ASIAN J. MATH.
Vol. 11, No. 1, pp. 167–178, March 2007

013

ON THE CONTINUITY PRINCIPLE*

BURGLIND JÖRICKE† AND EGMONT PORTEN‡

To Salah Baouendi on the occasion of his seventieth birthday

Key words. Continuity principle, envelopes of holomorphy, polynomial hulls

AMS subject classifications. Primary 32D10; Secondary 32E20, 32D20, 32D26

The name continuity principle is related to one of the first basic observations in several complex variables. It goes back to the work of Hartogs and Cartan–Thullen. This principle provides us the classical tool for obtaining compulsory analytic continuation in the following sense. There are domains in $\mathbb{C}^n$, $n > 1$, with the property that all analytic functions in the domain have analytic extension to a larger domain. The following classical form of the continuity principle uses continuous families of analytic discs. We will formulate it only for the case of complex dimension $n = 2$ and we will restrict ourselves to this case throughout the paper.

Denote by $c \subset \mathbb{C}^2$ the set $c = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, z_2 = 0\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = 1, z_2 \in [0, 1]\}$ and by C its convex hull, $C = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, z_2 \in [0, 1]\}$. Note that c is topologically a half-sphere with boundary a circle (equivalently, a closed disc; for avoiding confusion with analytic discs we prefer to speak about half-spheres). For a small positive number ϵ we denote by c_ϵ, respectively,. C_ϵ, the corresponding ϵ-neighbourhoods. Let $F : C_\epsilon \to \mathbb{C}^2$ be a locally biholomorphic mapping. The continuity principle says that any analytic function f in $F(c_\epsilon)$ has analytic continuation to a neighbourhood of any point in $F(C_\epsilon)$. (But in general, there is no analytic function in $F(C_\epsilon)$ which is an extension of the original function f).

The advantage of the continuity principle is that it is geometric in its nature. It is well-known that the envelope of holomorphy of a domain D in $\mathbb{C}^n$ (the "largest" Riemann domain over $\mathbb{C}^n$ to which all analytic functions in D have analytic extension) can be obtained by a successive procedure, consisting in gluing to the preceding set families of immersed analytic discs (Levi-flat 3-balls) $F(C)$ along half-spheres $F(c)$. Even this simple approach allows to make some statements about the envelope of holomorphy of some domains.

For further applications, in particular for making guesses concerning envelopes of holomorphy in more subtle situations, more general and flexible versions of the continuity principle are helpful. It is not easy to find such more general versions and mistakes have been made in the literature. One would like to replace families of immersed analytic discs by families of Riemann surfaces with boundary (bordered Riemann surfaces for short), or, more generally, by families of one-dimensional analytic varieties with boundary or by 1-chains, provided a suitable condition of semi-continuity of these objects and their boundaries is given.

Here we will state the version of continuity principle given in [JP].

Start with the following observation (which we explain just for the sake of simplicity only in case the mapping F is injective on C_ϵ). Namely, the aforementioned

*Received July 25, 2006; accepted for publication March 19, 2007.
†Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden (joericke@math.uu.se).
‡Department of Engineering, Physics and Mathematics, Mid Sweden University, Campus Sundsvall, S-85170 Sundsvall, Sweden (egmont.porten@miun.se).

procedure of gluing Levi-flat 3-balls along half-spheres (part of their boundaries) is equivalent to gluing such 3-balls along 2-spheres (their full boundaries).

Indeed, the topological half-sphere $F(c)$ can always be extended to a full (embedded) 2-sphere S which is contained in $F(c_\epsilon)$ and bounds an embedded 3-ball which is foliated by analytic discs. To see this, consider a copy of c which is translated away from c by a small constant vector which is transversal to the discs in C. Give this copy the opposite orientation and join the two copies by a collar consisting of a family of circles which are parallel translations of the unit circle in the z_1-plane. We obtain a 2-sphere $\mathcal{S}$ which bounds a 3-ball consisting of the union of a family of flat discs. We may assume that the sphere $\mathcal{S}$ is contained in c_ϵ and the related family of discs is contained in C_ϵ. The mapping F takes $\mathcal{S}$ to the required 2-sphere S. As in the classical form of continuity principle any analytic function in a neighbourhood of S extends analytically to a neighbourhood of the ball.

For many purposes it is more convenient to think about gluing of Levi-flat 3-balls along their full boundaries.

Consider now the more general situation when the family of analytic discs is replaced by a family of bordered Riemann surfaces (not necessarily connected) and the boundaries of the Riemann surfaces constitute an arbitrary oriented connected closed surface. (By a surface we always mean a real two-dimensional manifold). We assume that the Levi-flat manifold constituted by the family of Riemann surfaces is embedded into two-dimensional complex space.

Notice the following abuse of language. Speaking about bordered Riemann surfaces and their boundaries we have not necessarily the notion of a manifold with boundary in mind. We mean any relatively compact open subset of an open Riemann surface and its boundary in the open Riemann surface. We may take for example any open planar set and its boundary with respect to Euclidean topology.

The following definition (see [JP]) gives a precise description of the required conditions.

Definition. *Let M^2 be a two-dimensional Stein manifold. Let $\bar{h}$ be a smooth embedded compact three-manifold with boundary contained in M^2 such that the open manifold h is Levi-flat and ∂h is connected. Suppose that $\bar{h}$ extends to a larger smooth embedded Levi-flat hypersurface H in M^2 the Levi-leaves of which are closed in M^2 (but not necessarily connected). Assume that the parameter space of the Levi-leaves of H is the interval $(0,1)$. Then h is called a Hartogs manifold.*

Thus, as a set, the Hartogs manifold h is the union of connected open Riemann surfaces embedded into M^2. h has the structure of a smooth manifold. Moreover, on h a smooth function with values in a subset of $(0,1)$ is given, such that the level sets of this function are finite unions of connected Riemann surfaces. These level sets are the Levi-leaves of the Hartogs manifold and the function realizes a parametrization of the set of Levi-leaves of h. The union of the boundaries of the Levi-leaves constitute a connected surface. Moreover, it is required, that the Levi-leaves of h extend to a continuous family of relatively closed (not necessarily connected) one-dimensional analytic manifolds in a Stein manifold.

