\mathbb{P}^4} = \langle \tilde{\ell} \rangle_{\mathbb{P}^4} = \mathbb{P}^2$, and if we pull back Z to the canonical model $\Gamma \subset \mathbb{P}^6$ then we will have $\langle Z+u+v \rangle_{\mathbb{P}^6} = \mathbb{P}^4$. The latter linear span cannot be smaller than $\mathbb{P}^4$, because Γ has no g^3_7 (see [Mu-1]). Hence $Z+u+v$ is an element of a g^2_7 and $D_q = K - Z - u - v$ belongs to a g^1_5 on Γ. Thus we have constructed a map

$$\mu_q : \tau(X) \longrightarrow W^1_5(\Gamma)\ , \quad \ell \mapsto [D_q],$$

where the brackets denote the class of a divisor in the Picard group.

Now let us verify that the inverse map μ_q^{-1} is well defined. Take a point $z \in W_5^1(\Gamma)$ representing a linear series $g_5^1(z)$. Then $|K-z|$ is a g_7^2 and we have two cases:

Case A. $|K-z-u-v|$ is a single effective divisor Z.

Case B. $|K-z-u-v|$ is a pencil $g_5^{1\prime} = \{Z(t)\}_{t\in\mathbb{P}^1}$.

In the case A, projecting down to $\mathbb{P}^4$, we get a single conic $C_2^0(z) = \langle Z\rangle \cap Q^3$ meeting Γ_q in 5 points. Here we have two subcases: either $C_2^0(z)$ is irreducible, or it is a reducible conic $\ell_i' \cup m$, where ℓ_i' is one of the trisecant lines of Γ_q, and m is a bisecant line. When $C_2^0(z)$ is irreducible, we define μ_q^{-1} at z by $\mu_q^{-1}(z) = \Psi_q^{-1}(C_2^0(z))$, where Ψ_q^{-1} applied to a curve denotes the proper transform of this curve. In the other subcase, ℓ_i' is a flopping curve. It has no proper transform in X, so $\mu_q^{-1}(z)$ should be determined by considering a limit of the curves $\mu_q^{-1}(w)$ when $w\to z$. We use the following general observation concerning any flop φ: when a member C_z of some algebraic family of curves $\{C_w\}_{w\in T}$ acquires an irreducible component which is a flopping curve, say ℓ', then the limiting curve $D_z = \lim_{w\to z}\varphi^{-1}(C_w)$ of the flopped family $\{D_w\}_{w\in T}$ does not contain the flopping curve ℓ, corresponding to ℓ', and is the proper transform of the remaining components of C_z:

$$D_z = \lim_{w\to z}\varphi^{-1}(C_w) = \varphi^{-1}(C_z\setminus \ell').$$

Moreover, D_z meets ℓ in this case. Thus, when $C_2^0(z) = \ell_i'\cup m$, we should put $\mu_q^{-1}(z) = \Psi_q^{-1}(m)$.

Now we will eliminate Case B. Assume that $|K-z-u-v|$ is a pencil. Then we can associate to z a pencil of conics $C_2^0(z,t) = \langle Z(t)\rangle\cap Q^3$, and a pencil of lines $\ell(z,t)$ in X, so that μ_q^{-1} is not defined at z. The pair $u+v$ is determined as the unique effective divisor in $|K-z-Z(t)|$. On the other hand, $u+v=\lambda(q)$. The generic point of $W_{10}^4 = K-\Gamma^{(2)}$ is not contained in the image of the sum map $W_5^1(\Gamma)\times W_5^1(\Gamma)\longrightarrow \mathrm{Pic}^{10}(\Gamma)$. By dimension reasons, to see this, it is sufficient to verify that for any $z\in W_5^1(\Gamma)$ there are finitely many $w\in W_5^1(\Gamma)$ such that $|K-z-w|$ is effective. This is stated in the following lemma.

Lemma 2.6. *For the generic $z\in W_5^1(\Gamma)$ there are exactly 8 distinct points $w\in W_5^1(\Gamma)$ such that $|K-z-w|$ is effective.*

Proof. The image $\overline{\Gamma}$ of Γ under the map given by the linear system $g_7^2 = |K-z|$ is a plane septic without triple points. Hence $\overline{\Gamma}$ has exactly 8 double points, defining 8 linear subseries g_5^1 in the given g_7^2. ◻

Now we see that for a generic conic q, $\lambda(q)$ cannot be represented as the sum of two g_5^1's, hence Case B is impossible.

To compute the genus of $\tau(X)$, we will use the approach and the notation from § 8 of [RS].

Let M be the base of the family of lines $\ell\subset\Sigma$ on the spinor 10-fold Σ. By loc. cit., the incidence family

$$G = \{(x,L) : x\in L\}\subset \Sigma\times M$$

together with the natural projection $\mathrm{pr}_1 : G\to\Sigma$ is nothing else but the Grassmannization $G = G(3,\mathcal{B})\to\Sigma$ of the universal subbundle $\mathcal{B}\to\Sigma\subset G(5,10)$.

Let h be the class of the hyperplane section of $\Sigma\subset\mathbb{P}^{15}$ let $b_i = c_i(\mathcal{B}), i=1,...,5$ be the Chern classes of $\mathcal{B}$, and let $u_i = c_i(\mathcal{U})$, $i=1,2,3$ be the Chern classes of the universal subbundle $\mathcal{U}\subset\mathcal{B}_G$ on $G=G(3,\mathcal{B})$; in particular $-u_1 = -c_1(\mathcal{U})$ is the class of the hyperplane section of the Plücker embedding $M \subset G(3,10)$. Then $h^{10} = \deg\ \Sigma = 12 \in \mathbb{Q} = H^{20}(\Sigma, \mathbb{Q})$,

$$H^*(\Sigma, \mathbb{Q}) \cong \mathbb{Q}[h, b_3]/(b_3^2 + 8b_3h^3 + 8h^6, 6h^5b_3 + 7h^8), \tag{1}$$

and the cohomology ring $H^*(G, \mathbb{Q})$ is generated as a $H^*(\Sigma, \mathbb{Q})$-algebra by u_1 and u_2:

$$H^*(G, \mathbb{Q}) \cong H^*(\Sigma, \mathbb{Q})[u_1, u_2]/(f, g), \tag{2}$$

where

$$f = h^4 - h^2u_2 - \frac{1}{2}u_2^2 - \frac{1}{2}b_3u_1 + 2h^3u_1 - 2hu_2u_1 + 3h^2u_1^2 - \frac{1}{2}u_2u_1^2 + 2hu_1^3 + \frac{1}{2}u_1^4$$

and

$$g = b_3h^2 - \frac{1}{2}b_3u_2 + 2h^3u_2 - hu_2^2 + b_3hu_1 + 3h^2u_2u_1 - 2u_2^2u_1 - \\ \frac{1}{2}b_3u_1^2 + 2h^2u_1^3 + \frac{1}{2}u_2u_1^3 + 2hu_1^4 + \frac{1}{2}u_1^5.$$

In particular, the definition of the universal subbundle $\mathcal{U} \to G = G(3, \mathcal{B})$ yields

$$u_1^6h^{10} = (-u_1)^6h^{10} = \deg G(3,5) \cdot \deg \Sigma = 5 \cdot 12 = 60 \in \mathbb{Q} = H^{32}(G, \mathbb{Q}). \tag{3}$$

The second projection $\mathrm{pr}_2 : G \to M$ is a projectivization of the rank-2 vector bundle $\mathcal{E} = \mathrm{pr}_{2*}\,\mathrm{pr}_1^*\,\mathcal{O}(h)$, and

$$H^*(G, \mathbb{Q}) \cong H^*(M, \mathbb{Q})[h]/(h^2 - c_1h + c_2)$$

where c_1, c_2 are the Chern classes of $\mathcal{E}$. Thus $c_1 = -u_1$ and $c_2 = -h^2 - u_1h$. We have also $K_M = 6u_1$.

Since $\tau(X) \subset M$ is the common zero locus of 7 general sections of $\mathcal{E}$, then $[\tau(X)] = c_2(\mathcal{E})^7 = (-h^2 - u_1h)^7$ and $K_{\tau(X)} = (K_M + 7c_1(\mathcal{E}))|_{\tau(X)} = -u_1|_{\tau(X)}$. Therefore $\tau(X) \subset M \subset G(3,10)$ is a canonical curve, and it remains to compute the degree

$$d = (-h^2 - u_1h)^7(-u_1)h \in H^*(G, \mathbb{Q})$$

of $\tau(X)$ with respect to the Plücker hyperplane class $-u_1$. This is done by reducing d modulo the relations specified in (1), (2), (3), and the answer is $d = 84$. Hence $\tau(X) \subset G(3,10)$ is a canonical curve of genus $g_{\tau(X)} = \frac{1}{2}d + 1 = 43$.

To prove (iii), note that $\deg R(X) = \deg \tau(X) = 84$, hence $R(X) \sim 7H$. For any line ℓ, $\deg \mathcal{N}_{\ell/X} = -1$, so the contribution of ℓ to the intersection number $\ell \cdot R(X)$ is -1, hence ℓ meets $R(X)$ in eight isolated points counted with multiplicities. As X is generic, neither of the lines on X is a double curve of $R(X)$ and the multiplicity of a point of $R(X)$ equals the number of lines passing through this point. Hence any line ℓ meets exactly 8 other lines. ∎

3. Abel–Jacobi map. Let $X = X_{12}$ be any transversal linear section $\mathbb{P}^8 \cap \Sigma$. Let $J^d(X)$ denote the set of classes of algebraic 1-cycles of degree d in X modulo rational equivalence. It has a natural structure of a principal homogeneous space under $J^0(X)$, and according to [BM], $J^0(X) = J(X)$ is nothing else but the intermediate Jacobian of X. Either of the birational isomorphisms Φ_y, Ψ_q can be used to identify $J(X)$ with the Jacobian $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$. It is more convenient to use Ψ_q. With the notation from the proof of Proposition 2.1, the identification goes as follows: $J(Q^3) = 0$, and the passage from Q^3 to X consists in blowing up only one irrational curve Γ_q followed by blowups of rational curves and their inverses. By [CG], only the blowup with nonrational center modifies the intermediate Jacobian, therefore $J(X) \simeq J(Q^3) \times J(\Gamma_q) \simeq J(\Gamma)$. This isomorphism is induced by the map $\Gamma_q \longrightarrow J^d(X)$, $u \mapsto [\Psi_q^{-1}(u)]$, where $d = \deg \Psi_q^{-1}(u)$. Here $\Psi_q^{-1}(u)$ is the image of the exceptional fiber $\sigma_Q^{-1}(u) \simeq \mathbb{P}^1$ of σ_Q over a point $u \in \Gamma_q$ under the map $\sigma_X \circ \varphi^{-1}$, where φ is the flop (see Diagram 2). It is irreducible for generic u and has a flopping curve as one of its components for a finite set of values of u corresponding to the points of intersection of trisecants with Γ_q. According to Theorem 5.5 of [IM-3], the curves $\Psi_q^{-1}(u)$ are the rational cubics meeting q twice. Applying the Abel–Jacobi functors provides the desired isomorphism $a_q^1 : \mathrm{Pic}^1(\Gamma) \xrightarrow{\sim} J^3(X)$.

As in loc. cit., we use the symbol $\mathcal{C}_d^g[k]_Z$ to denote the family of all the connected curves of genus g and degree d meeting k times a given subvariety Z of a given variety V. More precisely, let $Z \subset V$ be a nonsingular curve (resp. a point). Then $\mathcal{C}_d^g[k]_Z$ is the closure in the Chow variety of V of the family of reduced connected curves C of degree d such that length $(\mathcal{O}_X/(\mathcal{I}_C + \mathcal{I}_Z)) = k$ (resp. $\mathrm{mult}_Z\, C = k$) and $p_a(\tilde{C}) = g$, where $\tilde{C}$ is the proper transform of C in the blowup of Z in V.

We will summarize the above in the following lemma:

Lemma 3.1. *Let q be a generic conic in X. Then for any $k \in \mathbb{Z}$, there is a natural isomorphism*

$$a_q^k : \mathrm{Pic}^k(\Gamma) \xrightarrow{\sim} J^{3k}(X), \quad \Big[\sum n_i u_i\Big] \mapsto \Big[\sum n_i \Psi_q^{-1}(u_i)\Big],$$

depending on q.

All the curves $C \in \mathcal{C}_3^0[2]_q$, except for finitely many of them, are irreducible and their images $\Psi_q(C)$ are points of Γ_q. This yields a map $b_q : \mathcal{C}_3^0[2]_q \longrightarrow \mathrm{Pic}^1(\Gamma)$. With the identification $Pic^1(\Gamma) \xrightarrow{\sim} J^3(X)$ given by a_q^1, the map b_q is the Abel-Jacobi map of the family $\mathcal{C}_3^0[2]_q$.

