ASIAN J. MATH.
Vol. 11, No. 1, pp. 019–038, March 2007

PAIRS OF INVOLUTIONS OF GLANCING HYPERSURFACES*

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Dedicated to Salah M. Baouendi on the occasion of his seventieth birthday

Key words. moduli space, normal form, glancing hypersurfaces

AMS subject classifications. Primary 32B10, 32F25

1. Introduction. Let ω be the standard symplectic 2-form on $\mathbf{R}^{2n}$, given by

$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j, \quad (\xi, x) = (\xi_1, \cdots, \xi_n, x_1, \cdots, x_n) \in \mathbf{R}^{2n}.$$

Consider two real analytic hypersurfaces in $\mathbf{R}^{2n}(n \geq 2)$ defined by

$$F : f(\xi, x) = 0, \quad G : g(\xi, x) = 0,$$

where f, g are real analytic functions. F and G are said to be *glancing* at $p \in F \cap G$ if

$$\{f, g\}(p) = 0, \quad df \wedge dg\,(p) \neq 0,$$
$$\{f, \{f, g\}\}(p) \neq 0 \neq \{g, \{g, f\}\}(p),$$

in which $\{f, g\}$ is the Poisson bracket of f, g with respect to ω, defined by

$$\{f, g\} = X_f g, \quad X_f = \sum \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j} - \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j}.$$

A (local) map from $\mathbf{R}^{2n}$ to $\mathbf{R}^{2n}$ is said to be symplectic if it preserves ω. Given two pairs of hypersurfaces $\{F_j, G_j\}$ glancing at $p_j (j = 1, 2)$ respectively, they are equivalent if there exists a real analytic symplectic mapping ϕ defined near p_1 such that

$$\phi(p_1) = p_2, \quad \phi(F_1) = F_2, \quad \phi(G_1) = G_2.$$

Since we consider local equivalence only, we assume that $p_1 = p_2 = 0$.

In [5], Melrose showed that each pair of glancing smooth hypersurfaces in $\mathbf{R}^{2n}(n \geq 2)$ is equivalent to the pair

$$\widehat{F} : x_1 = 0, \quad \widehat{G} : \xi_2 = \xi_1^2 + x_1 \tag{1.1}$$

under a (C^∞) smooth change of coordinates; Melrose's argument also shows that all real analytic glancing hypersurfaces are equivalent to the above normal form by formal symplectic maps. It was proved by Oshima [6] for $n \geq 3$ and by the second author [3] for $n \geq 2$ that for some pairs of real analytic glancing hypersurfaces, the normal form cannot be achieved by any convergent symplectic map.

*Received August 2, 2006; accepted for publication March 9, 2007.
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‡Research of the second author is supported in part by NSF grant DMS-0305474.

A pair of glancing hypersurfaces F and G generates a pair of involutions on $J = F \cap G$. This pair of involutions plays an important role in Melrose's approach, which we now describe. Melrose first showed that in suitable real analytic symplectic coordinates, $F = \widehat{F}\colon x_1 = 0$ and G is of the form

$$\xi_2 = \xi_1^2 + x_1 b(\xi_2, \ldots, \xi_n, x), \quad b(0) = 1. \tag{1.2}$$

In particular $J = F \cap G\colon x_1 = 0, \xi_2 = \xi_1^2$. Put $'\xi = (\xi_3, \ldots, \xi_n)$, $'x = (x_3, \ldots, x_n)$. Choose $\xi_1, x_2, {'\xi}, {'x}$ as coordinates on J. We have $\omega|_J = d\xi_1^2 \wedge dx_2 + \sum_{j=3}^n d\xi_j \wedge dx_j$. Each solution curve of Hamiltonian vector field X_f on F is tangent to $K \subset J\colon \{f, g\} = 0$ or intersects $J \setminus K$ at two distinct points. It turns out that the map, which interchanges two intersection points, extends to a real analytic involution I_F on J, fixing K pointwise. Note that $\omega^{n-1}|_J$ vanishes precisely on $K \subset J$ and that K is defined by $\xi_1 = 0$ on J.

Analogously, one can define glancing holomorphic hypersurfaces of $\mathbf{C}^{2n}$ $(n \geq 2)$, for which $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the holomorphic symplectic 2-form.

In fact, we will treat the complex case, and the real case is treated via a reality condition. From now on (ξ, x) will be the coordinates of $\mathbf{C}^{2n}$. We regard $\mathbf{R}^{2n}$ as the set of fixed points of anti-holomorphic involution $\rho\colon \xi \to \overline{\xi}, x \to \overline{x}$.

As mentioned above, there are examples of pairs of real analytic glancing hypersurfaces for which the normal form cannot be achieved by any convergent symplectic change of coordinates (although such a formal change of coordinates always exists). In [8], Voronin describes a method of showing that divergence not only can happen but it is *generic*. His results are based on his theory of moduli space in several variables [7]. As usual when this method is applicable it actually shows not only that the generic pair of glancing hypersurfaces is not convergently equivalent to the normal form but that the set of equivalence classes is *infinite dimensional* (not just infinite).

Paper [7] is rich in detail but [8] has no proofs at all. It is our opinion that providing the details for Voronin's program, [8], requires some more ideas than those included in his earlier paper, [7]. The purpose of this paper is to give a self contained and detailed proof about the infinite dimensionality of equivalence classes of glancing hypersurfaces, part of results announced in [8].

2. Realizing pairs of involutions for glancing hypersurfaces. In this section, we will show that two pairs of glancing hypersurfaces are equivalent if (and only if) their pairs of involutions are equivalent under some holomorphic mapping preserving the degenerate 2-form $d\xi_1^2 \wedge dx_2 + d\xi_3 \wedge dx_3 + \cdots + d\xi_n \wedge dx_n$. We will also find a pair of glancing hypersurfaces for a given pair of involutions satisfying some conditions. See Proposition 2.4 for conditions on the involutions. Therefore, we identity the classification of glancing hypersurfaces in $\mathbf{C}^{2n}$ with that of pairs of involutions in $\mathbf{C}^{2n-2}$ equipped with the degenerate 2-form.

Recall that $'\xi = (\xi_3, \ldots, \xi_n)$, $'x = (x_3, \ldots, x_n)$. When a pair of glancing hypersurfaces is given by

$$\hat{F}\colon x_1 = 0, \quad \hat{G}\colon \xi_2 = \xi_1^2 + x_1,$$

its pair of involutions, defined on $(\xi_1, x_2, '\xi, 'x)$-space, is

$$\widehat{I}_1\colon \begin{cases} \xi_1' = -\xi_1, \\ x_2' = x_2, \\ \xi_\alpha' = \xi_\alpha, \quad 2 < \alpha \le n, \\ x_\alpha' = x_\alpha \end{cases} \qquad \widehat{I}_2\colon \begin{cases} \xi_1' = -\xi_1, \\ x_2' = x_2 - 2\xi_1, \\ \xi_\alpha' = \xi_\alpha, \\ x_\alpha' = x_\alpha. \end{cases}$$

(Throughout the note the α runs from 3 through n.) The composition $\widehat{I}_2\widehat{I}_1$ is

$$\widehat{\sigma}\colon \begin{cases} \xi_1' = \xi_1, \quad x_2' = x_2 + 2\xi_1, \\ \xi_\alpha' = \xi_\alpha, \quad x_\alpha' = x_\alpha. \end{cases}$$

For an arbitrary pair of glancing hypersurfaces, we always assume, after a preliminary change of coordinates ([5], [3]), that it is in the form

$$F = \widehat{F}\colon x_1 = 0, \quad G\colon \xi_2 = \xi_1^2 + x_1 b(\xi_2, \ldots, \xi_n, x), \quad b(0) = 1.$$

Their involutions have the form

$$I_1 = \widehat{I}_1, \quad I_2 = \widehat{I}_2 + O(2), \quad I_2|_{\xi_1=0} = \widehat{I}_2|_{\xi_1=0}.$$

Hence $\sigma = I_2 I_1 = I_1\sigma^{-1}I_1$ has the form

$$\sigma\colon \begin{cases} \xi_1' = \xi_1 + \xi_1^2 p_1, \\ x_2' = x_2 + 2\xi_1 + \xi_1 q_1, & q_1(0) = 0, \\ \xi_\alpha' = \xi_\alpha + \xi_1 p_\alpha, & p_\alpha(0) = 0, \\ x_\alpha' = x_\alpha + \xi_1 q_\alpha, & q_\alpha(0) = 0, \end{cases}$$

where p_j, q_j are holomorphic functions defined near $0 \in J = F \cap G$. On J, we also have a holomorphic two-form

$$\omega|_J = (\sum_{1\le j\le n} d\xi_j \wedge dx_j)|_J = 2\xi_1 d\xi_1 \wedge dx_2 + \sum_{2<\alpha\le n} d\xi_\alpha \wedge dx_\alpha.$$

It is obvious that $I_1^*\omega|_J = \omega|_J$. We also have $I_2^*\omega|_J = \omega|_J$, since by a change holomorphic symplectic coordinates, we can transform $\widehat{F}, G$ into $\widetilde{G}, \widehat{F}$.

In the real case the restriction of ρ on J is

$$\rho\colon \xi_1' = \overline{\xi}_1, \quad x_2' = \overline{x}_2, \quad \xi_\alpha' = \overline{\xi}_\alpha, \quad x_\alpha' = \overline{x}_\alpha.$$

The corresponding holomorphic involutions then satisfy the reality condition

$$I_j = \rho I_j \rho, \quad \sigma = \rho\sigma\rho.$$

We also have $\rho^*\omega|_J = \overline{\omega}|_J$.

It is obvious that if two pairs of glancing hypersurfaces $\{F_j, G_j\}, j = 1, 2$ are equivalent by some holomorphic map f preserving ω, their corresponding pairs of involutions are also equivalent by a holomorphic map preserving $\omega|_J$.

Next we want to show the converse is true.

Throughout the note, $\omega|_F$ stands for the pull-back of a differential form ω on $\mathbf{C}^{2n}$ by the inclusion $F \hookrightarrow \mathbf{C}^{2n}$. We need the following version of relative Darboux lemma.

LEMMA 2.1. *Let ω_0, ω_1 be two closed holomorphic 2-forms defined in a neighborhood of the origin in $\mathbf{C}^{2n}$. Assume that $(1-t)\omega_0 + t\omega_1$ are non-degenerate at the origin for all $t \in [0,1]$.*

(*i*) *If $S \subset \mathbf{C}^{2n}$ is a germ of holomorphic submanifold at 0 with $\omega_1|_S = \omega_0|_S$ there exists a holomorphic mapping f, defined near the origin and fixing S pointwise, such that $f^*\omega_1 = \omega_0$.*

(*ii*) *If F and G are two smooth holomorphic hypersurfaces intersecting transversely at the origin and if $\omega_1|_F = \omega_0|_F$ and $\omega_1|_G = \omega_0|_G$, there exists a germ of holomorphic mapping f at 0, fixing F pointwise, such that $f(G) = G$ and $f^*\omega_1 = \omega_0$.*

In both cases, if all coefficients of $\omega_1 - \omega_0$ vanish at the origin, one can achieve $f = \mathrm{id} + O(2)$ additionally.

Proof. The proof is based on Moser's homotopy method. The first part is due to Givental'. We shall modify the proof in [2] to show the second part. Note that one would not expect to find f that fixes both F and G pointwise.

(i) Without loss of generality, one may assume that S is given by $x_1 = \cdots = x_k = 0$, with $(x_1, \ldots, x_{2n})$ being coordinates of $\mathbf{C}^{2n}$. Write $\omega_j = d\alpha_j$ on $\mathbf{C}^{2n}$. We need to find the flow ϕ_t of a time-dependent holomorphic vector field v_t defined near $0 \in \mathbf{C}^{2n}$ such that

$$0 = \frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(L_{v_t}\omega_t + \frac{d}{dt}\omega_t) = \phi_t^* d(\iota_{v_t}\omega_t + \alpha_1 - \alpha_0),$$

where $L_{v_t} = d\iota_{v_t} + \iota_{v_t} d$ is the Lie derivative. Then we will set $\iota_{v_t}\omega_t + \alpha_1 - \alpha_0 = 0$. We also need the coefficients of v_t to vanish on S, i.e. the vanishing of the coefficients of $\alpha_1 - \alpha_0$ on S, so $\phi_t|_S = \mathrm{id}$ as required.

