ASIAN J. MATH.
Vol. 11, No. 1, pp. 039–046, March 2007

003

EXCEPTIONAL BLOWUP SOLUTIONS TO QUASILINEAR WAVE EQUATIONS II*

SERGE ALINHAC†

Key words. Quasilinear wave equations, blowup, linearly degenerate eigenvalue, unstable solutions

AMS subject classifications. 35L40

1. Introduction. This Note is a continuation of the paper "Exceptional Blowup Solutions to Quasilinear Wave Equations" [1]. In this previous paper, we constructed, for some quasilinear wave equations, solutions blowing up at the origin like t^{-2}, which we considered to be an exceptional rate (the standard one in this context being t^{-1}, see [6]). We were encouraged by questions of the Referee (whom we thank) to investigate more precisely the stability of such solutions (an issue vaguely touched upon in [1]). It turns out that, depending on the perturbation of the Cauchy data, we can make the singularity of the solution either disappear, or go back to the generic t^{-1} case.

Since this paper is dedicated to M. S. Baouendi, we are happy to underline the similarity in spirit between previous constructions of conterexamples [7], [8], and the present work : in both cases, the insight is obtained through a careful self-contained construction.

2. Notation and main result. The notation and the framework is the same as in [1]. For simplicity, we restrict our attention to $n = 2$, and do not handle the 1D case (though it is straightforward). Thus the variables and dual variables are

$$x = (x_1, x_2, x_3), y = x_2, t = x_3, \xi = (\xi_1, \xi_2, \xi_3), \eta = \xi_2, \tau = \xi_3.$$

We consider a quasilinear wave equation with real *analytic* coefficients

$$P(u) = \Sigma p_{ij}(\partial u)\partial^2_{ij} u = 0, p_{ij} = p_{ji}, p_{3,3} = 1.$$

We denote here

$$\partial u = (\partial_1 u, \partial_2 u, \partial_3 u), p(\partial u; \xi) = \Sigma p_{ij}(\partial u)\xi_i\xi_j.$$

We assume given a point $(\overline{\partial u}, \bar\xi)$ where

$$p(\overline{\partial u}; \bar\xi) = 0, (\partial_\tau p)(\overline{\partial u}; \bar\xi) \neq 0, \bar\xi_1 = -1,$$

and the frozen operator $\Sigma p_{ij}(\overline{\partial u})\partial^2_{ij}$ is strictly hyperbolic with respect to t. Noting $D_j p = \partial_{(\partial_j u)} p$, we assume moreover that the given point is *linearly degenerate*, that is

$$(\bar\xi.D)p(\overline{\partial u}; \bar\xi) = 0.$$

*Received June 20, 2006; accepted for publication September 13, 2006.

†Département de Mathématiques, Université Paris-Sud, 91405 Orsay, France (serge.alinhac@math.u-psud.fr).

In this Note, our aim is to improve the results of [1] by constructing a family of solutions u^ϵ, containing the solution constructed in [1] as u^0, and such that,

i) For $\epsilon > 0$, the solution does not blow up,

ii) For $\epsilon < 0$, the solution blows up at time t_ϵ like $(t_\epsilon - t)^{-1}$.

The assumptions of our main result below are the same as those of Theorem 2.3 of [1].

Theorem. *Assume that, at the given point* $(\overline{\partial u}, \bar\xi)$, $\partial_\eta p \neq 0$, *and one of the following three quantities is not zero :*

$$-(D_2p)((\bar\xi D)\partial_\eta p + D_2p) - 1/2(\partial^2_\eta p)(\bar\xi D)^2 p + (\partial_\eta p)(\bar\xi D)D_2p,$$

$$-(D_3p)((\bar\xi D)\partial_\eta p + D_2p) - (D_2p)((\bar\xi D)\partial_\tau p + D_3p) -$$
$$-(\partial^2_{\tau\eta}p)(\bar\xi D)^2p + (\partial_\tau p)(\bar\xi D)D_2p + (\partial_\eta p)(\bar\xi D)D_3p,$$
$$-(D_3p)((\bar\xi D)\partial_\tau p + D_3p) - (\bar\xi D)^2p + (\partial_\tau p)(\bar\xi D)D_3p.$$