It will be convenient to have also the following definition.

Definition. *We will say that for a relatively compact Levi-flat 3-dimensional manifold h in a 2-dimensional Stein manifold the continuity principle holds if for any analytic function f defined on a neighbourhood of ∂h there exists an analytic (single-valued) function on a neighbourhood of $\overline{h}$ that coincides with f on a (possibly smaller)* *neighbourhood of ∂h.*

The following theorem is proved in [JP].

THEOREM 1. *Let M^2 be a 2-dimensional Stein manifold. Then the following holds:*

1) *Let H be a smooth embedded Levi-flat hypersurface in M^2 consisting of the union over the parameter set $(0,1)$ of closed (in M^2) Levi-leaves. Then any smooth connected closed surface $\mathfrak{S} \subset H$ bounds a uniquely defined Hartogs manifold h contained in H.*

2) *Suppose h is a Hartogs manifold in M^2. Then the continuity principle holds for h.*

Note that in theorem 1 we allow situations which do not appear in case the boundary ∂h is contained in the boundary of a pseudoconvex domain. Namely, we allow bifurcation of the Riemann surfaces which can be described as “drilling holes”. For example, we may consider a family of analytic discs which bifurcates to a punctured disc (by removing a point) and then to an annulus. This gives more flexibility for applications.

Part 1 of the theorem can be reformulated in the following way.

1') *Let H be a smooth embedded Levi-flat hypersurface in a two-dimensional Stein manifold M^2 with closed (in M^2) Levi-leaves. If H does not contain circles transverse to the Levi-foliation then the second homology of H vanishes.*

We refer also to the paper [Fo] where it is shown that in general the Hartogs-Bochner theorem can not be proved by “successively pushing analytic discs”. More precisely, there is a domain $\Omega \subset \mathbb{C}^2$ with connected boundary such that the following procedure does not exhaust the domain Ω. Consider a neighbourhood D_0 of the boundary $\partial\Omega$. Starting with D_0, subsequent domains D_k are obtained by gluing to the preceding domain D_{k-1} a connected subset of $F(C_\epsilon)$ (F biholomorphic and $\epsilon > 0$) along a connected neighbourhood of $F(c_\epsilon)$. At each step gluing is carried out in such a way that the added set does not leave Ω and does not meet the preceding domain along more than one connected component. The latter requirement is crucial for controlling that analytic continuation obtained by this method is single-valued.

Theorem 1 is a method to bypass this difficulty. For instance, the theorem has the following application. Let $\Omega \subset \mathbb{C}^2$ be a bounded domain. Denote by $A(\Omega)$ the space of analytic functions on Ω which are continuous on the closure $\overline{\Omega}$. For a compact set $K \subset \overline{\Omega}$ we consider the following notion of hull:

$$A(\Omega)\text{-hull}(K) = \{z \in \overline{\Omega} : |f(z)| \leq \max_K |f| \text{ for all } f \in A(\Omega)\}.$$

Let $\tilde{\Omega}$ be the envelope of holomorphy of Ω and let $\iota : \Omega \to \tilde{\Omega}$ be the natural embedding.

THEOREM 2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with smooth connected boundary. Let $K \subset \partial\Omega$ be a compact set for which $\partial\Omega \backslash K$ is connected and $\partial\Omega \cap A(\Omega)$-hull$(K) = K$. Let D be a connected neighbourhood of $\partial\Omega\backslash K$ in Ω which does not meet $A(\Omega)$-hull(K). Then for each $z \in \Omega\backslash A(\Omega)$-hull$(K)$ and for each generic $g \in A(\Omega)$ with $g(z) = 1$ and $\max_K |g| < 1$ the analytic manifold $A_g = \{z \in \Omega : g(z) = 1\}$ can be lifted to a Levi-leaf of a Hartogs manifold h in $\tilde{\Omega}$ with $\partial h \subset \iota D$.*

We call the function g with the aforementioned properties generic if 1 is a regular value for its extension to the envelope of holomorphy $\tilde{\Omega}$. Notice that for Ω, K and z as above such functions always exist (see [JP]). The theorem implies that, given Ω,

K, D, g, A_g and z as in the statement of the theorem, for any analytic function f in D there exists an analytic function in a neighbourhood of A_g which coincides with the previous function on the part of D which is close to ∂A_g. Denote the restriction of this analytic function to a neighbourhood of z by f_g. The following proposition holds.

PROPOSITION [Jö]. *The function f_g does not depend on the choice of the generic function g.*

As a corollary we obtain that under the conditions of Theorem 2 analytic functions in D have single-valued analytic extension to the whole $\Omega \backslash A(\Omega)$-hull$(K)$ (see [JP]). In particular, putting $K = \emptyset$ we obtain a proof of the Hartogs-Bochner theorem using only continuity principle.

As in the example of Fornaess, to carry out the analytic continuation stated in theorem 2 we leave the set $\iota\Omega$ and work on $\tilde{\Omega}$.

As for the proposition and the mentioned corollary we restrict the analytic continuation obtained in theorem 2 to a suitable neighbourhood of a leaf ιA_g of the Hartogs manifold described in theorem 2. These are sets contained in $\iota\Omega$. Analytic extension to $\Omega \backslash A(\Omega)$-hull$(K)$ is carried out by collecting the analytic continuations to subsets of $\iota\Omega$ of the form ιA_g and controlling that they match together to a single-valued analytic function. This can be done staying inside the set $\iota\Omega$.