Starting with $d(\alpha_1 - \alpha_0)|_S = 0$, we get a holomorphic function g_0 in $x_{k+1}, \ldots, x_{2n}$ such that $(\alpha_1 - \alpha_0)|_S = dg_0$. Thus on $\mathbf{C}^{2n}$ we can write

$$\begin{aligned}\alpha_1 - \alpha_0 &= dg_0 + \sum_{i=1}^k x_i\theta_i + \sum_{i=1}^k b_i(x_{k+1}, \ldots, x_{2n})\, dx_i \\ &= d(g_0 + \sum_i x_i b_i) + \theta, \quad \theta = \sum_{i=1}^k x_i(\theta_i - db_i).\end{aligned}$$

Thus $\omega_1 - \omega_0 = d\theta$. Since $\omega_t = \sum_{i,j} u_{ij}(x,t)dx_i \wedge dx_j$ and $(u_{ij}) = -(u_{ji})$ is non-degenerate, there is a unique holomorphic vector field $v_t = \sum v_j \frac{\partial}{\partial x_k}$ on $\mathbf{C}^{2n}$ such that

$$\iota_{v_t}\omega_t = 2\sum_{j,k} u_{jk}v_j\, dx_k = -\theta.$$

Since the coefficients of θ vanish on S, the coefficients of vector field v_t vanish on S too.

(ii) Without loss of generality, one may assume that F and G are hyperplanes given by $x_1 = 0, x_2 = 0$, respectively. Again we are looking for a particular vector field v_t such that its flow ϕ_t will fulfill the requirements.

As above, write $\omega_j = d\alpha_j$. We first want to find a holomorphic function g such that

$$\alpha_1 - \alpha_0 = dg + x_1x_2\beta + x_1c(x)\,dx_2, \tag{2.1}$$

where c is holomorphic near $0 \in \mathbf{C}^{2n}$. Starting with $d(\alpha_1 - \alpha_0)|_{x_1=0} = 0$, we get a holomorphic function g_0 in $x_2, \dots, x_{2n}$ such that $(\alpha_1 - \alpha_0)|_F = dg_0$. Write

$$\begin{aligned}\alpha_1 - \alpha_0 &= dg_0 + x_1 \sum a_k(x)\,dx_k + b(x_2, \dots, x_{2n})\,dx_1\\ &= d(g_0 + x_1 b) + x_1 \sum a_k^*(x)\,dx_k\\ &= dg_1 + x_1x_2 \sum \widetilde{a}_k(x)\,dx_k + x_1 \sum b_k(x_1, x_3, \dots, x_{2n})\,dx_k\\ &= dg_2 + x_1x_2 \sum \widetilde{a}_k(x)\,dx_k + x_1 b_2(x_1, x_3, \dots, x_{2n})\,dx_2,\end{aligned}$$

where $x_1 \sum_{k\neq 2} b_k(x_1, x_3, \dots, x_{2n})\,dx_k$ is absorbed into g_2 via

$$d(\alpha_1 - \alpha_0)|_{x_2=0} = d\{x_1 \sum_{k\neq 2} b_k(x_1, x_3, \dots, x_{2n})\,dx_k\} = 0.$$

Thus, the decomposition (2.1) is obtained. As before there is a unique holomorphic vector field $v_t = \sum v_j \frac{\partial}{\partial x_k}$ on $\mathbf{C}^{2n}$ such that

$$2\sum_{jk} u_{jk}v_j\,dx_k = \iota_{v_t}\omega_t = -x_1x_2\beta - x_1c(x)\,dx_2.$$

The diagonal elements of $(u_{kj})^{-1} = (\tilde{u}_{kj})$ are zero. Hence

$$v_t = x_1 \frac{c}{2} \sum_{j\neq 2} \tilde{u}_{j2}(x,t)\frac{\partial}{\partial x_j} + x_1x_2 \sum_j q_j(x,t)\frac{\partial}{\partial x_j}.$$

Therefore the flow of v_t fixes the hyperplane $x_1 = 0$ pointwise and preserves $x_2 = 0$.

Assume now that $\omega_1 - \omega_0 = O(1)$, i.e. it vanishes at the origin. Let $\tilde{\theta}$ be the linear part of $\sum_{i=1}^k x_i(\theta_i - db_i)$ for (i), and $\tilde{\theta} = x_1c(0)dx_2$ for (ii). Then $d\tilde{\theta} = 0$. Replace θ by $\theta - \tilde{\theta} = O(2)$. Then $v_t = O(2)$ and $\phi_t = \mathrm{id} + O(2)$. ◻

LEMMA 2.2. *Let φ be a local biholomorphic mapping of J that preserves $\omega|_J$. Assume that φ commutes with $I_{\widehat{F}}$ and its linear part $\varphi'(0)$ commutes with $I_{\widehat{G}}$. Write*

$$\varphi\colon \begin{cases} \xi_1' = \xi_1 \widetilde{A}_1(\xi_1^2, x_2, {}'\xi, {}'x),\\ \xi_\alpha' = \widetilde{A}_\alpha(\xi_1^2, x_2, {}'\xi, {}'x), \quad 2 < \alpha \leq n,\\ x_k' = \widetilde{B}_k(\xi_1^2, x_2, {}'\xi, {}'x), \quad 2 \leq k \leq n,\end{cases}$$

where $\tilde{A}_1, \tilde{A}_\alpha, \tilde{B}_k$ are holomorphic functions. Then φ extends to a biholomorphic mapping φ_1 of $\widehat{F}$ such that φ_1 preserves $\omega|_{\widehat{F}}$ and has the linear part

$$\varphi_1'(0)\colon \begin{cases} \xi_1' = \mu\xi_1, \quad \xi_2' = \mu^2\xi_2, \quad \mu = \tilde{A}_1(0),\ \mu^3 = 1,\\ x_2' = B_2(x_2, {}'\xi, {}'x)\\ \quad = \mu x_2 + \sum_{2<\alpha\leq n}(a_\alpha\xi_\alpha + b_\alpha x_\alpha),\\ \xi_\alpha' = A_\alpha({}'\xi, {}'x), \quad x_\alpha' = B_\alpha({}'\xi, {}'x),\end{cases}$$

in which $B_2(x_2, {}'\xi, {}'x)$, $A_\alpha({}'\xi, {}'x)$, and $B_\alpha({}'\xi, {}'x)$ are the linear parts of $\tilde{B}_2(0, x_2, {}'\xi, {}'x)$, $\tilde{A}_\alpha(0, x_2, {}'\xi, {}'x)$ and $\tilde{B}_\alpha(0, x_2, {}'\xi, {}'x)$, respectively.

Proof. Since $\varphi'(0)$ commutes with $I_{\widehat{G}}\colon \xi_1' = -\xi_1, x_2' = x_2 - 2\xi_1, \xi_\alpha' = \xi_\alpha, x_\alpha' = x_\alpha$ then the linear parts of $\varphi I_{\widehat{G}} = I_{\widehat{G}}\varphi$ yield

$$\begin{aligned}
\widetilde{B}_2(0, x_2, {}'\xi, {}'x) &= \mu x_2 + \sum_{2<\alpha\leq n} a_\alpha \xi_\alpha + b_\alpha x_\alpha + O(2),\\
\widetilde{A}_\alpha(0, x_2, {}'\xi, {}'x) &= A_\alpha({}'\xi, {}'x) + O(2), \quad 2 < \alpha \leq n,\\
\widetilde{B}_\alpha(0, x_2, {}'\xi, {}'x) &= B_\alpha({}'\xi, {}'x) + O(2).
\end{aligned}$$

We have

$$\begin{aligned}
(2.2)\qquad d\xi_1^2 \wedge dx_2 + \sum_{2<\alpha\leq n} d\xi_\alpha \wedge dx_\alpha &= d(\xi_1 \widetilde{A}_1(\xi_1^2, x_2, {}'\xi, {}'x))^2 \wedge d\widetilde{B}_2(\xi_1^2, x_2, {}'\xi, {}'x)\\
&+ \sum_{2<\alpha\leq n} d\widetilde{A}_\alpha(\xi_1^2, x_2, {}'\xi, {}'x) \wedge d\widetilde{B}_\alpha(\xi_1^2, x_2, {}'\xi, {}'x).
\end{aligned}$$

Note that the last summation does not contribute $\xi_1 d\xi_1 \wedge dx_2$ due to the absence of x_2 in the linear parts of $\widetilde{A}_j, \widetilde{B}_j$ for $2 < \alpha \leq n$. Comparing the coefficients of $\xi_1 d\xi_1 \wedge dx_2$ gives us $\mu^3 = 1$. Define φ_1 by

$$\begin{cases}
\xi_1' = \xi_1 \widetilde{A}_1(\xi_2, x_2, {}'\xi, {}'x),\\
\xi_2' = \xi_2 \widetilde{A}_1^2(\xi_2, x_2, {}'\xi, {}'x),\\
\xi_\alpha' = \widetilde{A}_\alpha(\xi_2, x_2, {}'\xi, {}'x),\\
x_k' = \widetilde{B}_k(\xi_2, x_2, {}'\xi, {}'x), \quad 2 \leq k \leq n.
\end{cases}$$

Replacing ξ_1^2 by ξ_2 in (2.2), we obtain $\varphi_1^*\omega|_{\widehat{F}} = \omega|_{\widehat{F}}$. □

Lemma 2.3. *Let $\{F_j, G_j\}, j = 1, 2$ be two pairs of holomorphic glancing hypersurfaces with $J_j = F_j \cap G_j$. Let $\varphi\colon J_1 \to J_2$ be a local biholomorphic mapping satisfying $\varphi^*(\omega|_{J_2}) = \omega|_{J_1}$, $I_{F_2} = \varphi I_{F_1} \varphi^{-1}$ and $I_{G_2} = \varphi I_{G_1} \varphi^{-1}$. Then φ extends to a holomorphic symplectic mapping on $\mathbf{C}^{2n}$, sending F_1, G_1 into F_2, G_2, respectively.*

Proof. By Melrose's preliminary normalization (see also [3]) and by two changes of symplectic coordinates, we may assume that $F_1 = F_2 = \widehat{F}\colon x_1 = 0$, and

$$G_j\colon \xi_2 = \xi_1^2 + x_1 b_j(\xi_2, \dots, \xi_n, x), \quad b_j(0) = 1.$$

By applying Lemma 2.2 twice, we first extend φ to some biholomorphic map $\varphi_1\colon F_1 \to F_2$ and to some biholomorphic map $\varphi_2\colon G_1 \to G_2$ satisfying $\varphi_1^*\omega|_{F_2} = \omega|_{F_1}$ and $\varphi_2^*\omega|_{G_2} = \omega|_{G_1}$. We then extend φ_1, φ_2 to some biholomorphic mapping φ_3 on $\mathbf{C}^{2n}$. The existence of such extension φ_3 is elementary, which can be verified by two changes of holomorphic coordinates sending both $\{F_1, G_1\}$ and $\{F_2, G_2\}$ to $x_1 = 0$ and $x_2 = 0$.

Let $\omega = d\xi_1 \wedge dx_1 + \cdots + d\xi_n \wedge dx_n$ and $\widetilde{\omega} = \varphi_3^{-1*}\omega$. We want to show that $(1-t)\omega + t\widetilde{\omega}$ is non-degenerate at the origin. At the origin, i.e. as 2-forms on $T_0\hat{F} \times T_0\hat{F}$ we have $\omega|_{\hat{F}} = d\xi_2 \wedge dx_2 + \cdots + d\xi_n \wedge dx_n$. Since $\tilde{\omega} = \varphi_3^{-1*}\omega = \omega$ on $\hat{F}$, then $\tilde{\omega} = \omega + dx_1 \wedge \theta$ at $0 \in \mathbf{C}^{2n}$, where θ is a 1-from with constant coefficients. Note that $T_0 G_j = T_0\hat{G}$. Hence $\tilde{\omega} = \omega$, i.e., $dx_1 \wedge \theta = 0$ on $T_0\hat{G} \times T_0\hat{G}$. $T_0\hat{G} \subset T_0\mathbf{C}^{2n}$ is given by $d(\xi_2 - x_1) = 0$. We obtain $dx_1 \wedge \theta = c\,dx_1 \wedge d(\xi_2 - x_1) = c\,dx_1 \wedge d\xi_2$ for some constant c. It is obvious that $t\tilde{\omega}+(1-t)\omega = d\xi_2\wedge d(x_2-tcx_1)+\sum_{j\neq 2} d\xi_j\wedge dx_j$ is non-degenerate.