Then there exist

a. *A domain D defined by*

$$-t_1 \le t \le t_2, x_1^2 + x_2^2 \le (R - kt)^2, t_1 > 0, t_2 > 0, R > 0, k > 0.$$

We call the disk $t = -t_1, ||(x_1, x_2)|| \le R + kt_1$ the base of D.

b. *A family of solutions u^ϵ (depending continuously on ϵ along with its derivatives), defined for ϵ close to zero such that*

i) *For $\epsilon = 0$, let*

$$D_0 = D \cap \{t < 0\}.$$

Then $u^0 \in C^1(\bar D_0)$, u^0 is analytic in $\bar D_0$ away from the origin, D_0 is an influence domain of its base, and $\partial^2 u^0$ blows up at the origin like $(-t)^{-2}$, as explained precisely in [1].

ii) *For $\epsilon > 0$, u^ϵ is defined and analytic in D,*

iii) *For $\epsilon < 0$ and some $t_\epsilon < 0$, let*

$$D_\epsilon = D \cap \{t < t_\epsilon\}.$$

Then $u^\epsilon \in C^1(\bar D_\epsilon)$, u^ϵ is analytic in $\bar D_\epsilon$ away from the origin, D_ϵ is an influence domain of its base, and $\partial^2 u^\epsilon$ blows up at a point $m_\epsilon = (p_\epsilon, t_\epsilon)$, close to zero, like $(t_\epsilon - t)^{-1}$ (more precisely, m_ϵ is a geometric blowup point of cusp type, in the terminology of [2], [3]).

Remark 1. We cannot describe which modifications of the Cauchy data on $t = -t_1$ lead to which singularities for the solution. What we do is show that *some modifications* of the data make the singularity disappear, while *some other* transform the singularity back to the generic type.

Remark 2. What we call "geometric blowup of cusp type", or more rapidly "generic blowup", is indeed stable, as shown in [5] in the more general context of quasilinear symmetric systems.

This Note is closely related to [1] : though the statements and the strategy of the proof are understandable without [1], the actual details are based on computations of [1], and can only be understood in connection with [1].

3. Proof of the main result.

3.1. Outline of the strategy. We recall here the basic facts from [1], section 3, keeping the same notation. For a broader introduction to the concepts and tools used here, see for instance [2], [3], [4]. We introduce a change of variables Φ (depending on a still unknown function ϕ)

$$(s, y, t) \mapsto (x_1 = \phi(s, y, t), y, t),$$

and set

$$w(s, y, t) = u(\phi(s, y, t), y, t), v(s, y, t) = (\partial_1 u)(\phi(s, y, t), y, t).$$

Setting

$$\bar{\partial} = (0, \partial_y, \partial_t), \hat{\phi} = (-1, \partial_y \phi, \partial_t \phi),$$

we define now

$$\mathcal{A} = w_s - v\phi_s,$$

$$\mathcal{E} = \Sigma p_{ij}(v, w_y - v\phi_y, w_t - v\phi_t)\hat{\phi}_i\hat{\phi}_j,$$

$$\mathcal{R} = \Sigma p_{ij}(v, w_y - v\phi_y, w_t - v\phi_t)[\bar{\partial}^2_{ij} w - v\bar{\partial}^2_{ij}\phi - (\hat{\phi}_i\bar{\partial}_j v + \hat{\phi}_j\bar{\partial}_i v)].$$