Note that in [CS] a version of continuity principle is stated for families of one-dimensional analytic varieties (even more generally, for holomorphic 1-chains) instead of Riemann surfaces. The conditions in [CS] allow in particular the aforementioned "drilling of holes" without further conditions on the analytic varieties. Unfortunately, in such situations analytic continuation cannot be guaranteed. Theorem 3 below provides, in particular, a counterexample to the statement in [CS]. A counterexample to the statement in [CS] was given before by Rosay in the simpler case of $\mathbb{C}^3$ ([Ro]).

Our theorem 3 below proves, moreover, the following. The requirement in Theorem 1 that the Levi-leaves of the Levi-flat manifold extend to a continuous family of closed 1-dimensional complex manifolds in a Stein manifold, can not be omitted. It will not be even enough to require that they extend to analytic varieties of dimension one which are closed in a Stein manifold.

Here is another application of Theorem 1.

COROLLARY. *Let $(S, \partial S)$ be a compact surface with boundary, contained in the unit bidisc $\overline{\mathbb{D}}^2$ in $\mathbb{C}^2$ and attached to a face of the bidisc, i.e. $\partial S \subset \partial\mathbb{D} \times \overline{\mathbb{D}}$. Suppose that S is contained in the real hyperplane $\{(z, w) \in \mathbb{C}^2 : \mathrm{Im} w = 0\}$. If ∂S is a circle which is not contractible in $\partial\mathbb{D} \times \overline{\mathbb{D}}$, then the envelope of holomorphy of $\partial\mathbb{D} \times \overline{\mathbb{D}} \cup S$ is the closed bidisc. In other words, for any function which is analytic in a neighbourhood of the latter set there exists an analytic function in a neighbourhood of $\overline{\mathbb{D}}^2$ which coincides with the previous function in a (maybe smaller) neighbourhood of $\partial\mathbb{D} \times \overline{\mathbb{D}} \cup S$.*

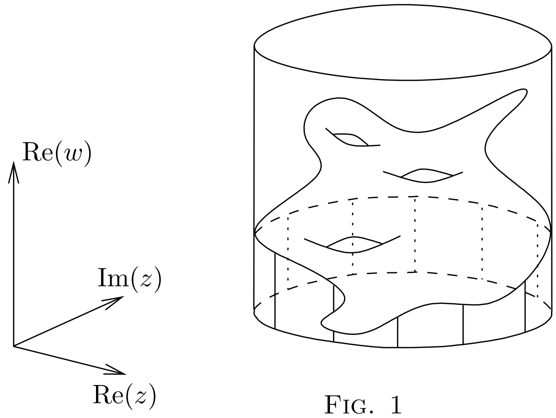

FIG. 1

Note that S may have arbitrary genus (see figure 1). For this generality we pay with the very restrictive condition that the surface is contained in a real hyperplane. The corollary complements results of Ivashkovitch-Shevchishin [IS], Chirka-Rosay [Ch], [CR1], [CR2], and Nemirovski [Ne].

We would like to emphasis here that though the scope of the present topic seems to be somewhat wider, the interest of the authors in the continuity principle arose in connection with global problems in CR-geometry. In particular theorem 2 applies to the problem of analytic extension of continuous CR-functions from parts of non-pseudoconvex hypersurfaces in $\mathbb{C}^2$ (see [JP]). Also, the results in [CS] were motivated by the problem of analytic extension of CR-functions from hypersurfaces.

Proof of corollary. Let f be an analytic function in a connected neighbourhood U of $\partial\mathbb{D}\times\overline{\mathbb{D}}\cup S$. Deforming S slightly inside U we may assume that $\overline{S}\subset\overline{\mathbb{D}}\times\mathbb{D}$. The boundary ∂S of the surface S is an embedded circle, contained in the set $\partial\mathbb{D}\times\mathbb{D}\cap\{\mathrm{Im}w=0\}=\partial\mathbb{D}\times(-1,1)$. The latter set is an annulus and ∂S is not contractible in this annulus. Hence ∂S is homologous to the circle $\partial\mathbb{D}\times\{1-\varepsilon\}$ for some small $\varepsilon>0$. We may assume that the circle is disjoint from ∂S. Attach to S the relatively compact subset Ω of $\partial\mathbb{D}\times(0,1)$ bounded by ∂S and the circle $\partial\mathbb{D}\times\{1-\varepsilon\}$. The union $S\cup\partial S\cup\Omega$ is a topological surface contained in $\{\mathrm{Im}w=0\}$ of the same genus as S. Its boundary is the flat circle $\partial\mathbb{D}\times\{1-\varepsilon\}$. Smoothen the surface to obtain a surface $S_0\subset\{\mathrm{Im}w=0\}\cap U$ with boundary $\partial S=\partial\mathbb{D}\times\{1-\varepsilon\}$. For small $\delta>0$ the surface S_δ which is a parallel copy of S_0 on $\{\mathrm{Im}w=\delta\}$ is contained in U. Join S_0 and S_δ by a collar contained in U which consists of the union of flat circles (i.e. circles contained in complex lines parallel to the first coordinate axis). We obtain a smooth surface $\mathcal{S}\subset U$. The surface $\mathcal{S}$ is contained in a smooth Levi-flat hypersurface the leaves of which are lines parallel to the first coordinate axis.

By theorem 1 $\mathcal{S}$ bounds a Hartogs manifold and the function f extends to a holomorphic function in a neighbourhood of the Hartogs manifold. Being relatively compact the Hartogs manifold contains the disc $\mathbb{D}\times\{1-\varepsilon\}$. By the classical version of the continuity principle the function f extends to a holomorphic function in a neighbourhood of $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$. $\square$

Failure of the continuity principle. We will now give the aforementioned example of failure of continuity principle, namely, we will show that there exists a smoothly embedded relatively compact Levi-flat hypersurface h in C^2 with connected boundary ∂h for which the continuity principle fails. More precisely, the following theorem holds.