Now we will study the Abel–Jacoby map of more general families of curves on X. We will use without mention the identification of $J^k(X)$ and $\mathrm{Pic}^d(\Gamma)$, which is determined by Lemma 3.1 uniquely modulo a constant translation. Remark also that $J(X) = J(\tilde{Q^3})$ in a natural way.

Lemma 3.2. *Let q be a generic conic in X. Let T be the base of an irreducible family of curves on X whose generic member is a reduced curve which intersects neither q, nor any of the flopping curves of Ψ_q. Assume that Ψ_q transforms the family parameterized by T into a subfamily of $\mathcal{C}_d^g[k]_{\Gamma_q}$ on Q^3. For generic $C \in T$, denote by Z_C or Z_C^q the intersection cycle $\Psi_q(C) \cap \Gamma_q$ considered as a degree-k divisor on Γ. It can be defined by the formula $Z_C = \sigma_{\Gamma *}(\tilde{C} \cdot E_Q)$, where $\tilde{C} = \varphi\sigma_X^{-1}(C)$ is the image of C in $\tilde{Q}^3$ and $\sigma_\Gamma : E_Q \longrightarrow \Gamma_q$ is the restriction of σ_Q. Then the Abel–Jacobi map for the family T is given, up to a constant translation, by $C \mapsto -[Z_C] \in \mathrm{Pic}(\Gamma)$.*

Proof. As $J(Q^3) = 0$, the Abel–Jacobi class of the pullback of any family of curves on Q^3 is a point. Hence the class of $\sigma_Q^{-1}\Psi_q(C)$ in $J^\bullet(\tilde{Q}^3)$ is a constant, say c. If $Z_C = \sum n_i u_i$ ($n_i \in \mathbb{N}$, $u_i \in \Gamma$), then $[\sigma_Q^{-1}\Psi_q(C))] = [C] + \sum n_i[\sigma_Q^{-1}(u_i)]$ and $[C] = c - \sum n_i[u_i]$, as was to be proved. □

Now we will invoke the exceptional curves of σ_X. By [IM-3], Theorem 5.5, their images in Q^3 are the elements of the family $\mathcal{C}_3^0[8]_{\Gamma_q}$. Hence to each curve $\sigma_X^{-1}(x)$ with $x \in q$ we can associate a degree-8 divisor on Γ, defined by $\sigma_Q \circ \varphi(\sigma_X^{-1}(x)) \cap \Gamma_q$. Its class in $\mathrm{Pic}^8(\Gamma)$ does not depend on $x \in q$, because q is rational. Denote it by d_8^q.

Lemma 3.3. *In the hypotheses of the previous lemma, assume that the generic curve C_t of T is of degree d and does not meet q. Let C_0 be a special member of T such that the scheme-theoretic intersection $C_0 \cap q = M$ is of length r. Let $\tilde{C}_t$ be the pullback of C_t to $\tilde{X}$ for $t \neq 0$, and $\tilde{C}_0$ the limit of $\tilde{C}_t$ as $t \to 0$. Assume that C_0 does not meet any of the flopping curves. Then the flop φ is locally an isomorphism in the neighbourhood of $\tilde{C}_0$ and all the nearby curves $\tilde{C}_t$, and the limit of $[Z_C]$ when $t \to 0$ is $\sigma_{\Gamma *}(\varphi(\tilde{C}_0) \cdot E_Q)$. This coincides with $[Z_{C_0}^q] + rd_8^q$ in the case when neither of the components of $\tilde{C}_0$ is contracted by σ_Q.*

Proof. Let $M = \sum n_i x_i$. Then $\tilde{C}_0 = C_0' + \sum n_i \sigma_X^{-1}(x_i)$, where C_0' is the proper transform of C_0. The result follows by applying σ_{Q*} to $\varphi(\tilde{C}_t) \cdot E_Q$ as $t \to 0$. □

Remark that $\sigma_{\Gamma *}(\sigma_q^{-1}(u) \cdot E_Q) = -u$, so the Abel–Jacobi image of $\sigma_Q^{-1}(u)$ is $[u]$, which agrees with Lemma 3.1.

In the proof of Propositions 2.1 and 2.2, we introduced the maps $\mu_q : \tau(X) \longrightarrow W_5^1(\Gamma)$ and $\lambda : \mathcal{F}(X) \longrightarrow \Gamma^{(2)} = W_2^0(\Gamma)$. They can be considered as maps to $\mathrm{Pic}(\Gamma)$.

Lemma 3.4. *The map $\mu = \mu_q$ does not depend on the choice of a generic conic q and is, up to a constant translation, the Abel-Jacobi map of the family of lines on X.*

Proof. For generic X, μ_q is an isomorphism of two nonsingular curves of genus 43. A curve of genus ≥ 2 has only finitely many automorphisms, hence μ_q does not depend on q for generic X. As we saw in the proof of Proposition 2.1, $\tau(X)$ remains a l. c. i. curve and is a zero locus of a section of a vector bundle for all nonsingular varieties X. Hence all of the components of $\tau(X)$ for the special (but still smooth) 3-folds X are in the limit of the family of curves $\tau(X)$ for nearby general 3-folds X. Hence μ_q does not depend on q by continuity on the special X, too.

The Ψ_q-image of a line ℓ not meeting q is a conic meeting Γ_q in a degree-5 divisor Z_ℓ^q, and

$$\mu_q(\ell) = K - \lambda(q) - [Z_\ell^q]. \tag{4}$$

By Lemma 3.2, μ_q is, up to a constant translation, the Abel–Jacobi map of the family of lines on X. □

In the following definition we generalize the formula (4) to curves of any degree.

Definition 3.5. Let $C \subset X$ be a curve of degree d, and q a sufficiently generic conic in X. This means that q is not a component of C, Ψ_q exists and C does not meet any of the flopping curves of Ψ_q. In this case the scheme-theoretic inverse image $\tilde{C} = \sigma_X^*(C)$ is mapped isomorphically by the flop φ to a curve in $\tilde{Q}^3$. Let length$(C \cap q) = r$ and $\Psi_q(C) \cap \Gamma = Z^q_C$. Define

$$AJ(C) = dK - d\lambda(q) - \sigma_{\Gamma *}(\varphi(\tilde{C}) \cdot E_Q) = \\ d(K - \lambda(q)) - rd_8^q - [Z^q_C] \in \mathrm{Pic}^{5d}(\Gamma). \quad (5)$$

We call $AJ(C)$ the canonical Abel–Jacobi image of C in $\mathrm{Pic}^{5d}(\Gamma)$.

Now we will determine the canonical Abel–Jacobi image of a conic.

LEMMA 3.6. *For a generic pair of conics q, q' on X,*

$$[Z^q_{q'}] = K - 2\lambda(q) + \lambda(q').$$

Proof. The Ψ_q-image of q' in Q^3 is a rational quartic $C^q_4(q') \subset Q^3$ intersecting Γ_q in a divisor $Z^q_{q'}$ of degree 10. From the ideal sheaf sequence for $C^q_4(q') \subset Q^3$ we obtain

$$h^0(\mathcal{I}_{C^q_4(q'), Q^3}(2)) \geq h^0(\mathcal{O}_{Q^3}(2)) - h^0(\mathcal{O}_{C^q_4(q')}(2)) = 14 - 9 = 5.$$

Therefore there exists a $\mathbb{P}^4$-family of quadric sections $S(t)$ of Q^3 through $C^q_4(q')$. Each of these $S(t)$ intersects Γ_q in a divisor $D_{20}(t) \sim 2K - 2\lambda(q)$ of degree 20 such that $D_{20}(t) = Z^q_{q'} + D_{10}(t)$ for an effective divisor $D_{10}(t)$ of degree 10 on Γ. Therefore $h^0(D_{10}(t)) \geq 5$. Since deg $D_{10}(t) = 10$ (and Γ is non-hyperelliptic), we have $h^0(D_{10}(t)) \geq 5$ and $D_{10}(t) = K - D_2(t)$ for some divisor $D_2(t)$ of degree 2. Again, as Γ is non-hyperelliptic, $D_2(t)$ does not depend on $t \in \mathbb{P}^4$.

Therefore $D_2(t) = D_2(q, q')$ depends only on q and q', and $Z^q_{q'} = 2H - D_{10}(t) = (2K - 2\lambda(q)) - (K - D_2(q, q')) = K - 2\lambda(q) + D_2(q, q')$.

If one regards q as a fixed conic and q' as a general one, then the map $q' \mapsto -[Z^q_{q'}]$ is, up to translation, the Abel–Jacobi map of the family of conics. It is obviously nonconstant. Indeed, assume the contrary. Then any two conics are rationally equivalent. Hence the sums $\ell + m$ of intersecting lines are all rationally equivalent. This implies that $W^1_5(\Gamma)$ is hyperelliptic and the curve F of pairs of intersecting lines is a g^1_2 on it, hence F is rational. This is absurd, for $F \subset \tau(X)^{(2)}$ is mapped injectively into $\mathcal{F}(X)$ and $\mathcal{F}(X) \simeq \Gamma^{(2)}$ does not contain rational curves. Therefore the Abel–Jacobi map of conics is nonconstant, and hence the map $q' \mapsto D_2(q, q')$ is nonconstant as well.

Thus the composition of this map with λ is a nonconstant self-map of $\Gamma^{(2)}$. By Proposition 8.1, it is the identity. Hence $D_2(q, q') = \lambda(q')$. ☐

COROLLARY 3.7. *The canonical Abel–Jacobi map $AJ|_{\mathcal{F}(X)}$ of the family of conics on X is given by the formula*

$$AJ(q) = K - \lambda(q) \quad \forall\, q \in \mathcal{F}(X).$$

PROPOSITION 3.8. *The map AJ defined by formula (5) does not depend on q, hence AJ induces a canonical isomorphism $J^d(X) \xrightarrow{\sim} \mathrm{Pic}^{5d}(\Gamma)$ such that $a^k_q \circ AJ$ is the translation by a constant depending only on k, q, d. For any two curves C_1, C_2 on X, we have*

$$AJ(C_1 + C_2) = AJ(C_1) + AJ(C_2).$$

Proof. By Lemma 3.4 and Corollary 3.7, the first statement of the proposition is true for lines and conics. The statement on the additivity of AJ is an immediate consequence of the definition, and we can use it to extend the first statement from lines and conics to curves of all degrees.

The Abel-Jacobi image of Γ in $J(\Gamma)$ (defined up to a translation) generates $J(\Gamma)$, hence the same is true for the Abel-Jacobi image of $\Gamma^{(2)}$. Hence the AJ-image of the family of conics generates $J(X) = J(\Gamma)$. This means that any algebraic 1-cycle on X is rationally equivalent to a linear combination of conics, and we are done. ∎

LEMMA 3.9. *For a generic conic $q \subset X$, the divisors of the linear system d_8^q on Γ, defined by the intersections of the extremal rational cubics $C_3 \in \mathcal{C}_3^0[8]_{\Gamma_q}$ with Γ_q, belong to the linear system $|K - 2\lambda(q)|$.*

Proof. We can assume Γ (or X) generic; the result for any Γ will follow by continuity. Consider the curve $D_q \subset \mathcal{F}(X)$ of conics q' in X intersecting q, defined as the closure of the set $\{q' \in \mathcal{F}(X) \mid \ q \cap q' \neq \varnothing,\ \#(q \cap q') < \infty\}$. Let $q' \in D_q$. Then $\Psi_q(q')$ is generically a bisecant line to Γ_q, so that $Z^q_{q'}$ is a pair of points. Using Corollary 3.7, Proposition 3.8 and Lemma 3.3, we can express the canonical Abel Jacobi image of q' in two different ways:

$$AJ(q') = K - \lambda(q') = 2K - 2\lambda(q) - d_8^q - [Z^q_{q'}],$$

where $Z^q_{q'} \in \Gamma^{(2)}$. Hence $[Z^q_{q'}] = c + \lambda(q')$ for some constant $c = c(q) \in \operatorname{Pic}^0(\Gamma)$ and for generic $q' \in D_q$.