Using the formula for the first and second order derivatives of u in terms of (v, ϕ, w), we see easily that

$$(Pu)(\Phi) = \frac{\mathcal{E}}{\phi_s} + \mathcal{R}.$$

Hence we associate to P the blowup system on (ϕ, v, w)

$$\mathcal{A} = 0, \mathcal{E} = 0, \mathcal{R} = 0.$$

Rotating the variables by

$$T = s + t, S = t - s, y = y,$$

the write the subsystem $\mathcal{A} = 0, \mathcal{E} = 0$ as

$$w_T = w_S - 2v\phi_S + v\lambda, \phi_T = -\phi_S + \lambda,$$

where

$$\lambda \equiv \lambda(v, w_y - v\phi_y, 2(w_S - v\phi_S); -1, \phi_y).$$

Using these equations, and introducing the new unknowns ϕ_S, ϕ_y, w_S, w_y, we can view the blowup system as a fully nonlinear first order system in the unknowns $v, \phi_S, \phi_y, w_S, w_y$, resolved with respect to the T-derivative. Our aim is to construct, using the Cauchy-Kovalevski theorem, a family of (smooth) solutions (near the origin) of the blowup system.

Heuristically, the construction of the family u^ϵ is based on a simple geometric deformation argument which goes as follow : we will construct ϕ such that, very roughly,

$$\phi_s = \epsilon + s^2 + y^2 + t^2.$$

Then

i) For $\epsilon = 0$, we have the exceptional blowup described in [1],

ii) For $\epsilon > 0$, we have no blowup at all in a fixed neighbourhood of the origin,
iii) For $\epsilon < 0$, $\phi_s = 0$ at the point $M_\epsilon = (0, 0, t_\epsilon = -(-\epsilon)^{1/2})$ and is stricly positive for $t < t_\epsilon$.

3.2. Choice of the jets at the origin. We modify now the construction of [1] by choosing a larger family of jet conditions at the origin. We still impose

$$\phi_y = \bar{\eta}, (v, w_y - v\phi_y, 2(w_S - v\phi_S)) = \bar{\partial}u,$$

but we take now, for ϵ small enough (positive and negative)

$$\lambda - 2\phi_S = \epsilon.$$

Remark that, from Lemma 3.2 de [1], $T = 0$ is still non characteristic for the blowup system. Note also that, at the origin, Λ is no longer zero, but only $O(\epsilon)$.

We choose now the values of the blocks B_{yy}, B_{Sy}, B_{SS} just as before for $\epsilon = 0$, in order to ensure that $Q \neq 0$. With $\phi_{yy}(0) = 0$, we then choose successively $\phi_{Sy}(0)$ and $\phi_{SS}(0)$ to obtain from Lemma 4.1

$$F_y(0) = 0, F_S(0) = 0,$$

but now of course F_T need not be zero, but only $O(\epsilon)$.

For the third order jets, we proceed exactly as in 4.4.3 of [1], choosing the blocks B_{SSS}, B_{SSy}, etc. to be zero ; then

$$\phi_{yyy}(0) = 0, A = -\phi_{Syy}(0) >> 0, 2\phi_{SSy}(0) = -\partial_\eta \lambda A, \phi_{SSS} = -B = -\frac{1}{2}(\partial_\eta \lambda)^2 A.$$

Since, according to Lemma 4.2 of [1], second order derivatives of F involve also second order derivatives of v, we fix for clarity, say,

$$v_{yy}(0) = v_{Sy}(0) = v_{SS}(0) = 0.$$

We fix now the value of A as for the case $\epsilon = 0$. For small enough ϵ, the hessian of F at the origin will still be positive definite.

3.3. Uniformity. Once the jets of ϕ, v, w are fixed as above, we solve the blowup system taking for ϕ, v, w polynomials of degree respectively $3, 2, 3$ with these jets. Using a precise version of the Cauchy-Kovalevski theorem (for instance that of Baouendi and Goulaouic [9]), we see that we obtain a family of solutions defined in a fixed neighbourhood of the origin, whose derivatives are continuous in ϵ.