By Lemma 2.1 (ii) there is a holomorphic map φ_4 fixing F_2 pointwise and sending G_2 into itself so that $\varphi_4^*\omega = \varphi_3^{-1*}\omega$. Now $\varphi_4\varphi_3$ is a holomorphic symplectic extension of φ, transforming F_1 into F_2 and G_1 into G_2. □

We now prove a realization result.

PROPOSITION 2.4. *Let $J=\widehat{F}\cap\widehat{G}$ and $K\subset J\colon \xi_1=0$. Let $\omega|_J = d\xi_1^2\wedge dx_2 + d\xi_3\wedge dx_3+\cdots+d\xi_n\wedge dx_n$. Let I_1, I_2 be a pair of holomorphic involutions on J satisfying $I_j\neq \mathrm{id}$, $I_2\neq I_1+O(2)$, $I_j|_K=\mathrm{id}$ and $I_j^*\omega|_J=\omega|_J$. Then $\{I_1,I_2\}$ is the pair of involutions of some glancing holomorphic hypersurfaces F,G.*

Proof. The realization is outlined as follows: We shall first find φ_1 which is symplectic on $\mathbf{C}^{2n}$, preserves J and its restriction to J transforms I_1 into $I_{\widehat{F}}$. Then F is the pull-back of $\hat{F}$ by φ_1. Construct G in the same way. We will verify that F,G form a pair of glancing hypersurfaces with $F\cap G=J$, by ensuring $T_0F=T_0\hat{F}$ and $T_0G=T_0\hat{G}$.

Let $I=I_1$. The linear part $\widehat{I}$ of I fixes K pointwise. So $\widehat{I}$ is given by

$$\xi_1'=a\xi_1,\ x_2'=x_2+b\xi_1,\ \xi_\alpha'=\xi_\alpha+p_\alpha\xi_1,\ x_\alpha'=x_\alpha+q_\alpha\xi_1,\ 2<\alpha\leq n.$$

Since I preserves $\omega|_J$, $\widehat{I}=I'(0)$ preserves $d\xi_3\wedge dx_3+\cdots+d\xi_n\wedge dx_n$, i.e.

$$\sum_{2<\alpha\leq n} d(\xi_\alpha+p_\alpha\xi_1)\wedge d(x_\alpha+q_\alpha\xi_1)=\sum_{2<\alpha\leq n} d\xi_\alpha\wedge dx_\alpha,$$

which implies that $p_\alpha=q_\alpha=0$. Now coefficients of $I^*\omega|_J=\omega|_J$ that are linear in $\xi_1,x_2,\xi_\alpha,x_\alpha$ give us

$$d(a\xi_1)^2\wedge d(x_2+b\xi_1)+\sum_{\alpha=3}^n(d\xi_\alpha\wedge\theta_\alpha+dx_\alpha\wedge\theta_\alpha')=d\xi_1^2\wedge dx_2,$$

which implies that $a^2=1$. Since $I^2=\mathrm{id}\neq I$ then $a=-1$ and the linear part of I_1 is $\xi_1'=-\xi_1, x_2'=x_2+b_1\xi_1, \xi_\alpha'=\xi_\alpha, x_\alpha'=x_\alpha$. This also shows that the linear part of I_2 is $\xi_1'=-\xi_1, x_2'=x_2+b_2\xi_1, \xi_\alpha'=\xi_\alpha, x_\alpha'=x_\alpha$.

By a change of coordinates $\xi_1'=\xi_1, x_2'=x_2+c_1\xi_1, \xi_\alpha'=\xi_\alpha, x_\alpha'=x_\alpha$, one may assume that $b_1=0$. Then $b_2\neq 0$, since linear parts of I_1,I_2 are distinct. By a further change of coordinates of the form $\xi_1'=c\xi_1, x_2'=c^{-2}x_2, \xi_\alpha'=\xi_\alpha, x_\alpha'=x_\alpha$, we obtain $b_2=-2$. Note that both changes of coordinates for J extend to maps preserving ω. For the first map is the restriction of $\xi_1'=\xi_1, x_1'=x_1+c_1(\xi_2-\xi_1^2), \xi_2'=\xi_2, x_2'=x_2+c_1\xi_1$, $'x'='x$ to $J=\hat{F}\cap\hat{G}$. The second map is the restriction of $\xi_1'=c\xi_1, x_1'=c^{-1}x_1, \xi_2'=c^2\xi_2, x_2'=c^{-2}x_2$, $'\xi'='\xi$, $'x'='x$ to J.

Therefore, we may assume that I_1,I_2 are tangent to $I_{\widehat{F}}, I_{\widehat{G}}$, respectively.

Return to $I_1=\widehat{I}+O(2)$ with $\widehat{I}=I_{\widehat{F}}$. On J define $\psi_0=(\widehat{I}I+\mathrm{id})/2$. Then $\widehat{I}\psi_0=\psi_0 I$. Since $\psi_0=\mathrm{id}+O(2)$ fixes $K\subset J\colon\xi_1=0$ pointwise, then

$$\psi_0^{-1}\colon \begin{cases} \xi_j'=\xi_j+\xi_1A_j, & A_j(0)=0, \quad j=1,3,\dots n,\\ x_j'=x_j+\xi_1B_j, & B_j(0)=0, \quad j=2,\dots,n, \end{cases} \tag{2.3}$$

where A_j, B_j are convergent power series in $\xi_1, x_2, \xi_\alpha, x_\alpha$. Let $\widetilde{\omega} = \psi_0^{-1*}\omega|_J$. Then

$$\begin{aligned}\widetilde{\omega} &= d(\xi_1 + \xi_1 A_1)^2 \wedge d(x_2 + \xi_1 B_2) + \sum_{2<\alpha\le n} d(\xi_\alpha + \xi_1 A_\alpha) \wedge d(x_\alpha + \xi_1 B_\alpha) \\ &= \xi_1\omega_0 + d\xi_1 \wedge \sum_{2<\alpha\le n} (p_\alpha d\xi_\alpha + q_\alpha dx_\alpha) + \sum_{2<\alpha\le n} d\xi_\alpha \wedge dx_\alpha,\end{aligned} \tag{2.4}$$

where $p_\alpha = -B_\alpha|_{\xi_1=0}, q_\alpha = A_\alpha|_{\xi_1=0}$. Since $I^*\omega|_J = \omega|_J$ then $\widehat{I}^*\widetilde{\omega} = \widetilde{\omega}$. Hence $p_\alpha = q_\alpha = 0$ and $\widehat{I}^*\omega_0 = -\omega_0$. The former implies that $A_\alpha = \xi_1\tilde{A}_\alpha$ and $B_\alpha = \xi_1\tilde{B}_\alpha$ for $\alpha > 2$, and the latter implies that

$$\begin{aligned}\xi_1\omega_0 &= d\xi_1^2 \wedge \{\sum_{j\ge 2} a_j(\xi_1^2, x_2, {}'\xi, {}'x)dx_j + \sum_{j>2} b_j(\xi_1^2, x_2, {}'\xi, {}'x)d\xi_j\} \\ &+ \ \xi_1^2\{\sum_{i>2,j>1} \gamma_{ij}(\xi_1^2, x_2, {}'\xi, {}'x)d\xi_i \wedge dx_j + \sum_{i>j>2} \gamma'_{ij}(\xi_1^2, x_2, {}'\xi, {}'x)d\xi_i \wedge d\xi_j \\ &+ \sum_{i>j>1} \gamma''_{ij}(\xi_1^2, x_2, {}'\xi, {}'x)dx_i \wedge dx_j\}.\end{aligned}$$

Looking at (2.4) again, we see that

$$a_\alpha(0) = \tilde{A}_\alpha(0), \quad b_\alpha(0) = -\tilde{B}_\alpha(0), \quad a_2(0) = (1 + A_1(0))^2 = 1.$$

This shows that using the two-to-one branched covering $T\colon (\xi_2', x_2', {}'\xi', {}'x') = (\xi_1^2, x_2, {}'\xi, {}'x)$ we can write $\tilde{\omega} = T^*\omega_1$, where

$$\begin{aligned}\omega_1(\xi_2, x_2, {}'\xi, {}'x) \equiv\ & d\xi_2 \wedge \{\sum_{j\ge 2} a_j(\xi_2, x_2, {}'\xi, {}'x)dx_j + \sum_{j>2} b_j(\xi_2, x_2, {}'\xi, {}'x)d\xi_j\} \\ &+ \ \xi_2\{\sum_{i>2,j>1} \gamma_{ij}(\xi_2, x_2, {}'\xi, {}'x)d\xi_i \wedge dx_j + \sum_{i>j>2} \gamma'_{ij}(\xi_2, x_2, {}'\xi, {}'x)d\xi_i \wedge d\xi_j \\ &+ \sum_{i>j>1} \gamma''_{ij}(\xi_2, x_2, {}'\xi, {}'x)dx_i \wedge dx_j\} + d\xi_3 \wedge dx_3 + \cdots + d\xi_n \wedge dx_n \\ &= d\xi_2 \wedge dx_2 + \cdots + d\xi_n \wedge dx_n \\ &\quad + d\xi_2 \wedge (\sum_{\alpha>2} \tilde{A}_\alpha(0)dx_\alpha - \tilde{B}_\alpha(0)d\xi_\alpha) + e, \quad e|_{\xi_2=0} = 0\end{aligned}$$

and e is a 2-form in $\xi_2, x_2, {}'\xi, {}'x$ whose coefficients vanish at the origin. Let $\psi_1(\xi_2, {}'\xi, x_2, {}'x) = (\xi_2, {}'\xi, x_2 - \sum_{\alpha>2}(\tilde{A}_\alpha(0)x_\alpha - \tilde{B}_\alpha(0)\xi_\alpha), {}'x)$. Then $\psi_1^*\omega_1 = d\xi_2 \wedge dx_2 + \cdots + d\xi_n \wedge dx_n + \psi_1^* e = d\xi_2 \wedge dx_2 + \cdots + d\xi_n \wedge dx_n + 0(1)$. Since ψ_1 preserves $\xi_2 = 0$ and $e|_{\xi_2=0} = 0$ then $\psi_1^*\omega_1|_{\xi_2=0} = (d\xi_2 \wedge dx_2 + \cdots + d\xi_n \wedge dx_n)|_{\xi_2=0}$. By the result of Givental' (Lemma 2.1 (i)), there exists a biholomorphic mapping ψ_2 on $\mathbf{C}^{2n-2}$ such that $\psi_2^*\psi_1^*\omega_1 = d\xi_2 \wedge dx_2 + \cdots + d\xi_n \wedge dx_n$. Moreover, ψ_2 is tangent to the identity and fixes $\xi_2 = 0$ pointwise. Thus we can write

$$\psi_1\psi_2\colon \begin{cases} \xi_2 = \xi_2 u_2^2(\xi_2, x_2, {}'\xi, {}'x), \\ \xi_j' = \xi_j + \xi_2 u_j(\xi_2, x_2, {}'\xi, {}'x), \quad j > 2, \\ x_2' = x_2 - \sum_{\alpha>2}(\tilde{A}_\alpha(0)x_\alpha - \tilde{B}_\alpha(0)\xi_\alpha) + \xi_2 v_2(\xi_2, x_2, {}'\xi, {}'x), \\ x_j' = x_j + \xi_2 v_j(\xi_2, x_2, {}'\xi, {}'x), \quad j > 2 \end{cases}$$

with $u_2(0)=1, u_3(0)=\cdots=u_n(0)=v_2(0)=\cdots=v_n(0)=0$. Define $\psi_3\colon J\to J$ by

$$\psi_3\colon \begin{cases} \xi_1'=\xi_1u_2(\xi_1^2,x_2,{}'\xi,{}'x),\\ \xi_j'=\xi_j+\xi_1^2u_j(\xi_1^2,x_2,{}'\xi,{}'x), \quad j>2,\\ x_2'=x_2-\sum_{\alpha>2}(\tilde A_\alpha(0)x_\alpha-\tilde B_\alpha(0)\xi_\alpha)+\xi_1^2v_2(\xi_1^2,x_2,{}'\xi,{}'x),\\ x_j'=x_j+\xi_1^2v_j(\xi_1^2,x_2,{}'\xi,{}'x), \quad j>2.\end{cases}$$

