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ON PROPER HARMONIC MAPS BETWEEN STRICTLY PSEUDOCONVEX DOMAINS WITH KÄHLER METRICS OF BERGMAN TYPE*

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Dedicated to Salah Baouendi for his seventieth birthday

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1. Introduction. Let M and N be Kähler manifolds with respective Kähler metrics $h = h_{i\bar{j}} dz_i \otimes d\bar{z}_j$ and $g = g_{\alpha\bar{\beta}} dw^\alpha \otimes d\overline{w}^\beta$, respectively. A map $u : M^m \to N^n$ is said to be harmonic if the tension field $\tau^s[u]$ satisfies

$$\tau^s[u] = \Delta_M u^s + \sum_{i,j=1}^{m} \sum_{t,\gamma=1}^{n} \Gamma^s_{t\gamma} \partial_i u^t \partial_{\bar{j}} u^\gamma h^{i\bar{j}} = 0 \text{ for } 1 \le s \le n, \tag{1.1}$$

where $(h^{i\bar{j}})^t$ is the inverse of the matrix $(h_{i\bar{j}})$, $\Delta_M = \sum_{i,j} h^{i\bar{j}} \partial_{i\bar{j}}$ and $\Gamma^s_{t\gamma}$ denote the Christoffel symbols of the Hermitian metric g on N. It follows from (1.1) that if u is holomorphic, then u must be harmonic. Thus, it is natural to ask under what circumstances a harmonic map is holomorphic or antiholomorphic. Under the assumption that both M and N are compact, Siu [31] demonstrated that if the curvature tensor of N is strongly negative and the rank of du is greater than or equal to four at a point of M, then a harmonic map u must be holomorphic or antiholomorphic. The proof follows from Siu's Bochner type identity together with the compactness assumption on M.

If M is a complete noncompact manifold of strongly negative curvature with infinite volume, the previous Bochner type identity technique fails and not much is known about the rigidity of u. In general, the answer to the above posed question is negative: one needs to add some natural conditions to the map such as being a proper map. Along this direction, when M and N are unit balls in $\mathbb{C}^n$ endowed with Bergman metrics (the simplest case of Kähler manifolds with strongly negative curvature) progress was made by Li and Ni in [25]. They showed that for $m > 1$, if $u : (B^m, h) \to (B^n, g)$ is a C^2 up to the boundary pluriharmonic proper map, where h and g are respective Bergman metrics on B^m and B^n, then u must be holomorphic or antiholomorphic. In addition to this, several other equivalent conditions were given (cf. [25]).

The main purpose of this paper is to use a similar approach to the one given in (cf. [25]) to generalize their theorem from unit balls to smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^m$ and $\mathbb{C}^n$ for $m > 1$ with more general metrics of Bergman type. More precisely, we consider two smoothly bounded strictly pseudoconvex domains Ω_m and Ω_n in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. Let ρ and r be C^4 respective strictly

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plurisubharmonic defining functions for Ω_m and Ω_n. We consider the complex Kähler metric

$$h = h_{i\bar{j}} dz_i \otimes d\bar{z}_j = -\frac{\partial^2 \log(-\rho)}{\partial z_i \otimes \partial \bar{z}_j} dz_i \otimes d\bar{z}_j \tag{1.2}$$

for Ω_m and the Kähler metric

$$g = g_{\alpha\bar{\beta}} dw^\alpha \otimes d\overline{w}^\beta = -\frac{\partial^2 \log(-r)}{\partial w^\alpha \partial \overline{w}^\beta} dw^\alpha \otimes d\overline{w}^\beta \tag{1.3}$$

for Ω_n. By the asymptotic expansion of the Bergman kernel function given by C. Fefferman in [15] the Bergman metric is a special case of the above setting, and so is the Kähler-Einstein metric given by Cheng and Yau in [8].

Let $(\rho^{i\bar{j}})^t$ be the inverse matrix of the matrix $(\rho_{i\bar{j}})$. Let

$$\rho^i = \sum_j \rho^{i\bar{j}} \rho_{\bar{j}}, \quad \rho^{\bar{j}} = \sum_i \rho^{i\bar{j}} \rho_i \text{ and } |\partial\rho|^2_\rho = \sum_i \rho^i \rho_i = \sum_j \rho^{\bar{j}} \rho_{\bar{j}}; \tag{1.4}$$

a complex normal derivative R, a tangential complex derivative X_j and an elliptic operator $\mathcal{L}$ be defined as follows:

$$R := \rho^j \partial_j, \quad X_j := \sum_{i=1}^m (\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho}) \partial_i, \quad \mathcal{L} := (\rho^{i\bar{j}} - \frac{\rho^i \rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho}) \partial_{i\bar{j}}. \tag{1.5}$$

Let $e[u]$ be the energy density function associated to the map $u : (M, h) \to (N, g)$ defined as

$$e[u](z) := \sum_{\alpha,\beta=1}^n \sum_{i,j=1}^m h^{i\bar{j}} g_{\alpha\bar{\beta}} (\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\bar{j}} u^\alpha \overline{\partial_{\bar{i}} u^\beta}) \tag{1.6}$$

Our first theorem, which is a generalization of the main theorem for the case when $\Omega_m = B_m$ and $\Omega_n = B_n$ are balls given by Li and Ni in [25], is as follows:

THEOREM 1.1. *Assume that $m > 1$, and let $\Omega_m \subset \mathbb{C}^m$, $\Omega_n \subset \mathbb{C}^n$ be bounded strictly pseudoconvex domains with strictly plurisubharmonic defining functions $\rho \in C^4(\overline{\Omega}_m)$, $r \in C^4(\overline{\Omega}_n)$, respectively. Let $u : \Omega_m \to \Omega_n$ be a proper map so that $u \in C^2(\overline{\Omega}_m, \overline{\Omega}_n)$. Then the following statements are equivalent.*

(i) The map u is either holomorphic or antiholomorphic.
(ii) The map u is pluriharmonic.
(iii) The map u is harmonic and $r_s \mathcal{L} u^s + r_{st} X_j u^t u^s_{\bar{j}} = 0$ on $\partial\Omega_m$.
(iv) The map u is harmonic and the energy density function $e[u](z) = m$ on the set $\{z \in \partial\Omega_m : E_b[u](z) > 0\}$, where $E_b[u](z) = |\overline{\partial}_b u|^2 + |\partial_b u|^2$ and $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator.
(v) The map u is harmonic and $\sum_{\gamma,s=1}^n (r_s R u^s(z))(r_\gamma \overline{R} u^\gamma(z)) = 0$ on $\partial\Omega_m$.

Another problem we want to explore is the existence and regularity of proper harmonic maps. More precisely, if $\phi : \partial\Omega_m \to \partial\Omega_n$ is a smooth map, can one find a harmonic map u that when restricted to $\partial\Omega_m$ equals ϕ? If so, what type of regularity statement can we offer?

In a series of papers [19]–[21], P. Li and L-F. Tam explored the existence, uniqueness and regularity of proper harmonic maps between real hyperbolic spaces. They showed that if $\phi : D^m \to D^n$ (here D^m is the unit ball in $\mathbb{R}^m$ with the Poincaré metric and similarly D^n is the unit ball in $\mathbb{R}^n$ with the Poincaré metric) is a C^1 map with nonvanishing energy density $e(\phi)(x)$ at every $x \in S^{m-1}$, then there exists a unique proper harmonic map $u : D^m \to D^n$ which equals ϕ when restricted to S^{m-1}. If in addition, the boundary map ϕ is in $C^{k,\alpha}(S^{m-1}, S^{n-1})$, where $1 \le k \le m-1$ and $0 < \alpha \le 1$, then u belongs to $C^{k,\gamma}(\overline{D}^m)$ for $0 < \gamma < \alpha$. They also proved that if $u : \overline{D}^m \to \overline{D}^n$ and $v : \overline{D}^m \to \overline{D}^n$ are C^1 proper harmonic maps such that they are equal on S^{m-1} and the energy density of the boundary map does not vanish anywhere, then $u = v$. As a corollary, they obtained that if $u : \overline{D}^m \to \overline{D}^n$ is a C^1, proper harmonic map with non-vanishing energy density on S^{m-1}, then the energy density equals m at the boundary.

The case where both M and N are rank one symmetric spaces was tackled by Donnelly in [11]. He was able to generalize the existence and regularity results of Li and Tam under the assumption that the boundary map ϕ satisfies some contact conditions. The problem was also studied by S-Y. Li and L. Ni in [25] where they formulated a simpler contact condition and provided an existence theorem. The second purpose of this paper is to generalize their theorem on unit balls to strictly pseudoconvex domains. Our result is:

THEOREM 1.2. *Assume that $\phi : \partial\Omega_m \to \partial\Omega_n$ belongs to $C^{k,\alpha}(\partial B_m)$ for $k \ge 2$ and $0 < \alpha \le 1$. In addition, suppose that $E_b[\phi](z) = |\overline{\partial}_b\phi|^2 + |\partial_b\phi|^2 > 0$ on $\partial\Omega_m$, where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator, and the necessary condition*

$$\sum_s r_s(\phi)(z)\, X_j\phi^s(z) = 0, \quad z \in \partial\Omega_m, \quad 1 \le j \le m. \tag{1.7}$$

Then for all $0 < l + \beta < \min\{m, k+\alpha\}$ there exists a unique proper harmonic map $u \in C^{l,\beta}(\overline{\Omega}_m)$ such that $u = \phi$ on $\partial\Omega_m$.

2. Cauchy-Riemann functions. A complex-valued C^1 function u in a domain Ω in $\mathbb{C}^m$ is said to be CR if $\overline{\partial}u = 0$, which is the same as u being holomorphic. Since $\frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_k} = \frac{\partial}{\partial \overline{z}_k}\frac{\partial}{\partial z_j}$ for all $1 \le j, k \le m$, it is easy to show that if for each $z \in \Omega$ we have that either $\overline{\partial}u(z) = 0$ or $\overline{\partial}\overline{u}(z) = 0$, then we must have that either $\overline{\partial}u \equiv 0$ on Ω or $\overline{\partial}\overline{u} \equiv 0$ on Ω.

It was proved by Li and Ni [25] that the above phenomenon remains true for functions on the unit sphere in $\mathbb{C}^m$ $(m > 1)$ where the problem is much more difficult since the tangent vector fields are not commutative.

Let Ω be a smoothly bounded domain in $\mathbb{C}^m$. Let $u \in C^1(\partial\Omega)$. We say that u is a CR function on $\partial\Omega$ if u satisfies the tangential Cauchy-Riemann equation: $\overline{\partial}_b u = 0$ on $\partial\Omega$, which is equivalent to $\overline{X}_j u = 0$ on $\partial\Omega$ for all $1 \le j \le m$ where $X_1, \cdots, X_m$ are holomorphic tangent vector fields which span the holomorphic tangent bundle on $\partial\Omega$. Based on the main idea in [25], Li and Zhang [27] proved the following theorem.

THEOREM 2.1. *Let $m > 1$ and let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^3 boundary. Let $g \in C^2(\partial\Omega)$ so that for any point $z \in \partial\Omega$, we have that either $\overline{\partial}_b g(z) = 0$ or $\overline{\partial}_b\overline{g}(z) = 0$. Then either g is CR or $\overline{g}$ is CR on $\partial\Omega$.*

3. Preliminary results. Let $u = (u^1, u^2, \ldots, u^n) : (M, h) \to (N, g)$ be a map. We need the following definitions:

(i) u is pluriharmonic if

$$\partial_{i\bar{j}}u^s + \sum_{t,\gamma=1}^{n} \Gamma^s_{t\gamma}\partial_i u^t \partial_{\bar{j}} u^\gamma = 0 \text{ for } 1 \le i, j \le m \text{ and } 1 \le s \le n. \tag{3.1}$$

(ii) u is holomorphic if $\partial_{\bar{i}} u^s = 0$ for $1 \le i \le m$ and $1 \le s \le n$.

(iii) the energy density function of u denoted $e[u](z)$ is defined by (1.6).