Now extend this construction to the whole incidence 3-fold D, the closure in $\mathcal{F}(X) \times \mathcal{F}(X)$ of the set $\{(q,q') \mid q \cap q' \neq \varnothing,\ \#(q \cap q') < \infty\}$. Then we obtain the maps $h : D \longrightarrow \Gamma^{(2)}$, $(q,q') \mapsto Z^q_{q'}$, and $c : \mathcal{F}(X) \longrightarrow J(\Gamma)$, $q \mapsto c(q)$, such that $h(D) = \bigcup_{q \in \mathcal{F}(X)} (c(q) + \lambda(D_q)) \subset \Gamma^{(2)}$. Assume that $c(q) \neq 0$ for some q. Then there is a one-parameter family of distinct representations of $c(q)$ as the difference $w(t) - z(t)$ of points $z(t), w(t) = z(t) + c(q) \in \Gamma^{(2)}$, parameterized by $t \in D_q$. Hence $w(t) + z(t') = w(t') + z(t)$ in $\operatorname{Pic}^4(\Gamma)$ for t, t' moving in the same connected component of D_q. This either implies the existence of a linear series g^1_4 on Γ, or $D_q = u + \Gamma$, $c(q) = v - u$ for some $u, v \in \Gamma$. The first alternative is impossible, see [Mu-1]. The second one is also false. Indeed, the lines spanned by the pairs $Z^q_{q'}$ for $q' \in D_q$ are secant lines of Γ_q contained in Q, but not all such secant lines pass through a given point $v \in \Gamma_q$. Hence $c(q) \equiv 0$ and we are done. ∎

COROLLARY 3.10. *On the family $\mathcal{C}_3^0[2]_q$, the canonical Abel–Jacobi map is given by*

$$AJ(C) = K + \lambda(q) + [\Psi_q(C)] \textit{ for generic } q \in \mathcal{F}(X) \textit{ and } C \in \mathcal{C}_3^0[2]_q.$$

Proof. In the notation of Proposition 3.8, $\tilde{C} = \sigma_X^{-1}(x_1) + \sigma_X^{-1}(x_2) + \sigma_\Gamma^{-1}(u)$ for some $x_1, x_2 \in q$, $u \in \Gamma_q$. Then $\sigma_{\Gamma *}(\tilde{C} \cdot E_Q) = 2d_8^q - u$. The result now follows from Proposition 3.8 and Lemma 3.9. ∎

This still holds for a special cubic C_3^0 of the form $q_0' + \ell$, where q, q_0', ℓ intersect each other with multiplicity 1. Then ℓ is a flopping line of Ψ_q, and q_0' is a special element of D_q (notation from the proof of Lemma 3.9). The flopping curve in Q^3 corresponding to ℓ is a trisecant ℓ' to Γ_q, and if $\ell' \cap \Gamma_q = u_1 + u_2 + u_3$, then the image of q_0 in $\tilde{Q}^3$ is the exceptional curve $\sigma_Q^{-1}(u_i)$ for one of the values of $i = 1, 2, 3$, say $i = 3$. The limit of the proper transforms of the curves $q' \in D_q$ as $q' \to q'_0$ is the reducible curve $\sigma_Q^{-1}(u_3) + \tilde{\ell}'$, so that $AJ(q'_0)$ is given by the same formula as above with $Z^q_{q'_0} = u_1 + u_2$. This implies:

COROLLARY 3.11. *If, in the above notation, q, q'_0, ℓ intersect each other with multiplicity 1, then $AJ(\ell) = u_1 + u_2 + u_3 + \lambda(q)$ and $AJ(q'_0) = K - u_1 - u_2$.*

We can apply the results of this section to obtain some additional information on lines, conics and the map Ψ_q. First, we can characterize the curve of reducible conics in $\mathcal{F}(X)$.

LEMMA 3.12. *Let ℓ, m be two distinct lines in X. Then $\ell \cap m \neq \varnothing$ if and only if $|K - \mu(\ell) - \mu(m)|$ is nonempty. In this case $\ell \cup m$ is a reducible conic and $\lambda(\ell \cup m) = K - \mu(\ell) - \mu(m)$.*

Proof. This follows immediately from the existence of the canonical Abel–Jacobi map AJ such that $AJ(\ell \cup m) = AJ(\ell) + AJ(m)$ and from Lemma 3.4 and Corollary 3.7. □

The next lemma answers the question, which lines should be considered as lines "intersecting themselves".

LEMMA 3.13. *Let ℓ be a line in X. Then there is a double structure on ℓ making it a conic in X if and only if ℓ is a singular point of $\tau(X)$.*

Proof. Assume that the normal sheaf of ℓ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let C be a *plane* double structure on a line ℓ , that is, a double structure embeddable into $\mathbb{P}^2$. Any Gorenstein doubling ℓ is given by Ferrand's construction [F], [BanF] and is associated to a surjective morphism of $\mathcal{O}_\ell$-modules $\mathcal{N}^\vee_{\ell/X} \longrightarrow \mathcal{L}$, where $\mathcal{L}$ is some invertible sheaf on ℓ. The kernel of the surjection can be represented in the form $\mathcal{J}/\mathcal{I}^2_\ell$ for an ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$, and this ideal sheaf defines the Ferrand's double structure C on ℓ: $\mathcal{J} = \mathcal{I}_C$. The dualizing sheaf of Ferrand's double structure satisfies $\omega_C|_\ell \simeq \omega_\ell \otimes \mathcal{L}^{-1}$. Applying this to our situation, we see that $\mathcal{L} \simeq \mathcal{O}_\ell(k)$ for some $k \geq 0$, hence $\omega_C|_\ell \simeq \mathcal{O}_\ell(-2-k)$, which contradicts the property $\omega_C|_\ell \simeq \mathcal{O}_\ell(-1)$ verified for a *plane* doubling of ℓ.

For a line ℓ with normal sheaf $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, the surjection $\mathcal{N}^\vee_{\ell/X} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2)$ defines a unique plane double structure on ℓ. □

COROLLARY 3.14. *Let ℓ be a generic line on X. Then there are exactly* 8 *distinct lines ℓ_i such that $\ell + \ell_i$ is a conic. They satisfy the condition $K - \mu(\ell) - \mu(\ell_i) \in \Gamma^{(2)}$.*

If the normal bundle of ℓ is of type $(0, -1)$, *then the lines ℓ_i meet ℓ and are different from ℓ.*

If the normal bundle of ℓ is of type $(1, -2)$, *then $K - 2\mu(\ell) \in \Gamma^{(2)}$. In this case, only one of the ℓ_i coincides with ℓ and the* 7 *others are distinct and different from ℓ.*

Proof. This follows from Lemmas 2.6 and 3.12. □

4. Rational normal curves in X**.** Let $X = X_{12} = \mathbb{P}^8 \cap \Sigma$ be a Fano 3-dimensional linear section of the spinor tenfold Σ and $\Gamma = \check{X}$ its orthogonal curve. We will use the symbol $\mathcal{C}^g_d(X)$, or simply $\mathcal{C}^g_d$, to denote some families of degree-d curves of genus g in X, whose precise definitions will be given in the context, and C^g_d to denote a member of such a family. A *rational normal curve* of degree d in X is an irreducible nonsingular curve C in X such that $\deg C = d$ and $\dim\langle C\rangle = d$. Let $\mathcal{C}^0_d$ be the family of rational normal curves of degree d in X.

Lemma 4.1. *The family $\mathcal{C}_3^0(X)$ of rational normal cubics in X is irreducible,* 3-*dimensional and is birational to the symmetric cube $\Gamma^{(3)}$ of the curve Γ. The normal bundle of a generic $C_3^0 \in \mathcal{C}_3^0(X)$ is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

Proof. The family of rational normal cubics in X was studied in [Isk-P], 4.6.1–4.6.4. The authors determined the normal bundle and constructed the birational transformations $\Psi_{C_3^0}$ associated to sufficiently general rational normal cubics $C_3^0 \in \mathcal{C}_3^0(X)$. We will explain how the existence of these transformations implies the irreducibility of $\mathcal{C}_3^0(X)$. Theorem 4.6.4 in loc. cit. provides the inverse construction, which permits to reconstruct $C_3^0 \subset X$ starting from any sufficiently general $\Gamma_9^7 \subset \mathbb{P}^3$. Recall that Γ_9^7 is a projection of $\Gamma \subset \mathbb{P}^6$ from a unique triple of points $u, v, w \in \Gamma$. Hence the open part of $\mathcal{C}_3^0(X)$ consisting of those cubics C_3^0 for which the map $\Psi_{C_3^0}$ exists is birational to the symmetric cube $\Gamma^{(3)}$ and is irreducible. ∎

Lemma 4.2. *Let C be a connected Cohen–Macaulay curve of degree* 4 *in X. Then the following assertions hold:*

(i) $\dim\langle C\rangle = 4$.

(ii) If C is reduced and irreducible, then it is a rational normal quartic.

(iii) If C is the union of two conics $q \cup q'$ such that $q \cap q' \neq \varnothing$ and $\#(q \cap q') < \infty$, then q, q' meet each other quasitransversely at a single point.

(iv) C has no singular points of multiplicity ≥ 3 and $p_a(C) = 0$.

Proof. All the assertions are easy consequences of the fact that $\langle C\rangle = \mathbb{P}^4$. The latter follows from the non-existence of 2-planes that are 4-secant to X. Indeed, assume that $\langle C\rangle = \mathbb{P}^3$. Then for any plane $\mathbb{P}^2$ in this $\mathbb{P}^3$, $\mathbb{P}^2 \cap X$ contains at least the 4 points of $\mathbb{P}^2 \cap C$ (counted with multiplicities). By Lemma 1.3, we obtain a three-dimensional family of conics or lines in X, which is absurd. If $\langle C\rangle = \mathbb{P}^2$, then $\langle C\rangle \cap X$ is an intersection of quadrics, hence coincides with $\langle C\rangle = \mathbb{P}^2$. This is absurd, as X does not contain planes. Hence $\langle C\rangle = \mathbb{P}^4$. ∎

Lemma 4.3. *Let X be generic. Then the family $\mathcal{C}_4^0(X)$ of rational normal quartics in X is irreducible.*

Proof. Consider the family I of all pairs (C_4^0, X), where X is a Fano 3-fold section of the spinor 10-fold Σ and $C_4^0 \in \mathcal{C}_4^0(X)$. It has two natural projections $p : I \longrightarrow \mathcal{C}_4^0(\Sigma)$ and $q : I \longrightarrow G(9,16)$, $q : X \mapsto \langle X\rangle = \mathbb{P}^8 \subset \mathbb{P}^{15}$, where $\mathcal{C}_4^0(\Sigma)$ is the family of rational normal quartics in Σ. A nonempty fiber $q^{-1}(u)$ is the family $\mathcal{C}_4^0(X_u)$, where $X_u = \mathbb{P}^8_u \cap \Sigma$, and $p^{-1}(C_4^0)$ is an open subset of the Grassmannian $G(4,11)$ parametrizing the subspaces $\mathbb{P}^8 \subset \mathbb{P}^{15}$ which contain $\mathbb{P}^4 = \langle C_4^0\rangle$. By a standard monodromy argument, the irreducibility of the generic fiber $q^{-1}(u)$ will follow from the following two facts: (1) I is irreducible; (2) simultaneously for all sufficiently general u, one can choose in the fiber $q^{-1}(u)$ one distinguished irreducible component depending rationally on u. As the fibers of p are irreducible, the first fact is equivalent to the irreducibility of $\mathcal{C}_4^0(\Sigma)$. The latter follows from [P-1], where the author proves that the Hilbert scheme $\mathrm{Hilb}^\alpha_\Sigma$ of irreducible nonsingular rational curves of class α in a complex projective homogeneous manifold Σ is smooth and irreducible when $\dim \Sigma \geq 3$ and α is strictly positive. The last condition holds in our situation, because $\mathrm{Pic}\,\Sigma \simeq \mathbb{Z}$.

Now we will produce a distinguished component $\mathcal{C}_4^{0*}$ of $\mathcal{C}_4^0(X)$ for a fixed X. Let C_3^0 be a generic rational normal cubic in X. It intersects the surface $R(X)$ swept by the lines in X at a finite number of points. Hence there is at least one line ℓ in X

meeting C_3^0. Such a line cannot intersect C_3^0 in a scheme of length ≥ 2, for then the quartic $C_3^0 \cup \ell$ will span $\mathbb{P}^3$, which is impossible by Lemma 4.2. Therefore the family $\mathcal{C}_{3,1}^0$ of reducible quartics $C_4^0 = C_3^0 \cup \ell$, where $C_3^0 \in \mathcal{C}_3^0$, ℓ is a line and $\mathrm{length}(C_3^0 \cap \ell) = 1$, is a finite cover of $\mathcal{C}_3^0$. It is 3-dimensional. By the standard normal bundle sequence for a reducible nodal curve, $\chi(\mathcal{N}_{C_4^0/X}) = 4$, so $\dim_{[C_4^0]} \mathrm{Hilb}_X \geq 4$ and hence C_4^0 can be deformed into a smooth rational normal quartic. We define $\mathcal{C}_4^{0*}$ to be the component containing the smoothings of curves from $\mathcal{C}_{3,1}^0$, but for this we need to prove the irreducibility of $\mathcal{C}_{3,1}^0$.