Theorem 3. *There exists a smooth embedded two-sphere S in $\mathbb{C}^2$ which bounds a smooth embedded relatively compact Levi-flat three-ball $\mathcal{B}$ with the following properties.*

1) *$\mathcal{B}$ is foliated into schlicht-like Riemann surfaces (i.e. Riemann surfaces that are equivalent to domains in the complex plane). Each leaf is contained in an algebraic hypersurface in $\mathbb{C}^2$ (but some of the hypersurfaces have singularities and are reducible. In such case the corresponding Riemann surface is contained in one of the irreducible branches). The algebraic hypersurfaces form a continuous one-parameter family (in the sense that there are defining polynomials depending continuously on the parameter).*

2) *There exists an analytic function in a domain $\mathcal{D} \subset \mathbb{C}^2$ containing S which does not extend to an analytic function in a neighbourhood of $\overline{\mathcal{B}}$. Moreover the projection of the envelope of holomorphy of a small neighbourhood of S does not cover $\mathcal{B}$.*

3) *S can be lifted to a two-sphere $\mathcal{S}$ contained in a (two-sheeted) pseudoconvex Riemann domain $\mathcal{R}$ over $\mathbb{C}^2$. $\mathcal{S}$ bounds a Levi-flat three-ball in $\mathcal{R}$ which is foliated into analytic discs. Hence S can be filled by (not necessarily embedded and disjoint) analytic discs.*

Note that by property 1) the three-ball $\mathcal{B}$ is contained in the polynomial hull $\hat{S}$ of S. Property 2) shows that the envelope of holomorphy of S does not cover the polynomial hull $\hat{S}$. Moreover $\hat{S}$ has non-empty interior (see below).

In order to contrast theorem 3 to theorem 1 we will first sketch the scheme of proof of theorem 1 and explain why the effect described in statement 2) of theorem 3 cannot occur if the Riemann surfaces extend to smooth closed complex curves in a Stein manifold the union of which forms a smooth (not necessarily closed) embedded real hypersurface. After that we will prove theorem 3.

Scheme of proof of Theorem 1. Let $\mathcal{S}$ be as in statement 1 of theorem 1. We have to prove that $\mathcal{S}$ bounds a Hartogs manifold h and statement 2 holds. We may assume that the parametrization m of leaves of $\mathcal{S}$ defines a smooth Morse function m on $\mathcal{S}$. For all real numbers $a > \min_{\mathcal{S}} m$ we denote by $\mathcal{S}_j^a$, $j = 1, \dots, N$, the connected components of $\mathcal{S}^a \overset{\text{def}}{=} \mathcal{S} \cap \{m < a\}$. Put $H^a = H \cap \{m < a\}$. The following proposition is the key of the proof.

Proposition 1. *For each j the set $H^a \backslash \mathcal{S}_j^a$ has exactly one connected component which is relatively compact in M^2. This component is denoted by $\mathcal{B}_j^a$ and is called a bowl (more precisely a-bowl). The boundary $\partial \mathcal{B}_j^a$ is the union of $\mathcal{S}_j^a$ and a subset of the analytic manifold $H \cap \{m = a\}$. Each function that is holomorphic in a neighbourhood of $\mathcal{S}_j^a$ extends holomorphically to a neighbourhood of $\mathcal{B}_j^a \cup \mathcal{S}_j^a$.*

Here is a sketch of proof of the proposition. Uniqueness of relatively compact connected components of $H^a \backslash \mathcal{S}_j^a$ is based on the fact that the Stein manifold M^2 does not contain compact analytic manifolds. Existence of the bowls and analytic continuation is proved successively for increasing a, passing through critical values.

1) For $a > \min_{\mathcal{S}} m$ and close to a there is exactly one bowl $H \cap \{m < a\}$.

2) When the parameter increases over regular values a, new slices of the same topology as the top slices $\mathcal{B}_j^a \cap \{m = a'\}$, $a' < a$ and close to a , will be added to the bowl. This is possible because the leaves of H are closed manifolds in a Stein manifold.

Analytic continuation follows from continuity principle applied to a continuous family of bordered Riemann surfaces of the same topological type.

3) When a goes through a local minimum, a new bowl appears (it may or may not be a subset of the continuation of a bowl previously existing).

4) For a passing through a local maximum the topology of the top slice changes. Either a connected component disappears or a hole in the slice disappears.

In both cases 3) and 4) analytic continuation to a neighbourhood of the bowls is obtained without difficulty.

5) Passing through saddle points is more subtle. In a neighbourhood of a saddle point the following three cases may occur.

a) Two disjoint local connected components of previous bowls (i.e. connected components of the intersection of the previous bowls with a neighbourhood of the saddle point) touch each other at the saddle point and join to a single bowl. Globally either two bowls may join or some connected component of slices may obtain more complicated topology. Analytic continuation is straightforward.

b) This case is obtained from a) by inverting the direction of the parameter: A single local bowl splits into two local connected components of bowls. The number of bowls is preserved but either the number of connected components of slices increases or the topology of some of the components simplify. Analytic continuation is straightforward.

c) Two local bowls intersect and their boundaries touch each other at the saddle point. Then one of the global bowls is contained in the other one. Remove the closure of the smaller bowl from the larger bowl. This difference of the two bowls continues as a single bowl.