Since

$$h_{i\bar{j}} = -\frac{\partial}{\partial z_i}(\frac{\rho_{\bar{j}}}{\rho}) = \frac{\rho_{i\bar{j}}}{-\rho} + \frac{\rho_i\rho_{\bar{j}}}{\rho^2} = \frac{1}{-\rho}[\rho_{i\bar{j}} + \frac{\rho_i\rho_{\bar{j}}}{-\rho}], \tag{3.2}$$

we have

$$h^{i\bar{j}} = (-\rho)[\rho^{i\bar{j}} - \frac{\rho^i\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho}] \tag{3.3}$$

where

$$\rho^i = \sum_j \rho^{i\bar{j}}\rho_{\bar{j}},\ \rho^{\bar{j}} = \sum_i \rho^{i\bar{j}}\rho_i \text{ and } |\partial\rho|^2_\rho = \sum_i \rho^i\rho_i = \sum_j \rho^{\bar{j}}\rho_{\bar{j}}. \tag{3.4}$$

Therefore

$$r_{i\bar{l}}r^{\bar{l}} = r_{i\bar{l}}r^{p\bar{l}}r_p = \delta_{ip}r_p = r_i, \quad r_{j\bar{l}}r^{\bar{l}} = r_{j\bar{l}}r^{p\bar{l}}r_p = \delta_{jp}r_p = r_j. \tag{3.5}$$

Let $\mathcal{L}$, R and X_j be defined by (1.5). Then

$$\overline{X}_i = \sum_{j=1}^{m}(\rho^{i\bar{j}} - \frac{\rho^i\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho})\partial_{\bar{j}} \quad \text{and} \quad \overline{R} = \sum_{j=1}^{m}\rho^{\bar{j}}\partial_{\bar{j}}. \tag{3.6}$$

For $u \in C^2(\overline{\Omega}_m, \overline{\Omega}_n)$, let

$$E_b(u) := |\partial_b u|^2 + |\overline{\partial}_b u|^2 \tag{3.7}$$

where

$$|\partial_b u|^2 = \sum_{j=1}^{m}\sum_{s=1}^{n}|X_j u^s|^2,\ |\overline{\partial}_b u|^2 = \sum_{j=1}^{m}\sum_{s=1}^{n}|\overline{X}_j u^s|^2. \tag{3.8}$$

Finally, since $\rho(z) < 0$ on Ω_m, let

$$a[u](z) = \frac{r(u(z))}{\rho(z)}, \quad z \in \Omega_m. \tag{3.9}$$

For each $z_0 \in \partial\Omega_m$, $a[u](z_0) = \lim_{z\to z_0} a[u](z)$. We will apply this convention for the rest of the paper.

Now we proceed to compute $\tau^s[u]$ explicitly. Then using the properness assumption on u, we obtain an expression for $\tau^s[u]$ that allows us to understand under what circumstances either u is CR or $\overline{u}$ is CR.

Let us first obtain an explicit expression for the Christoffel symbols Γ^k_{ij} for $N = (\Omega_n, g)$. By definition we know that

$$
\begin{aligned}
\Gamma^k_{ij} &= \sum_\ell g^{k\bar\ell}\frac{\partial g_{i\bar\ell}}{\partial u^j} \qquad (3.10)\\
&= (-r)(r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r})\frac{\partial}{\partial u^j}\Big[\frac{1}{(-r)}(r_{i\bar\ell} + \frac{r_i r_{\bar\ell}}{-r})\Big]\\
&= (-r)(r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r})\Big[\frac{r_j}{r^2}(r_{i\bar\ell} + \frac{r_i r_{\bar\ell}}{-r}) + \frac{1}{(-r)}(r_{ij\bar\ell} + \frac{r_j}{r^2} r_i r_{\bar\ell} + \frac{r_{ij}r_{\bar\ell} + r_i r_{j\bar\ell}}{-r})\Big]\\
&= (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r})\Big[\frac{r_j}{-r}(r_{i\bar\ell} + \frac{r_i r_{\bar\ell}}{-r}) + r_{ij\bar\ell} + \frac{r_j}{r^2} r_i r_{\bar\ell} + \frac{r_{ij}r_{\bar\ell} + r_i r_{j\bar\ell}}{-r}\Big]\\
&= (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r})\Big[r_{ij\bar\ell} + \frac{2}{r^2} r_i r_j r_{\bar\ell} + \frac{r_{i\bar\ell} r_j + r_{ij}r_{\bar\ell} + r_i r_{j\bar\ell}}{-r}\Big]\\
&= (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r_r|^2 - r}) r_{ij\bar\ell} + (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r})(\frac{r_{i\bar\ell} r_j + r_{ij}r_{\bar\ell} + r_i r_{j\bar\ell}}{-r})\\
&\quad + \frac{2}{r^2}(r_i r_j r^k - \frac{|\partial r|_r^2}{|\partial r|_r^2 - r} r^k r_i r_j)\\
&= (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r}) r_{ij\bar\ell} + \frac{1}{(-r)}(\delta_{ik} r_j + r^k r_{ij} + r_i \delta_{jk})\\
&\quad - \frac{r^k r^{\bar\ell}}{(-r)(|\partial r|_r^2 - r)}(r_{i\bar\ell} r_j + r_{ij}r_{\bar\ell} + r_i r_{j\bar\ell}) + \frac{2}{(-r)}\frac{r^k r_i r_j}{(|\partial r|_r^2 - r)}\\
&= (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r}) r_{ij\bar\ell} + \frac{1}{(-r)}(\delta_{ik} r_j + r^k r_{ij} + r_i \delta_{jk})\\
&\quad - \frac{1}{(-r)(|\partial r|_r^2 - r)}(r_i r^k r_j + r_{ij} r^k |\partial r|_r^2 + r_i r^k r_j) + \frac{2}{(-r)}\frac{r^k r_i r_j}{(|\partial r|_r^2 - r)}\\
&= (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r}) r_{ij\bar\ell} + \frac{1}{(-r)}(\delta_{ik} r_j + r^k r_{ij} + r_i \delta_{jk})\\
&\quad - \frac{|\partial r|_r^2}{(-r)(|\partial r|_r^2 - r)} r_{ij} r^k\\
&= \frac{1}{-r}(\delta_{ik} r_j + r_i \delta_{jk}) + (r^{k\bar\ell} - \frac{r^k r^{\bar\ell}}{|\partial r|_r^2 - r}) r_{ij\bar\ell} + \frac{r_{ij} r^k}{|\partial r|_r^2 - r}.
\end{aligned}
$$

Substituting the expression we just found for Γ^k_{ij} in the definition of $\tau^s[u]$, we obtain

$$
\begin{aligned}
\tau^s[u] &= \Delta_M u^s + \frac{1}{(-r(u))} h^{i\bar j}(\delta_{ts} r_\gamma(u) + r_t(u)\delta_{\gamma s}) u_i^t u_{\bar j}^\gamma \qquad (3.11)\\
&\qquad + [(r^{s\bar\ell}(u) - \frac{r^s(u) r^{\bar\ell}(u)}{|\partial r|_r^2(u) - r(u)}) r_{t\gamma\bar\ell}(u) + \frac{r_{t\gamma}(u) r^s(u)}{(|\partial r|_r^2 - r(u))}] h^{i\bar j} u_i^t u_{\bar j}^\gamma\\
&= \Delta_M u^s + \frac{1}{(-r(u))} h^{i\bar j}(r_\gamma(u) u_i^s u_{\bar j}^\gamma + r_t(u) u_i^t u_{\bar j}^s)\\
&\qquad + \Big[(r^{s\bar\ell}(u) - \frac{r^s r^{\bar\ell}(u)}{|\partial r|_r^2 - r(u)}) r_{t\gamma\bar\ell}(u) + \frac{r_{t\gamma}(u) r^s(u)}{|\partial r|_r^2 - r(u)}\Big] h^{i\bar j} u_i^t u_{\bar j}^\gamma.
\end{aligned}
$$

Let $u = (u_1, \cdots, u_n) \in C^2(\overline{\Omega}_m)$ be a map from $\Omega_m \to \Omega_n$. Define

$$E^s[u] := \frac{1}{(-\rho)} h^{i\bar{j}}[r_\gamma(u) u_i^s u_{\bar{j}}^\gamma + r_t(u) u_i^t u_{\bar{j}}^s]. \tag{3.12}$$

Then using the expression for $h^{i\bar{j}}$ in (3.3), we have

$$\begin{aligned} E^s[u] &= (\rho^{i\bar{j}} - \frac{\rho^i\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho})[r_t(u) u_i^s \overline{\partial}_j u^t + r_t(u)\partial_i u^t u_{\bar{j}}^s] \\ &= u_i^s\, r_t(u)\overline{X}_i u^t + u_{\bar{i}}^s\, r_t(u) X_i u^t. \end{aligned} \tag{3.13}$$

Next we will express $E^s[u]$ in terms of the vector fields $X_j, R, \overline{X}_j$ and $\overline{R}$. This is carried out in the following lemma.

Lemma 3.1. *Let $u = (u^1, \cdots, u^n) : \Omega_m \to \Omega_n$ be a map with $u \in C^2(\overline{\Omega}_m)$. Then*

$$\begin{aligned} E^s[u] &= \rho_{i\bar{j}}[X_i u^s\, r_t(u)\overline{X}_j u^t + \overline{X}_j u^s\, r_t(u) X_i u^t] \\ &\quad + \frac{-\rho}{(|\partial\rho|^2_\rho - \rho)^2}[Ru^s\, r_t(u)\overline{R}u^t + \overline{R}u^s\, r_t(u) Ru^t] \end{aligned} \tag{3.14}$$

for all $z \in \overline{\Omega}_m$.

Proof. For any point $z_0 \in \overline{\Omega}_m$, by a rotation if necessary, we may assume that the complex Hessian matrix of ρ at z_0 is diagonal. In other words, we may assume that $H(\rho)(z_0) = \mathrm{diag}(\rho_{1\bar{1}}, \cdots, \rho_{m\bar{m}})$. Thus $\rho^{i\bar{j}}(z_0) = \delta_{ij}\rho_{i\bar{i}}(z_0)^{-1}$ and

$$\begin{aligned} E^s[u] &= u_i^s\, r_t(u)\overline{X}_i u^t + u_{\bar{i}}^s\, r_t(u) X_i u^t \\ &= \rho_{i\bar{i}} X_i u^s\, r_t(u)\overline{X}_i u^t + \rho_{i\bar{i}}\frac{\rho^\ell\rho^{\bar{i}}}{|\partial\rho|^2_\rho - \rho} u_\ell^s\, r_t(u)\overline{X}_i u^t \\ &\quad + \rho_{i\bar{i}}\overline{X}_i u^s\, r_t(u) X_i u^t + \rho_{i\bar{i}}\frac{\rho^i\rho^{\bar{\ell}}}{|\partial\rho|^2_\rho - \rho} u_{\bar{\ell}}^s\, r_t(u) X_i u^t \\ &= \rho_{i\bar{i}}\Big[X_i u^s\, r_t(u)\overline{X}_i u^t + \overline{X}_i u^s\, r_t(u) X_i u^t\Big] \\ &\quad + \rho_{i\bar{i}}\Big[\frac{\rho^{\bar{i}}}{|\partial\rho|^2_\rho - \rho} Ru^s\, r_t(u)\overline{X}_i u^t + \frac{\rho^i}{|\partial\rho|^2_\rho - \rho}\overline{R}u^s\, r_t(u) X_i u^t\Big]. \end{aligned} \tag{3.15}$$

Note that at $z = z_0$, we have that $\rho_{i\bar{i}}\rho^{\bar{i}} = \rho_i$ and $\rho_i\rho^{i\bar{i}} = \rho^{\bar{i}}$. Thus

$$\begin{aligned} \sum_{i=1}^m \rho_{i\bar{i}}\rho^{\bar{i}}\overline{X}_i = \sum_{i=1}^m \rho_i\overline{X}_i &= \sum_{i=1}^m \rho_i \sum_{j=1}^m (\rho^{i\bar{j}} - \frac{\rho^i\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho})\partial_{\bar{j}} \\ &= \sum_{i=1}^m \rho^{\bar{i}}\partial_{\bar{i}} - \sum_{j=1}^m \frac{|\partial\rho|^2_\rho\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho}\partial_{\bar{j}} \\ &= \overline{R} - \frac{|\partial\rho|^2_\rho\overline{R}}{|\partial\rho|^2_\rho - \rho} \\ &= \frac{-\rho}{|\partial\rho|^2_\rho - \rho}\overline{R}. \end{aligned} \tag{3.16}$$

Therefore,

$$E^s[u] = \rho_{i\bar{i}}[X_i u^s\, r_t(u)\overline{X}_i u^t + \overline{X}_i u^s\, r_t(u) X_i u^t] \\ + \frac{-\rho}{(|\partial\rho|_\rho^2 - \rho)^2}[Ru^s\, r_t(u)\overline{R}u^t + \overline{R}u^s\, r_t(u) Ru^t].$$

The proof is complete. ▯

Next, we want to understand the behavior of $r_s X_j u^s$ and $r_s \overline{X}_j u^s$ on $\partial\Omega_m$. This is the content of Lemma 3.2.