Let $C_3^0 \in \mathcal{C}_3^0$ be sufficiently generic. Then the lines ℓ such that $C_3^0 \cup \ell \in \mathcal{C}_{3,1}^0$ are the flopping curves of $\Psi_{C_3^0}$ (see Diagram 3 of Section 1). Hence they are in a bijective correspondence with the quadrisecants of Γ_9^7. Let $u+v+w \in \Gamma^{(3)}$ be the triple of points of Γ associated to Γ_9^7. Let L be a quadrisecant of Γ_9^7 and $L \cap \Gamma_9^7 = u_1 + u_2 + u_3 + u_4$. Then the span of the divisor $D = u_1+u_2+u_3+u_4+u+v+w$ in $\mathbb{P}^6$ is $\mathbb{P}^4$, hence D belongs to a linear series g_7^2. Let us denote by G_d^r the subset of $\Gamma^{(r)}$ which is the union of all the linear series g_7^2. As a generic Γ has no g_7^3, the natural map $\pi : G_7^2 \longrightarrow W_7^2$ is a $\mathbb{P}^2$-bundle over the smooth curve $W_7^2 \simeq W_5^1$ and the quadrisecants of Γ_9^7 are in a bijective correspondence with the elements of the subset $\{D \in G_7^2 \mid D - u - v - w \text{ is effective}\}$.

Let $I^{(k)} = \{(F, D) \in \Gamma^{(k)} \times G_7^2 \mid D - F \text{ is effective}\}$ $(1 \leq k \leq 7)$, and let $q_k : I^{(k)} \longrightarrow G_7^2$ be the natural projection. We have identified a dense open subset of $\mathcal{C}_{3,1}^0$ with that of $I^{(3)}$. So we have to show that $I^{(3)}$ is irreducible. The map $q_3 : I^{(3)} \longrightarrow G_7^2$ is a 35-sheeted covering obtained by applying the relative symmetric cube to the 7-sheeted covering q_1. Hence it suffices to prove that the monodromy group M permuting the sheets of q_1 is the whole of S_7. This follows from two facts: (a) M is transitive, that is $I^{(1)}$ is irreducible, and (b) M is generated by transpositions.

To verify (a), restrict q_1 to the fiber $\mathbb{P}^2$ of π over a general $g_7^2 \in W_7^2$. An orbit of length k of M gives rise to a k-valued multisection of $q_1|_{q_1^{-1}(\mathbb{P}^2)}$, or equivalently, to a map $\mathbb{P}^2 \longrightarrow \Gamma^{(k)}$. But $\Gamma^{(k)}$ does not contain rational curves for $k < 5$, since Γ has no linear series of degree $k < 5$. If we assume that M is not transitive, then there is an orbit of length $k < 4$ and the above map $\mathbb{P}^2 \longrightarrow \Gamma^{(k)}$ is constant, which immediately leads to a contradiction. Hence M is transitive.

To verify (b), one can show that the ramification of $q_1|_{q_1^{-1}(\mathbb{P}^2)}$ is simple in codimension 1. This follows from the observation that all the divisors from the linear series g_7^2 are obtained as the intersections $L \cap \Gamma_0$, where $\Gamma_0 \subset \mathbb{P}^2$ is the image of Γ under the map given by the g_7^2 and L runs over the lines in $\mathbb{P}^2$. The ramification points of q_1 correspond to the points of tangency of L to Γ_0, and the ramification is simple when L is a simple tangent to Γ_0. But for g_7^2 generic, Γ_0 is a nodal septic of genus 7 having only finitely many flexes or bitangents. Hence the ramification of q_1 is simple in codimension 1. $\square$

LEMMA 4.4. *Let X be generic. Then the family $\mathcal{C}_4^0(X)$ of rational normal quartics in X is 4-dimensional and the normal bundle of a generic quartic $C_4^0 \in \mathcal{C}_4^0(X)$ is either $2\mathcal{O}_{\mathbb{P}^1}(1)$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.*

Proof. Take a generic pair of intersecting conics $q \cup q'$. Both q and q' have normal bundle $2\mathcal{O}$. The strong smoothability of $q \cup q'$ is proved by a standard application of the Hartshorne–Hirschowitz techniques [HH], so $q \cup q'$ is represented by a smooth point of the closure $\overline{\mathcal{C}}_4^0$ of $\mathcal{C}_4^0$ in Hilb_X and $\dim \mathcal{C}_4^0 = \chi(\mathcal{N}_{q \cup q'/X}) = 4$. For an example of such argument see Lemma 1.2 of [MT-2].

From the semicontinuity of $h^1(\mathcal{N}_{C/X})$ for $C \in \overline{\mathcal{C}}_4^0$ and the fact that

$h^1(\mathcal{N}_{q\cup q'/X}(-x)) = 0$ for a point $x \in q \smallsetminus q'$, we deduce that $h^1(\mathcal{N}_{C/X}(-x)) = 0$ for generic $C \in \mathcal{C}_4^0$ and $x \in C$. This implies the assertion on the normal bundle. ☐

LEMMA 4.5. *Let X and $C_4^0 \subset X$ be generic. Then $\mathcal{N}_{C_4^0/X} \simeq 2\mathcal{O}_{\mathbb{P}^1}(1)$.*

Proof. Assume that $\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ for generic $C = C_4^0$. Let $p \in C$ be a generic point. Let $H(p) \subset \mathrm{Hilb}_X$ be the the closure of the family of rational normal quartics in X passing through p. It can be identified with a closed subscheme of $\mathrm{Hilb}_{\tilde{X}}$, where $\tilde{X}$ is the blowup of p in X. Let $\tilde{C}$ be the proper transform of C in $\tilde{X}$. We have $\mathcal{N}_{\tilde{C}/\tilde{X}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and the tangent space to $H(p)$ at $[C]$ is identified with $H^0(\mathcal{N}_{\tilde{C}/\tilde{X}})$. Since $H^1(\mathcal{N}_{\tilde{C}/\tilde{X}}) = 0$, $H(p)$ is smooth at $[C]$ of dimension 2, and there is a unique component $H(C,p)$ of $H(p)$ containing C. By our assumption, the proper transform $\tilde{C}'$ of a generic quartic C' in $H(C,p)$ has the same normal sheaf. Let F be the universal family of curves $\tilde{C}'$ over $H(C,p)$ and $\pi : F \longrightarrow \tilde{X}$ the natural map. For generic C', considered as a fiber of F over the point $[C'] \in H(C,p)$, we have $\mathcal{N}_{C'/F} \simeq 2\mathcal{O}$, and for its image $\tilde{C}' = \pi(C')$ in $\tilde{X}$, $\mathcal{N}_{\tilde{C}'/\tilde{X}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$. As any map from $2\mathcal{O}$ to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ has its image in the second factor $\mathcal{O}(1)$, the differential of π is degenerate at the points of C', hence also at the generic point of F. By the Sard theorem, $\pi(F)$ is a surface. Hence all the rational normal quartics in $H(C,p)$ sweep out a surface in X, say $S(C,p)$.

Let $p' \neq p$ be another generic point of C. Then there is a 1-dimensional family of rational normal quartics in $S(p)$ passing through both p, p'. The curves of this 1-dimensional family cover an open set of $S(C,p)$ and of $S(C,p')$. This implies that $S(C,p') = S(C,p)$. We can also replace C by a generic curve C' in $H(C,p)$, then take generic $p'' \neq p'$ in C' and see that $S(C,p) = S(C',p') = S(C',p'')$. This implies, in particular, that $S(C,p)$ is generically smooth at p and that $S(C,p)$ contains a 3-dimensional family of rational normal quartics. The 4-dimensional family of rational normal quartics in X is thus rationally fibered over some irreducible curve B into 3-dimensional families H_t, $t \in B$, such that the curves parameterized by H_t for generic t cover a surface $S_t \subset X$ ($S_t = S(C,p)$ for some $p \in C$, $C \in H_t$). The existence of a three-dimensional covering family of rational curves implies the rationality of S_t for generic t.

Let $S = S_t$ for generic $t \in B$ and T the minimal desingularization of S. We have already seen that there are 1-dimensional families of rational normal quartics passing through two generic points p, p' of S. By the "bend and break argument" ([Kol-2], Corollary II.5.6), there is a reducible member in every such pencil. Thus S is covered by conics. As it is a rational surface, there is a linear pencil of conics on S. This contradicts the non-existence of rational curves in the symmetric square of Γ and Proposition 2.2. ☐

LEMMA 4.6. *Let X and $C_4^0 \subset X$ be generic. Then C_4^0 is the scheme-theoretic intersection $\mathbb{P}^4 \cap X$, where $\mathbb{P}^4 = \langle C_4^0 \rangle$ is the linear span of C_4^0.*

Proof. If $\mathbb{P}^4 \cap X$ contains a point p of the secant 3-fold of C_4^0, then $\mathbb{P}^4 \cap X$ contains also the secant line ℓ of C_4^0 passing through p, because X is an intersection of quadrics. But a generic $C = C_4^0$ in X has no secant lines contained in X. Indeed, if p, p' are the points of $\ell \cap C$ for a line $\ell \subset X$, then the infinitesimal deformations of C with fixed point p are given by $H^0(\mathcal{N}_{\tilde{C}/\tilde{X}})$ in the notation of Lemma 4.5. We have $\mathcal{N}_{\tilde{C}/\tilde{X}} \simeq 2\mathcal{O}$, so the infinitesimal deformations lift to algebraic ones and $H^0(\mathcal{N}_{\tilde{C}/\tilde{X}})$ generates $\mathcal{N}_{\tilde{C}/\tilde{X}}$ at p', hence C can be moved off ℓ near p' inside the family of quartics passing through p. The line ℓ cannot deform with C, for C meets the surface swept by lines in a finite number of points, and there are only finitely many lines through p (Proposition 4.2.2, (iv) of [Isk-P]).

Assume now that $\mathbb{P}^4 \cap X$ contains a point p not on the secant threefold of C. Then there is a 1-dimensional family of 3-secant planes $\mathbb{P}^2_t$ to C through p, parameterized by the points t of some curve B. These planes are 4-secant to X, hence, by Lemma 1.3, they contain conics q_t lying in X and passing through p. All these conics lie in $\mathbb{P}^4 = \langle C \rangle$ and sweep out a surface in X. But $\operatorname{Pic} X \simeq \mathbb{Z}$, so the linear span of a surface in X is at least $\mathbb{P}^6$. The obtained contradiction proves that $\mathbb{P}^4 \cap X = C$ set-theoretically.

Assume now that $\mathbb{P}^4 \cap X$ has an embedded component supported at $z \in C$. Then there is a line $L \neq T_pC$ in $\mathbb{P}^4$ passing through z and tangent to X. Choose any $p \in L \smallsetminus \{z\}$. Then there exists a 3-secant $\mathbb{P}^2$ to C passing through p and z. It is 4-secant to X, as the intersection of $\mathbb{P}^2$ with X at z is multiple. Hence $\mathbb{P}^2 \cap X$ is a conic q passing through z in the direction of L. Then $\mathbb{P}^4 \cap X$ contains a point p' of q which does not lie in the secant variety of C, which contradicts to what we have proved. $\square$

5. Elliptic sextics in X. An *elliptic sextic* in X is a nonsingular irreducible curve $C \subset X$ of genus 1 and of degree 6. We will also deal with degenerate "elliptic" sextics, which we will call just *quasi-elliptic sextics*. A quasi-elliptic sextic is a locally complete intersection curve C of degree 6 in X, such that $h^0(\mathcal{O}_C) = 1$ and the canonical sheaf of C is trivial: $\omega_C = \mathcal{O}_C$. A reduced quasi-elliptic sextic will be called a *good sextic*.

Lemma 5.1. *Let q be a generic conic on X. Then X contains a 2-dimensional family of good sextics of the form $C_4^0 \cup q$, such that C_4^0 is a rational normal quartic and* $\operatorname{length}(C_4^0 \cap q) = 2$*, that is C_4^0, q meet each other quasitransversely in 2 distinct points or are mutually tangent at a single point. For a generic sextic of this form, $C_4^0 \cap q$ is a pair of distinct points.*

If we let q vary, then the family $\mathcal{C}^1_{4,2}$ of good sextics of type $C_4^0 \cup q$ is irreducible and 4-dimensional.

For any good sextic C in X, $\langle C \rangle = \mathbb{P}^5$.