Recall the map $T\colon(\xi_2',x_2',{}'\xi',{}'x')=(\xi_1^2,x_2,{}'\xi,{}'x)$. Then $\psi_1\psi_2T=T\psi_3$. Now $\psi_2^*\psi_1^*\omega_1=d\xi_2\wedge dx_2+\cdots+d\xi_n\wedge dx_n$ and $\tilde\omega=T^*\omega_1$ imply that $\psi_3^*\psi_0^{-1*}\omega|_J=\psi_3^*\tilde\omega=\psi_3^*T^*\omega_1=T^*\psi_2^*\psi_1^*\omega_1=T^*(d\xi_2\wedge dx_2+\cdots+d\xi_n\wedge dx_n)=\omega|_J$. Return to (2.3) and recall that $A_\alpha=\xi_1\tilde A_\alpha$ and $B_\alpha=\xi_1\tilde B_\alpha$ for $\alpha>2$. We extend ψ_0^{-1} to $\mathbf{C}^{2n}$ by

$$\tilde\psi_0^{-1}\colon \begin{cases} \xi_1'=\xi_1+\xi_1A_1(\xi_1,{}'\xi,x_2,{}'x), \quad x_1'=x_1,\\ \xi_2'=\xi_2(1+A_1(\xi_1,{}'\xi,x_2,{}'x))^2,\\ x_2'=x_2+\xi_1B_2(\xi_1,{}'\xi,x_2,{}'x),\\ \xi_\alpha'=\xi_\alpha+\xi_2\tilde A_\alpha(\xi_1,{}'\xi,x_2,{}'x),\\ x_\alpha'=x_\alpha+\xi_2\tilde B_\alpha(\xi_1,{}'\xi,x_2,{}'x).\end{cases}$$

Extend ψ_3 to $\tilde\psi_3$ in $\mathbf{C}^{2n}$ by

$$\tilde\psi_3\colon \begin{cases} \xi_1'=\xi_1u_2(\xi_1^2,x_2,{}'\xi,{}'x), \quad x_1'=x_1,\\ \xi_2=\xi_2u_2^2(\xi_1^2,x_2,{}'\xi,{}'x),\\ x_2'=x_2-\sum_{\alpha>2}(\tilde A_\alpha(0)x_\alpha-\tilde B_\alpha(0)\xi_\alpha)+\xi_1^2v_2(\xi_1^2,x_2,{}'\xi,{}'x),\\ \xi_j'=\xi_j+\xi_1^2v_j(\xi_1^2,x_2,{}'\xi,{}'x), \quad j>2,\\ x_j'=x_j+\xi_1^2v_j(\xi_1^2,x_2,{}'\xi,{}'x), \quad j>2.\end{cases}$$

Recall that $A_1(0)=B_2(0)=0$ and $u_2(0)=1$. A simple computation shows that

$$\tilde\psi_3^*\tilde\psi_0^{-1*}\omega=\omega+O(1).$$

Since $\tilde\psi_3,\tilde\psi_0$ are extensions, we still have $\tilde\psi_3^*\tilde\psi_0^{-1*}\omega|_J=\omega|_J$. By the result of Givental' (Lemma 2.1 (i)), there exists a biholomorphic mapping $\psi_4=\mathrm{id}+O(2)$ on $\mathbf{C}^{2n}$ such that ψ_4 fixes J pointwise and $\psi_4^*\tilde\psi_3^*\tilde\psi_0^{-1*}\omega=\omega$.

Set $\varphi_1=\psi_4^{-1}\tilde\psi_3^{-1}\tilde\psi_0$. Since ψ_4 is tangent to the identity, looking at the above formulas of $\tilde\psi_0,\tilde\psi_3$ we conclude that $F=\varphi_1^{-1}(\hat F)$ is tangent to $\hat F\colon x_1=0$. Since $\psi_4|_J=\mathrm{id}$ and $\tilde\psi_3|_J=\psi_3$ commute with $I_{\hat F}$ and $\tilde\psi_0|_J=\psi_0$ transforms I into $I_{\hat F}$, we have $\varphi_1I\varphi_1^{-1}=I_{\hat F}$. It is obvious that $I_F=I_1$, for any $\tilde G$ such that $F,\tilde G$ form a pair of glancing hypersurfaces with $F\cap\tilde G=J$.

Applying the above to I_2, we find G, tangent to $\hat G$, such that $I_G=I_2$ for any $\tilde F$ such that $\tilde F,G$ form a pair of glancing hypersurfaces with $\tilde F\cap G=J$.

Let us show that F,G form a pair of glancing hypersurface. We have $J\subset F\cap G$. Let f,g with $df\neq0,dg\neq0$ be some defining functions of F,G respectively. Let $\hat f,\hat g$ be the defining functions of $\hat F,\hat G$ respectively. Since f,g vanish on $J=\hat F\cap\hat G$, then $f=a\hat f+b\hat g$ and $g=c\hat f+d\hat g$. Since F is tangent to $\hat F$, then $b(0)=0$. Also $c(0)=0$. Without loss of generality, we may assume that $f=\hat f+b\hat g$ and $g=\hat g+c\hat f$. Since

$b\hat{g} = O(2)$ and $c\hat{f} = O(2)$, then at the origin we have $df \wedge dg = d\hat{f} \wedge d\hat{g} \neq 0$. Recall that $\{f, g\} = X_f g$. At the origin, we have $\{f, g\} = \{\hat{f} + b\hat{g}, \hat{g} + c\hat{f}\} = \{\hat{f}, \hat{g}\} = 0$, and

$$\begin{aligned}\{f, \{f, g\}\} &= \{\hat{f}, \{\hat{f}, \hat{g}\}\} + \{\hat{f}, \{\hat{f}, c\hat{f}\} + \{b\hat{g}, \hat{g}\}\} \\ &= 2 + \{\hat{f}, \hat{f}\{\hat{f}, c\} - \hat{g}\{\hat{g}, b\}\} \\ &= 2 + \hat{f}\{\hat{f}, \{\hat{f}, c\}\} - \{\hat{f}, \hat{g}\}\{\hat{g}, b\} - \hat{g}\{\hat{f}, \{\hat{g}, b\}\} = 2.\end{aligned}$$

A similar computation shows $\{g, \{g, f\}\}(0) = -2$. □

3. Realizing moduli functions for pairs of involutions ($n = 2$). The realization for moduli functions by pairs of involutions is essentially contained in [7], with some obvious changes. The one-dimension case is due to Malgrange [4].

Let $V_{\alpha,\beta,r} = \{x\colon \arg x \in (\alpha, \beta), 0 < |x| < r\} \subset \mathbf{C}^2$ and $S_{\alpha,\beta,r} = V_{\alpha,\beta,r} \times \Delta_r \subset \mathbf{C}^2$, where $\beta - \alpha < 2\pi$ is called the opening of $V_{\alpha,\beta,r}$ or $S_{\alpha,\beta,r}$. A *semi-formal* power series $F(x, y)$ on $S_{\alpha,\beta,r}$ is a formal power series in x whose coefficients are holomorphic in y on disc Δ_r. A holomorphic function f defined on $S_{\alpha,\beta,r}$ is said to admit an *asymptotic expansion* by a semi-formal power series $F(x, y) = \sum_k F_k(y)x^k$, denoted by $f \sim F$, if for each positive integer N

$$\lim_{V_{\alpha,\beta,r} \ni x \to 0} x^{-N}\{f(x, y) - \sum_{k=0}^{N} F_k(y)x^k\} = 0$$

uniformly for $|y| < r'$ for some $0 < r' < r$. We say that a holomorphic map H on $S_{\alpha,\beta,r}$ admits an asymptotic expansion Φ of semi-formal map if each component of Φ is the asymptotic expansion of the corresponding component of H on $S_{\alpha,\beta,r}$. It is an elementary result that if a holomorphic map H is asymptotic to the identity map on $S_{\alpha,\beta,r}$. Then for each $\epsilon \in (0, \frac{\beta-\alpha}{6})$ there exists $0 < r' < r$ such that $H\colon S_{\alpha+2\epsilon,\beta-2\epsilon,r'/2} \to S_{\alpha+\epsilon,\beta-\epsilon,r'}$ is injective and $H(S_{\alpha+2\epsilon,\beta-2\epsilon,r'/2}) \supset S_{\alpha+3\epsilon,\beta-3\epsilon,r'/4}$ (see [1]).

Let $0 < \alpha < \frac{\pi}{32}$. Consider 4 sectorial domains $S_{j\,j+1} = S_{\alpha_j,\beta_j,r}$ with

$$\begin{aligned}\alpha_1 &= -\frac{\pi}{2} + 2\alpha, \quad \beta_1 = \frac{\pi}{2} - 2\alpha, \quad \alpha_2 = -\frac{\pi}{2} - \alpha, \quad \beta_2 = -\frac{\pi}{2} + \alpha, \\ \alpha_3 &= -\frac{3\pi}{2} + 2\alpha, \quad \beta_3 = -\frac{\pi}{2} - 2\alpha, \quad \alpha_4 = -\frac{3\pi}{2} - \alpha, \quad \beta_4 = -\frac{3\pi}{2} + \alpha.\end{aligned} \tag{3.1}$$

Let $H_{j\,j+1} (= H_{j+4\,j+5})$ be holomorphic maps which are asymptotic to the identity on $S_{j\,j+1}$. Suppose also that

$$H_{j\,j+1}\hat{\sigma} = \hat{\sigma}H_{j\,j+1}, \quad H_{j+2\,j+3} = IH_{j\,j+1}I, \tag{3.2}$$
$$\hat{\sigma}(x, y) = (x, y + 2x),\ I(x, y) = (-x, y),\ \rho(x, y) = (\overline{x}, \overline{y}),$$
$$H^*_{j\,j+1}dx^2 \wedge dy = dx^2 \wedge dy, \tag{3.3}$$
$$H_{1\,2} = \rho H_{1\,2}\rho, \quad H_{2\,3} = \rho H_{4\,1}\rho. \tag{3.4}$$

By the realization, we mean a biholomorphic map σ, defined in a neighborhood of the origin in J and satisfying $\sigma = \hat{\sigma} + O(2)$, $\sigma|_{x=0} = \mathrm{id}$, $\sigma = I\sigma^{-1}I = \rho\sigma\rho$ and $\sigma^* dx^2 \wedge dy = dx^2 \wedge dy$, and biholomorphic mappings H_j defined on some sectorial domains and satisfying $H_j^{-1}\sigma H_j = \hat{\sigma}$, $H_2 = \rho H_1 \rho$, $H_4 = \rho H_3 \rho$, and $H_{j+2} = IH_jI, H_j^* dx^2 \wedge dy = dx^2 \wedge dy$. Moreover, H_j are asymptotic to the same semi-formal biholomorphic map Φ, and finally $H_j^{-1}H_{j+1} = H_{j\,j+1}$ on a sectorial domain $S_{\alpha'_j,\beta'_j,r'}$ with opening shrunk slightly from the sectorial domain $S_{\alpha_j,\beta_j,r}$ on which $H_{j\,j+1}$ is defined and with $0 < r' \leq r$. Without the reality conditions (3.4), one drops $\rho\sigma\rho = \sigma$, $H_2 = \rho H_1 \rho$ and $H_4 = \rho H_3 \rho$.