LEMMA 3.2. *Let $u = (u_1, \cdots, u_n) : \Omega_m \to \Omega_n$ be a proper harmonic map so that $u \in C^2(\overline{\Omega}_m)$. Then*

$$\sum_{s=1}^n r_s(u)\overline{\partial}_b u^s = \sum_{s=1}^n r_s(u)\partial_b u^s \equiv 0 \quad on \quad \partial\Omega_m. \tag{3.17}$$

Proof. By Lemma 3.1

$$r_s(u)E^s[u] = 2\rho_{i\bar{j}} r_s(u) X_i u^s\, r_t(u)\overline{X}_j u^t + \frac{2(-\rho)}{(|\partial\rho|_\rho^2 - \rho)^2} r_s(u) Ru^s\, r_t(u)\overline{R}u^t.$$

Since $r[u] = 0$ on $\partial\Omega_m$, and X_i is a tangential vector field, we have that

$$0 = X_i r[u](z_0) = r_s(u) X_i u^s + r_{\bar{s}}(u) X_i \overline{u}^s = r_s(u) X_i u^s + \overline{r_s(u)\overline{X}_i u^s}.$$

Thus,

$$r_s(u) X_i u^s\, r_t(u)\overline{X}_i u^t(z_0) = -\Big|\sum_{s=1}^n r_s(u) X_i u^s(z_0)\Big|^2 = -\Big|\sum_{s=1}^n r_s(u)\overline{X}_i u^s(z_0)\Big|^2$$

for all $1 \le i \le m$. For any $z_0 \in \partial\Omega_m$, by a rotation, we may assume $H(\rho)(z_0)$ is diagonal. Since $H(\rho)$ is positive definite, there is a positive constant ϵ so that $H(\rho) \ge \epsilon I_n$ for all $z \in \overline{\Omega}_m$. Therefore,

$$\begin{aligned} r_s(u)E^s[u](z_0) &= -2\sum_{i=1}^m \rho_{i\bar{i}}(z_0)|\sum_{s=1}^n r_s(u) X_i u^s(z_0)|^2 \\ &= -2\sum_{i=1}^m \rho_{i\bar{i}}(z_0)|\sum_{s=1}^n r_s(u)\overline{X}_i u^s(z_0)|^2. \end{aligned} \tag{3.18}$$

Since u is proper harmonic ($\tau^s[u] = 0$ in Ω_m), and $u \in C^2(\overline{\Omega}_m)$, one can easily see that $\sum_{s=1}^n r_s(u)E^s[u] = 0$ on $\partial\Omega_m$. Combining this with the above identity, we obtain

$$\sum_s r_s(u) X_i u^s(z_0) = \sum_{s=1}^n r_s(u)\overline{X}_i u^s = 0, \quad 1 \le i \le m$$

for all $z_0 \in \partial\Omega_m$.
By the fact that $X_1, \cdots, X_m$ generate $T^{1,0}(\partial\Omega_m)$, we conclude that

$$\sum_{s=1}^n r_s(u)\overline{\partial}_b u^s(z) = \sum_{s=1}^n r_s(u)\partial_b u^s(z) = 0 \quad \text{for all} \;\; z \in \partial\Omega_m.$$

The proof of the lemma is complete. ∎

Let

$$Y_i = (\delta_{ij} - \frac{\rho_i \rho^j}{|\partial\rho|^2_\rho - \rho})\partial_j. \tag{3.19}$$

Note that $Y_i = \sum_j \rho_{i\bar{j}} X_j$. Thus $Y_i \in T^{(1,0)}(\partial\Omega_m)$. The following lemma expresses $\mathcal{L}$ in terms of the vector fields $X_j, \overline{X}_j, Y_j, \overline{Y}_j, R$ and $\overline{R}$.

Lemma 3.3. *With the notation above, we have*

$$\mathcal{L} = X_j\overline{Y}_j + \frac{\mathcal{L}(\rho)}{|\partial\rho|^2_\rho - \rho}\overline{R} - \frac{\rho}{|\partial\rho|^2_\rho - \rho}R\frac{1}{|\partial\rho|^2_\rho - \rho}\overline{R} \tag{3.20}$$

$$= \overline{X}_jY_j + \frac{\mathcal{L}(\rho)}{|\partial\rho|^2_\rho - \rho}R - \frac{\rho}{|\partial\rho|^2_\rho - \rho}\overline{R}\frac{1}{|\partial\rho|^2_\rho - \rho}R. \tag{3.21}$$

Proof. The proof is just a simple computation. By definition

$$\begin{aligned}
\mathcal{L} &= \sum_{ij}(\rho^{i\bar{j}} - \frac{\rho^i\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho})\partial_{i\bar{j}} \\
&= \sum_j X_j\partial_{\bar{j}} \\
&= \sum_j X_j\overline{Y}_j + \sum_j X_j(\frac{\rho^{\bar{\ell}}\rho_{\bar{j}}}{|\partial\rho|^2_\rho - \rho}\partial_{\bar{\ell}}) \\
&= \sum_j X_j\overline{Y}_j + \sum_{j,\ell=1}^m \frac{X_j(\rho_{\bar{j}})}{|\partial\rho|^2_\rho - \rho}\rho^{\bar{\ell}}\partial_{\bar{\ell}} + \sum_{j,\ell=1}^m \rho_{\bar{j}}X_j(\frac{\rho^{\bar{\ell}}}{|\partial\rho|^2_\rho - \rho}\partial_{\bar{\ell}}) \\
&= \sum_j X_j\overline{Y}_j + \frac{\mathcal{L}(\rho)}{|\partial\rho|^2_\rho - \rho}\overline{R} + \frac{-\rho}{|\partial\rho|^2_\rho - \rho}R\frac{1}{|\partial\rho|^2_\rho - \rho}\overline{R}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathcal{L} &= \sum_{ij}(\rho^{i\bar{j}} - \frac{\rho^i\rho^{\bar{j}}}{|\partial\rho|^2_\rho - \rho})\partial_{i\bar{j}} \\
&= \sum_j \overline{X}_j\partial_j \\
&= \sum_j \overline{X}_jY_j + \sum_j \overline{X}_j(\frac{\rho^{\ell}\rho_j}{|\partial\rho|^2_\rho - \rho}\partial_\ell) \\
&= \sum_j \overline{X}_jY_j + \sum_{j,\ell=1}^m \frac{\overline{X}_j(\rho_j)}{|\partial\rho|^2_\rho - \rho}\rho^\ell\partial_\ell + \sum_{j,\ell=1}^m \rho_j\overline{X}_j(\frac{\rho^\ell}{|\partial\rho|^2_\rho - \rho}\partial_\ell) \\
&= \sum_j \overline{X}_jY_j + \frac{\mathcal{L}(\rho)}{|\partial\rho|^2_\rho - \rho}R - \frac{\rho}{|\partial\rho|^2_\rho - \rho}\overline{R}\frac{1}{|\partial\rho|^2_\rho - \rho}R.
\end{aligned}$$

The proof is complete. ∎

Since $Y_i \in T^{(1,0)}(\partial\Omega_m)$, it follows from Lemma 3.2 that

$$\sum_{s=1}^{n} r_s(u) Y_j u^s(z) = 0 \text{ on } \partial\Omega_m, \quad 1 \le j \le m.$$

A similar reasoning shows that

$$\sum_{s=1}^{n} r_s(u) \overline{Y}_j u^s(z) = 0 \text{ on } \partial\Omega_m, \quad 1 \le j \le m.$$

This implies that for each $1 \le j \le m$ we have that

$$0 = \overline{X}_j(r_s(u)Y_j u^s) = r_{st}\overline{X}_j u^t\, Y_j u^s + r_{s\bar{t}}\overline{X}_j \overline{u}^t\, Y_j u^s + r_s(u)\overline{X}_j Y_j u^s$$

and

$$0 = X_j(r_s(u)\overline{Y}_j u^s) = r_{st}X_j u^t\, \overline{Y}_j u^s + r_{s\bar{t}}X_j \overline{u}^t\, \overline{Y}_j u^s + r_s(u)X_j \overline{Y}_j u^s.$$

Thus

$$-r_s(u)\overline{X}_j Y_j u^s = r_{s\bar{t}}(u)\overline{X}_j \overline{u}^t\, Y_j u^s + r_{st}(u)\overline{X}_j u^t\, Y_j u^s$$

and

$$-r_s(u)X_j \overline{Y}_j u^s = r_{s\bar{t}}(u)X_j \overline{u}^t\, \overline{Y}_j u^s + r_{st}(u)X_j u^t\, \overline{Y}_j u^s.$$

Therefore, it follows from Lemma 3.3 that on $\partial\Omega_m$

$$\begin{aligned} &\frac{\mathcal{L}(\rho)}{|\partial\rho|^2_\rho}\sum_s r_s(u)Ru^s \qquad (3.22)\\ &= \sum_{s=1}^{n} r_s(u)\mathcal{L}u^s - \sum_{s=1}^{n}\sum_{j=1}^{m} r_s(u)\overline{X}_j Y_j u^s + r_s(u)\frac{\rho(z)}{|\partial\rho|^2_\rho - \rho}\overline{R}\frac{1}{|\partial\rho|^2_\rho - \rho}Ru^s\\ &= r_s(u)\mathcal{L}u^s(z) + r_{s\bar{t}}(u)\overline{X}_j\overline{u}^t\, Y_j u^s + r_{st}(u)\overline{X}_j u^t\, Y_j u^s\\ &= r_s(u)\mathcal{L}u^s(z) + r_{st}(u)\overline{X}_j u^t Y_j u^s + r_{s\bar{t}}(u)\overline{X}_j \overline{u}^t Y_j u^s. \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{\mathcal{L}(\rho)}{|\partial\rho|^2_\rho}\sum_s r_s(u)\overline{R}u^s \qquad (3.23)\\ &= \sum_{s=1}^{n} r_s(u)\mathcal{L}u^s - \sum_{s=1}^{n}\sum_{j=1}^{m} r_s(u)X_j \overline{Y}_j u^s + r_s(u)\frac{\rho(z)}{|\partial\rho|^2_\rho - \rho}R\frac{1}{|\partial\rho|^2_\rho - \rho}\overline{R}u^s\\ &= r_s(u)\mathcal{L}u^s(z) + r_{st}(u)X_j u^t\overline{Y}_j u^s + r_{s\bar{t}}(u)X_j\overline{u}^t\overline{Y}_j u^s. \end{aligned}$$