Proof. Let q be a generic conic. Assume that there exists a reduced quartic C_4^0 passing through two distinct points x, y of q, or which is tangent to q at one point $x = y$. We have $l = \operatorname{length}(C_4^0 \cap q) = 2$, for if $l \geq 3$, then $\deg \Psi_q(C_4^0) = 2 \deg C_4^0 - 3l = 8 - 3l < 0$, which is absurd. In fact, the only irreducible curves $C \subset X$ whose degree with respect to the linear system defining Ψ_q is negative are components of the reducible members of the family $\mathcal{C}_3^0[2]_q$ contracted by Ψ_q, so $\deg C \leq 2$.

The birational map Ψ_q transforms C_4^0 into a conic meeting Γ_q at 4 points, u_1, u_2, u_3, u_4. These points span a plane $\mathbb{P}^2$. As in the proof of Proposition 2.1, consider Γ_q as the projection of the canonical curve Γ from the line $\overline{uv} \subset \mathbb{P}^6$, where $\lambda(q) = u + v$. Then $\langle u_1, u_2, u_3, u_4, u, v \rangle = \mathbb{P}^4$ and, by the geometric Riemann–Roch Theorem, $\sum u_i + u + v \in G^1_6 = G^1_6(\Gamma)$, where G^r_d denotes the union of all linear series g^r_d on Γ; we keep the notation W^r_d for the Brill–Noether locus of classes of such divisors in $\operatorname{Pic}^d(\Gamma)$.

Assume that Γ (or equivalently, X) is generic. By [ACGH], G^1_6 is a $\mathbb{P}^1$-bundle over W^1_6, both G^1_6 and W^1_6 are nonsingular, irreducible, and $\dim W^1_6 = 3$.

Thus we have constructed a map $\mathcal{C}_4^0[2]_q \longrightarrow G_q$, where $G_q \subset G^1_6$ is the subset of divisors D with $D - u - v$ effective. It is obvious that G_q is 2-dimensional. In fact, for generic $k \leq 4$ points $z_1, \dots, z_k \in \Gamma$, the dimension of $G_{z_1,\dots,z_k} = \{D \in G^1_6 \mid D - \sum z_i \text{ is effective}\}$ is equal to $4-k$.

It is easy to construct the inverse map: take a divisor $D \in G_q$ and let $D-u-v = u_1+u_2+u_3+u_4$. Then, after projecting to $\mathbb{P}^4$ from $\overline{uv}$, we have $\langle u_1, u_2, u_3, u_4\rangle_{\mathbb{P}^4} = \mathbb{P}^2$. As Q^3 does not contain planes, $\mathbb{P}^2 \cap Q^3$ is a conic, say C_2, and $C^0_4 := \Psi^*_q(C_2) \in \mathcal{C}^0_4[2]_q$. The scheme-theoretic intersection $C^0_4 \cap q$ is either two distinct points, or one point with multiplicity 2.

We have seen that $\mathcal{C}^0_4[2]_q$ is nonempty, 2-dimensional and birational to G_q. Take another generic conic q', and let $\lambda(q') = u' + v'$. Then $G_q \cap G_{q'} = G_{u,v,u',v'}$ is finite. Hence the union of $\mathcal{C}^0_4[2]_q$ when q runs over an appropriate open subset $U \subset \mathcal{F}(X)$ is 4-dimensional. This implies that the generic quartic from this union is irreducible, for the family of reducible quartics in X is 3-dimensional. For the pairs of intersecting conics, this follows from the fact that for a generic $x \in X$, there are only finitely many (namely, 24) conics passing through x, see Section 1. For the pairs of type a cubic plus a line, use Lemma 4.1.

We have seen that the family of good sextics of the form $C^0_4 + q$ is birational to $I^{(2)}$, where $I^{(k)} = \{(F, D) \in \Gamma^{(k)} \times G^1_6 \mid D - F \text{ is effective}\}$. The irreducibility of $I^{(2)}$ is proved in the same way as in Lemma 4.3. Denote by q_k the natural projection to G^1_6 and restrict to a generic pencil $\mathbb{P}^1 = g^1_6 \subset G^1_6$. The 6-sheeted covering $q_1^{-1}(\mathbb{P}^1) \longrightarrow \mathbb{P}^1$ has only simple ramifications, hence its monodromy is the whole of S_6 and all the $I^{(k)}$ for $k = 1, \dots, 6$ are irreducible.

The fact that $C^0_4 \cap q$ is generically a pair of distinct points follows from the degeneration of C^0_4 to a curve of the form $C^0_3 + \ell$, where $\text{length}(C^0_3 \cap q) = 2$, that is $C^0_3 \in \mathcal{C}^0_3[2]_q$ in the notation of Section 3. But the family $\mathcal{C}^0_3[2]_q$ is well understood: all its members are smooth rational curves contracted by σ_Q, except for 14 reducible members of the form $q_i + \ell_i$, where ℓ_i are the flopping lines of Ψ_q, and q_i, ℓ_i are unisecant to q. Hence the generic $C^0_3 \in \mathcal{C}^0_3[2]_q$ meets q at two distinct points, and the same is true for a generic $C^0_4 \in \mathcal{C}^0_4[2]_q$.

Now, let C be any good sextic in X. Assume that the linear span of C is strictly smaller than $\mathbb{P}^5$. Let, for example, $\langle C\rangle = \mathbb{P}^4$. Then the projection from a general secant line $< x, y >$, $x, y \in C$ sends C to a quartic curve $\overline{C} \subset \mathbb{P}^2$ with at least two double points giving rise to two 4-secant planes to C passing through $< x, y >$. By Lemma 1.3, these planes meet X along two conics passing through x, y, which contradicts Lemma 4.2, (iii). ◻

Proposition 5.2. *There is a distinguished* 6*-dimensional irreducible component* $\mathcal{C}^{1*}_6(X)$ *of the family of elliptic sextics in* X *satisfying the following properties:*

(i) The closure $\overline{\mathcal{C}}^{1*}_6(X)$ *of* $\mathcal{C}^{1*}_6(X)$ *in* Hilb_X *contains the* 4*-dimensional family* $\mathcal{C}^1_{4,2}$ *of reducible good sextics of the form* $C^0_4 + q$ *introduced in Lemma* 5.1.

(ii) A generic good sextic of the form $C^0_4 + q$ *is a smooth point of* Hilb_X.

(iii) A generic good sextic of the form $C^0_4 + q$ *can be partially smoothed to an irreducible rational curve with only one node, and such partial smoothings fill a five-dimensional subfamily of* $\overline{\mathcal{C}}^{1*}_6(X)$.

Proof. For $C = C_1 \cup C_2$ with $C_1 = q$, $C_2 = C^0_4$, we have the following exact sequences [HH]:

$$0 \to \mathcal{N}_{C/X} \to \bigoplus_{i=1}^{2} N_{C/X}|_{C_i} \xrightarrow{\alpha} \mathcal{N}_{C/X}|_Z \to 0, \qquad \text{length}(\mathcal{N}_{C/X}|_Z) = 4,$$

$$0 \to \mathcal{N}_{C_i/X} \to \mathcal{N}_{C/X}|_{C_i} \xrightarrow{\varepsilon_i} T^1_Z \to 0, \quad i = 1, 2,$$

$$0 \to \mathcal{N}_{C/X}|_{C_i}(-Z) \to \mathcal{N}_{C/X} \xrightarrow{R_i} \mathcal{N}_{C/X}|_{C_{2-i}} \to 0, \quad i = 1, 2,$$

where $Z = C_1 \cap C_2$, α is the difference map $(s_1, s_2) \mapsto (s_2 - s_1)|_Z$ and T^1_Z is the Schlesinger sheaf of infinetisimal deformations of singularities of C. For generic C, Z is a pair of distinct points and T^1_Z is a sky-scraper sheaf with 1-dimensional fibers at points of Z, so that $\mathrm{length}(T^1_Z) = 2$.

A sufficient condition for the smoothness of Hilb_X at C is $h^1(\mathcal{N}_{C/X}) = 0$. If it is verified, then the smoothability of C is equivalent to the following condition: the image of the composition

$$H^0(\mathcal{N}_{C/X}) \xrightarrow{H^0 R_i} H^0(\mathcal{N}_{C/X}|_{C_{2-i}}) \xrightarrow{H^0 \varepsilon_{2-i}} H^0(T^1_Z) \tag{6}$$

generates the sheaf T^1_Z for at least one value of i. The property (iii) is equivalent to saying that one can smooth by a small analytic deformation only one node in a general curve of type $C^0_4 + q$. A sufficient condition which assures the existence of such a partial smoothing is the surjectivity of the map (6) for at least one value of i.

The three conditions are obviously verified if $\mathcal{N}_{C/X}|_{C_i} \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $a > 0, b > 0$ for one value of i and $a \geq 0, b \geq 0$ for the other. The second exact sequence, Proposition 2.2, (ii), and Lemma 4.5 imply that $\mathcal{N}_{C/X}|_{C_1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and $\mathcal{N}_{C/X}|_{C_2} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$. This proves the proposition. ☐

Corollary 5.3. *The family of elliptic sextics on a generic $X = X_{12}$ is irreducible: $\mathcal{C}^1_6(X) = \mathcal{C}^{1*}_6(X)$.*

Proof. The proof is similar to that of Lemma 4.3. A result of [P-2] is used, which states that the family of elliptic curves $\mathcal{C}^1_d(\Sigma)$ of given degree $d \geq 4$ on the spinor tenfold is irreducible. ☐

Lemma 5.4. *For generic $C \in \mathcal{C}^1_6(X)$, $\langle C \rangle \cap X = C$ scheme-theoretically.*

Proof. Notice that this is definitely false for some special C, for there are elliptic sextics in X having a secant line contained in X. But one can show that a generic elliptic sextic from $\mathcal{C}^{1*}_6(X)$ has no secant lines. Indeed, if we assume the contrary, then the generic quasi-elliptic sextic of the form $C^0_4 + q$ has also a secant line, say ℓ. This line is not a secant to C^0_4, because by Lemma 4.6, $\langle C^0_4 \rangle \cap X = C^0_4$ for a generic quartic C^0_4. Hence ℓ is one of the 14 lines meeting q, which are the flopping curves of Ψ_q. Degenerate now C^0_4 to a curve of the form $C^0_3 + \ell'$, where $C^0_3 \in \mathcal{C}^0_3[2]_q$ and ℓ' is a unisecant to C^0_3. Then ℓ' is movable, hence generically different from ℓ, and both C^0_3 and ℓ' meet ℓ. This is absurd, for the generic member of $\mathcal{C}^0_3[2]_q$ is an exceptional curve of σ_Q which does not meet any of the flopping curves.

So, assume that C has no secant lines and there is a point $p \in \mathbb{P}^5 \cap X \setminus C$. The 3-secant planes $\mathbb{P}^2$ of C sweep over all the projective space $\langle C \rangle = \mathbb{P}^5$, so there is a 3-secant $\mathbb{P}^2$ to C passing through p. By Lemma 1.3, there is a conic q in X passing through the 4 points of $C \cap \mathbb{P}^2$, so X contains the octic $C + q$ of arithmetic genus 3. Except for C, q, there are no other curves in $\langle C \rangle \cap X$, for otherwise the residual curves to $\langle C \rangle \cap X$ in the linear sections $\mathbb{P}^6 \cap X$ through $\langle C \rangle \cap X$ will form a rational net of cubics, conics or lines in X, which is absurd. But in the case when the 1-dimensional locus of $\langle C\rangle \cap X$ is $C \cup q$, we also obtain a contradiction: the residual quartic curve D in a generic $\mathbb{P}^6$-section of X through $C+q$ satisfies length$(D\cap(C+q)) = 5$. As D is reduced, it is a rational normal quartic, hence $\langle D\cap(C+q)\rangle = \langle D\rangle = \mathbb{P}^4$, which is absurd, as $\langle C\rangle \cap \langle D\rangle = \mathbb{P}^3$.