This is the only case where analytic continuation is more subtle. The argument is as follows. Denote the critical point by z, put $a = m(z)$ and denote the two bowls by $\mathcal{B}_1^a$ and $\mathcal{B}_2^a$. The bowls correspond to connected components $\mathcal{S}_1^a$ and $\mathcal{S}_2^a$ resp. of the sublevel set of $\mathcal{S}$. $\overline{\mathcal{S}_1^a}$ and $\overline{\mathcal{S}_2^a}$ meet at the point z and join to a single component of the sublevel set of $\mathcal{S}$ for $b > a$. Consider an analytic function in the neighbourhood of $\overline{\mathcal{S}_1^a} \cup \overline{\mathcal{S}_2^a}$. Its restriction to $\overline{\mathcal{S}_j^a}$ has analytic continuation to a neighbourhood of $\mathcal{B}_j^a \cup \mathcal{S}_j^a$, $j = 1, 2$. The obtained analytic functions near the $\mathcal{B}_j^a$ coincide in a neighbourhood of z. Hence the function near the smaller bowl must be the restriction of the analytic function near the bigger bowl. In particular there is an analytic function near the difference of the bowls which coincides with the previous one near both boundary components $\mathcal{S}_1^a$ and $\mathcal{S}_2^a$. Now analytic continuation to a neighbourhood of the continuation of the difference of the bowls is obtained in the standard way.

Theorem 1 is obtained by applying the proposition to values exceeding $\max_{\mathcal{S}} m$. In this case there will be a single bowl, which will be the desired Hartogs manifold h. The statement of proposition 1 concerning analytic continuation is just statement 2 of the theorem. $\square$

Note that the validity of the statement of theorem 1 is essentially based on the properties of Hartogs manifolds. First, the existence of the smooth Levi-flat manifold H which extends h and has closed Levi leaves in a Stein manifold allows to add slices of the same topology as the top slice (see case 2)) or, respectively, allows to associate to "drilling of holes" (i.e. to the change of connectivity of the leaves of h) the appearance of a new bowl inside the continuation of an already existing one (case 3)).

Connectedness of the boundary $\mathcal{S} = \partial h$ guarantees that the functions providing analytic continuation from a neighbourhood of $\mathcal{S} = \partial h$ to respective neighbourhoods of different nested a-bowls are related. Indeed, case 5c) shows that these functions coincide on a neighbourhood of the smaller of the a-bowls if suitable continuations (and reconstructions) of the nested a-bowls have a common boundary point on S. Since $\mathcal{S} = \partial h$ is connected this is the case for any pair of nested a-bowls.

The situation in theorem 3 is different. For the Levi-flat ball $\mathcal{B}$ some "drilled holes" can be filled by bowls only up to some level a and not further. Given an analytic function in a neighbourhood of $\mathcal{S}$, for nested a-bowls $\mathcal{B}_0^a \subset \mathcal{B}_1^a$ corresponding to sublevel sets $\mathcal{S}_0^a$ and $\mathcal{S}_1^a$ of $\mathcal{S}$ the analytic continuation of f from a neighbourhood of $\mathcal{S}_1^a$ to a neighbourhood of $\mathcal{B}_1^a$ may not coincide with f near $\mathcal{S}_0^a$. In such case the function may not possess analytic continuation to a neighbourhood of the difference $\mathcal{B}_1^b \setminus \overline{\mathcal{B}_0^b}$ for $b > a$.

Proof of Theorem 3. The sphere S will be the connected sum of two spheres S_0 and S_1 (i.e. a disc will be removed from each sphere and a two-dimensional handle will be glued along the circles). Start with the description of S_0. Denote coordinates in $\mathbb{C}^2$ by $(z,w) = (x + \mathrm{i}y, u + \mathrm{i}v)$. S_0 will be the graph over a two-sphere in $\{(z,w) \in \mathbb{C}^2 : \mathrm{Im}w = 0\} \simeq \mathbb{C} \times \mathbb{R}$ and will have exactly two elliptic points. More precisely, consider the following algebraic hypersurfaces $X_t = \{(z,w) \in \mathbb{C}^2 : (w-t)z = s(t)\}$, $t \in \mathbb{R}$. Here s is a smooth function, $s(t) = 0$ for $|t| \geq 1$ and $s(t) > 0$ for $|t| < 1$. The family X_t is defined by a continuous family of polynomials. The family of analytic varieties is continuous in the sense of Gromov. For $t = \pm 1$ bubbling occurs. The limit in the Gromov topology of the irreducible curves X_t, $|t| < 1$, at ± 1 consists of the union of the two irreducible curves $\{z = 0\}$ and $\{w = \pm 1\}$.

For all t with $|t| \geq 1$ the variety X_t is reducible and its irreducible branch $\tilde{X}_t = \{(z,w) \in \mathbb{C}^2 : w = t\}$ is contained in $\{\mathrm{Im}w = 0\}$. Moreover, $\bigcup_{t \geq 1} \tilde{X}_t = \{\mathrm{Im}w = 0, \mathrm{Re}w \geq 1\}$, and the respective relation holds for the union over $t \leq -1$.

We will require further that $|s'(t)| < 1$ on $\mathbb{R}$. This condition implies that the mapping

$$(z,t) \to (z, \mathrm{Re}(t + \frac{s(t)}{z})),\ |z| \geq 1,\ t \in \mathbb{R},$$

is a diffeomorphism onto the set $\{(z,w) \in \mathbb{C}^2 : \mathrm{Im}w = 0, |z| \geq 1\}$ which maps the set $\{|z| \geq 1, |t| \leq 1\}$ onto the set $\{|z| \geq 1, w \in [-1,1]\}$. Put $g(z,t) = (z, t + \frac{s(t)}{z})$, $|z| \geq 1$, $|t| < 1$, and $g(z,t) = (z,t)$, $z \in \mathbb{C}$, $|t| > 1$. The mapping g is defined on the set $E \stackrel{\mathrm{def}}{=} \{(z,t) \in \mathbb{C} \times \mathbb{R} : |z| \geq 1, |t| \leq 1\} \cup \{(z,t) \in \mathbb{C} \times \mathbb{R} : z \in \mathbb{C}, |t| > 1\}$. It maps each subset of E to a graph over the corresponding subset of $\{\mathrm{Im}w = 0\}$ in $\mathbb{C}^2$. Moreover the full image $\mathcal{F} \stackrel{\mathrm{def}}{=} g(E) = \bigcup_{|t| \leq 1}(X_t \cap \{|z| \geq 1\}) \cup \bigcup_{|t|>1} \tilde{X}_t$ is a smooth Levi-flat hypersurface in $\mathbb{C}^2$, the leaves of which are parts of the algebraic hypersurfaces X_t.