Recall that $a[u](z)$ is given by

$$r(u) = a[u](z)\rho(z), \quad z \in \overline{\Omega}_m.$$

It is easy to see that $a[u] \ge 0$ on $\overline{\Omega}_m$, $a[u](z) > 0$ on Ω_m and $a[u] \in C^1(\overline{\Omega}_m)$. Thus,

$$\tau^s[u] = (-\rho)\mathcal{L}u^s + (-\rho)\Big[r^{s\overline{\ell}}(u)r_{t\gamma\overline{\ell}} + \frac{r_{t\gamma}(u)r^s(u) - r^s r^{\overline{\ell}} r_{t\gamma\overline{\ell}}}{|\partial r|_r^2 - r}\Big] X_j u^t\, u^{\gamma}_{\overline{j}} \tag{3.24}$$
$$+ \frac{\rho(z)}{r(u)} E^s[u]$$
$$= (-\rho)\mathcal{L}u^s + (-\rho)\Big[r^{s\overline{\ell}}(u)r_{t\gamma\overline{\ell}}(u) + \frac{r_{t\gamma}(u)r^s(u) - r^s r^{\overline{\ell}} r_{t\gamma\overline{\ell}}}{|\partial r|_r^2 - r}\Big] X_j u^t u^{\gamma}_{\overline{j}}$$
$$+ \frac{1}{a[u]}\rho_{i\overline{j}}[X_i u^s r_t(u)\overline{X}_j u^t + \overline{X}_j u^s r_t(u) X_i u^t]$$
$$+ \frac{1}{a[u]}\frac{(-\rho)}{(|\partial\rho_\rho|^2 - \rho)^2}[R u^s r_t(u)\overline{R}u^t + \overline{R}u^s r_t(u) R u^t]$$
$$= (-\rho)\Big[\mathcal{L}u^s + (r^{s\overline{\ell}}(u)r_{t\gamma\overline{\ell}} + \frac{r_{t\gamma}(u)r^s(u) - r^s(u)r^{\overline{\ell}}(u)r_{t\gamma\overline{\ell}}(u)}{|\partial r|_r^2 - r}) X_j u^t\, u^{\gamma}_{\overline{j}}$$
$$+ \frac{1}{a[u]}\frac{1}{(|\partial\rho|_\rho^2 - \rho)^2}(R u^s r_t(u)\overline{R}u^t + \overline{R}u^s r_t(u) R u^t)\Big]$$
$$+ \frac{1}{a[u]}\rho_{i\overline{j}}[X_i u^s r_t(u)\overline{X}_j u^t + \overline{X}_j u^s r_t(u) X_i u^t].$$

Since $\sum_{s=1}^n r_s(u)r^s(u) = |\partial r|_r^2$, we have that on $\partial\Omega_m$

$$\sum_{s=1}^n \Big(r^{s\overline{\ell}}(u)r_{t\gamma\overline{\ell}} + \frac{r_{t\gamma}(u)r^s(u) - r^s(u)r^{\overline{\ell}} r_{t\gamma\overline{\ell}}}{|\partial r|_r^2}\Big) r_s(u) X_j u^t u^{\gamma}_{\overline{j}}$$
$$= (r^{\overline{\ell}}(u)r_{t\gamma\overline{\ell}} + r_{t\gamma}(u) - r^{\overline{\ell}} r_{t\gamma\overline{\ell}}(u)) X_j u^t u^{\gamma}_{\overline{j}}$$
$$= r_{t\gamma}(u) X_j u^t u^{\gamma}_{\overline{j}}.$$

Thus on $\partial\Omega_m$,

$$r_s(u)\tau^s[u] = \frac{2}{a[u]}\rho_{i\overline{j}}(r_s(u)X_i u^s)(r_t(u)\overline{X}_j u^t) \tag{3.25}$$
$$+ (-\rho)\Big[(r_s(u)\mathcal{L}u^s + r_{t\gamma}(u)X_j u^t u^{\gamma}_{\overline{j}}) + 2\frac{r_s(u) R u^s\, r_t(u)\overline{R}u^t}{a[u](|\partial\rho|_\rho^2 - \rho)^2}\Big].$$

Since $u \in C^2(\overline{\Omega}_m)$, we know by Lemma 3.2 that

$$(\sum_s r_s(u) X_i u^s)(\sum_s r_s(u)\overline{X}_j u^s) = O(\rho(z)^2).$$

From the harmonicity of u, it follows that $r_s(u)\tau^s[u] = 0$. This implies that for any $z \in \partial\Omega_m$

$$a[u](r_s(u)\mathcal{L}u^s + r_{t\gamma}(u)X_j u^t u^{\gamma}_{\overline{j}}) + 2\frac{r_s(u) R u^s\, r_t(u)\overline{R}u^t}{|\partial\rho|_\rho^4} = 0. \tag{3.26}$$

THEOREM 3.4. *Let $\Omega_m \subset \mathbb{C}^m$ $(m > 1)$ and $\Omega_n \subset \mathbb{C}^n$ be smoothly bounded strictly pseudoconvex domains with metric h and g, respectively. Let $u = (u^1, \cdots, u^n) \in C^2(\overline{\Omega}_m)$ be a proper harmonic map from (Ω_m, h) to (Ω_n, g). If*

$$\lim_{w\to z}(r_s\mathcal{L}u^s(w) + r_{st}(u)X_j u^t u^{s}_{\overline{j}}) = 0, \quad \textit{for}\ \ z \in \partial\Omega_m, \tag{3.27}$$

then either u is CR or $\overline{u}$ is CR.

Proof. Notice that

$$\sum_{j=1}^{m} \rho_{\overline{j}} X_j = \frac{(-\rho)R}{|\partial\rho|_\rho^2 - \rho}.$$

By a rotation if necessary, we may assume without loss of generality that for any $z_0 \in \partial\Omega_m$, we have that $\rho_{i\overline{j}}(z_0) = \rho_{i\overline{i}}(z_0)\delta_{ij}$. Therefore,

$$\begin{aligned} r_{st}(u) X_j u^t\, u^s_{\overline{j}}(z_0) &= r_{st}(u) X_j u^t \overline{Y}_j u^s(z_0) + \frac{r_{st}(u) X_j u^t \rho_{\overline{j}} \rho^k \partial_{\overline{k}} u^s(z_0)}{|\partial\rho|_\rho^2 - \rho} \\ &= r_{st}(u) X_j u^t \overline{Y}_j u^s(z_0). \end{aligned} \tag{3.28}$$

Now (3.26) and (3.27) imply that

$$\Big(r_s(u) R u^s(z_0)\Big)\Big(r_t(u)\overline{R} u^t(z_0)\Big) = 0. \tag{3.29}$$

Also

$$\begin{aligned} Y_j u^s(z_0) &= \rho_{j\overline{j}}(z_0)(\rho^{j\overline{j}}\partial_j - \rho^{j\overline{j}} \frac{\rho_j \rho^k}{|\partial\rho|_\rho^2}\partial_k) u^s(z_0) \qquad (3.30) \\ &= \rho_{j\overline{j}}(z_0)(\rho^{k\overline{j}}\partial_k - \frac{\rho^{\overline{j}}\rho^k}{|\partial\rho|_\rho^2}\partial_k) u^s(z_0) \\ &= \rho_{j\overline{j}}(z_0) X_j u^s(z_0). \end{aligned}$$

Similarly, we have

$$\overline{Y}_j u^s(z_0) = \rho_{j\overline{j}}(z_0)\overline{X}_j u^s(z_0). \tag{3.31}$$

Combining (3.29) with (3.22) and (3.23), and using (3.30) and (3.31), we obtain

$$(r_{s\overline{t}}\rho_{j\overline{j}}\overline{X}_j u^s X_j \overline{u}^t)(r_{p\overline{q}}\rho_{k\overline{k}} X_k u^s \overline{X}_k \overline{u}^t) = 0.$$

Since $(r_{p\overline{q}}(u(z_0))$ and $H(\rho)(z_0)$ are positive definite, we have

$$\Big[\sum_{j=1}^{m}\sum_{s=1}^{n} |X_j u^s(z)|^2\Big]\Big[\sum_{j=1}^{m}\sum_{t=1}^{n} |\overline{X}_j u^t(z_0)|^2\Big] = 0.$$

Therefore, either $\overline{\partial}_b \overline{u}^s(z_0) = 0$ or $\partial_b \overline{u}^s(z_0) = 0$ for all $1 \le s \le n$. The proof is complete by applying Theorem 2.1. ▯

The following lemma gives an expression for $a[u](z)$ in terms of the vector fields R and $\overline{R}$ and provides a sufficient condition for two proper harmonic maps with the same boundary data to be equal.

Lemma 3.5. *Let $u : \Omega_m \to \Omega_n$ be a proper harmonic map so that $u \in C^2(\overline{\Omega}_m)$.*
(i) Then $r_s(u)Ru^s$ and $r_s(u)\overline{R}u^s$ are non-negative on $\partial\Omega_m$. In particular, for any $z \in \partial\Omega_m$ with $|\partial\rho|_0^2 = \sum_{j=1}^n |\partial_j\rho|^2$, we have

$$a[u](z)|\partial\rho|_0^2 = \frac{r_s(u)(R+\overline{R})u^s(z)}{|\partial\rho|_\rho^2}. \tag{3.32}$$

(ii) On $\{z \in \partial\Omega_m : a[u](z) > 0\}$ *we have*

$$r_s \mathcal{L} u^s(w) + r_{st}(u) X_j u^t u^s_{\bar{j}} = \frac{-(m+1)E[u](z) + D[u](z)}{4m} \tag{3.33}$$

$$a[u](z)|\partial\rho|_0^2 = \frac{E(u)(z)}{2m} + \frac{D[u]}{2m(m-1)}, \tag{3.34}$$

where

$$E[u](z) = r_{s\bar{t}}(u)\overline{X_j u^t} Y_j u^s(z_0) + r_{s\bar{t}}(u)\overline{\overline{X}_j u^t}\,\overline{Y}_j u^s(z_0)$$

and

$$D[u] = \left[(m+1)^2 E[u]^2 - 16m(r_{s\bar{t}}(u)\overline{X_j u^t} Y_j u^s(z_0))(r_{s\bar{t}}(u)\overline{\overline{X}_j u^t}\,\overline{Y}_j u^s(z_0))\right]^{1/2}.$$

(iii) For $z \in \partial\Omega_m$*, we have that* $E_b(u)(z) > 0$ *if and only if* $a[u](z) > 0$*.*

(iv) If $u(z) = v(z)$ *and* $a[u](z) > 0$ *on* $\partial\Omega_m$*, then* $u \equiv v$ *on* Ω_m*.*

Proof. For any $z_0 \in \partial\Omega_m$, we may assume that $|\partial\rho(z_0)|^2 = 1$; otherwise, we may use $\tilde{\rho}(z) = \rho(z)/|\partial\rho(z_0)|$ to replace ρ and use $\tilde{r}(w) = r(w)/|\partial\rho(z_0)|_0$ to replace $r(w)$. By a rotation if necessary, we may assume without loss of generality that $H(\rho)(z_0)$ is diagonal.

First we prove (i). Let

$$A = \sum_{s=1}^n r_s(u)\mathcal{L}u^s(z_0) + \sum_{s,t=1}^n r_{st}(u)\, u^t_{\bar{j}} X_j u^s(z_0),$$

$$A_1 = r_s(u) R u^s(z_0), \quad A_2 = r_s(u)\overline{R} u^s(z_0)$$

$$\begin{aligned} E_1 &= r_{s\bar{t}}(u)\overline{X_j u^t} Y_j u^s(z_0) \\ &= r_{s\bar{t}}(u)\rho_{j\bar{j}} X_j u^s \overline{X_j u^t}(z_0), \text{ and} \end{aligned} \tag{3.35}$$

$$\begin{aligned} E_2 &= r_{s\bar{t}}(u)\overline{\overline{X}_j u^t}\,\overline{Y}_j u^s(z_0) \\ &= r_{s\bar{t}}(u)\rho_{j\bar{j}} \overline{X}_j u^s \overline{\overline{X}_j u^t}(z_0), \end{aligned} \tag{3.36}$$

since $Y_j u^s = \rho_{j\bar{j}} X_j u^s$ by (3.30).