The above argument works as well if p is an embedded component of $\langle C\rangle \cap X$ whose tangent space is not contained in the tangent space to the secant variety of C. Hence C is a scheme-theoretic intersection $\langle C\rangle \cap X$ for generic C. $\square$

6. The Abel–Jacobi map on elliptic sextics. Let $X = X_{12}$ be a generic linear section $\Sigma^{10}_{12} \cap \mathbb{P}^8$. Exactly as in [IM-3] in the case of quasi-elliptic quintics, we can associate to any quasi-elliptic sextic $C \subset X$ a rank-2 vector bundle $\mathcal{E} = \mathcal{E}_C$ on X with Chern classes $c_1(\mathcal{E}) = H$ and $c_2(\mathcal{E}) = 6[\ell]$, where H is the class of a hyperplane section and $[\ell]$ the class of a line. It is obtained as the middle term of the following nontrivial extension of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(1) \longrightarrow 0\ , \tag{7}$$

where $\mathcal{I}_C = \mathcal{I}_{C/X}$ is the ideal sheaf of C in X. One can easily verify (see [MT-1] for a similar argument) that, up to isomorphism, there is a unique nontrivial extension (7), thus C determines the isomorphism class of $\mathcal{E}$. This way of constructing vector bundles is called Serre's construction. The vector bundle $\mathcal{E}$ has a section s whose scheme of zeros is exactly C. Conversely, for any section $s \in H^0(X,\mathcal{E})$ such that its scheme of zeros $C_s = (s)_0$ is of codimension 2, the vector bundle obtained by Serre's construction from C_s is isomorphic to $\mathcal{E}$. The normal sheaf $\mathcal{N}_{C_s/X}$ is naturally isomorphic to $\mathcal{E}|_{C_s}$. As $\det\mathcal{E} \simeq \mathcal{O}_X(1)$, we have $\mathcal{E} \simeq \mathcal{E}^\vee(1)$.

Let us denote by $M_X(2;m,n)$ the moduli space of stable rank-2 vector bundles with fixed Chern classes $c_i \in H^{2i}(X,\mathbb{Z})$: $c_1 = mH$ and $c_2 = n[\ell]$. Recall also some notation from Section 5: $\mathcal{C}^1_{4,2}(X)$, the 4-dimensional family of reducible good sextics of the form $C^0_4 + q$ introduced in Lemma 5.1, and $\mathcal{C}^1_6(X)$, the 6-dimensional irreducible family of elliptic sextics in X.

Lemma 6.1. *For generic $C \in \mathcal{C}^1_6(X)$, the associated vector bundle $\mathcal{E}_C$ is generated by global sections.*

Proof. By (7), it suffices to verify that $\mathcal{I}_C(1)$ is generated by global sections, or equivalently, that $\mathbb{P}^5 \cap X = C$ scheme-theoretically, where $\mathbb{P}^5 = \langle C\rangle$. This follows from Lemma 5.4. $\square$

The following proposition is proved in the same way as similar statements for the (quasi-)elliptic quintics and associated vector bundles in Section 3 of [IM-3].

Proposition 6.2. *For any good sextic $C \subset X$, the associated vector bundle $\mathcal{E}$ possesses the following properties:*

(i) $h^0(\mathcal{E}) = 4$, $h^i(\mathcal{E}(-1)) = 0\ \forall\ i \in \mathbb{Z}$, and $h^i(\mathcal{E}(k)) = 0\ \forall\ i > 0$, $k \geq 0$.

(ii) $\mathcal{E}$ is stable and the local dimension of the moduli space of stable vector bundles at $[\mathcal{E}]$ is at least 3.

(iii) The scheme of zeros $(s)_0$ of any nonzero section $s \in H^0(X,\mathcal{E})$ is a quasi-elliptic sextic with linear span $\mathbb{P}^5$.

(iv) If s, s' are two nonproportional sections of $\mathcal{E}$, then $(s)_0 \neq (s')_0$. This means that $(s)_0$ and $(s')_0$ are different subschemes of X.

(v) The following three conditions are equivalent:

(a) for some (and hence for any) nonzero section $s' \in H^0(X,\mathcal{E})$, the Hilbert scheme of curves Hilb_X is nonsingular and 6-dimensional at $[C']$, where $C' = (s')_0$ is the zero locus of s';

(b) the moduli space of stable rank-2 vector bundles $M_X(2;1,6)$ is nonsingular and 3-dimensional at $\mathcal{E}$;

(c) for some (and hence for any) nonzero section $s' \in H^0(X,\mathcal{E})$, $h^1(\mathcal{N}_{C'/X}) = 0$, where $C' = (s')_0$.

If, moreover, the zero loci $(s)_0$ for $s \in H^0(X,\mathcal{E})$ have no base points, then (a), (b), (c) are equivalent to:

(d) for some (and hence for any) nonzero section $s' \in H^0(X,\mathcal{E})$, $\mathcal{N}_{C/X}$ is a nontrivial extension of $\mathcal{O}_C$ by $\mathcal{O}_C(1)$, that is, there is an exact triple

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

and $\mathcal{N}_{C/X} \not\simeq \mathcal{O}_C \oplus \mathcal{O}_C(1)$.

The Serre construction can be relativized to provide a rational map $\overline{\mathcal{C}}{}^1_6(X) \dashrightarrow M_X = M_X(2;1,6)$, which we will call the Serre map. Let M_X^0 be the image of the smooth locus of $\mathcal{C}^1_6(X)$ in M_X and $\mathcal{C}_X$ its inverse image in $\overline{\mathcal{C}}{}^1_6(X)$. Propositions 5.2, 6.2 and Lemma 6.1 imply the following corollary:

Corollary 6.3. *(i) $\mathcal{C}_X$, resp. M_X^0 is an open subset in the smooth locus of $\overline{\mathcal{C}}{}^1_6(X)$, resp. M_X; $\dim \mathcal{C}_X = 6$ $\dim M_X^0 = 3$, and the Serre map $\mathcal{S} : \mathcal{C}_X \longrightarrow M_X^0$ is a locally trivial $\mathbb{P}^3$-bundle.*

(ii) $\mathcal{C}_X$ contains a 4-dimensional family $\mathcal{C}^1_{4,2} \cap \mathcal{C}_X$ of reducible good sextics of the form $C_4^0 + q$.

(iii) The fiber $\mathcal{S}^{-1}([\mathcal{E}]) \simeq \mathbb{P}H^0(X,\mathcal{E})$ is identified with the family of zero loci $(s)_0$ of the sections $s \in H^0(X,\mathcal{E})$ and consists of quasi-elliptic sextics C satisfying the condition $h^1(\mathcal{N}_{C/X}) = 0$.

Let now $\Gamma = \Sigma^{10}_{12} \cap \mathbb{P}^6$ be the dual curve of genus 7 associated to X. The Brill–Noether locus W^1_6 of Γ is identified with the singular locus of the canonical theta divisor $\Theta \subset \mathrm{Pic}^6(\Gamma)$ (see [GH], Riemann–Kempf Theorem, Section 2.7). Denote by $\alpha : \mathcal{C}_X \longrightarrow \mathrm{Pic}^{30}(\Gamma)$ the restriction of the canonical Abel–Jacobi map to $\mathcal{C}_X$, $[C] \mapsto AJ(C)$ (see Definition 3.5).

Theorem 6.4. *The Abel–Jacobi map $\alpha : \mathcal{C}_X \longrightarrow \mathrm{Pic}^{30}(\Gamma)$ factors through the Serre map $\mathcal{S}$, that is there exists a morphism $\beta : M_X^0 \longrightarrow \mathrm{Pic}^{30}(\Gamma)$ such that $\alpha = \beta \circ \mathcal{S}$. The map β is a birational isomorphism of M_X^0 onto the singular locus of the divisor $3K - \Theta \subset \mathrm{Pic}^{30}(\Gamma)$, where $K = K_\Gamma$ is the canonical class of Γ.*

Proof. The fibers of $\mathcal{S}$ are projective spaces, so they are contracted to points by the Abel–Jacobi map. Thus β exists as a set-theoretic map. The fact that it is a morphism can be proved along the lines of the proof of Theorem 5.6 in [MT-1].

Consider the restriction of α to the reducible sextics from $\mathcal{C}^1_{4,2}$. In the proof of Lemma 5.1, we described a birational isomorphism of $\mathcal{C}^1_{4,2}$ with $I^{(2)} = \{(F,D) \in \Gamma^{(2)} \times G^1_6 \mid D - F \text{ is effective}\}$, where G^1_6 is the union of all the linear series g^1_6 in $\mathrm{Pic}^6(\Gamma)$. Let $C_4^0 + q$ be a generic curve from $\mathcal{C}^1_{4,2}$, represented by a point $(F,D) \in I^{(2)}$. By Lemma 3.3 and Proposition 3.8, $AJ(C_4^0) = 4K - 4\lambda(q) - [Z^q_{C_4^0}] - 2d^q_8$. By construction, $Z^q_{C_4^0} = D - F$, $\lambda(q) = [F]$. By Corollary 3.7 and Lemma 3.9, $AJ(q) = K - \lambda(q)$ and

$d_8^q = K - 2\lambda(q)$. This implies that $\alpha(C_4^0 + q) = AJ(C_4^0) + AJ(q) = 3K - [D]$. As D runs over G_6^1, the classes $3K - [D]$ fill the divisor $3K - W_6^1 = 3K - \operatorname{Sing}\Theta$. The image of α coincides with that of β, and hence is at most 3-dimensional, for $\dim M_X^0 = 3$. As $\dim W_6^1 = 3$, $\alpha(\mathcal{C}_{4,2}^1 \cap \mathcal{C}_X)$ is dense in M_X^0 and β is quasifinite.

It remains to prove that β is birational onto its image, or equivalently, that the generic fiber of α is one copy of $\mathbb{P}^3$. As the 4-dimensional family $\mathcal{C}_{4,2}^1 \cap \mathcal{C}_X$ dominates M_X^0, the fibers $\mathbb{P}^3$ of $\mathcal{S}$ contain generically a 1-dimensional family of curves from $\mathcal{C}_{4,2}^1$. So, if there were several fibers of $\mathcal{S}$ in one fiber of α, then the generic fiber of the restriction $\mathcal{C}_{4,2}^1 \cap \mathcal{C}_X \longrightarrow 3K - W_6^1$ of α would be a disjoint union of several curves. But we have seen in the proof of Lemma 5.1 that this fiber is an irreducible 15-sheeted covering of $\mathbb{P}^1$, so the generic fiber of α is connected. ∎

7. Irreducibility of $M_X(2;1,6)$. Let $X = X_{12}$ be a Fano 3-dimensional linear section of the spinor tenfold and $M_X = M_X(2;1,6)$. We will prove that M_X is irreducible for generic X. This will follow from the irreducibility of the family of elliptic sextics on a generic X as soon as we have proved that a generic $\mathcal{E}$ in any component of M_X is obtained by Serre's construction from an elliptic sextic.

LEMMA 7.1. *Let $\mathcal{E} \in M_X$, S a generic hyperplane section of X, $E = \mathcal{E}|_S$ the restriction of $\mathcal{E}$ to S. Then the following assertions hold:*

(i) $\chi(\mathcal{E}) = 4$, $h^3(\mathcal{E}) = 0$.

(ii) E is stable and the scheme of zeros $Z_s = (s)_0$ of any nonzero section s of E is 0-dimensional and of length 6. E can be obtained by Serre's construction on S from a 0-dimensional subscheme $Z \subset S$ of length 6:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow \mathcal{I}_Z(1) \longrightarrow 0. \tag{8}$$

For such a Z, $\dim\langle Z\rangle = 4$ and $\langle Z\rangle = \langle Z'\rangle$ for any $Z' \subset Z$ of length 5.

(iii) E is generated by global sections at the generic point of S.

Proof. (i) We have $\chi(\mathcal{E}) = 4$ by Riemann–Roch, and $h^3(\mathcal{E}) = h^0(\mathcal{E}(-2)) = 0$ by stability.

(ii) $E = \mathcal{E}|_S$ is slope-semistable by Theorem 3.1 of [Ma]. The semistability implies the stability because $\operatorname{Pic} S = \mathbb{Z}H$ and $\det E = \mathcal{O}(H)$ is odd. Hence $h^2(E) = h^0(E(-1)) = 0$ and $\chi(E) = 4$ implies $h^0(E) \geq 4$. The zero locus $(s)_0$ of any non-zero section of E is finite, for otherwise it would be a curve from the linear system $|kH|$ and then $h^0(E(-k)) \neq 0$, which is absurd. Hence it is a subscheme Z of length equal to $c_2(E) = 6$, and there is an exact triple (8) with the inclusion $\mathcal{O}_S \longrightarrow E$ defined by s. We have $h^1(\mathcal{I}_Z(1)) = 5 - m$, where $m = \dim\langle Z\rangle$. By Serre duality, $\dim \operatorname{Ext}^i(\mathcal{I}_Z(1), \mathcal{O}_S) = h^{2-i}(\mathcal{I}_Z(1))$, hence the triple (8) can be nonsplit only if $m \leq 4$. The values $m \leq 2$ are impossible by Lemma 1.3. Hence $m = 3$ or 4.