Fix $0 < \epsilon < \frac{\alpha}{20}$. Choose $0 < r_3 < r_2 < r_1$ sufficiently small such that the first component $h_{j\,j+1}$ of $H_{j\,j+1}$ satisfies $\arg\{x^{-1}h_{j\,j+1}(x,y)\} < \epsilon$ on S_{α_j,β_j,r_1},

$$H_{j\,j+1}\colon A_j \equiv S_{\alpha_j+2\epsilon,\beta_j-2\epsilon,r_2} \to \widetilde{C}_j \equiv H_{j\,j+1}(A_j) \tag{3.5}$$

is biholomorphic and A_j is now the domain of $H_{j\,j+1}$. Moreover,

$$S_{\alpha_j+3\epsilon,\beta_j-3\epsilon,r_3} \subset \tilde{C}_j \subset S_{\alpha_j+\epsilon,\beta_j-\epsilon,r_1}. \tag{3.6}$$

Set $\alpha_0 = \alpha_4 + 2\pi$ and $\beta_0 = \beta_4 + 2\pi$. For $j = 1, 2, 3, 4$, let $S_j = A_{j-1} \cup B_j \cup \tilde{C}_j$ with $B_j = S_{\alpha_j+3\epsilon,\beta_{j-1}-3\epsilon,r_3}$. Let X_0 be the disjoint union $\sqcup_{j=1}^4 S_j$. We identify $p \in A_j$ with $H_{j\,j+1}(p) \in \widetilde{C}_j$, which defines an equivalence relation on X_0 since $\widetilde{C}_j$ does not intersect A_k for $k \neq j \mod 4$ by the choice of ϵ and by (3.5) and (3.6). Let X be the quotient space of X_0 by the equivalence relation, and $\pi\colon X_0 \to X$ be the projection. So $U \subset X$ is open if and only if $\pi^{-1}(U) \cap S_j$ are open for all j; in particular, if V is open in S_j then $\pi^{-1}(\pi(V)) = V \cup H_{j-1\,j}(V \cap A_{j-1}) \cup H_{j\,j+1}^{-1}(V \cap \widetilde{C}_j)$ is open and hence $\pi(V)$ is open. We need to show that X is Hausdorff. Let p, q be in X_0 with $\pi(p) \neq \pi(q)$. If p, q are in the same S_j, take disjoint open sets $U_p \ni p$, $U_q \ni q$ in S_j. Since $H_{j\,j+1}$ is one-to-one then $\pi(U_p), \pi(U_q)$ are also disjoint open sets. If p is in S_j and q is in S_k for $k \neq j, j-1, j+1 \mod 4$, then $\pi(S_j), \pi(S_k)$ separate p and q. Finally it remains to check the case that $p \in S_j$ and $q \in S_{j+1}$. If $q \in A_j$, then p and $H_{j\,j+1}(q)$ are both in S_j, which is reduced to a previous case. The same argument applies if $p \in \widetilde{C}_j$. Assume now that $p = (p_1, p_2)$ is in $S_j \setminus \widetilde{C}_j$ and $q = (q_1, q_2)$ is in $S_{j+1} \setminus A_j$. Since $|\arg\{x^{-1}h_{j\,j+1}(x,y)\}| < \epsilon$ on S_{α_j,β_j,r_1} and $|\arg\{q_1^{-1}p_1\}| > \epsilon$, we can choose open sets $U_p \ni p$ and $U_q \ni q$ such that $H_{j\,j+1}(U_q \cap A_j)$ does not intersect U_p. Therefore, $\pi(U_p) \cap \pi(U_q)$ is empty and X is Hausdorff.

Now X is a complex manifold with the coordinate map $\pi_j^{-1} = (x_j, y_j)$ defined on $\pi(S_j)$ and with value in $S_j \subset \mathbf{C}^2$, and we also have its inverse $\pi_j\colon S_j \hookrightarrow X_0 \xrightarrow{\pi} X$. Note that $H_{j\,j+1} = \pi_j^{-1}\pi_{j+1}$ on A_j. On $\pi(X_0/4)$ define $\tilde{\sigma}, \tilde{I}_1, \tilde{\omega}, \tilde{\rho}$ in coordinates as follows

$$\tilde{\sigma}\colon (x_j, y_j) \to (x_j, y_j + 2x_j), \quad \tilde{\omega} = dx_j^2 \wedge dy_j,$$
$$\widetilde{I}_1\colon (x_j, y_j) \to (x_{j+2}, y_{j+2}) = (-x_j, y_j),$$
$$\widetilde{\rho} = \widetilde{\rho}^{-1}\colon \begin{cases} (x_1, y_1) \to (x_2, y_2) = (\overline{x}_1, \overline{y}_1), \\ (x_3, y_3) \to (x_4, y_4) = (\overline{x}_3, \overline{y}_3). \end{cases}$$

Take a smooth non-negative smooth function $\chi_j(x,y) \equiv \chi_j(x/|x|)$ such that it equals 1 for $\arg x \in ((1-j)\frac{\pi}{2} + \epsilon, (2-j)\frac{\pi}{2} - \epsilon)$ and zero for $\arg x \notin ((1-j)\frac{\pi}{2} - \epsilon, (2-j)\frac{\pi}{2} + \epsilon)$, and such that $\chi_1 + \cdots + \chi_4 = 1$. Set $\chi_k(\pi_k(p)) = 0$ when $p \in X \setminus \pi(S_k)$ and define

$$K(p) = \sum_{k=1}^{4} \chi_k(x_k(p), y_k(p))(x_k(p), y_k(p)).$$

Then $K(X) = D \cap (\mathbf{C}^* \times \mathbf{C})$, where D is an open neighborhood of the origin in $\mathbf{C}^2$, and K is a diffeomorphism for possibly smaller r_2, r_3. Thus one gets a complex structure on $K(X)$ defined by $K_{j*}\frac{\partial}{\partial x_j}, K_{j^*}\frac{\partial}{\partial y_j}$, where $K_j \circ \pi_j^{-1} = K$ on $\pi(S_j)$. Note that $(x_k(p), y_k(p)) = \pi_k^{-1}(p) = H_{k\,j}(x_j(p), y_j(p))$ when $\chi_k(x_k(p), y_k(p))\chi_j(x_j(p), x_j(p)) \neq 0$. Thus

$$\begin{aligned} K_j(t) &= \sum_{k=1}^{4} \chi_k(H_{k\,j}(t))H_{k\,j}(t) \sim \sum_{k=1}^{4} \chi_k(t)H_{k\,j}(t) \\ &\sim \sum_{k=1}^{4} \chi_k(t)t = t, \quad t = (x_j(p), y_j(p)) \in S_j. \end{aligned}$$

Hence the complex structure extends to D and agrees with the standard one along $x = 0$ to infinitely order. By the Newlander-Nirenberg theorem, there is a diffeomorphism $\psi\colon \widetilde{D}(\subset D) \to \Omega \subset \mathbf{C}^2$ with $\psi(0) = 0$ such that ψK is biholomorphic. Now the inverse ψ^{-1}, expanded as formal power series in $x, \overline{x}$, is a formal power series in x only and has coefficients holomorphic in y in a fixed domain. Using a finite order Taylor expansion of ψ^{-1} if necessary, one may also assume that $\psi(x, y) = (x, y) + O(|x|^2)$. On $\Omega \cap (\mathbf{C}^* \times \mathbf{C})$ define $\sigma' = \psi K\widetilde{\sigma}K^{-1}\psi^{-1}$, $I' = \psi K\widetilde{I}_1K^{-1}\psi^{-1}, \omega' = (\psi K)^{-1*}dx^2 \wedge dy$ and $\rho' = \psi K\widetilde{\rho}K^{-1}\psi^{-1}$. Again, since $H_{j\,j+1} \sim \mathrm{id}$ then $\sigma', I_j', \omega', \rho'$ extend to Ω with $\sigma'(x, y) = \hat{\sigma}(x, y) + O(|x|^2)$ and $\hat{\sigma}(x, y) = (x, y + 2x)$, $I'(x, y) = I(x, y) + O(|x|^2), \omega' = A_0(x, y)dx^2 \wedge dy, A_0(0) = 1$ and $\rho'(x, y) = \rho(x, y) + O(|x|^2)$. We need to apply holomorphic changes of coordinates that are tangent to the identity and preserve $x = 0$ to transform $\{\sigma', I', \omega', \rho'\}$ into $\{\sigma, I, \omega, \rho\}$.

Let $\varphi_0 = (\mathrm{id} + \rho\rho')/2$. Then φ_0 is tangent to the identity and fixes $x = 0$ pointwise, and $\rho = \varphi_0\rho'\varphi_0^{-1}$. Put $I^+ = \varphi_0 I'\varphi_0^{-1}$ and $\sigma^+ = \varphi_0\sigma'\varphi_0^{-1}$. Note that $\tilde{I}\tilde{\rho} = \tilde{\rho}\tilde{I}$ implies that $I'\rho' = \rho' I'$. Hence $I^+\rho = \rho I^+$. Let $\varphi_1 = (\mathrm{id} + II^+)/2$. Then $\varphi_1\rho = \rho\varphi_1$ and $\varphi_1 I^+ = I\varphi_1$. Since $\tilde{I}^*\tilde{\omega} = \tilde{\omega} = \overline{\tilde{\rho}^*\tilde{\omega}}$, then $\omega_1 = (\varphi_1\varphi_0\psi K)^{-1*}\tilde{\omega}$ satisfies $I^*\omega_1 = \omega_1 = \overline{\rho^*\omega_1}$. Moreover, $\omega_1 = A_1(x, y)dx^2 \wedge dy$ with $A_1(0) = 1$. Thus $A_1(-x, y) = A_1(x, y) = \overline{A}_1(x, y)$. Hence $A_1(x, y)dx^2 \wedge dy = d(xA(x^2, y))^2 \wedge dy$ with $A(x, y) = \overline{A}(x, y)$ and $A(0) = 1$. Let $\varphi_2(x, y) = (xA(x^2, y), y)$. Then φ_2 preserves I, ρ. Now $\varphi_2^* dx^2 \wedge dy = \omega_1$.

Take $H_j = \varphi_2\varphi_1\varphi_0\psi K_j = \varphi_2\varphi_1\varphi_0\psi K\pi_j$, which is holomorphic on S_j. As formal power series in x, y, H_j preserves $x = 0$. On S_j recall that $K_j \sim \mathrm{id}$ and we have

$$H_j(t) = \varphi_2\varphi_1\varphi_0\psi K_j(t) \sim \varphi_2\varphi_1\varphi_0\hat{\psi}(t) \equiv \Phi(t) = \mathrm{id} + O(2), \quad t = (x_j, y_j),$$

where $\hat{\psi}(x, y)$ is the Taylor series expansion of $\psi(x, y)$ in $x, \overline{x}$. As mentioned above, $\hat{\psi}^{-1}(x, y)$ and hence $\hat{\psi}(x, y)$ is a power series in x only and whose coefficients are holomorphic in y on a fixed domain. Finally, $H_j^{-1}\sigma H_j = \hat{\sigma}$, $H_j^* dx^2 \wedge dy = dx^2 \wedge dy$, $IH_jI = H_{j+2}$, $\rho H_1\rho = H_2, \rho H_2\rho = H_4$, and $H_j^{-1}H_{j+1} = H_{j,j+1}$ on A_j. When the reality condition (3.4) is not imposed on $H_{j\,j+1}$, one drops the correction map φ_0 and all requirements involving anti-holomorphic involutions. The proof of the realization is complete.

Note that the realization for $H_{j\,j+1}$ is achieved by shrinking the openings of sectorial domains slightly. (The radius of the sectorial domains could be small.) In particular, if the opening of the sectorial domain is larger than $\frac{\pi}{2}$, the opening of the shrunk sectorial domain is still bigger than $\frac{\pi}{2}$.

Let us recall a special family of moduli functions [7]: $H_{j\,j+1}$ are defined on sectorial domains $S_{j\,j+1}$, and $H_{4\,1} = \mathrm{id} = H_{2\,3}$. And the opening of $S_{1\,2}, S_{3\,4}$ is $\pi - 4\alpha > \frac{\pi}{2}$ by fixing $\alpha < \frac{\pi}{32}$. In the real case, the last requirement is not needed. We still assume that $H_{j\,j+1}$ satisfy (3.2)-(3.4) (in the complex case we drop (3.4)).