Two facts remain to be proved :

i) If $\epsilon > 0$, there is a fixed neighbourhood of 0 where F remains positive.
ii) If $\epsilon < 0$, there is a point close to 0 where $F = 0$, and the corresponding solution blows up.

4. Case $\epsilon > 0$. This is the easy case. In fact, expanding F by Taylor up to second order and using the fact that the hessian is uniformly positive definite in a fixed neighbourhood of the origin, we obtain, for some $C' > 0$,

$$F(S, y, T) \geq F(0) - C\epsilon||(S, y, T)|| + C'(S^2 + y^2 + T^2), F(0) = \epsilon.$$

Hence $F > 0$ for $\epsilon < 4C'/C^2$.

5. Case $\epsilon < 0$.

a. For given (ϵ, T), we look first for a point where $\phi_{ss} = \phi_{sy} = 0$, that is

$$F_T - F_S = 0, F_y = 0.$$

For $\epsilon = 0$ and $T = 0$, the origin is such a point. To apply the implicit function theorem, it is enough to check that, at the origin,

$$d = F_{yy}(F_{ST} - F_{SS}) - F_{Sy}(F_{Ty} - F_{Sy}) \neq 0.$$

With the above choices of jets, we have

$$d = -\delta + F_{ST}F_{yy} - F_{Sy}F_{Ty} = -2(\partial_\eta \lambda)^2 A^2 + O(A).$$

We can always assume that A has been already chosen so big that $d \neq 0$. So we have now functions $S(\epsilon, T)$ and $y(\epsilon, T)$, smoothly defined near zero, such that $\phi_{ss} = \phi_{sy} = 0$ at the point

$$M(\epsilon, T) = (S = S(\epsilon, T), y = y(\epsilon, T), T).$$

b. We set now

$$\tilde{F}(\epsilon, T) = F(S(\epsilon, T), y(\epsilon, T), T),$$

and look for $T = T_\epsilon$ such that

$$\tilde{F}(\epsilon, T_\epsilon) = 0.$$

Lemma 5.1. *For $\epsilon = 0, T = 0$, if we choose $(v_T - v_S)Q > 0$ and $v_T - v_S$ small enough, then*

$$\tilde{F}_{TT}(0, 0) > 0.$$

Proof. We have

$$\tilde{F}_T(\epsilon, T) = F_T + F_S S_T + F_y y_T.$$

Since at $M(\epsilon, T)$, $F_T = F_S$ and $F_y = 0$, we obtain

$$F_T = F_S = (1/2)(v_T - v_S)\Lambda + *F,$$

hence

$$\tilde{F}_T = (S_T + 1)F_S = (1/2)(S_T + 1)(v_T - v_S)\Lambda + *F,$$

$$\tilde{F}_{TT}(0) = (1/2)(S_T + 1)(v_T - v_S)(\partial_T \Lambda + \partial_S \Lambda S_T + \partial_y \Lambda y_T).$$

Now, differentiating the identities which define S and y, we obtain

$$F_{TT} - F_{ST} + S_T(F_{TS} - F_{SS}) + y_T(F_{Ty} - F_{Sy}) = 0,$$

$$F_{yT} + S_T F_{yS} + y_T F_{yy} = 0.$$

Replacing F_{TT}, F_{TS}, F_{Ty} using the formula of Lemma 4.2 of [1], we see that the equalities are satisfied, up to some term $O(v_T - v_S)$, if we take $S_T = 1, y_T = -\partial_\eta \lambda$. Hence

$$S_T = 1 + O(v_T - v_S), y_T = -\partial_\eta \lambda + O(v_T - v_S),$$

$$\Lambda_T + S_T\Lambda_S + y_T\Lambda_y = Q + O(v_T - v_S).$$

This proves the claim. ∎

LEMMA 5.2. *There exists, for $\epsilon < 0$ small enough, a smooth function $T_\epsilon = T((-\epsilon)^{1/2})$, such that*

$$T_\epsilon = -c(-\epsilon)^{1/2} + O(\epsilon), c > 0,$$
$$\tilde{F}(\epsilon, T_\epsilon) = 0.$$