Assume that $m = 3$. By Lemma 1.3, for any subscheme $Z' \subset Z$ of length 5, we have $\dim\langle Z'\rangle = \dim\langle Z\rangle = 3$. Hence $h^1(\mathcal{I}_{Z'}(1)) = 1$, and there is a unique nontrivial extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E' \longrightarrow \mathcal{I}_{Z'}(1) \longrightarrow 0.$$

Again by Lemma 1.3, for any $Z'' \subset Z'$ of length 4, $\langle Z''\rangle = \langle Z'\rangle$, which implies the local freeness of E' (see, for example, [Tyu], Lemma 1.2). Thus the Serre construction applied to Z' provides a rank-2 vector bundle E' with $c_1(E') = [H]$, $c_2(E') = 5$. It is easy to see that E' is stable. Indeed, if we assume that it is unstable, then any destabilizing subsheaf should be of the form $\mathcal{I}_W(k)$, where $k > 0$ and W is a 0-dimensional subscheme of S. If we replace $\mathcal{I}_W(k)$ by its saturate $\mathcal{I}_W(k)^{\vee\vee} = \mathcal{O}_S(k)$, we get an inclusion $\mathcal{O}_S(k) \hookrightarrow E'$, which is absurd, since $h^0(\mathcal{O}_S(k)) \geq h^0(\mathcal{O}_S(1)) = 8 > h^0(E') = 5$. By Corollary 5.8 of [IM-3], E' is generated by global sections. From Serre's exact triple for E', we conclude that $\mathcal{I}_{Z'}(1)$ is generated by global sections. Hence $\langle Z'\rangle \cap S = Z'$ scheme-theoretically, which contradicts the equality $\langle Z'\rangle = \langle Z\rangle$. Thus we have proved that $m = 4$, that is, $\langle Z\rangle \simeq \mathbb{P}^4$.

Suppose now that there is a subscheme $Z' \subset Z$ of length 5 with $\langle Z'\rangle \subsetneqq \langle Z\rangle$. Then $\dim\langle Z'\rangle = 3$. The exact triple $0 \to \mathcal{I}_Z(1) \xrightarrow{\iota} \mathcal{I}_{Z'}(1) \to \mathbb{C}_p \to 0$, where $\{p\}$ is the support of $\mathcal{I}_{Z'}/\mathcal{I}_Z$, and the local-to-global spectral sequence provide the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^1(\mathcal{I}_Z(1), \mathcal{O}_S) \arrow[r, hook] & H^0(\mathcal{E}xt^1(\mathcal{I}_Z(1), \mathcal{O}_S)) \\
\operatorname{Ext}^1(\mathcal{I}_{Z'}(1), \mathcal{O}_S) \arrow[r, hook] \arrow[u, "\iota^*"', "\wr"] & H^0(\mathcal{E}xt^1(\mathcal{I}_{Z'}(1), \mathcal{O}_S)) \arrow[u, hook]
\end{tikzcd}$$

From its right column we see that the extension class of (8) does not generate the stalk of $\mathcal{E}xt^1(\mathcal{I}_Z(1), \mathcal{O}_S)$, which contradicts the local freeness of E at p by Serre's Lemma (see e. g. Lemma 5.1.2 in [OSS]). Hence $\langle Z'\rangle = \langle Z\rangle$ and we are done.

(iii) Let s_1, s_2 be two non-proportional sections of E. If they do not generate E at any point of S, then there is a rank-1 subsheaf of E with at least 2 linearly independent sections, which contradicts the stability. ∎

LEMMA 7.2. *In the assumptions of Lemma 7.1, the following statements hold:*
(i) $h^1(E(k)) = 0$ *for all* $k \in \mathbb{Z}$ *and* $\chi(E(k)) = h^0(E(k)) = h^2(E(-k-1)) = 12k(k+1) + 4$ *for* $k \geq 0$.
(ii) $h^i(\mathcal{E}(k)) = 0$ *for all* $k \in \mathbb{Z}$, $i = 1, 2$; $\chi(\mathcal{E}(k)) = h^0(\mathcal{E}(k)) = h^3(\mathcal{E}(-k-2)) = 4(k+1)^3$ *for* $k \geq -1$.

Proof. This is standard; use the exact triples

$$0 \longrightarrow \mathcal{O}_S(k) \longrightarrow E(k) \longrightarrow \mathcal{I}_Z(k+1) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{I}_Z(k) \longrightarrow \mathcal{O}_S(k) \longrightarrow \mathcal{O}_Z(k) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{E}(k-1) \longrightarrow \mathcal{E}(k) \longrightarrow E(k) \longrightarrow 0,$$

the Serre duality and the Kodaira vanishing $h^1(\mathcal{E}(k)) = 0$ for $k \ll 0$. ∎

COROLLARY 7.3. *Let* $\mathcal{E} \in M_X$, S *any nonsingular hyperplane section of* X, $E = \mathcal{E}|_S$ *the restriction of* $\mathcal{E}$ *to* S. *Then the restriction map* $H^0(\mathcal{E}) \longrightarrow H^0(E)$ *is an isomorphism and the assertion (i) of Lemma 7.2 holds for the cohomology* $h^i(E(k))$.

If in addition $\operatorname{Pic} S = \mathbb{Z}H$, *then* E *is stable and any two nonproportional sections of* E *define distinct* 0*-dimensional length-*6 *subschemes of* S.

Proof. The assertions on the restriction map and on $h^i(E(k))$ are obvious. If $\operatorname{Pic} S = \mathbb{Z}H$, $\operatorname{rk} E = 2$ and $c_1(E) = H$, then the stability of E is equivalent to $h^0(E(-1)) = 0$. Hence E is stable. This imples that $\operatorname{Hom}(E, E) = H^0(E \otimes E(-1)) = \mathbb{C}$. From the exact triple (8) tensored by $E(-1)$, we deduce that $H^0(E \otimes \mathcal{I}_Z) \simeq H^0(E \otimes E(-1)) = \mathbb{C}$, hence a section of E having Z as its zero locus is unique up to proportionality. ∎

PROPOSITION 7.4. *Assume X generic, and let $\mathcal{E} \in M_X$. Then $\mathcal{E}$ is generated by global sections at the generic point of X and can be obtained by Serre's construction from a quasi-elliptic sextic lying in the closure of the family of elliptic sextics in the Hilbert scheme of X.*

Proof. Let C_s denote the curve $(s)_0$ of zeros of a nonzero section $s \in H^0(\mathcal{E})$. It is a l. c. i. sextic curve with trivial canonical sheaf for any $s \neq 0$. Moreover, it is connected, that is $h^0(\mathcal{O}_{C_s}) = 1$, and $\mathcal{N}_{C_s} \simeq \mathcal{E}|_{C_s}$, so $\chi(\mathcal{N}_{C_s}) = 6$. This implies that the dimension of the Hilbert scheme of curves in X at the point $\{C_s\}$ representing C_s is at least 6. Moreover, the properties of being a l. c. i. curve and to have trivial canonical sheaf are open, so any small deformation of a l. c. i. curve with trivial canonical sheaf is of the same type. We will use this observation to show that C_s is in the closure of the family of smooth elliptic sextics in X.

The outline of the proof is the following. First, we decompose C_s into the sum of the fixed part F and the movable part M_s. Second, we show that $\deg M_s \geq 4$. Finally, in assuming s generic, we examine the possible types of decomposition of M_s and F in irreducible components to show that $F + M_s$ deforms to a smooth sextic curve.

By Lemma 7.2 and Corollary 7.3, the curves C_s form a family with base $\mathbb{P}^3$, and for two nonproportional sections s, s' of $\mathcal{E}$, the curves C_s, $C_{s'}$ are distinct as subschemes in X. Let F be the sum of the fixed components of this family, and M_s the movable part, so that $C_s = F + M_s$ as an algebraic cycle. By Bertini Theorem, both F and the singular loci of M_s for generic s (if nonempty) are contained in the base locus $\mathrm{BL}(\mathcal{E})$ of $\mathcal{E}$, defined as the locus of points $x \in X$ in which the stalk $\mathcal{E}_x$ is not generated by $H^0(\mathcal{E})$. According to Lemma 7.1, (iii) and Corollary 7.3, $\mathrm{BL}(\mathcal{E})$ is a proper closed subset of X, so M_s is reduced for generic s. Taking any 3 nonproportional sections s_1, s_2, s_3 of $\mathcal{E}$ and a generic point $x \in X$, we can find a nontrivial linear combination $s = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3$ vanishing at x. Hence the family $\{M_s\}_{[s]\in\mathbb{P}^3}$ is a covering family of curves on X: there is at least one curve M_s passing through a generic point of x.

Let us show that the curves M_s are different for nonproportional sections s. If C_s has multiple components, this does not follow directly from the above observation that C_s, $C_{s'}$ are distinct whenever $s \not\sim s'$, for C_s, $C_{s'}$ may differ, a priori, by the nilpotent structure along the multiple components whilst the associated algebraic cycles $F + M_s$, $F + M_{s'}$ are the same. Thus, assuming that $s \not\sim s'$, we will verify that the supports of C_s and $C_{s'}$ are distinct.

By the stability of $\mathcal{E}$, the subsheaf $\mathcal{O}_X \cdot s + \mathcal{O}_X \cdot s' \subset \mathcal{E}$ cannot be of rank 1. Hence s, s' are generically linearly independent and the section $s \wedge s' \in H^0(\det \mathcal{E}) = H^0(\mathcal{O}_X(1))$ is nonzero. As $\operatorname{Pic} X \simeq \mathbb{Z}$, the zero locus $S = (s \wedge s')_0$ is a possibly singular, but reduced and irreducible surface from the linear series of hyperplane sections of X. Obviously $C_s \subset S$, $C_{s'} \subset S$. The restrictions $\sigma = s|_S$, $\sigma' = s'|_S$ are sections of the rank-1 torsion-free sheaf $\mathcal{L} = \mathcal{O}_S \cdot \sigma + \mathcal{O}_S \cdot \sigma' \subset \mathcal{E}|_S$. They are nonproportional, for if $\lambda\sigma + \lambda'\sigma' = 0$ for some nonzero constants $\lambda, \lambda' \in \mathbb{C}$, then $\lambda s + \lambda' s'$ is a nonzero section of $\mathcal{E}$ which vanishes exactly on S, and this is impossible by the stability of $\mathcal{E}$.

If we assume that $\operatorname{Supp} C_s = \operatorname{Supp} C_{s'}$, then all the nontrivial linear combinations $\lambda s + \lambda' s'$ have the same zero set. This is absurd, for if $x \in S$ is generic, then the fiber $\mathcal{L}(x) := \mathcal{L} \otimes \mathbb{C}(x)$ is one-dimensional, so there exists a nontrivial linear combination $\lambda s + \lambda' s'$ vanishing at x and the curve $C_{\lambda s + \lambda' s'}$ passes through x. This implies that $M_{\lambda s + \lambda' s'}$ is a movable curve, and hence $M_s \neq M_{s'}$.

Since the family of lines is not covering for X and since the one of conics contains no rational subvarieties, we have $\deg M_s \geq 3$. Suppose that $\deg M_s = 3$. Then we get a 3-dimensional family of cubic curves $\mathcal{M} = \{M_s\}$, bijectively parameterized by $\mathbb{P}^3 = \mathbb{P}H^0(\mathcal{E})$. Let $s \in \mathbb{P}H^0(\mathcal{E})$ be generic. We have seen that then M_s is reduced. Let us show that it is also irreducible. Indeed, if M_s is a line plus a conic, then by projecting $\mathcal{M}$ to the families $\tau(X)$, $\mathcal{F}(X)$ of lines and conics in X, we get a nonconstant rational map $\mathbb{P}^3 \dashrightarrow \tau(X) \times \mathcal{F}(X)$. But $\tau(X)$ is a smooth curve of genus 43, and $\mathcal{F}(X) \simeq \Gamma^{(2)}$ for the orthogonal genus-7 curve $\Gamma = \check{X}$, so $\tau(X)$, $\mathcal{F}(X)$ do not contain rational subvarieties. Further, if M_s is a union of three lines, we get a generically injective rational map $\mathbb{P}^3 \dashrightarrow \tau(X)^{(3)}$ which is also absurd, since $\tau(X)^{(3)}$ is irreducible and nonrational.

Thus M_s is a reduced and irreducible cubic curve in X for generic s. As X is an intersection of quadrics, the span of M_s is $\mathbb{P}^3$ and M_s is nonsingular. We obtain a family of rational normal cubics in X, bijectively parameterized by an open set of $\mathbb{P}^3$. This contradicts Lemma 4.1, saying that $\mathcal{C}_3^0(X)$ is irreducible and birational to $\Gamma^{(3)}$.