Then by (3.22)

$$\frac{\mathcal{L}(\rho)}{|\partial\rho|_\rho^2} A_1 = A + r_{s\bar{t}}(u)\rho_{j\bar{j}} X_j u^s \overline{X_j u^t} = A + E_1, \tag{3.37}$$

by (3.23)

$$\frac{\mathcal{L}(\rho)}{|\partial\rho|_\rho^2} A_2 = A + r_{s\bar{t}}(u)\rho_{j\bar{j}} \overline{X}_j u^s \overline{\overline{X}_j u^t} = A + E_2, \tag{3.38}$$

and by (3.26)

$$a[u]A + \frac{2}{|\partial\rho|_\rho^4} A_1 A_2 = 0. \tag{3.39}$$

Thus,

$$\mathcal{L}(\rho)^2 a[u]A + 2[A+E_1][A+E_2] = 0$$

or

$$2A^2 + (2E_1 + 2E_2 + \mathcal{L}(\rho)^2 a[u])A + 2E_1E_2 = 0. \tag{3.40}$$

Therefore, A is non-positive since E_j is non-negative for $j=1,2$. Thus A_j is real for $j=1,2$. Moreover, $A<0$ when both $E_1>0$ and $E_2>0$.
Define $\mathcal{R} = \rho_{\overline{j}}\partial_j$, $\overline{\mathcal{R}} = \rho_j \partial_{\overline{j}}$. Since $(\mathcal{R}-\overline{\mathcal{R}})r(u(z)) = 0$ for $z \in \partial\Omega_m$, we have that

$$a[u] = \frac{\mathcal{R}+\overline{\mathcal{R}}}{2} r(u) = \mathcal{R}r(u) = \overline{\mathcal{R}}r(u).$$

Also $R = \langle R, \mathcal{R}\rangle \mathcal{R} + T$ where T is a tangential vector. Since $\langle R, \mathcal{R}\rangle = |\partial\rho|^2_\rho$

$$a[u] = \mathcal{R}r(u) = \frac{Rr(u)}{|\partial\rho|^2_\rho} - \frac{T\,r(u)}{|\partial\rho|^2_\rho} = \frac{R\,r(u)}{|\partial\rho|^2_\rho}$$

and

$$a[u] = \overline{\mathcal{R}}r(u) = \frac{\overline{R}r(u)}{|\partial\rho|^2_\rho} - \frac{\overline{T}r(u)}{|\partial\rho|^2_\rho} = \frac{\overline{R}r(u)}{|\partial\rho|^2_\rho}.$$

Since A_2 is real, we obtain

$$\begin{aligned} 0 \le a[u] = \frac{Rr(u)}{|\partial\rho|^2_\rho} &= \frac{1}{|\partial\rho|^2_\rho}(r_s(u)Ru^s + r_{\overline{s}}(u)R\overline{u}^s) \\ &= \frac{1}{|\partial\rho|^2_\rho}(r_s(u)Ru^s + \overline{r_s(u)\overline{R}u^s}) \\ &= \frac{1}{|\partial\rho|^2_\rho}(A_1 + \overline{A}_2) \\ &= \frac{1}{|\partial\rho|^2_\rho}(A_1 + A_2), \end{aligned} \tag{3.41}$$

which is (3.32). Also by (3.39) and the nonpositivity of A,

$$A_1A_2 = -\frac{a[u]A|\partial\rho|^4_\rho}{2} \ge 0.$$

Thus we have established that $A_1 + A_2 \ge 0$ and $A_1A_2 \ge 0$. As a result, $A_j \ge 0$ for $j=1,2$. This finishes the proof of (i).

Next we prove (ii). Using the fact that $\rho^i\rho^{\overline{j}}\rho_{i\overline{j}} = \rho^{i\overline{k}}\rho_{\overline{k}}\rho^{\overline{j}}\rho_{i\overline{j}} = \rho^{i\overline{k}}\rho_{i\overline{j}}\rho_{\overline{k}}\rho^{\overline{j}} = \delta_{kj}\rho_{\overline{k}}\rho^{\overline{j}} = \rho_{\overline{j}}\rho^{\overline{j}} = |\partial\rho|^2_\rho$ we have that on $\partial\Omega_m$

$$\begin{aligned} \mathcal{L}(\rho) &= \sum_{ij}(\rho^{i\overline{j}} - \frac{\rho^i\rho^{\overline{j}}}{|d\rho|^2 - \rho})\rho_{i\overline{j}} \\ &= \sum_i \delta_{ii} - \sum_{ij}\frac{\rho^i\rho^{\overline{j}}}{|d\rho|^2-\rho}\rho_{i\overline{j}} \end{aligned} \tag{3.42}$$

$$= m - \frac{|\partial\rho|_\rho^2}{|\partial\rho|_\rho^2 - \rho}$$
$$= m - 1.$$

Adding (3.37) to (3.38) and using (3.41) together with (3.42), we have that

$$a[u] = \frac{2A + E[u]}{m-1} \text{ since } E[u] = E_1 + E_2. \tag{3.43}$$

Substituting (3.43) into (3.40) and using (3.42), we obtain

$$mA^2 + \Big(\frac{m+1}{2}\Big)E[u]A + E_1E_2 = 0. \tag{3.44}$$

Since $A_i \geq 0$ for $i = 1, 2$, (3.37) and (3.38) implies that if $a[u] > 0$ or $E_1 \neq E_2$, then

$$A = \frac{-(m+1)E[u] + D[u]}{4m}, \tag{3.45}$$

where $D[u] = \sqrt{(m+1)^2E[u]^2 - 16mE_1E_2}$. This establishes (3.33). By (3.43) and (3.45)

$$\begin{aligned} a[u] &= \frac{2A + E[u]}{m-1} \\ &= \frac{1}{m-1}\Big(\frac{-(m+1)E[u] + D[u]}{2m} + E[u]\Big) \\ &= \frac{E[u]}{2m} + \frac{D[u]}{2m(m-1)} \end{aligned}$$

which is (3.34).

Now we prove (iii). By (ii) if $a[u] > 0$, then $E[u] > 0$ which implies that $E_b(u) > 0$ since $r_{s\bar{t}}$ is positive definite. In order to establish the converse, we must show that for every $z_0 \in \partial\Omega_m$ such that $a[u](z_0) = 0$, it holds that $E_b(u)(z_0) = 0$. Since $r_{s\bar{t}}$ is positive definite, this is equivalent to showing that $E[u] = 0$. If z_0 is a boundary point of the zero set of $a[u]$ on $\partial\Omega_m$, then $E[u] = 0$ by (3.34) and passing to the limit. So assume that z_0 is an interior point of the zero set of $a[u]$ on $\partial\Omega_m$. Since $r(u) = a[u]\rho(z)$, we have that $Rr(u) = Ra[u]\rho(z) + a[u]R\rho(z)$. Therefore,

$$Rr(u) = r_sRu^s + r_{\bar{s}}R\overline{u}^s = r_sRu^s + \overline{r_s\overline{R}u^s} = Ra[u]\rho(z) + a[u]|\partial\rho|_\rho^2.$$

Thus,

$$2\mathrm{Re}(r_sRu^s)(r_s\overline{R}u^s)a[u]^{-1} \tag{3.46}$$
$$= |Ra[u]\rho(z) + a[u]|\partial\rho|_\rho^2|^2a[u]^{-1} - (|r_sRu^s|^2 + |r_{\bar{s}}R\overline{u}^s|^2)a[u]^{-1} \tag{3.47}$$
$$\begin{aligned} &\leq |Ra[u]\rho(z) + a[u]|\partial\rho|_\rho^2|^2a[u]^{-1} \\ &= |Ra[u]|^2\rho(z)^2a[u]^{-1} + a[u]|\partial\rho|_\rho^4 + 2\mathrm{Re}(Ra[u]\rho(z)|\partial\rho|_\rho^2) \\ &= |Ra[u]|^2\rho(z)^2a[u]^{-1} + a[u]|\partial\rho|_\rho^4 + (R + \overline{R})a[u]\rho(z)|\partial\rho|_\rho^2 \\ &= 4(Ra[u]^{1/2})(\overline{R}a[u]^{1/2})\rho(z)^2 + a[u]|\partial\rho|_\rho^4 + (R + \overline{R})a[u]\rho(z)|\partial\rho|_\rho^2. \end{aligned}$$

Since $X_ir(u) = X_ia[u]\rho(z) + a[u]X_i(\rho)$ and $X_i(\rho) = \frac{-\rho\rho^{\bar{i}}}{|\partial\rho|_\rho^2 - \rho}$, we have

$$X_ir(u) = r_sX_iu^s + \overline{r_s\overline{X}_iu^s} = X_ia[u]\rho(z) - a[u]\frac{\rho\rho^{\bar{i}}}{|\partial\rho|_\rho^2 - \rho}.$$

Thus,

$$\begin{aligned}&2\mathrm{Re}[(r_sX_iu^s)(r_s\overline{X}_iu^s)a[u]^{-1}\rho(z)^{-1} \qquad (3.48)\\
&=|X_ia[u]\rho(z)-a[u]\frac{\rho\rho^{\bar{i}}}{|\partial\rho|^2_\rho-\rho}|^2a[u]^{-1}\rho(z)^{-1}-\frac{(|r_sX_iu^s|^2+|r_s\overline{X}_iu^s|^2)}{a[u]\rho(z)}\\
&\le|X_ia[u]\rho(z)-a[u]\frac{\rho\rho^{\bar{i}}}{|\partial\rho|^2_\rho-\rho}|^2a[u]^{-1}\rho(z)^{-1}\\
&=|X_ia[u]|^2\rho(z)a[u]^{-1}-\frac{2\mathrm{Re}(X_ia[u]\rho^i)}{|\partial\rho|^2_\rho-\rho}\rho+\frac{|\rho^{\bar{i}}|^2a[u]\rho(z)}{(|\partial\rho|^2_\rho-\rho)^2}\\
&=4(X_ia[u]^{1/2})(\overline{X}_ia[u]^{1/2})\rho-\frac{2\mathrm{Re}(X_ia[u]\rho^i)}{|\partial\rho|^2_\rho-\rho}\rho+\frac{|\rho^{\bar{i}}|^2a[u]\rho(z)}{(|\partial\rho|^2_\rho-\rho)^2}.\end{aligned}$$

Since $a[u]\in C^1(\overline{\Omega}_m)$ and $a[u]=0$ on $\partial\Omega_m\cap B(z_0,\delta)$ for some $\delta>0$, it is clear that $a[u]^{1/2}\in C^{1/2}(\overline{\Omega}_m)$. Therefore,

$$\lim_{z\to z_0}[(Ra[u]^{1/2})(\overline{R}a[u]^{1/2})\rho(z)^2]=0. \qquad (3.49)$$

At the same time, since X_i and $\overline{X}_i$ are tangential

$$\lim_{z\to z_0}[(X_ia[u]^{1/2})(\overline{X}_ia[u]^{1/2})\rho(z)]=0. \qquad (3.50)$$

Thus, (3.25) and the previous computations show that

$$\begin{aligned}0&=\limsup_{z\to z_0}\Big[\mathrm{Re}(r_s(u)\mathcal{L}u^s+r_{st}(u)X_ju^tu^s_{\bar{j}})+2\mathrm{Re}\frac{r_s(u)Ru^s\,r_t(u)\overline{R}u^t}{a[u]|(d\rho|^2-\rho)^2}\\
&\quad+\frac{2\mathrm{Re}}{a[u]|\rho|}\rho_{i\bar{j}}(r_s(u)X_iu^s)(r_t(u)\overline{X}_ju^t)\Big]\\
&=\limsup_{z\to z_0}\mathrm{Re}(r_s(u)\mathcal{L}u^s+r_{st}(u)X_ju^tu^s_{\bar{j}}).\end{aligned}$$

This implies that $(r_s(u)\mathcal{L}u^s+r_{st}(u)X_ju^tu^s_{\bar{j}})\ge0$ at z_0. By (3.43), we know that $0=a[u]=\frac{2A+E[u]}{m-1}$. Thus $0\le2A=-E$ which implies that $E=0$.