Next we want to discuss the equivalence relation on moduli functions. Let σ and $\tilde\sigma$ be two realizations, constructed above, corresponding to $\{H_{j\,j+1}\}$ on $S_{j\,j+1} = S_{\alpha_j,\beta_j,r}$, $\{\tilde H_{j\,j+1}\}$ on $\tilde S_{j\,j+1} = S_{\tilde\alpha_j,\tilde\beta_j,\tilde r}$, respectively, where $\alpha_j, \beta_j, \tilde\alpha_j, \tilde\beta_j$ are of the form (3.1). We still assume that the openings of $S_{1\,2}, \tilde S_{1\,2}, S_{3\,4}, \tilde S_{3\,4}$ are bigger than $\frac{\pi}{2}$. Suppose also that $H_{2\,3} = H_{4\,1} = \tilde H_{2\,3} = \tilde H_{4\,1} = \mathrm{id}$. So there exist normalizing transformations H_j such that $H_j^{-1}\sigma H_j = \hat\sigma$. Moreover, $H_j^{-1}H_{j+1} = H_{j\,j+1}$ on $S_{j\,j+1}$ (by shrinking the opening slightly and by choosing a smaller radius). Also $\tilde H_j^{-1}\tilde H_{j+1} = \tilde H_{j\,j+1}$ on a sector $\tilde S_{j\,j+1}$. Assume now that $g\sigma g^{-1} = \tilde\sigma$ and $gI = Ig$. Then g preserves $x = 0$, since the latter is the set of fixed points of $\sigma, \tilde\sigma$. Write $g(x,y) = (xg_1(x,y), g_2(x,y))$. Let $g_1(0) = \mu = |\mu|e^{i\gamma}$. There are two cases: $\operatorname{Im}\mu \geq 0$ and $\operatorname{Im}\mu < 0$. When $\operatorname{Im}\mu \geq 0$, say $0 \leq \gamma \leq \frac{\pi}{2}$, we take sectorial domains $S_j^* = i^{1-j}S_1^*, \widetilde S_j^*$ with

$$S_1^* = \{x\colon -\epsilon < \arg x + \frac{\gamma}{2} < \frac{\pi}{2} + \epsilon, 0 < |x| < r'\} \times \Delta_{r'}, \ \widetilde S_j^* = \mu S_j^*. \tag{3.7}$$

Note that $S_1^* \subset S_1$ and $S_2^* \subset S_2 \cup S_3$, if ϵ and r' are sufficiently small. Also $S_3^* \subset S_3$ and $S_4^* \subset S_4 \cup S_1$. Since $H_1 = H_4$ and $H_2 = H_3$ we can define $H_j^* = H_j|_{S_j^*}$ and we still have $H_4^* = H_1^*$ and $H_2^* = H_3^*$ on the overlaps. We can also define $\tilde H_j^* = \tilde H_j|_{\tilde S_j^*}$. When $\frac{\pi}{2} \leq \gamma < \pi$, we take $S_j^* = i^{1-j}S_1^*, \widetilde S_j^*$ with

$$S_1^* = \{x\colon -\epsilon < \arg x - \frac{\gamma}{4} < \frac{\pi}{2} + \epsilon, 0 < |x| < r'\} \times \Delta_{r'}, \quad \widetilde S_{j+2}^* = \mu S_j^*. \tag{3.8}$$

We still define $H_j^* = H_j|_{S_j^*}$ and $\tilde H_j = \tilde H_j|_{\tilde S_j^*}$. With the above choice of $S_j^*, \tilde S_j^*$, the restriction of $H_{j\,j+1}$ to a possibly smaller intersection is still a transition function. Rename H_j^* by H_j and $\tilde H_j^*$ by $\tilde H_j$. We retain $H_{2\,3} = H_{4\,1} = \tilde H_{2\,3} = \tilde H_{4\,1} = \mathrm{id}$. When $\Im\mu < 0$, one can rearrange the intersections to meet this requirement (by reversing the roles of $H, \tilde H$). Recall $H_j\hat\sigma H_j^{-1} = \sigma$ on S_j^* and $H_{j\,j+1} = H_j^{-1}H_{j+1}$ on $S_j^* \cap S_{j+1}^*$. (As usual, it holds on a smaller sector.) Let $G_j = \tilde H_j^{-1}gH_j$ when $\Re\mu \geq 0$, and let $G_j = \tilde H_{j+2}^{-1}gH_j$ when $\Re\mu < 0$. Then for both cases of (3.7) and (3.8) we have $G_1 = G_4$, $G_2 = G_3$, and $IG_jI = G_{j+2}$. For the real case we have $G_2 = \rho G_1\rho, G_3 = \rho G_2\rho$ additionally. Then we get the equivalence relation

$$G_jH_{j\,j+1}G_{j+1}^{-1} = \tilde H_{j\,j+1}, \ \forall j; \quad \text{or} \quad G_jH_{j\,j+1}G_{j+1}^{-1} = \tilde H_{j+2\,j+3}, \forall j.$$

Recall that $I(x,y) = (-x,y)$, $\hat\sigma(x,y) = (x, y+2x)$, and $\rho(x,y) = (\overline{x}, \overline{y})$. To deal with mappings, defined on a sectorial domain $S = V \times \Delta_r$, that commute with $\hat\sigma(x,y) = (x, y+2x)$, it is convenient to work on the quotient space $S/\hat\sigma$ obtained by the projection $(x,t) = \pi(x,y) = (x, e^{\frac{\pi i y}{x}})$. More specifically, if H commutes with $\hat\sigma$ then it has the form $H(x,y) = (xa(x,y), ya(x,y) + b(x,y))$ with $a\hat\sigma = a$ and $b\hat\sigma = b$, which yields a mapping in the (x,t)-space defined for $x \in V$ and $e^{-\frac{\pi r}{|x|}} < |t| < e^{\frac{\pi r}{|x|}}$ by

$$\tilde H\colon x' = x\tilde a(x,t), \quad t' = t\lambda(x,t),$$

$$\tilde a(x,t) = a(x, \frac{x\log t}{\pi i}), \quad \lambda(x,t) = e^{d(x,t)}, \quad d(x,t) = \frac{\pi i b(x, \frac{x\log t}{\pi i})}{xa(a, \frac{x\log t}{\pi i})}.$$

When H is asymptotic to the identity on the sectorial domain $V \times \Delta_r$, such as a mapping $H_{j\,j+1}$ in $\{H_{j\,j+1}\}$, we have $|a(x,y)-1| < c|x|$ and $|y(a(x,y)-1)+b(x,y)| < c|x|^2$ for $x \in V \cap \Delta_\delta$ and $y \in \Delta_\epsilon$, which implies that

$$|d(x,t)| \leq 2\pi c|x| + 2\pi c|y| < \pi$$

for $|x|, |y|$ sufficiently small. Hence $\tilde{H}$ determines H uniquely. We will also consider mappings G, such as a mapping G_j appeared in the equivalence relation of moduli space, defined on a sectorial domain $V \times \Delta_r$, which commutes with $\hat{\sigma}$ and admits an asymptotic expansion $\Psi(x,y) = (xA(x,y), yA(x,y) + B(x,y))$, $A(0) \neq 0 = B(0)$. Note that the semi-formal map Ψ still commutes with $\hat{\sigma}$, so $A\hat{\sigma} = A$ and $B\hat{\sigma} = B$. However, G is not uniquely determined by $\tilde{G}$; $\tilde{G} = \tilde{G}'$ if and only if

$$A' = A, \quad B'(x,y) = B(x,y) + 2kxA(x,y), \quad k \in \mathbf{Z},$$

i.e. $G' = \hat{\sigma}^k G$. Therefore, the asymptotic expansion of G determines k; in particular, the equivalence class of $\{H_{j\,j+1}\}$ is determined by its equivalence class in the (x,t)-space. Of course, on the (x,t)-space the moduli functions $\{H_{j\,j+1}\}$ and mappings $\{G_j\}$ are required to satisfy asymptotic expansion conditions, and by definition those asymptotic expansion conditions mean the ones described in the (x,y)-space. Note that H, or G, preserves $dx^2 \wedge dy$ if and only if in the quotient space it preserves

$$dx^3 \wedge d\log t \equiv \hat{\omega}.$$

In (x,t)-space, define $I(x,t) = (-x, t^{-1})$, and $\rho(x,t) = (\overline{x}, \overline{t}^{\,-1})$. Then moduli functions $H_{j\,j+1}, j = 1, \ldots, 4$ will still satisfy the conditions (3.2) and (3.4) (with the new I and ρ). Condition (3.3) becomes $H_{j\,j+1}^* \hat{\omega} = \hat{\omega}$. For the above moduli functions, if they are equivalent by $\{G_j\}$ then $G_1 = G_4, G_2 = G_3$ satisfy

$$G_{j+2} = IG_jI, \quad G_2 = \rho G_1 \rho, \quad G_4 = \rho G_2 \rho, \quad G_j^* \hat{\omega} = \hat{\omega}.$$

Moreover, $G_j(x,t) = (xa_j(x), t\lambda_j(x))$, where $a_j(x)$ admits the same asymptotic expansion $a(x)$ with $a(0) \neq 0$, and $\lambda_j(x)$ admit the same asymptotic expansion $\lambda(x)$ with $\lambda(0) \neq 0$. (See [7], Corollary 3, p. 207.)

4. A family of non-equivalent glancing hypersurfaces. We want to show that the space of equivalence classes is infinite dimensional. We will also drop the 2-form in the equivalence relations for pairs of involutions. This is needed in order to obtain our results in higher dimension.

Recall 4 sectorial domains $S_{1\,2} = S_{1\,2}(\alpha, r) = S_{-\frac{\pi}{2}+2\alpha, \frac{\pi}{2}-2\alpha, r}$, $S_{2\,3} = S_{-\frac{\pi}{2}-\alpha, -\frac{\pi}{2}+\alpha, r}$, $S_{3\,4} = -S_{1\,2}$ and $S_{4\,1} = -S_{2\,3}$. We will choose $\alpha \in (0, \frac{\pi}{32})$ later.

We want to find a family of $\{H_{j\,j+1}\}$ on $S_{j\,j+1}$ in the (x,t)-space, which are not equivalent. We will take $H_{4\,1} = H_{2\,3} = \mathrm{id}$ and $H_{3\,4} = IH_{1\,2}I$. So we need only to describe $H_{1\,2}$. Now $H_{1\,2}$ needs to satisfy

$$H_{1\,2}^* \hat{\omega} = \hat{\omega}, \quad \hat{\omega} = dx^3 \wedge d\log t.$$

Also $H_{1\,2}$ must be asymptotic to the identity in the (x,y)-space, and for the real case we need $H_{1\,2} = \rho H_{1\,2} \rho$ additionally.

Complex case. Using the local generating function $x^3\log\hat t+\hat t p(x)e^{-\frac{1}{x}}$ with a meromorphic function $p(x)$ on $\mathbf{C}^*$, we want to define $H_{1\,2}=K$ and $K(x,t)=(\hat x,\hat t)$ by the identity

$$\log t\,dx^3+\hat x^3\,d\log\hat t=d\{x^3\log\hat t+\hat t p(x)e^{-1/x}\}.$$

Equivalently,

$$\hat x^3=x^3+\hat t p(x)e^{-1/x},$$
$$\log t=\log\hat t+\hat t p^*(x)e^{-1/x},\quad p^*(x)=\frac{1}{3x^4}p(x)+\frac{1}{3x^2}p'(x).$$

So K preserves $dx^3\wedge d\log t$, if K defines a biholomorphic map. We will consider meromorphic functions on $\mathbf{C}^*$ of the form

$$p(x)=\sum_{k=1}^{\infty}\frac{\epsilon_k}{(k^2x^2+1)^k},\quad 0<|\epsilon_k|<\frac{k^{2k}}{k!}. \tag{4.1}$$

Thus we need to find where K and K^{-1} are defined. We also need to find coefficients of its Laurent series expansion in t. We first rewrite the above identities as

$$\hat x=x(1+\hat t p_1(x)e^{-\frac{1}{x}})^{1/3},\quad p_1(x)=x^{-3}p(x), \tag{4.2}$$
$$\hat t=te^{-\hat t p^*(x)e^{-\frac{1}{x}}}. \tag{4.3}$$

If $|\arg x|<\frac{\pi}{2}-\alpha$ then $|k^2x^2+1|\geq k^2|x|^2\sin 2\alpha$. Hence for $|\epsilon_k|<\frac{\delta^{2k}k^{2k}}{(2k)!}$, we have

$$|p(x)|<e^{\frac{\delta}{|x|\sqrt{\sin 2\alpha}}},\quad |\arg x|<\frac{\pi}{2}-\alpha.$$

Fix $\alpha=\frac{\pi}{100}$. Note that $\frac{1}{|x|^N}\leq N!e^{\frac{1}{|x|}}$. There exists δ_* depending only on $\epsilon>0$ such that p is meromorphic on $\mathbf{C}^*$ and

$$\max\{|p_1(x)|,|p_1'(x)|,|p^*(x)|,|p^{*\prime}(x)|\}<|x|^3e^{\frac{\epsilon}{|x|}}, \tag{4.4}$$

if

$$|\arg x|<\frac{\pi}{2}-\alpha,\quad 0<|x|<r=r_\epsilon,\quad \epsilon_k<\frac{\delta_*^k k^{2k}}{(2k)!}.$$

Using identities (4.2)-(4.3), we first define a map K on $\{(x,y)\colon 0<|x|<r, |\arg x|<\frac{\pi}{2}-\alpha, e^{-\frac{\epsilon}{|x|}}<|t|<e^{\frac{\epsilon}{|x|}}\}$, and a map K^{-1} on $\{(x,y)\colon 0<|x|<r, |\arg x|<\frac{\pi}{2}-2\alpha, e^{-\frac{\epsilon}{|x|}}<|t|<e^{\frac{\epsilon}{|x|}}\}$ for some positive constant r, where r is sufficiently small but dependent of ϵ. The two maps are inverses of each other, when restricted to suitable sectorial domains. We take $\epsilon=\frac{\sin\alpha}{100}$ such that $|e^{\frac{100\epsilon}{|x|}-\frac{1}{x}}|<1$ for $|\arg x|<\frac{\pi}{2}-\alpha$.