We have proved that $\deg M_s \geq 4$. From now on we assume $s \in H^0(\mathcal{E})$ generic. We will treat several cases differing by the degree of $M = M_s$ and the type of its decomposition in irreducible components.

Case 1: $\deg M = 6$, that is, $F = 0$. Then $C = M$ is a good sextic.

Subcase 1.1: M is irreducible. Either it is an elliptic sextic, and we are done, or it is a rational sextic with one double point whose contribution to the arithmetic genus is 1, that is a node or a cusp. An argument as in the proof of Lemma 4.3 shows that when X is generic, then all the components of the family of rational sextics in X are 6-dimensional, and the singular rational sextics fill a codimension-1 locus. As the local deformation space of C in X is at least 6-dimensional, we conclude that C deforms to a smooth elliptic sextic in X.

Subcase 1.2: at least one of the components of M is a line. By the same argument as we used for $\deg M = 3$, the number of line components is ≥ 4. But then the remaining component cannot be a conic, for then this conic should be fixed and $\deg M = 4$, which is absurd. So, M has to be a connected union of 6 lines. As any line meets only finitely many lines in X, the dimension of the family of connected unions of 6 lines in X is ≤ 1, but the family of different M's is 3-dimensional, so M is not of this type.

Subcase 1.3: M has a conic component. Then, as above, M is a connected union of three smooth conics, $M = q_1 \cup q_2 \cup q_3$. The sextuples $\sum \lambda(q_i)$ of points of Γ, where $\lambda : \mathcal{F}(X) \xrightarrow{\sim} \Gamma^{(2)}$ was defined in the proof of Proposition 2.2, sweep out a unirational 3-dimensional subvariety of $\Gamma^{(6)}$. This is impossible, for Γ is a generic genus-7 curve and hence it has no g_6^3 (and even g_6^2).

Subcase 1.4: M has a cubic component. Then it is a union of two rational normal cubics $C_1 \cup C_2$ with $\operatorname{length}(C_1 \cap C_2) = 2$. As $\dim \mathcal{C}_3^0(X) = 3$ and two generic rational normal cubics in X are disjoint, we see that the family of pairs of intersecting rational normal cubics is at most 5-dimensional. Hence M deforms to an irreducible sextic, and this reduces the problem to Subcase 1.1, which we have already settled.

Case 2: $\deg M = 5$, then $F = \ell$ is a line. Similarly to the above, we can prove that M is a smooth rational quintic and $\operatorname{length}(\ell \cap M_s) = 2$. An argument as in the proof of Lemma 4.3 shows that the rational quintics in a generic X fill a 5-dimensional family, and those meeting a line twice lie in codimension 1. Hence $\ell + M$ deforms to an irreducible good sextic, which brings us to Subcase 1.1.

Case 3: $\deg M = 4$. We have two subcases.

Subcase 3.1: $C = q + M$, where q is a reduced conic. Then the result follows by the same argument as in Case 2.

Subcase 3.2: $C = F + M$, where F is a Cohen–Macaulay double structure on a line ℓ. As before, we can prove that M is a rational normal quartic such that $\mathrm{length}(F \cap M) = 2$. We have an exact triple

$$0 \longrightarrow \mathcal{O}_F(-Z) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_M \longrightarrow 0,$$

where Z is the intersection scheme $F \cap M$.

The Cohen–Macaulay double structures on a smooth curve are completely described, for example, in [BanF]. They all are obtained by Ferrand's construction as in the proof of Lemma 3.13; one can think of F as ℓ together with a cross section ξ of the projectivized normal bundle $\mathbb{P}(\mathcal{N}_{\ell/X})$ over ℓ. The multiplicity of the intersection $F \cap M$ can be interpreted via the relative position of the proper transform $\tilde{M}$ of M and ξ on the blowup $\tilde{X}$ of X with center ℓ. We have the following three possibilities for the intersection $F \cap M$ of total multiplicity 2: (1) ℓ intersects M quasi-transeversely at one point p, and $\tilde{M}$ passes through ξ_p; (2) $\ell \cap M = \{p_1, p_2\}$, $p_1 \neq p_2$, and $\tilde{M}$ does not pass through any one of the points ξ_{p_1}, ξ_{p_2}; (3) M is simply tangent to ℓ at p and $\tilde{M}$ does not pass through ξ_p. The singular points of C are analytically equivalent to $(x^2, z) \cap (x, y)$ in the case (2) and $(x^2, z) \cap (x, z - y^2)$ in the case (3). These singularities are not Gorenstein, so the only possible case is (1). But in this case we have $\omega_F \simeq \omega_C|_F(-Z)$ and $\omega_C \simeq \mathcal{O}_C$. Restricting ω_F to ℓ, we obtain a contradiction as in the proof of Lemma 3.13: on the one hand, $\omega_F|_\ell = \mathcal{O}_F(-Z)|_\ell = \mathcal{O}_\ell(-1)$, on the other hand $\omega_F|_\ell \simeq \omega_\ell \otimes \mathcal{L}^{-1}$, where $\mathcal{L} \simeq \mathcal{O}_\ell(k)$ for some $k \geq 0$, which is impossible. Thus the Subcase 3.2 does not occur. □

Corollary 7.5. *If X is generic, then M_X is irreducible.*

Proof. This is an immediate consequence of Proposition 7.4 and Corollary 5.3. □

8. Appendix: Maps from the symmetric square of a curve. Here we prove the following assertion:

Proposition 8.1. *Let Γ be a generic curve of genus $g \geq 4$ and $S = \Gamma^{(2)}$ the symmetric square of Γ. Then the following assertions hold:*

(i) If $g \neq 4$, then for any nonrational irreducible curve C, there are no nonconstant rational maps $\varphi : S \dashrightarrow C$.

(ii) Let $\varphi : S \dashrightarrow S$ be a nonconstant rational map. Then $\varphi = \mathrm{id}_S$.

We fix for the sequel the notations Γ and S for a *generic* curve of genus g and its symmetric square respectively. The proposition follows from a sequence of lemmas. Before stating them, we need to describe the Mori cone and the ample cone of S.

Let $g \geq 2$. Let $\pi : \Gamma \times \Gamma \longrightarrow S = \Gamma^{(2)}$ be the quotient map and $\Delta \subset \Gamma \times \Gamma$ the diagonal. The Neron–Severi group $NS(S)$ contains 3 natural classes: the first one is f, the class of a fiber $\pi(\{x\} \times \Gamma)$, where $x \in \Gamma$ is a point, the second one is $\delta = \frac{1}{2}\pi(\Delta)$, and the third one is $\Theta|_S$, the pullback of the theta-divisor via the Abel-Jacobi map $S \longrightarrow J(X)$ defined up to translations. There is one relation among them, $\delta = (g+1)f - \Theta|_S$, and $NS(S)$ is freely generated by f and δ (see [ACGH], Sect. 5 of Ch. VIII, and [GH], Sect. 5 of Ch. II). We have also:

$$\delta^2 = 1 - g,\ \delta f = f^2 = 1,\ K_S = -\delta + (2g-2)f,\ K_S^2 = (2g-3)^2 - g,$$

$$c_2(S) = (2g-3)(g-1),\ \chi(\mathcal{O}_S) = \frac{(g-1)(g-2)}{2}.$$

If $g \geq 3$, then S contains no rational curves and $K_S^2 > 0$, so S is of general type, and moreover, K_S is ample.

Let $N(S)$ be the real vector plane $NS(S) \otimes \mathbb{R}$, $\overline{NE}(S) \subset N(S)$ the smallest closed cone containing the classes of effective curves (the Mori cone of S), and $\overline{NA}(S)$ the dual cone with respect to the intersection product on $N(S)$; this is the smallest closed cone containing the classes of ample curves. In our case, the cones are just angles in the plane. It is obvious that one of the rays bordering $\overline{NE}(S)$ is $\mathbb{R}_+\delta$ and the other is of the form $\mathbb{R}_+(-\delta + kf)$ for some real k, $1 < k < g+1$. Similarly, $\overline{NA}(S)$ is bordered by the rays $\mathbb{R}_+(\delta + (g-1)f)$ and $\mathbb{R}_+(-\delta + lf)$ with $k \leq l = \frac{k+g-1}{k-1} < g+1$ The following theorem, proved in [Kou], [CiKou], gives more precise estimates:

Theorem [Kouvidakis, Ciliberto–Kouvidakis]. *Assume that Γ is a generic curve of genus $g \geq 4$. Then*

$$\sqrt{g} \leq k \leq \sqrt{g} + 1 \leq l \leq \frac{g}{\sqrt{g}-1} + 1.$$

If $\sqrt{g} \in \mathbb{Z}$, then $k = l = \sqrt{g} + 1$. If moreover $g \neq 4$, then there are no classes of effective curves in the ray $\mathbb{R}_+(-\delta + (\sqrt{g}+1)f)$.

Lemma 8.2. *Let $g \geq 5$, and let C be a nonsingular complete curve. Then there are no nonconstant morphisms $\varphi : S \longrightarrow C$. If moreover C is nonrational, then every rational map $\varphi : S \dashrightarrow C$ is regular, hence constant.*

Proof. The fiber of such a morphism would provide a rational numerically effective class $h = a\delta + bf$ with $h^2 = 0$, which implies $\frac{b}{a} = -1 \pm \sqrt{g}$. Hence $\sqrt{g} \in \mathbb{Z}$. In the interior of $\overline{NA}(S)$, $h^2 > 0$, hence h is on the border and is proportional to $h_0 = -\delta + (\sqrt{g}+1)f$. This contradicts the non-existence of effective curves in the ray $\mathbb{R}_+h_0$. ◻

Remark 8.3. For $g = 4$, Γ has two g_3^1's. A g_3^1 defines the following curve on S:

$$D = \{x + y \in S \mid \exists z \in \Gamma : \ x + y + z \in g_3^1\}.$$

The two g_3^1's thus provide two curves D, D' in S in the same numerical class $-\delta + 3f$ such that $D^2 = D'^2 = 0$. Hence the border ray of $\overline{NE}(S)$ contains effective curves, and to extend the previous lemma to $g = 4$, one has to show that $\dim |nD| = \dim |nD'| = 0$ for all $n > 0$.

Lemma 8.4. *Let $g \geq 3$, and let $\varphi : S \dashrightarrow S$ be a rational map of degree $d > 0$. Then $d = 1$.*

Proof. Since C is not hyperelliptic, φ is regular. By [Beau-2], Proposition 2, if a compact complex manifold X admits an endomorphism of degree $d > 1$, then $\kappa(X) < \dim X$. But S is of general type, so it has no endomorphisms of degree > 1. ◻

Lemma 8.5. *Let $g \geq 4$ and let $\varphi : S \dashrightarrow S$ be a birational map. Then $\varphi = \mathrm{id}_S$.*

Proof. As S contains no rational curves, φ is biregular. The induced automorphism φ^* of $N(S)$ is given by an integer matrix in the basis δ, f. It preserves the intersection product and the cones $\overline{NE}(S)$, $\overline{NA}(S)$. The canonical class K_S is an eigenvector of φ^* with eigenvalue 1. Hence if φ^* preserves both border rays of $\overline{NE}(S)$, it is the identity map. As $\delta^2 < 0$, there is only one effective curve in the numerical class 2δ, the diagonal $\Delta' = \pi(\Delta)$, so Δ' is invariant under φ. But $\Delta' \simeq \Gamma$ and Γ has no nontrivial automorphisms, for it is a generic curve of genus g. Hence $\varphi|_{\Delta'} = \mathrm{id}$.

Now, any of the curves $F_x = \pi(\{x\} \times \Gamma)$, represented by the class f, is tangent to Δ' at a single point $2x = \pi(x,x)$. Hence its image $\varphi(F_x)$ is also tangent to Δ' at $2x$ and belongs to the same class f. Lifting it to $\Gamma \times \Gamma$, one immediately verifies that $\varphi(F_x) = F_x$ and $\varphi = \mathrm{id}$.

It remains to consider the second case, when φ^* permutes the border rays of $\overline{NE}(S)$. Then φ^* is an orthogonal reflection with mirror $\mathbb{R}K_S$. We have

$$\varphi^*(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha, \quad \alpha = -(2g-3)\delta + (3g-3)f.$$

This gives $\varphi^*(\delta) = -\frac{4g-3}{4g-9}\delta + \frac{12g-12}{4g-9}f$. The coefficient of δ is fractional for all $g \geq 4$, which contradicts the condition that φ^* is integer in the basis δ, f. Hence the second case is impossible. □

Remark 8.6. The previous lemma does not extend to $g = 3$, because in this case the formula $\varphi : x + y \mapsto K_\Gamma - x - y$ defines an involution on S.

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