Finally, we prove (iv). Since $a[u](z)>0$ for all $z\in\partial\Omega_m$, it follows from (ii) that $a[v]=a[u]$ on $\partial\Omega_m$. Next we show $d_{\Omega_n}(u(z),v(z))=0$ for $z\in\partial\Omega_m$. For each $z_0\in\partial\Omega_m$, after a holomorphically change of coordinates, we may assume that $z_0=0$, $w_0=u(z_0)=0$ and

$$r(w)=-\lambda_0\mathrm{Re}\,w^n+\sum_{j=1}^n\lambda_j|w^j|^2+o(|w|^2),\quad \lambda_j>0$$

for all $w\in\Omega_n\cap B(0,\epsilon)$ for some $0<\epsilon<<1$. For any $z\in\Omega_m$ with $|z|\le\delta$ so that $u(z),v(z)\in\Omega_n\cap B(0,\epsilon)$, let $S=\{\gamma:[0,1]\to\Omega_n:\gamma\text{ is piecewise differentiable curve with }\gamma(0)=v(z),\gamma(1)=u(z)\}$. Then

$$\begin{aligned}d_{\Omega_n}(u(z),v(z))&=\inf_{\gamma\in S}\int_0^1\sqrt{g_{i\bar{j}}(\gamma(t))\gamma^{i\prime}(t)\overline{\gamma^{j\prime}(t)}}\,dt\\
&=\inf_{\gamma\in S}\int_0^1\sqrt{[\frac{1}{-r}(r_{i\bar{j}}+\frac{r_ir_{\bar{j}}}{-r})](\gamma(t))\gamma^{i\prime}(t)\overline{\gamma^{j\prime}(t)}}\,dt\end{aligned}$$

$$\le C \inf_{\gamma\in S}\int_0^1 \Big(\frac{|\gamma'(t)|}{\sqrt{-r(\gamma(t))}} + \frac{|(r(\gamma(t))'|}{|r(\gamma(t))|}\Big)dt$$

Since $r(w)$ is convex in $\Omega_n \cap B(0,\epsilon)$ and assuming that $r(v(z)) < r(u(z)$, we have that $r(tu(z) + (1-t)v(z))$ is an increasing convex function in t and

$$\begin{aligned} a[tu(z) + (1-t)v(z)] &= \frac{r(tu(z) + (1-t)v(z))}{\rho(z)} \\ &\ge ta[u](z) + (1-t)a[v(z)] \\ &\ge a[u](z_0)/2 \end{aligned}$$

when $\epsilon > 0$ is small and $z \in \Omega_n \cap B(0,\epsilon)$. Let $\pi(z)$ be the radial projection of z onto $\partial\Omega_m$. Then

$$\begin{aligned} |u(z) - v(z)| &\le |u(z) - \phi(\pi(z))| + |\phi(\pi(z)) - v(z)| \\ &= |u(z) - u(\pi(z))| + |v(z) - v(\pi(z))| \\ &\le C(|u|_1 + |v|_1)|\rho(z)|. \end{aligned}$$

Let

$$\gamma(t) = tu(z) + (1-t)v(z) \in S.$$

Then

$$\begin{aligned} d_{\Omega_n}(u(z), v(z)) \le\; & \frac{2C}{\sqrt{a[u](z_0)}}\int_0^1 \frac{|\gamma'(t)|}{\sqrt{-\rho(z))}}dt + C\int_0^1 \frac{(r(\gamma(t))'}{-r(\gamma(t))}dt \\ =\; & \frac{2C}{\sqrt{a[u](z_0)}}\frac{|u(z) - v(z)|}{\sqrt{|\rho(z)|}} + C\log\frac{r(v(z))}{r(u(z))} \\ \le\; & \frac{2C}{\sqrt{a[u](z_0)}}C(|u|_1 + |v|_1)\sqrt{|\rho(z)|} + C\log\frac{a[v(z)]}{a[u(z)]} \\ \to\; & 0 \quad \text{as } z \to z_0. \end{aligned}$$

Using the fact established in [30] that $d_{\Omega_n}(u(z), v(z))$ is subharmonic whenever u and v are harmonic maps, we conclude by the maximum principle that $u = v$. Thus we have proved (iv). Therefore, the proof of the lemma is complete. □

4. The energy density function. The goal of this section is to calculate the energy density function on $\partial\Omega_m$.

LEMMA 4.1. *Assume that $u \in C^2(\overline{\Omega}_m)$ is a harmonic map from Ω_m to Ω_n. Then for any $z_0 \in \Omega_m$ such that $a[u](z_0) > 0$, we have that*

$$\lim_{z\to z_0} e[u](z) = m + 2\frac{(r_\alpha R u^\alpha)(z_0)(r_\beta \overline{R} u^\beta)(z_0)}{(a[u](z_0))^2(|\partial\rho|_\rho^4)}. \tag{4.1}$$

Proof. For any $z_0 \in \partial\Omega_m$, we may assume that $|\partial\rho(z_0)|^2 = 1$; otherwise, we may use $\tilde\rho(z) = \rho(z)/|\partial\rho(z_0)|$ to replace ρ and use $\tilde r(w) = r(w)/|\partial\rho(z_0)|_0$ to replace $r(w)$. By diagonalizing we can assume without loss of generality that at $z_0 \in \partial\Omega_m$ we have that $\rho_{i\overline{j}}(z_0) = \delta_{ij}\rho_{j\overline{j}}(z_0)$, $r_{\alpha\overline{\beta}}(u(z_0)) = \delta_{\alpha\beta} r_{\alpha\overline{\alpha}}(u(z_0))$. By definition the energy density function equals

$$
\begin{aligned}
&e[u](z_0)\\
&= h^{i\overline{j}} g_{\alpha\overline{\beta}} (\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta})\\
&= (-\rho)[\rho^{i\overline{j}} - \frac{\rho^i \rho^{\overline{j}}}{|\partial\rho|^2_\rho - \rho}](\frac{1}{-r})[r_{\alpha\overline{\beta}} + \frac{r_\alpha r_{\overline{\beta}}}{-r}](\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta})\\
&= (\frac{\rho}{r})\{X_j u^\alpha \overline{\partial_j u^\beta} + X_j \overline{u}^\beta \partial_{\overline{j}} u^\alpha\}[r_{\alpha\overline{\beta}} + \frac{r_\alpha r_{\overline{\beta}}}{-r}]\\
&= (\frac{\rho}{r})\{X_j u^\alpha \overline{Y_j u^\beta} + X_j \overline{u}^\beta \overline{Y}_j u^\alpha + X_j u^\alpha \frac{\rho_{\overline{j}} \overline{R}\overline{u}^\beta}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\beta \frac{\rho_{\overline{j}} \overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&\quad \times [r_{\alpha\overline{\beta}} + \frac{r_\alpha r_{\overline{\beta}}}{-r}]\\
&= (\frac{\rho}{r}) r_{\alpha\overline{\beta}} \{X_j u^\alpha \overline{Y_j u^\beta} + X_j \overline{u}^\beta \overline{Y}_j u^\alpha + X_j u^\alpha \frac{\rho_{\overline{j}} \overline{R}\overline{u}^\beta}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\beta \frac{\rho_{\overline{j}} \overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&\quad + (\frac{\rho}{r})(\frac{r_\alpha r_{\overline{\beta}}}{-r})\{X_j u^\alpha \overline{Y_j u^\beta} + X_j \overline{u}^\beta \overline{Y}_j u^\alpha + X_j u^\alpha \frac{\rho_{\overline{j}} \overline{R}\overline{u}^\beta}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\beta \frac{\rho_{\overline{j}} \overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&= (\frac{\rho}{r}) r_{\alpha\overline{\alpha}} \{X_j u^\alpha \overline{Y_j u^\alpha} + X_j \overline{u}^\alpha \overline{Y}_j u^\alpha + X_j u^\alpha \frac{\rho_{\overline{j}} \overline{R}\overline{u}^\alpha}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\alpha \frac{\rho_{\overline{j}} \overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&\quad + (\frac{\rho}{r})(\frac{r_\alpha r_{\overline{\beta}}}{-r})\{X_j u^\alpha \overline{Y_j u^\beta} + X_j \overline{u}^\beta \overline{Y}_j u^\alpha + X_j u^\alpha \frac{\rho_{\overline{j}} \overline{R}\overline{u}^\beta}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\beta \frac{\rho_{\overline{j}} \overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&= (\frac{\rho}{r}) r_{\alpha\overline{\alpha}} \{\rho_{j\overline{j}} X_j u^\alpha \overline{X_j u^\alpha} + \rho_{j\overline{j}} X_j \overline{u}^\alpha \overline{X}_j u^\alpha + X_j u^\alpha \rho_{\overline{j}} \frac{\overline{R}\overline{u}^\alpha}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\alpha \rho_{\overline{j}} \frac{\overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&\quad + (\frac{\rho}{r})(\frac{1}{-r})\{\rho_{j\overline{j}} r_\alpha X_j u^\alpha r_{\overline{\beta}} \overline{X_j u^\beta} + \rho_{j\overline{j}} r_{\overline{\beta}} X_j \overline{u}^\beta r_\alpha \overline{X}_j u^\alpha\}\\
&\quad + (\frac{\rho}{r})(\frac{r_\alpha r_{\overline{\beta}}}{-r})\{X_j u^\alpha \rho_{\overline{j}} \frac{\overline{R}\overline{u}^\beta}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\beta \rho_{\overline{j}} \frac{\overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}.
\end{aligned}
$$

Using the fact $X_j u^\alpha \rho_{\overline{j}} = \dfrac{-\rho R u^\alpha}{|\partial\rho|^2_\rho - \rho}$ and Lemma 3.2, we obtain

$$
\begin{aligned}
&e[u](z_0)\\
&= (\frac{\rho}{r}) r_{\alpha\overline{\alpha}} \{\rho_{j\overline{j}} X_j u^\alpha \overline{X_j u^\alpha} + \rho_{j\overline{j}} X_j \overline{u}^\alpha \overline{X}_j u^\alpha + X_j u^\alpha \rho_{\overline{j}} \frac{\overline{R}\overline{u}^\alpha}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\alpha \rho_{\overline{j}} \frac{\overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&\quad + (\frac{\rho}{r})(\frac{r_\alpha r_{\overline{\beta}}}{-r})\{X_j u^\alpha \rho_{\overline{j}} \frac{\overline{R}\overline{u}^\beta}{|\partial\rho|^2_\rho - \rho} + X_j \overline{u}^\beta \rho_{\overline{j}} \frac{\overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&= (\frac{\rho}{r}) r_{\alpha\overline{\alpha}} \{\rho_{j\overline{j}} |X_j u^\alpha|^2 + \rho_{j\overline{j}} |\overline{X}_j u^\alpha|^2 + \frac{(-\rho) R u^\alpha}{|\partial\rho|^2_\rho - \rho} \frac{\overline{R}\overline{u}^\alpha}{|\partial\rho|^2_\rho - \rho} + \frac{(-\rho) R\overline{u}^\alpha}{|\partial\rho|^2_\rho - \rho} \frac{\overline{R} u^\alpha}{|\partial\rho|^2_\rho - \rho}\}\\
&\quad + (\frac{\rho}{r})(\frac{r_\alpha r_{\overline{\beta}}}{-r})\{\frac{(-\rho) R u^\alpha}{(|\partial\rho|^2_\rho - \rho)} \frac{\overline{R}\overline{u}^\beta}{(|\partial\rho|^2_\rho - \rho)} + \frac{(-\rho) R\overline{u}^\beta}{(|\partial\rho|^2_\rho - \rho)} \frac{\overline{R} u^\alpha}{(|\partial\rho|^2_\rho - \rho)}\}\\
&= (\frac{\rho}{r}) r_{\alpha\overline{\alpha}} \{\rho_{j\overline{j}} |X_j u^\alpha|^2 + \rho_{j\overline{j}} |\overline{X}_j u^\alpha|^2\}
\end{aligned}
$$

$$+(\frac{\rho}{r})\frac{(-\rho)}{(-r)}\{\frac{r^\alpha Ru^\alpha}{(|\partial\rho|^2_\rho-\rho)}\frac{r^{\overline{\beta}}\overline{R}\overline{u}^\beta}{(|\partial\rho|^2_\rho-\rho)}+\frac{r^\alpha R\overline{u}^\alpha}{(|\partial\rho|^2_\rho-\rho)}\frac{r^{\overline{\beta}}\overline{R}u^\beta}{(|\partial\rho|^2_\rho-\rho)}\}.$$