Proof. Let us write the Taylor expansion of $\tilde{F}$ at $(0,0)$:

$$\tilde{F}(\epsilon,T) = \tilde{F}(0,0) + \epsilon\partial_\epsilon\tilde{F}(0,0) + T\tilde{F}_T(0,0) + O(\epsilon^2) + O(\epsilon T) + (T^2/2)\tilde{F}_{TT}(0,0) + O(T^3).$$

Since

$$\tilde{F}(0,0) = 0, \tilde{F}_T(0,0) = 0, \partial_\epsilon\tilde{F}(0,0) = (\partial_\epsilon F)(0,0) + *F_S(0,0) + *F_y(0,0) = 1,$$

we get

$$\tilde{F}(\epsilon,T) = \epsilon(1 + O(\epsilon) + O(T)) + (T^2/2)\tilde{F}_{TT}(0)(1 + O(T)).$$

Introducing $\mu = (-\epsilon)^{1/2}$ and applying Morse's Lemma, we obtain the claim, thanks to Lemma 5.1. ∎

We now finish the proof of the Theorem. Let M_ϵ be the point

$$M_\epsilon = M(\epsilon, T_\epsilon) = (S(\epsilon, T_\epsilon), y(\epsilon, T_\epsilon), T_\epsilon).$$

The $t = t_\epsilon$ coordinate of this point is

$$2t = S(\epsilon, T_\epsilon) + T_\epsilon = 2T_\epsilon + O(\epsilon),$$

and is negative for ϵ small enough. Consider now the set $\phi_s = F = 0$; in a small enough neighbourhood of the origin, it is a compact submanifold. At a point of this submanifold where t is minimum, we necessarily have

$$\phi_{ss} = \phi_{sy} = 0.$$

But, from the above considerations, we know that there is only one such point, namely M_ϵ. Since, for $t = t_\epsilon$, ϕ_s has at M_ϵ a critical point with definite positive hessian, it is non negative then, and $\phi_s > 0$ for $t < t_\epsilon$. Hence the image m_ϵ by Φ of the point M_ϵ corresponds to a blowup point of cusp type, as usual, for the solution of the equation.

REFERENCES

[1] ALINHAC S., *Exceptional Blowup Solutions to Quasilinear Wave Equations*, to appear, Int. Math. Res. Notices, (2006).

[2] ALINHAC S., *Blowup for Nonlinear Hyperbolic Equations*, Progr. Nonlinear Differential Equations Appl. 17, Birkhäuser Boston, Boston MA, (1995).

[3] ALINHAC S., *A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations*, Journées "Equations aux Dérivées partielles" (Forges les Eaux, 2002), Université de Nantes, Nantes, 2002. www.math.sciences.univ-nantes.fr/edpa.

[4] Alinhac S., *Explosion géométrique pour des systèmes quasi-linéaires*, Amer. J. Math., 117 (1995), pp. 987–1017.

[5] Alinhac S., *Stability of geometric blowup*, Arch. Rat. Mech. Analysis, 150 (1999), pp. 97–125.

[6] Alinhac S., *Remarks on the blowup rate of classical solutions to quasilinear multidimensional hyperbolic systems*, J. Math. Pure Appl., 79 (2000), pp. 839–854.

[7] Alinhac S. and Baouendi M. S., *A counterexample to strong uniqueness for Schrödinger's type partial differential equations*, Comm. Part. Diff. Eq., 19 (1994), pp. 1727–1733.

[8] Alinhac S. and Baouendi M. S., *A non uniqueness result for operators of principal type*, Math. Zeitsch., 220 (1995), pp. 561–568.

[9] Baouendi M. S. and Goulaouic C., *Remarks on the abstract form of nonlinear Cauchy-Kovalevsky theorems*, Comm. Part. Diff. Eq., 2 (1977), pp. 1151–1162.