Let us start with equation (4.3). By the contraction map theorem, for some small $r_0>0$ the equation $T=e^{-\omega T}$ admits a unique solution $T=T(\omega)$ which is holomorphic in ω for $|\omega|<r_0$, by requiring $|T|<8$. Note that

$$T=T(\omega)=1-\omega+O(\omega^2), \tag{4.5}$$
$$|T(\omega)-1|=|e^{-\omega T(\omega)}-1|\leq\frac{|\omega T(\omega)|}{1-|\omega T(\omega)|}\leq 2|\omega T(\omega)|.$$

Hence (4.3) admits a unique solution

$$\hat{t} = tT(tp^*(x)e^{-\frac{1}{x}}) \tag{4.6}$$

with $|T(tp^*(x)e^{-\frac{1}{x}}) - 1| \leq 16|tp^*(x)e^{-\frac{1}{x}}| \leq |e^{-\frac{1}{x}+\frac{2\epsilon}{|x|}}|$ for $|t| < e^{\frac{\epsilon}{|x|}}$. Substituting $tT(tp^*(x)e^{-\frac{1}{x}})$ for $\hat{t}$ in (4.2), we get

$$\hat{x} = x[1 + tT(tp^*(x)e^{-\frac{1}{x}})p_1(x)e^{-\frac{1}{x}}]^{1/3}.$$

Also $|[1 + tT(tp^*(x)e^{-\frac{1}{x}})p_1(x)e^{-\frac{1}{x}}]^{1/3} - 1| \leq |tT(tp^*(x)e^{-\frac{1}{x}})p_1(x)e^{-\frac{1}{x}}| < |e^{-\frac{1}{x}+\frac{2\epsilon}{|x|}}|$. Therefore, K is defined on

$$S_{\alpha,r,\epsilon} \equiv \{(x,t)\colon |x| < r, |\arg x| < \frac{\pi}{2} - \alpha, |t| < e^{\frac{\epsilon}{|x|}}\}.$$

Moreover, $K(S_{\alpha,r,\epsilon}) \subset S_{\alpha/2,2r,2\epsilon}$, if $r < r_0$.

From (4.6) and (4.5) we get

$$\hat{t} = t(1 - p^*(x)e^{-\frac{1}{x}}t + O(t^2e^{-\frac{2}{x}+\frac{2\epsilon}{|x|}})), \tag{4.7}$$

where $O(t^2e^{-\frac{2}{x}+\frac{2\epsilon}{|x|}})$ stands for a term with absolute value bounded by $c|t^2e^{-\frac{2}{x}+\frac{2\epsilon}{|x|}}|$ and its Laurent series (and hence Taylor series) expansion in t has no t^k terms for $k < 2$. Now (4.2) and (4.7) imply that

$$\hat{x} = x(1 + \frac{1}{3}p_1(x)e^{-\frac{1}{x}}t + O((te^{-\frac{1}{x}+\frac{2\epsilon}{|x|}})^2)).$$

By (4.4) we see that K is asymptotic to the identity on $\{x\colon |\arg x| < \frac{\pi}{2} - \alpha, |x| < r\} \times \{y\colon |y| < \frac{\epsilon}{\pi}\}$.

To find where K^{-1} is defined, we start with (4.2). Let $x = \hat{x}(1+u)$ and rewrite the equation as

$$u = \{1 + \hat{t}e^{-\frac{1}{\hat{x}}}p_1(\hat{x}(1+u))e^{\frac{u}{\hat{x}(1+u)}}\}^{-1/3} - 1 \equiv L(u).$$

Using (4.4), one can verify that for $|\hat{t}| < e^{\frac{\epsilon}{|\hat{x}|}}$, $|\arg \hat{x}| < \pi/2 - 2\alpha$ and $0 < |\hat{x}| < r'_\epsilon < r_\epsilon/2$, L is a contraction map sending the disc $\{u\colon |u| < \min\{\epsilon, \frac{2}{\pi}\alpha\}\}$ into itself. Hence there is a unique holomorphic solution $u = u(\hat{x}, \hat{t})$ satisfying $|u| < \min\{\epsilon, \frac{2}{\pi}\alpha\}$. Solving t in (4.3), we get

$$t = \hat{t}e^{\hat{t}p^*(\hat{x}(1+u(\hat{x},\hat{t})))e^{-\frac{1}{\hat{x}(1+u(\hat{x},\hat{t}))}}}.$$

We see that K^{-1} sends $S_{2\alpha,r/2,\epsilon/2}$ into $S_{\alpha,r,\epsilon}$. Recall that K sends $S_{\alpha,r,\epsilon}$ into $S_{\alpha/2,2r,2\epsilon}$. The uniqueness of solutions implies that $KK^{-1} = \mathrm{id}$ on $S_{2\alpha,r/2,\epsilon/2}$. Hence K is a biholomorphic map from $K^{-1}(S_{2\alpha,r/2,\epsilon})$ into $S_{2\alpha,r/2,\epsilon}$. We can also obtain $K^{-1}(S_{2\alpha,r/2,\epsilon}) \supset S_{3\alpha,r/4,\epsilon/2}$ by showing $K(S_{3\alpha,r/4,\epsilon/2}) \subset S_{2\alpha,r/2,\epsilon}$ and $K^{-1}K = \mathrm{id}$.

In summary, we define a biholomorphic map

$$H_{1\,2} = K\colon \begin{cases} \hat{x} = x(1 + \frac{t}{3}p_1(x)e^{-\frac{1}{x}} + O((te^{-\frac{1}{x}+\frac{2\epsilon}{|x|}})^2)), \\ \hat{t} = t(1 - tp^*(x)e^{-\frac{1}{x}} + O(t^2e^{-\frac{2}{x}+\frac{2\epsilon}{|x|}})). \end{cases}$$

Recall $I(x,t)=I(-x,t^{-1})$, and $p_1(-x)=-p_1(x)$. For $H_{3\,4}=IH_{1\,2}I$ we have

$$H_{3\,4}\colon\begin{cases}\hat x=x(1-\frac{t^{-1}}{3}p_1(x)e^{\frac{1}{x}}+O((t^{-1}e^{\frac{1}{x}+\frac{2\epsilon}{|x|}})^2)),\\ \hat t=t(1+t^{-1}p^*(-x)e^{\frac{1}{x}}+O(t^{-2}e^{\frac{2}{x}+\frac{2\epsilon}{|x|}})).\end{cases}$$

Let $\tilde H_{j\,j+1}$ have the same form with p being $\tilde p$. From section 3, we then find realizations $\sigma=I\sigma^{-1}I,\tilde\sigma=I\tilde\sigma^{-1}I$ for $H,\tilde H$ (defined on the (x,y)-space and being asymptotic to the identity), respectively. Assume that $g\sigma g^{-1}=\tilde\sigma$ for some biholomorphic map $g=IgI$. As discussed in section 3, we have $g(x,y)=(xg_1(x,y),g_2(x,y))$. Let $\mu=g_1(0)$. When $\operatorname{Im}\mu\geq 0$ define $S_j^*,\tilde S_j^*$ in the (x,y)-space by (3.7) or (3.8). When $\operatorname{Im}\mu<0$, reverse the roles of $H,\tilde H$ and define $S_j^*,\tilde S_j^*$ by (3.7) or (3.8) again. Then $H,\tilde H$ are equivalent by $\{G_j\}$ (see section 3). We have $G_1=G_4$, $G_2=G_3$ and $G_{j+2}=IG_jI$. We now return to the (x,t)-space. In the (x,t)-space we have

$$G_j(x,t)=(xa_j(x),t\lambda_j(x)),\quad a_j\sim a,\ a(0)=\mu\neq 0,\ \lambda_j\sim\lambda,\ \lambda(0)\neq 0.$$

Let us first consider $G_1H_{1\,2}=\tilde H_{1\,2}G_2$ on $S_1^*\cap S_2^*$. From x-components on both sides we get

$$\begin{aligned}(4.8)\quad &(1+\frac{t}{3}p_1(x)e^{-\frac{1}{x}}+O((te^{-\frac{1}{x}+\frac{2\epsilon}{|x|}})^2))a_1(x(1+\frac{t}{3}p_1(x)e^{-\frac{1}{x}}+O((te^{-\frac{1}{x}+\frac{2\epsilon}{|x|}})^2)))\\ &=a_2(x)(1+\frac{t\lambda_2(x)}{3}\tilde p_1(xa_2(x))e^{-\frac{1}{xa_2(x)}}+O((t\lambda_2(x)e^{-\frac{1}{xa_2(x)}+\frac{2\epsilon}{|xa_2(x)|}})^2)).\end{aligned}$$

The above identity holds for x in a sector V, and $e^{-\frac{\epsilon}{|x|}}<|t|<e^{\frac{\epsilon}{|x|}}$. Fix x and expand both sides as Laurent series in t (actually a Taylor series in t). The constant terms give us $a_1(x)=a_2(x)$. From the x-components of $G_3H_{3\,4}=\tilde H_{3\,4}G_4$ we get

(4.9)

$$\begin{aligned}&(1-\frac{t^{-1}}{3}p_1(x)e^{\frac{1}{x}}+O((t^{-1}e^{\frac{1}{x}+\frac{2\epsilon}{|x|}})^2))a_3(x(1-\frac{t^{-1}}{3}p_1(x)e^{\frac{1}{x}}+O((t^{-1}e^{\frac{1}{x}+\frac{2\epsilon}{|x|}})^2)))\\ &=a_4(x)(1-\frac{t^{-1}\lambda_4(x)^{-1}}{3}\tilde p_1(xa_4(x))e^{\frac{1}{xa_4(x)}}+O((t^{-1}\lambda_4(x)^{-1}e^{\frac{1}{xa_4(x)}+\frac{2\epsilon}{|xa_4(x)|}})^2)).\end{aligned}$$

The same argument yields $a_3=a_4$ on $-V$. Since a_j are bounded then $a_1=a_2=a$ is a holomorphic function defined near the origin. We fix x again and look at the coefficients of t^1 in (4.8). We get

$$p_1(x)e^{-\frac{1}{x}}(a(x)+xa'(x))=a(x)\lambda_2(x)\tilde p_1(xa(x))e^{-\frac{1}{xa(x)}}.$$

The identity holds on $S_1^*\cap S_2^*$ and hence on S_2^*. On S_2^*, λ_2 is holomorphic. If $a^2\not\equiv 1$, then the orders of poles of both sides indicate that $a+xa'\equiv 0$, which is impossible because the right hand side is not identically zero. Consequently, $a^2\equiv 1$, i.e. $a\equiv 1$ since $\Re a(0)\geq 0$. Now we have

$$p_1(x)=\lambda_2(x)\tilde p_1(x).$$

From coefficients of t^{-1} in (4.9) (for which we now know $a_3=a_1=1$) we get

$$p_1(x)=\lambda_4(x)^{-1}\tilde p_1(x).$$

Therefore, λ_j extend to meromorphic functions on a punctured neighborhood of the origin with $\lambda_1\lambda_2 = 1$. From the t-components of $G_1H_{1\,2} = \tilde{H}_{1\,2}G_2$ we get

$$(1 + O(t))\lambda_1(x(1 + O(t)) = \lambda_2(x)(1 + O(t)).$$

Hence $\lambda_1 = \lambda_2$ on a sector and hence in a punctured neighborhood of the origin since there are meromorphic on $\mathbf{C}^*$. Now $\lambda^2 = 1$, i.e. $\lambda = \pm 1$. Consequently $p = \pm\tilde{p}$. When $\tilde{p} = p$, we get $\lambda = 1$.