Adding (3.22) to (3.23) and using Lemma 3.5, we get

$$\begin{aligned} & r_{\alpha\overline{\alpha}}\rho_{j\overline{j}}(|X_ju^\alpha|^2+|\overline{X}_ju^\alpha|^2) \\ &= \mathcal{L}(\rho)\frac{r_s(u)(R+\overline{R})u^s}{|\partial\rho|^2_\rho}-2(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}}) \\ &= \mathcal{L}(\rho)a[u](z)-2(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}}). \end{aligned} \tag{4.2}$$

We know from (3.42) that $\mathcal{L}(\rho)=m-1$ on $\partial\Omega_m$. Thus,

$$\begin{aligned} e[u](z_0) &= (\frac{\rho}{r})\{\mathcal{L}(\rho)a[u](z)-2(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}})\} \\ &\quad +(\frac{\rho}{r})^2\{\frac{r^\alpha Ru^\alpha}{(|\partial\rho|^2_\rho-\rho)}\frac{r^{\overline{\beta}}\overline{R}\overline{u}^\beta}{(|\partial\rho|^2_\rho-\rho)}+\frac{r^\alpha R\overline{u}^\alpha}{(|\partial\rho|^2_\rho-\rho)}\frac{r^{\overline{\beta}}\overline{R}u^\beta}{(|\partial\rho|^2_\rho-\rho)}\} \\ &= (\frac{\rho}{r})\mathcal{L}(\rho)a[u](z)-2(\frac{\rho}{r})(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}}) \\ &\quad +(\frac{\rho}{r})^2\{\frac{|r_\alpha Ru^\alpha|^2}{(|\partial\rho|^2_\rho-\rho)^2}+\frac{|r_\alpha\overline{R}u^\alpha|^2}{(|\partial\rho|^2_\rho-\rho)^2}\} \\ &= \mathcal{L}(\rho)-2(\frac{\rho}{r})(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}}) \\ &\quad +\frac{|r_\alpha Ru^\alpha|^2+|r_\alpha\overline{R}u^\alpha|^2}{(a[u](z))^2(|\partial\rho|^4_\rho)}. \end{aligned}$$

Lemma 3.5 tells us that $(r_\alpha Ru^\alpha)$ and $(r_\alpha\overline{R}u^\alpha)$ are real. Thus,

$$\begin{aligned} \frac{|r_\alpha Ru^\alpha|^2+|r_\alpha\overline{R}u^\alpha|^2}{(a[u](z))^2(|\partial\rho|^4_\rho)} &= \Big(\frac{r_\alpha(R+\overline{R})u^\alpha}{(a[u](z))(|\partial\rho|^2_\rho)}\Big)^2-2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)} \\ &= 1-2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)} \end{aligned}$$

by (3.32). As a result, we obtain

$$\begin{aligned} e[u](z_0) &= (m-1)-2\frac{(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}})}{a[u]} \\ &\quad +1-2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)} \\ &= m-2\frac{(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{X}_ju^\alpha\rho_{j\overline{j}})}{a[u]}-2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)} \\ &= m-2\frac{(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta\overline{Y}_ju^\alpha)}{a[u]}-2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)} \\ &= m-2\frac{(r_\alpha\mathcal{L}u^\alpha+r_{\alpha\beta}X_ju^\beta u^{\alpha}_{\overline{j}})}{a[u]}-2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)} \\ &= m+2\frac{(r_\alpha Ru^\alpha)(r_\beta\overline{R}u^\beta)}{(a[u](z))^2(|\partial\rho|^4_\rho)}, \end{aligned}$$

where the third equality comes from (3.30), the fourth from (3.28) and the last one from applying (3.26). The proof is complete. ∎

5. The proof of Theorem 1.1. In this section we prove Theorem 1.1. It is obvious that (i) implies (ii). Next, we prove that (ii) implies (iii). Without loss of generality we may assume that $\rho_{i\bar{j}}(z_0) = \delta_{ij}\rho_{i\bar{i}}(z_0)$. Since u is pluriharmonic

$$\begin{aligned}0 &= \partial_{i\bar{j}}u^s + \sum_{t,\gamma=1}^{n} \Gamma^s_{t\gamma}\partial_i u^t \partial_{\bar{j}} u^\gamma \\ &= \partial_{i\bar{j}}u^s + \sum_{t,\gamma=1}^{n} \Big[\frac{\delta_{ts}r_\gamma + r_t\delta_{\gamma s}}{-r} + (r^{s\bar{\ell}} - \frac{r^s r^{\bar{\ell}}}{|\partial r|_r^2 - r})r_{t\gamma\bar{\ell}} + \frac{r_{t\gamma}r^s}{|\partial r|_r^2 - r)}\Big]\partial_i u^t \partial_{\bar{j}} u^\gamma.\end{aligned}$$

Multiplying by $(-\rho)$, we obtain that on $\partial\Omega_m$

$$\begin{aligned}0 &= \frac{1}{a[u]}\sum_{t,\gamma=1}^{n}(\delta_{ts}r_\gamma + r_t\delta_{\gamma s})\partial_i u^t \partial_{\bar{j}} u^\gamma \\ &= \frac{1}{a[u]}(\sum_{\gamma=1}^{n} r_\gamma \partial_i u^s \partial_{\bar{j}} u^\gamma + \sum_{t=1}^{n} r_t \partial_i u^t \partial_{\bar{j}} u^s) \\ &= \frac{1}{a[u]}(\sum_{\gamma=1}^{n} r_\gamma \partial_i u^s \partial_{\bar{j}} u^\gamma + r_\gamma \partial_i u^\gamma \partial_{\bar{j}} u^s).\end{aligned}$$

Multiplying by r_s and adding over s we have

$$\begin{aligned}0 &= \frac{1}{a[u]}(\sum_{\gamma,s=1}^{n} r_s\partial_i u^s r_\gamma \partial_{\bar{j}} u^\gamma + r_\gamma \partial_i u^\gamma r_s \partial_{\bar{j}} u^s) \\ &= \frac{2}{a[u]}(\sum_{\gamma,s=1}^{n} r_s\partial_i u^s r_\gamma \partial_{\bar{j}} u^\gamma) \\ &= \frac{2}{a[u]}\Big(\sum_{\gamma,s=1}^{n} r_s\rho_{i\bar{i}}(X_i u^s + \frac{\rho^{\bar{i}}}{|\partial\rho|_\rho^2 - \rho}Ru^s) r_\gamma \rho_{i\bar{i}}(\overline{X}_i u^\gamma + \frac{\rho^i}{|\partial\rho|_\rho^2 - \rho}\overline{R}u^\gamma)\Big) \\ &= \frac{2}{a[u]}\Big(\sum_{\gamma,s=1}^{n} r_s\rho_{i\bar{i}}(\frac{\rho^{\bar{i}}}{|\partial\rho|_\rho^2 - \rho}Ru^s) r_\gamma \rho_{i\bar{i}}(\frac{\rho^i}{|\partial\rho|_\rho^2 - \rho}\overline{R}u^\gamma)\Big) \\ &= \frac{2}{a[u]}\Big(\sum_{\gamma,s=1}^{n} r_s(\frac{\rho_i}{|\partial\rho|_\rho^2 - \rho}Ru^s) r_\gamma (\frac{\rho_{\bar{i}}}{|\partial\rho|_\rho^2 - \rho}\overline{R}u^\gamma)\Big) \\ &= \frac{2}{a[u]}\Big(\sum_{\gamma,s=1}^{n} |\rho_i|^2(\frac{r_s Ru^s}{|\partial\rho|_\rho^2 - \rho})(\frac{r_\gamma \overline{R}u^\gamma}{|\partial\rho|_\rho^2 - \rho})\Big)\end{aligned}$$

Thus, we have obtained that $\frac{2}{a[u]}\Big(\sum_{\gamma,s=1}^{n}(\frac{r_s Ru^s}{|\partial\rho|_\rho^2-\rho})(\frac{r_\gamma \overline{R}u^\gamma}{|\partial\rho|_\rho^2-\rho})\Big) = 0$ on $\partial\Omega_m$. It follows from (3.26) that $r_s\mathcal{L}u^s + r_{st}X_j u^t u^s_{\bar{j}} = 0$ on $\partial\Omega_m$, which is (iii).

Next we show that (iii) implies (i). By assumption, u is harmonic and $r_s\mathcal{L}u^s + r_{st}X_j u^t u^s_{\bar{j}} = 0$ on $\partial\Omega_m$. By Theorem 3.4, we find that either u or $\overline{u}$ is CR. Thus, there exists a holomorphic or antiholomorphic map v such that $v|_{\partial\Omega_m} = u|_{\partial\Omega_m}$. Since r is plurisubharmonic and v is holomorphic or antiholomorphic, $r(v(z))$ is plurisubharmonic. Thus, by Hopf's lemma $a[v](p) = D_\nu r(v(p)) > 0$ at every $p \in \partial\Omega_m$. Thus by (iv) of Lemma 3.5 we obtain that $u \equiv v$ on Ω_m.

Now we proceed to show that (iii) implies (iv). Lemma 3.5 tells us that $E_b[u] > 0$ on $\partial\Omega_m$ if and only if $a[u] > 0$ on $\partial\Omega_m$. If $r_s\mathcal{L}u^s + r_{st}X_j u^t u^s_{\bar{j}} = 0$ on $\partial\Omega_m$, by (3.26) we obtain that $(r_sRu^s)(r_t\overline{R}u^t) = 0$, which implies by Lemma 4.1 that $e[u](z) = m$ when $E_b[u] > 0$.

Next, we show that (iv) implies (v). Lemma 3.5 tells us that $E_b[u] > 0$ on $\partial\Omega_m$ if and only if $a[u] > 0$ on $\partial\Omega_m$. Thus by (4.1) we have that $(r_sRu^s)(r_t\overline{R}u^t) = 0$ on the set $\{z \in \partial\Omega_m : a[u](z) > 0\}$. On the other hand, by (3.26) we obtain that $(r_sRu^s)(r_t\overline{R}u^t) = 0$ on the set $\{z \in \partial\Omega_m : a[u](z) = 0\}$. Thus (iv) implies (v). Finally, we show that (v) implies (iii). Using the hypothesis of (v) together with (3.26) and (3.32) we obtain that $r_s\mathcal{L}u^s + r_{st}X_j u^t u^s_{\bar{j}} = 0$ on $\partial\Omega_m$. Since (iii) implies (i) we are done. ∎

6. The proof of Theorem 1.2. In this section we prove Theorem 1.2. Let $\phi \in C^{k,\alpha}(\partial\Omega_m)$ with $k \geq 2$ and $\alpha \geq 0$. Let $\phi(z)$ denote the 'radial' extension of ϕ from $\partial\Omega_m$ to $\overline{\Omega}_m$ in the sense that $r(\phi(z)) = 0$ for all $z \in \Omega_m$ near $\partial\Omega_m$. In order to apply Li-Tam's general existence theorem of [21], we first construct an approximating harmonic map similar to the construction in [25]. To do this, we define an extension $v(z)$ given by

$$v(z) = \phi(z) + \rho(z)b(z), \tag{6.1}$$

where $\rho(z)$ is a strictly plurisubharmonic defining function for Ω_m, which is the potential function for the metric h, and $b(z)$ is a vector valued function which will be given later. A computation shows that

$$\mathcal{L}v(z) = \mathcal{L}\phi(z) + \mathcal{L}\rho(z)b(z) + \rho(z)\mathcal{L}b + X_i\rho\,\partial_{\bar{i}}b + \overline{X}_j\rho\,\partial_j b(z), \tag{6.2}$$

which implies that on $\partial\Omega_m$

$$\mathcal{L}v(z) = \mathcal{L}\phi(z) + (m-1)b(z). \tag{6.3}$$

By (1.7) and (3.25), we have

$$r_s(v)\tau^s[v] = -\rho(z)[r_s(v)\mathcal{L}v^s + r_{st}X_j v^t v^s_{\bar{j}} + \frac{2r_s(v)Rv^s r_t(v)\overline{R}v^t}{a[v](|\partial\rho|^2_\rho - \rho)^2}] + O(\rho^2). \tag{6.4}$$