Choose now a smooth sphere S_0' in E with exactly two elliptic points which is not homologous to zero in E, e.g. a large round sphere in $\mathbb{C} \times \mathbb{R}$ centered at the origin. Let $S_0 \subset \mathcal{F}$ be the image of this sphere under the map g and let $\tilde{S}_0$ be the orthogonal projection of S_0 onto $\{\mathrm{Im}w = 0\}$. We may assume that the theorem of Bedford and Gaveau applies, and hence the polynomial hull $\widehat{S}_0$ (and also the envelope of holomorphy) of S_0 is a closed three-ball $\overline{b}_0$, such that the open ball b_0 is foliated by analytic discs. Note that b_0 is a graph over the bounded connected component $\tilde{b_0}$ of $\{\mathrm{Im}w = 0\} \setminus \tilde{S}_0$. We may also assume that $\tilde{b_0} \times i\mathbb{R}$ is strictly pseudoconvex.

Let $S_1 \subset \mathcal{F}$ be a two-sphere with two elliptic points, which is the graph over a large sphere $\tilde{S}_1$ in $\{\mathrm{Im}w = 0\} \setminus \tilde{b_0}$.

Given a number $\epsilon > 0$, by the choice of the mapping g the sphere S_1 may be chosen in such a way that for any point $(z, w) \in S_1$ the inequality $|\mathrm{Im} w| < \epsilon$ holds. Indeed, taking for $\tilde{S_1}$ a large round sphere in $\{\mathrm{Im} w = 0\}$ centered at the origin will suffice. We may again assume that the polynomial hull $\widehat{S}_1$ (and the envelope of holomorphy of S_1) is equal to a closed three-ball $\overline{b}_1$ with b_1 foliated by analytic discs. Note that $\overline{b}_1 \subset \{|\mathrm{Im} w| < \epsilon\}$.

The union of the two spheres $S_0 \cup S_1$ bounds a spherical shell Ω on $\mathcal{F}$ which is foliated into complex curves, either discs or annuli, and which is therefore contained in the polynomial hull $\widehat{S_0 \cup S_1}$ of $S_0 \cup S_1$. The complex curves constituting Ω are subsets of the set X_t (for $|t| < 1$) or $\tilde{X}_t$ (for $|t| \geq 1$). Moreover, the part of Ω which is contained in $\{|\mathrm{Re} w| \geq 1\}$ is foliated by flat discs or annuli (i.e. by curves along which w is constant). These discs or annuli, respectively, are also contained in the polynomial hull $\widehat{S}_1$.

The leaves of $\Omega \cap \{|\mathrm{Re} w| < 1\}$ are annuli contained in certain X_t with $|t| < 1$. They do not extend to analytic discs and therefore intersect each disc of the foliation of b_1 in a (possibly empty) discrete set. Moreover, by boundary uniqueness theorems for analytic functions the boundary of any disc in the foliation of b_1 cannot intersect some X_t, $|t| < 1$, along a set of positive linear measure.

Construction of $\mathcal{R}$. Before obtaining the sphere S as connected sum of S_0 and S_1 we construct a pseudoconvex Riemann domain $\mathcal{R}$ over $\mathbb{C}^2$ (see statement 3 of the theorem) to which we will lift the sphere S.

Start with the following observation. By Sard's lemma there is a dense open subset of $S_1 \cap \{|\mathrm{Re} w| < 1\}$, so that at each point of this set the boundary of the leaf $X_t \cap \Omega$ which passes through this point intersects transversally the boundary of the analytic disc of the foliation of b_1. (The respective fact also holds for S_0 instead of S_1).

This implies by the edge-of-the-wedge theorem [AH] that the polynomial hull $\widehat{S_0 \cup S_1}$ of $S_0 \cup S_1$ has non-empty interior.

Further, notice that no point in Ω is contained in $\widehat{S}_0$ (because these sets are graphs over disjoint subsets of $\{\mathrm{Im} w = 0\}$). Choose $\epsilon > 0$ small enough and choose S_1 so that $\widehat{S}_1$ is contained in $\{|\mathrm{Im} w| < \epsilon\}$. Then there are points in S_0 which are contained in $\{|\mathrm{Im} w| \geq \epsilon\}]$. This implies that for parameters t in some interval $I \subset (-1, 1)$ the set $X_t \cap \Omega$ is not contained in $\widehat{S}_0 \cup \widehat{S}_1$. Let $t_0 \in I$ and let γ be a simple curve in $X_{t_0} \cap \Omega$ which joins a point in S_0 with a point in S_1. We may assume that $\gamma \backslash \{|\mathrm{Im} w| < \epsilon\}$ is connected (take for instance for γ a connected component of $X_{t_0} \cap \{\mathrm{Re} z = 0\} \cap \Omega$). After a small change of the parameter t_0 and the curve $\gamma \subset X_{t_0} \cup \Omega$ we may assume that the endpoint p of γ on S_1 is in the aforementioned open set.