Consider the second case where $G_1H_{1\,2} = \tilde{H}_{3\,4}G_2$. From x-components we get

$$\begin{aligned}(1 + \frac{t}{3}p_1(x)e^{-\frac{1}{x}} &+ O((te^{-\frac{1}{x}+\frac{2\epsilon}{|x|}})^2))a_1(x(1 + \frac{t}{3}p_1(x)e^{-\frac{1}{x}} + O((te^{-\frac{1}{x}+\frac{2\epsilon}{|x|}})^2)))\\ &= a_2(x)(1 - \frac{t^{-1}\lambda_2(x)^{-1}}{3}\tilde{p}_1(xa_2(x))e^{\frac{1}{xa_2(x)}} + O((t^{-1}\lambda_2(x)^{-1}e^{\frac{1}{xa_2(x)}+\frac{2\epsilon}{|xa_2(x)|}})^2)).\end{aligned}$$

Expand both sides as Laurent series expansion in t. The coefficient of t^{-1} on the right-hand side is non-zero, while the coefficient on the left-hand side is zero. We rule out this case immediately.

From (4.2)-(4.3), one sees that σ_p is equivalent to σ_{-p} by $(x,t) \to (x,-t)$. When $p = \tilde{p} \not\equiv 0$, the above argument shows that $G_j = \mathrm{id}$ in the (x,t)-space. In the (x,y)-space, we conclude that $G_j(x,y) = \hat{\sigma}^k$, so $g = H_j\hat{\sigma}^kH_j^{-1} = \sigma^k$. Since I reverses σ, we conclude $g = \mathrm{id}$ if g preserves I and σ.

The following proposition gives our reduction from higher dimension case to the case of $\mathbf{C}^4$ at the expense of symplectic 2-form for the involutions.

PROPOSITION 4.1. *Let $\{\widehat{F}, G_j\}, j = 1,2$ be two pairs of glancing hypersurfaces in $\mathbf{C}^4$ given by*

$$G_j\colon \xi_2 = \xi_1^2 + x_1b_j(\xi_2, x), \quad b_j(0) = 1.$$

If $\{\widehat{F} \times \mathbf{C}^{2n-4}, G_1 \times \mathbf{C}^{2n-4}\}$ and $\{\widehat{F} \times \mathbf{C}^{2n-4}, G_2 \times \mathbf{C}^{2n-4}\}$ are equivalent under a holomorphic symplectic mapping ψ of $\mathbf{C}^{2n}$, the corresponding pairs of involutions of $\{\widehat{F}, G_1\}$ $\{\widehat{F}, G_2\}$ are equivalent under some biholomorphic map ϕ of $\mathbf{C}^2 \equiv \widehat{F} \cap \widehat{G}$. If ψ is a real map, the ϕ is real too.

Proof. Let $\mathbf{C}^4$ be the (ξ_1, ξ_2, x_1, x_2)-space and $\mathbf{C}^{2n-4}$ the $('\xi, 'x)$-space. Let $\mathbf{C}^{2n} = \mathbf{C}^4 \times \mathbf{C}^{2n-4}$. Let $\tilde{F} = \hat{F} \times \mathbf{C}^{2n-4}$ and $\tilde{G}_k = G_k \times \mathbf{C}^{2n-4}$. Let $\{I, I_k\}$ be the pair of involutions of $\{\widehat{F}, G_k\}$, and $\{\tilde{I}, \tilde{I}_k\}$ the pair of involutions of $\{\hat{F} \times \mathbf{C}^{2n-4}, G_k \times \mathbf{C}^{2n-4}\}$. Assume that a biholomorphic mapping $\tilde{\phi}$ in $(\xi_1, x_2, '\xi, 'x)$-space sends the pair of involutions $\{\tilde{I}, \tilde{I}_1\}$ into $\{\tilde{I}, \tilde{I}_2\}$. Let π_1 be the projection from the $(\xi_1, x_2, '\xi, 'x)$-space onto (ξ_1, x_2)-subspace. Looking at the flows of Hamiltonian vector fields of x_1 and $\xi_2 - \xi_1^2 - x_1b(\xi_2, x_1, x_2)$ we get $\tilde{I}(\xi_1, x_2, '\xi, 'x) = (I(\xi_1, x_2), '\xi, 'x)$ and $\tilde{I}_k(\xi_1, x_2, '\xi, 'x) = (I_k(\xi_1, x_2), '\xi, 'x)$. From $\tilde{\phi}\tilde{I} = \tilde{I}\tilde{\phi}$ and $\tilde{\phi}\tilde{I}_1 = \tilde{I}_2\tilde{\phi}$, we get easily that $\phi I = I\phi$ and $\phi I_1 = I_2\phi$. Note that ϕ is a biholomorphic map, since $\tilde{\phi}'(0)$ preserves the Jordan normal form

$$(\tilde{I}_1\tilde{I})'(0)\colon \xi_1 \to \xi_1,\ x_2 \to x_2 + 2\xi_1,\ \xi_\alpha \to \xi_\alpha,\ x_\alpha \to x_\alpha,\ \alpha > 2.$$

It is obvious that ϕ is real, if $\tilde{\phi}$ is real. □

Summarizing the above results we obtain the following.

Proposition 4.2. *Let $n \geq 2$. There exists $\delta > 0$ such that each meromorphic function*

$$p(x) = \sum \frac{\epsilon_k}{(k^2x^2+1)^k}, \quad 0 < |\epsilon_k| < \frac{\delta^k k^k}{k!}$$

gives arise to a pair of holomorphic glancing hypersurfaces $\hat{F}, G = G_p$ in $\mathbf{C}^{2n}$ with $\hat{F} \cap G_p \colon x_1 = \xi_n - \xi_1^2 = 0$ such that the pair of involutions $I_{\hat{F}}, I_{G_p}$ and $\sigma_p = I_{G_p} I_{\hat{F}}$ satisfy

(i) *σ_p and $\sigma_{\tilde{p}}$ are holomorphically equivalent on J, if and only if $\tilde{p} \equiv \pm p$. In particular the pairs $\{\hat{F}, G_p\}$ and $\{\hat{F}, G_{\tilde{p}}\}$ are not equivalent under holomorphic symplectic mappings of $\mathbf{C}^{2n}$ if $\tilde{p} \not\equiv \pm p$.*

(ii) *if $n = 2$ and $p \not\equiv 0$, σ_p^k are the only local biholomorphic maps on J that commute with σ_p, where $k = 0, 1, -1, 2, -2, \dots$; in particular, the identity map is the only biholomorphic map that preserves both $I_{\hat{F}}$ and I_{G_p}.*

Real case. Recall $I(x,y) = (-x,y)$, $\hat{\sigma}(x,y) = (x, y+2x)$, and $\rho(x,y) = (\overline{x}, \overline{y})$. In the (x,t)-space, we have $I(x,t) = (-x, t^{-1})$, and $\rho(x,t) = (\overline{x}, \overline{t}^{-1})$.

Consider

$$p(x) = \sum_{k=1}^{\infty} \frac{\epsilon_k}{(k^2x^2+1)^k}, \quad 0 < \epsilon_k < \frac{\delta^{2k} k^{2k}}{(2k)!}.$$

Note that $p(x)$ is a meromorphic function on $\mathbf{C} \setminus \{0\}$. If δ is sufficiently small,

$$H_{1\,2} \colon x' = x, \quad t' = te^{ip(x)e^{-\frac{1}{x}}}, \quad |\arg x| < \frac{\pi}{4}$$

is asymptotic to the identity. It is obvious that $H_{1\,2} = \rho H_{1\,2} \rho$ preserves $dx^3 \wedge d\log t$. Let

$$H_{3\,4} = I H_{1\,2} I \colon x' = x, \quad t' = te^{-ip(x)e^{\frac{1}{x}}}, \quad |\arg x| > \frac{3\pi}{4}.$$

Set $H_{4\,1} = I H_{2\,3} I = \mathrm{id}$ on $|\arg x - \frac{\pi}{2}| < \frac{\pi}{4}$. Let $\sigma = I\sigma^{-1}I = \rho\sigma\rho$ be a holomorphic map realizing moduli functions $\{H_{j\,j+1}\}$. Let $\tilde{\sigma}$ be another one corresponding to $\tilde{p}$ that still have the above form. Note that in the real case it is not necessary to have openings of $S_{1\,2}$ and $S_{3\,4}$ to be bigger than $\frac{\pi}{2}$.

We want to show that $\tilde{\sigma}$ and σ are equivalent by some real analytic map preserves I if and only if $\tilde{p}(x) = p(x)$.

Assume that there is a real analytic map $g = IgI$ such that $g\sigma g^{-1} = \tilde{\sigma}$. Since g is real we know that $g(x,y) = (xg_1(x,y), g_2(x,y))$ with $g_1(0) \in \mathbf{R}$. We still have $G_j^{-1} H_{j\,j+1} G_{j+1} = \tilde{H}_{j\,j+1}$ for all j or $G_j^{-1} H_{j\,j+1} G_{j+1} = \tilde{H}_{j+2\,j+3}$ for all j, where G_j have the form

$$G_j(x,t) = (xa_j(x), t\lambda_j(x)), \quad G_1 = G_4, \quad G_2 = G_3,$$
$$a_j \sim a, \quad a(0) \neq 0, \quad \lambda_j \sim \lambda, \quad \lambda(0) \neq 0,$$
$$a_2(x) = \overline{a_1(\overline{x})}, \quad a_3(x) = a_1(-x), \quad \lambda_2(x) = \overline{\lambda_1(\overline{x})}^{-1}, \quad \lambda_3(x) = \lambda_1(-x)^{-1}.$$

Let us look at the first case $a(0) > 0$. Then we must have $H_{1\,2}G_2 = G_1\tilde{H}_{1\,2}$, which implies that on $V = \{x \colon |\arg x| < \frac{\pi}{4} - \epsilon, 0 < |x| < r\}$ we have $a_1 = a_2$ and

$$\lambda_2(x) e^{ip(xa_2(x))e^{-\frac{1}{xa_2(x)}}} = \lambda_1(x) e^{i\tilde{p}(x)e^{-\frac{1}{x}}}. \tag{4.10}$$

By $H_{3\,4}G_4 = G_3\tilde{H}_{3\,4}$ on $-V$, we get $a_2 = a_3 = a_4 = a_1$ on $-V$ and hence all a_j are the same. By removable singularity, we get $a_j = a$ is holomorphic at the origin. In (4.10), we take $x > 0$ and conjugate both sides, and by $\lambda_2(x) = \overline{\lambda_1(x)}^{-1}$ we get

$$\lambda_1(x)^{-1}e^{-i\overline{p}(xa(x))e^{-\frac{1}{xa(x)}}} = \lambda_2(x)^{-1}e^{-i\overline{\tilde{p}}(x)e^{-\frac{1}{x}}}.$$

Using (4.10) again and eliminating λ_1, λ_2 from both sides, we get

$$-\overline{p}(xa(x))e^{-\frac{1}{xa(x)}} + \tilde{p}(x)e^{-\frac{1}{x}} = -\overline{\tilde{p}}(x)e^{-\frac{1}{x}} + p(xa(x))e^{-\frac{1}{xa(x)}}$$

Recall $p(x) = \overline{p}(x)$ and $\tilde{p}(x) = \overline{\tilde{p}}(x)$. We get

$$p(xa(x))e^{-\frac{1}{xa(x)}} = \tilde{p}(x)e^{-\frac{1}{x}},$$

which now holds on $\mathbf{C}^*$. Looking at the orders of the poles we see $a \equiv 1$ and then $\tilde{p} = p$.

Consider now the case $a(0) < 0$. We then have $(G_jI)^{-1}H_{j\,j+1}G_{j+1}I = \tilde{H}_{j\,j+1}$, which is reduced to the previous case. The conclusion is then $\tilde{p}(x) = p(-x) = p(x)$.

We have proved the following.

Proposition 4.3. *Let $n \geq 2$. There exists $\delta > 0$ such that each real analytic function*

$$p(x) = \sum \frac{\epsilon_k}{(k^2x^2+1)^k}, \quad x > 0, \quad 0 < \epsilon_k < \frac{\delta^k k^k}{k!}$$

gives arise to a pair of real analytic glancing hypersurfaces $\hat{F}, G = G_p$ in $\mathbf{R}^{2n}$ with $\hat{F} \cap G_p \colon x_1 = \xi_2 - \xi_1^2 = 0$ such that if $I_{\hat{F}}, I_{G_p}$ are the corresponding involutions on J, the pair $\{I_{\hat{F}}, I_{G_p}\}$ is equivalent to $\{I_{\hat{F}}, I_{G_{\tilde{p}}}\}$ by a real analytic mapping on J, if and only if $p = \tilde{p}$.

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