Let

$$I[v] = a[v]\Big(r_s(v)\mathcal{L}v^s + r_{st}X_j v^t v^s_{\bar{j}}\Big) + \frac{2}{(|\partial\rho|^2_\rho - \rho)^2} r_s(v)Rv^s r_t(v)\overline{R}v^t. \tag{4.5}$$

Since $\sum_j \rho_{\bar{j}}X_j = O(\rho)R$, we have

$$\sum_{j=1}^n X_j\phi^t v^s_{\bar{j}} = \sum_{j=1}^n X_j\phi^t\overline{Y_j}v^s_{\bar{j}} + O(\rho) = \sum_{j=1}^n X_j\phi^t\overline{Y_j}\phi^s + O(\rho).$$

Let $b_0 \geq 0$ and

$$b^s = b_0 r_{\bar{s}}(\phi), \quad 1 \leq s \leq n.$$

Note that on $\partial\Omega_m$,

$$a[v](z) = \frac{r(v(z))}{\rho(z)} = \frac{r(\phi + b\rho)}{\rho} = r_s(\phi + b\rho)b^s + r_{\bar{s}}(\phi + b\rho)\overline{b^s} = 2b_0\sum_s |r_s(\phi)|^2.$$

Since we extended ϕ from $\partial\Omega_m$ to $\overline{\Omega}_m$ so that

$$(R+\overline{R})\phi = 0, \quad \text{for } z \text{ near } \partial\Omega_m,$$

and $r(\phi)=0$ on $\partial\Omega_m$, we have that

$$2\,\mathrm{Im}\sum_s r_s(\phi)(\overline{R}-R)\phi^s = 0.$$

So $\sum_s r_s(\phi)\overline{R}\phi^s$ is real. By Lemma 3.3, we know that

$$\sum_{j=1}^{m}(X_j\overline{Y}_j - \overline{X}_jY_j) = \frac{m-1}{|\partial\rho|^2_\rho - \rho}(\overline{R}-R) + O(\rho)$$

and

$$\mathcal{L} = \frac{1}{2}(X_j\overline{Y}_j + \overline{X}_jY_j) + \frac{m-1}{|\partial\rho|^2_\rho}\frac{(R+\overline{R})}{2} + O(\rho).$$

Since $r(\phi)=0$ on $\partial\Omega_m$ and X_j is tangential we have that

$$X_jr(\phi) = r_sX_j\phi^s + r_{\overline{s}}X_j\overline{\phi}^s = 0 \text{ on } \partial\Omega_m. \tag{6.5}$$

Equation (6.5) together with (1.7) implies that

$$\sum_s r_s(\phi)\overline{Y}_j\phi^s = \sum_s r_s(\phi)X_j\phi^s = 0 \quad \text{on } \partial\Omega_m.$$

As a result, we obtain that

$$0 = \sum_s (r_{st}X_j\phi^t + r_{s\overline{t}}(\phi)X_j\overline{\phi^t})\overline{Y}_j\phi^s + \sum_s r_s(\phi)X_j\overline{Y}_j\phi^s \tag{6.6}$$

and

$$0 = \sum_s (r_{st}\overline{X}_j\phi^t + r_{s\overline{t}}(\phi)\overline{X}_j\overline{\phi^t})Y_j\phi^s + \sum_s r_s(\phi)\overline{X}_jY_j\phi^s. \tag{6.7}$$

Thus on $\partial\Omega_m$

$$\begin{aligned}
&r_s\mathcal{L}v^s + r_{st}X_jv^sv^t_{\overline{j}}\\
&= \frac{r_s}{2}(X_j\overline{Y}_j + \overline{X}_jY_j)\phi^s + (m-1)r_sb^s + r_{st}X_jv^sv^t_{\overline{j}}\\
&= (\frac{1}{2})\Big(r_sX_j\overline{Y}_j\phi^s + r_s\overline{X}_jY_j\phi^s\Big) + (m-1)r_sb^s + r_{st}X_jv^sv^t_{\overline{j}}\\
&= (-\frac{1}{2})\Big(r_{st}(X_j\phi^t)(\overline{Y}_j\phi^s) + r_{s\overline{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s) + r_{st}(\overline{X}_j\phi^t)(Y_j\phi^s)\\
&\quad + r_{s\overline{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s)\Big) + (m-1)r_sb^s + r_{st}X_jv^sv^t_{\overline{j}}\\
&= -(\frac{1}{2})\Big(r_{st}(X_j\phi^t)(\overline{Y}_j\phi^s) + r_{st}(\overline{X}_j\phi^t)(Y_j\phi^s) + r_{s\overline{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s)\\
&\quad + r_{s\overline{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s)\Big) + (m-1)r_sb^s + r_{st}X_jv^sv^t_{\overline{j}}.
\end{aligned}$$

It is easy to see that on $\partial\Omega_m$

$$r_s R v^s = r_s(R\phi^s + R(\rho)b^s) = r_s R\phi^s + |\partial\rho|^2_\rho r_s b^s = r_s R\phi^s + |\partial\rho|^2_\rho \frac{a[v]}{2}$$

and

$$r_s \overline{R} v^s = r_s \overline{R}\phi^s + |\partial\rho|^2_\rho \frac{a[v]}{2}.$$

Let $z_0 \in \partial\Omega_m$. By a rotation if necessary, we may assume without loss of generality that $\rho_{i\bar{j}}(z_0) = \rho_{i\bar{i}}(z_0)\delta_{ij}$. Then at z_0

$$Y_j\phi^s = \rho_{j\bar{j}} X_j\phi^s \text{ and } \overline{Y}_j\phi^s = \rho_{j\bar{j}}\overline{X}_j\phi^s.$$

Also

$$\begin{aligned} r_{st}(v)X_j v^t\, v^s_{\bar{j}}(z_0) &= r_{st}(v)X_j v^t \overline{Y}_j v^s(z_0) + r_{st}(v)X_j v^t \rho_{\bar{j}}\rho_k\partial_{\bar{k}} v^s(z_0) \\ &= r_{st}(v)X_j v^t \overline{Y}_j v^s(z_0) \\ &= r_{st}(v)X_j v^t \overline{Y}_j \phi^s(z_0) \\ &= r_{st}(\phi)(X_j\phi^t)(\rho_{j\bar{j}}\overline{X}_j\phi^s)(z_0). \end{aligned}$$

Therefore at z_0,

$$r_s\mathcal{L}v^s + r_{st}X_j v^s v^t_{\bar{j}} = -\frac{1}{2} r_{s\bar{t}}(\overline{X}_j\overline{\phi^t} Y_j\phi^s + X_j\overline{\phi}^t\overline{Y}_j\phi^s) + (m-1)r_s(\phi)b^s$$

and by (6.6) and (6.7)

$$\sum_{j=1}^m r_s(X_j\overline{Y}_j - \overline{X}_j Y_j)\phi^s = -r_{s\bar{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s) + r_{s\bar{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s).$$

Thus, we obtain

$$\begin{aligned} I[v] &= a[v]\Big(r_s\mathcal{L}v^s + r_{st}X_j v^s v^t_{\bar{j}}\Big) + 2\frac{(r_s R v^s)(r_t\overline{R}v^t)}{|d\rho|^4} \\ &= a[v]\Big(\frac{r_{s\bar{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s) + r_{s\bar{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s)}{-2} + (m-1)r_s b^s\Big) \\ &\quad + \frac{1}{2}\Big(\frac{|d\rho|^4 a[v]^2 - (r_s(R-\overline{R})\phi^s)^2}{|d\rho|^4}\Big) \\ &= 2|r_s|^2 b_0\Big(\frac{r_{s\bar{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s) + r_{s\bar{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s)}{-2} + (m-1)|r_s|^2 b_0\Big) \\ &\quad + \frac{1}{2}\Big((2|r_s|^2 b_0)^2 - \frac{(r_s(X_j\overline{Y}_j - \overline{X}_j Y_j)\phi^s)^2}{(m-1)^2}\Big) \\ &= 2|r_s|^2 b_0\Big(\frac{r_{s\bar{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s) + r_{s\bar{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s)}{-2} + (m-1)|r_s|^2 b_0\Big) \\ &\quad + \frac{1}{2}\Big((2|r_s|^2 b_0)^2 - \frac{-(r_{s\bar{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s) + r_{s\bar{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s))^2}{(m-1)^2}\Big). \end{aligned}$$

Let

$$A = r_{s\bar{t}}(X_j\overline{\phi}^t)(\overline{Y}_j\phi^s),$$
$$B = r_{s\bar{t}}(\overline{X}_j\overline{\phi}^t)(Y_j\phi^s),$$
$$E = A + B.$$

Then

$$\begin{aligned} I[v] &= 2|r_s|^2 b_0\left(\frac{-E}{2} + (m-1)|r_s|^2 b_0\right) + 2|r_s|^4 b_0^2 - \frac{1}{2(m-1)^2}(-A+B)^2 \\ &= 2m|r_s|^4 b_0^2 - E|r_s|^2 b_0 - \frac{(-A+B)^2}{2(m-1)^2}. \end{aligned}$$

Let

$$\begin{aligned} 4m|r_s|^2 b_0 &= E + \sqrt{E^2 + \frac{4m}{(m-1)^2}(-A+B)^2} \\ &= E + \frac{1}{m-1}\sqrt{(m-1)^2E^2 + 4m(E^2 - 4AB)} \\ &= E + \frac{1}{m-1}\sqrt{(m+1)^2E^2 - 16mAB} \\ &\geq E + \frac{1}{m-1}\sqrt{(m+1)^2E^2 - 4mE^2} \\ &= 2E. \end{aligned}$$

By assumption $E_b[\phi] > 0$ on $\partial\Omega_m$, which implies that $E > 0$ since $r_{s\bar{t}}$ is positive definite. This in turn implies that $b_0 > 0$ and $a[v] = 2b_0 > 0$ on Ω_m. Moreover,

$$I[v] = a[v](r_s\mathcal{L}v^s + r_{st}X_jv^sv^t_{\bar{j}}) + 2\frac{(r_sRv^s)(r_t\overline{R}v^t)}{|\partial\rho|_\rho^4} = 0 \text{ on } \partial\Omega_m.$$

Note that $\sum_s r_s\tau^s[v] = O(\rho^2)$ by (6.4). Thus,

$$\frac{|\sum_s r_s(v)\tau^s[v]|^2}{r(v)^2} = O(\rho^2).$$

By (3.24), we have

$$\tau^s[v] = O(\rho).$$

Therefore,

$$\begin{aligned} |\tau[v]|_g^2 &= g_{\alpha\overline{\beta}}\tau^\alpha[v]\overline{\tau^\beta[v]} \\ &= \frac{\sum_\alpha r_{\alpha\overline{\beta}}\tau^\alpha[v]\overline{\tau^\beta[v]}}{|r(v)|} + \frac{r_\alpha\tau^\alpha[v]\overline{r_\beta\tau^\beta[v]}}{r(v)^2} \\ &= O(\rho(z)). \end{aligned}$$

This implies $|\tau[v]|_g \in L^{2p}(\Omega_m, d\lambda_m)$ for $p > m$ where $d\lambda_m(z) = \det(h_{i\bar{j}})dv(z)$ and $\det(h_{i\bar{j}})(z) \approx |\rho(z)|^{-m-1}dv(z)$ for all $z \in \Omega_m$ since Ω_m is strictly pseudoconvex.

An application of the existence theorem of [19] and the regularity argument in [20] establishes our claim. The proof of Theorem 1.2 is complete. ∎

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