The Riemann domain $\mathcal{R}$ is now obtained in the following way. Let U_0 be a bounded (connected) and smoothly bounded strictly pseudoconvex neighbourhood of $\widehat{S}_0$, which is close to $\widehat{S}_0$, and let U_1 be a bounded and smoothly bounded strictly pseudoconvex domain which contains the product of a large ball in $\{\mathrm{Im} w = 0\}$ with the segment $\{w \in (-i\epsilon, +i\epsilon)\}$ and is close to it. We require also that U_1 contains $\widehat{S}_1$. We may assume that the curve $\gamma_1 \stackrel{\mathrm{def}}{=} \gamma \setminus (U_0 \cup U_1)$ is connected and meets $\overline{U_0} \cup \overline{U_1}$ only at the endpoints of γ_1.

Surround γ_1 by a thin cylindrical tube $\mathcal{T}$ of varying width (topologically the product of an open 3-ball with the curve) the closure of which does not meet X_{t_1} for another parameter $t_1 \in I$. The tube $\mathcal{T}$ may be chosen disjoint from $U_0 \cup U_1$ and is attached to ∂U_0, and to ∂U_1, respectively, along 3-balls corresponding to the tips of γ_1. The resulting set $U_0 \cup U_1 \cup \mathcal{T}$ may be lifted to a two-sheeted Riemann domain $\mathcal{R}$. The tubular neighbourhood $\mathcal{T}$ may be chosen in such a way that the Riemann domain $\mathcal{R}$ is pseudoconvex (for example, in the sense that $\mathcal{R}$ is a manifold with boundary and the natural projection $\pi : \overline{\mathcal{R}} \to \mathbb{C}^2$ maps the intersection with $\mathcal{R}$ of suitable neighbourhoods of boundary points of $\overline{\mathcal{R}}$ to pseudoconvex domains in $\mathbb{C}^2$).

The prescribed procedure of gluing pseudoconvex handles appeared in the literature first in [Sh]. For a more general approach see [El]. By results of Oka [Ok] and Grauert [Gr] $\mathcal{R}$ is a Stein manifold.

Construction of S. We start with considerations which allow to fulfill the requirement that the sphere S admits a lift $\mathcal{S} \subset \mathcal{R}$ which is contained in a strictly pseudoconvex boundary. Let $V_0 \subset U_0$ be a smoothly bounded strictly pseudoconvex domain the boundary of which contains S_0. We may assume that V_0 is contained in $\tilde{b_0} \times i\mathbb{R}$.

Similarly, let $V_1 \subset U_1$ be a smoothly bounded strictly pseudoconvex domain which contains S_1 in its boundary and has the following additional property: The curve γ meets the closure $\overline{V_1}$ exactly at the endpoint p of γ on S_1.

The latter property can always be achieved in the following way. Take a strictly pseudoconvex domain containing $\widehat{S}_1$ and close to $\widehat{S}_1$. Consider a smooth hypersurface $\mathcal{H}$ which contains S_1, is tangent to $\mathcal{F}$ at points of S_1 close to p and strictly pseudoconvex from the side which meets b_1 . (Recall that near p the leaves of $\mathcal{F}$ are transversal to the discs of the foliation of b_1.) It follows that near p the boundary of V_1 is on one side of the Levi-flat hypersurface $\mathcal{F}$. We assume that the hypersurface $\mathcal{H}$ is relatively closed in the aforementioned strictly pseudoconvex domain and forms a collar around S_1. This collar divides the pseudoconvex domain. Take the connected component which is pseudoconvex and smoothen its boundary fixing the part close to S_1.

By the choice of V_0 and V_1 the curve γ meets $\overline{V_0} \cup \overline{V_1}$ exactly at the endpoints. Surround γ by another strictly pseudoconvex handle T glued along three-balls to ∂V_0 and ∂V_1, respectively. We may choose T so that the union $V_0 \cup V_1 \cup T$ lifts to a strictly pseudoconvex domain G contained in the Riemann domain $\mathcal{R}$.

Notice that $T \cap \overline{\Omega}$ is glued to S_0 and S_1, respectively, along two-balls (discs). The result of the latter (two-dimensional) handle gluing is the required sphere S which is embedded into $\mathbb{C}^2$ (see figure 2, which shows the intersection of S with $\mathrm{Re} z = 0$). The handle T can be chosen so that S is smooth.

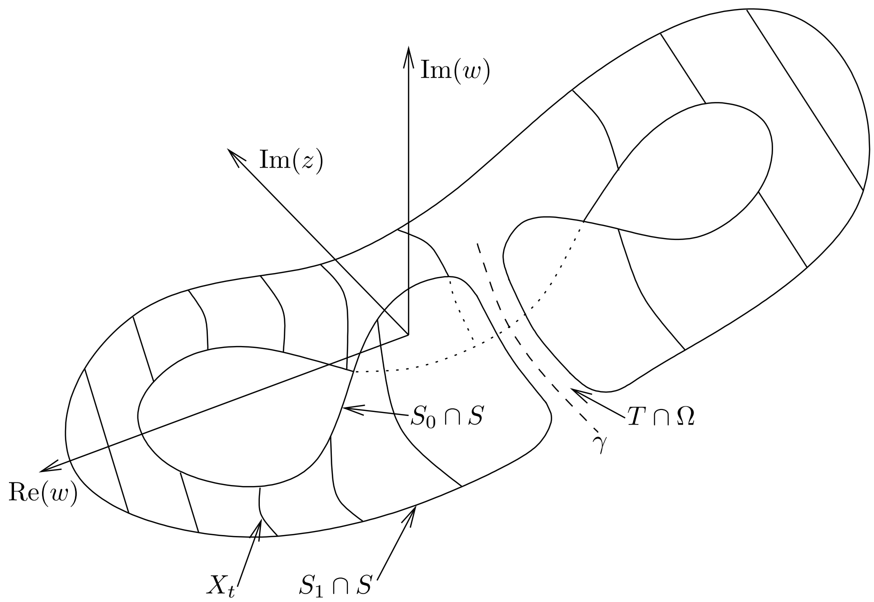

Fig. 2

The part of the sphere contained in $S_0 \cup S_1$ contains only elliptic complex tangencies. A point in the remaining part $\Omega \cap \partial T$ is a complex tangency iff at this point ∂T is tangent to a leaf of Ω. It is now easy to see that possibly after a small perturbation of ∂T the constructed sphere S has the following property: the lift $\mathcal{S} \subset \mathcal{R}$ of S satisfies the conditions of the theorem of Bedford-Klingenberg [BK]. Part 3 of the theorem is proved.

The set $\mathcal{B}$ of part 1 of theorem 3 equals $\Omega \backslash T$. This set is a three-ball obtained from a spherical shell by drilling a cylindrical hole around a curve joining the two boundary components. It is now clear that $\mathcal{B}$ has the properties required in assertion 1 of the theorem.

For the proof of assertion 2 of the theorem notice that the natural projection from $\mathcal{R}$ to $\mathbb{C}^2$ defines a biholomorphic mapping π from a (connected) neighbourhood of $\mathcal{S}$ in $\mathcal{R}$ onto a neighbourhood $\mathcal{D}$ of S in $\mathbb{C}^2$. Hence the envelope of holomorphy $\tilde{\mathcal{D}}$ of $\mathcal{D}$ can be considered as a Riemann domain over the pseudoconvex manifold $\mathcal{R}$ and hence the natural projection $\tilde{\pi}$ from $\tilde{\mathcal{D}}$ to $\mathbb{C}^2$ maps $\tilde{\mathcal{D}}$ into $\pi\mathcal{R}$. Since $\pi\mathcal{R} = U_0 \cup U_1 \cup \mathcal{T}$ does not contain $X_{t_1} \cap \Omega \subset \mathcal{B}$ assertion 2 of the theorem is proved. An analytic function on $\mathcal{D}$ which cannot be extended to an analytic function on a neighbourhood of $\overline{\mathcal{B}}$ can be obtained from a nowhere extendible analytic function on $\mathcal{R}$ by restricting to $\pi^{-1}(\mathcal{D})$ and taking the push-forward under π. The theorem is proved. $\square$

The authors are grateful to the referee for his critics, suggestions and questions

which led to a simplification of the proof and a strengthening of the statement of theorem 3.

REFERENCES

[AH] Airapetjan, R.A., Henkin, G.M., *Analytic continuation of CR-functions across the "edge of the wedge"*, Dokl. Akad. Nauk. SSSR, 259:4 (1981), pp. 777–781.

[BG] Bedford, E., Gaveau, B., *Envelopes of holomorphy of certain 2-spheres in* $\mathbb{C}^2$, Amer. J. Math., 105:4 (1983), pp. 975–1009.

[BK] Bedford, E., Klingenberg, W., *On the envelope of holomorphy of a 2-sphere in* $\mathbb{C}^2$, J. Amer. Math. Soc., 4:3 (1991), pp. 623–646.

[Ch] Chirka, E. M., *The generalized Hartogs lemma and the nonlinear* $\overline{\partial}$*-equation*, (Russian) Complex analysis in modern mathematics (Russian), pp. 19–31,

[CR1] Chirka, E. M., Rosay, J.-P., *Remarks on the proof of a generalized Hartogs lemma*, Ann. Polon. Math., 70 (1998), pp. 43–47.

[CR2] Chirka, E. M., Rosay, J.-P., *A correction to and a remark on our paper: "Remarks on the proof of a generalized Hartogs lemma"*, Ann. Polon. Math., 83 (2004), pp. 289–290.

[CS] Chirka, E. M., Stout, E. L., *A Kontinuitätssatz*, Topics in complex analysis (Warsaw, 1992), pp. 143–150, Banach Center Publ., 31, Polish Acad. Sci., Warsaw.

[El] Eliashberg, Y., *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math., 1 (1990), pp. 29–46.

[Fo] Fornaess,, J.-E., *The disc method*, Math. Z., 227 (1998), pp. 705–709.

[Gr] Grauert, H., *Charakterisierung der holomorph vollständigen komplexen Räume*, (German) Math. Ann., 129 (1955), pp. 233–259.

[IS] Ivashkovich, S., Shevchishin, V., *Structure of the moduli space in a neighborhood of a cusp-curve and meromorphic hulls*, Invent. Math., 136 (1999), pp. 571–602.

[Jö] Jöricke, B., *Some remarks concerning holomorphically convex hulls and envelopes of holomorphy*, Math. Z., 218 (1995), pp. 143–157.

[JP] Jöricke, B.,Porten, E., *Hulls and Analytic Extension from non-pseudoconvex boundaries*, Preprint, Uppsala (U.U.D.M.-reports 2002:19, Feb. 2002).

[Ne] Nemirovski, S. Y., *Complex analysis and differential topology on complex surfaces*, (Russian) Uspekhi Mat. Nauk., 54 (1999), pp. 47–74; translation in Russian Math. Surveys, 54 (1999), pp. 729–752.

[Ok] Oka, K., *Sur les fonctions analytiques de plusieurs variables. IX. Domaines finis sans point critique intérieur,* (French) Jap. J. Math., 23 (1953), pp. 97–155.

[Ro] Rosay, J.-P., private communication.

[Sh] Shcherbina, N., *Decomposition of a common boundary of two domains of holomorphy into analytic curves,* (Russian) Izv. Akad. Nauk. SSSR Ser. Mat., 46:5 (1982), pp. 1106–1123.