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ON THE GEOGRAPHY OF GORENSTEIN MINIMAL 3-FOLDS OF GENERAL TYPE*

MENG CHEN† AND CHRISTOPHER D. HACON‡

Abstract. Let X be a minimal projective Gorenstein 3-fold of general type. We give two applications of an inequality between $\chi(\omega_X)$ and $p_g(X)$:

1) Assume that the canonical map $\Phi_{|K_X|}$ is of fiber type. Let F be a smooth model of a generic irreducible component in the general fiber of $\Phi_{|K_X|}$. Then the birational invariants of F are bounded from above.

2) If X is nonsingular, then $c_1^3 \leq \frac{1}{27}c_1c_2 + \frac{10}{3}$.

Key words. Canonical map, 3-folds of general type, Albanese map.

AMS subject classifications. 14J30, 14C20.

1. Introduction. We work over the complex number field $\mathbb{C}$.

The main purpose of this note is to study the geometry of Gorenstein minimal 3-folds X of general type. We improve the inequality $\chi(\omega_X) \leq 2p_g(X)$ (see Proposition 2.1 for a precise statement), and we show how this leads to several applications which we explain below:

First, we improve the main theorem in [8]:

THEOREM 1.1. *Let X be a minimal projective Gorenstein 3-fold of general type. Assume that the canonical map $\Phi_{|K_X|}$ is of fiber type. Let F be a smooth model of a generic irreducible component in the general fiber of $\Phi_{|K_X|}$. Then the invariants of F are bounded from above as follows:*

(1) if F is a curve, then $g(F) \leq 487$;

(2) if F is a surface, then $p_g(F) \leq 434$.

REMARK 1.2. 1) Theorem 1.1 was verified by the first author in [8] under the assumption that $p_g(X)$ is sufficiently large.

2) When $\Phi_{|K_X|}$ is generically finite, the generic degree is bounded from above by the second author in [11].

3) In the surface case, the corresponding boundedness theorem was proved by Beauville in [1].

4) The numerical bounds in the above theorem might be far from sharp.

Our second application is an inequality of Noether type between c_1 and c_2 which improves the main theorem of [17].

THEOREM 1.3. *Let X be a nonsingular projective minimal 3-fold of general type. Then the following inequality holds:*

$$K_X^3 \geq \frac{8}{9}\chi(\omega_X) - \frac{10}{3}, \text{ or equivalently}$$

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†Institute of Mathematics, School of Mathematical Sciences, Fudan University, Shanghai, 200433, People's Republic of China (mchen@fudan.edu.cn). The first author was supported both by Program for New Century Excellent Talents in University (Ministry of Education, PR China) and by the National Natural Science Foundation of China (no: 10131010).

‡Department of Mathematics, University of Utah, 155 South 1400 East, Room 233, Salt Lake City, Utah 84112-0090, USA (hacon@math.utah.edu). The second author was partially supported by NSF research grant no: 0456363.

$$c_1^3 \le \frac{1}{27}c_1c_2 + \frac{10}{3}.$$

Chen is grateful to De-Qi Zhang for pointing out an inequality (see the proof of Lemma 2.1(3) in [19]) similar to the one in Proposition 2.1 and for an effective discussion.

2. Proof of Theorem 1.1. Throughout this note, *a minimal 3-fold X* is one with nef canonical divisor K_X and with only $\mathbb{Q}$-factorial terminal singularities.

2.1. Notations and the set up. Let X be a minimal projective 3-fold of general type. Since we are discussing the behavior of the canonical map, we may assume $p_g(X) \ge 2$. Denote by φ_1 the canonical map which is usually a rational map. Take the birational modification $\pi : X' \longrightarrow X$, which exists by Hironaka's big theorem, such that

(i) X' is smooth;

(ii) the movable part of $|K_{X'}|$ is base point free;

(iii) there exists a canonical divisor K_X such that $\pi^*(K_X)$ has support with only normal crossings.

Denote by h the composition $\varphi_1 \circ \pi$. So $h : X' \longrightarrow W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $h : X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of h. We can write

$$K_{X'} = \pi^*(K_X) + E = S + Z,$$

where S is the movable part of $|K_{X'}|$, Z is the fixed part and E is an effective $\mathbb{Q}$-divisor which is a sum of distinct exceptional divisors.

If $\dim \varphi_1(X) < 3$, f is a called an *induced fibration of* φ_1. If $\dim \varphi_1(X) = 2$, a general fiber F of f is a smooth curve C of genus $g := g(C) \ge 2$. If $\dim \varphi_1(X) = 1$, a general fiber F of f is a smooth projective surface of general type. Denote by F_0 the smooth minimal model of F and by $\sigma : F \longrightarrow F_0$ the smooth blow down map. Denote by b the genus of the base curve B.

Proposition 2.1. *Let V be a smooth projective 3-fold of general type with $p_g(V) > 0$. Then $\chi(\omega_V) \le p_g(V)$ unless a generic irreducible component in the general fiber of the Albanese morphism is a surface V_y with $q(V_y) = 0$, in which case one has the inequality*

$$\chi(\omega_V) \le (1 + \frac{1}{p_g(V_y)})p_g(V).$$

Proof. Since $\chi(\omega_V) = p_g(V) + q(V) - h^2(\mathcal{O}_V) - 1$, the result is clear if $q(V) \le 1$. So assume that $q(V) \ge 2$. Let $a : V \to Y$ be the Stein factorization of the Albanese morphism $V \to A(V)$. By the proof of Theorem 1.1 in [11], one sees that we may assume that $\dim Y = 1$ and hence Y is a smooth curve. Recall also that by [11], $p_g(V) \ge \chi(a_*\omega_V)$. Let $y \in Y$ be a general point and V_y the corresponding fiber. V_y is a smooth surface of general type. If $q(V_y) > 0$, then proceeding as in [11], one sees that $\chi(R^1a_*\omega_V) = \chi(R^1a_*\omega_{V/Y} \otimes \omega_Y)$. Since the genus of Y is $q(V)$, and $\deg R^1a_*\omega_{V/Y} \ge 0$, one sees by an easy Riemann-Roch computation that

$$\chi(R^1a_*\omega_V) \ge (q(V) - 1)q(V_y).$$

Recall that $R^2 a_*\omega_V \cong \omega_Y$ and so

$$\chi(\omega_V) = \chi(a_*\omega_V) - \chi(R^1 a_*\omega_V) + \chi(R^2 a_*\omega_V) \leq \chi(a_*\omega_V) \leq p_g(V)$$

whenever $q(V_y) > 0$.

We may therefore assume that $q(V_y) = 0$. Notice that by [14], the sheaf $R^1 a_*\omega_V$ is torsion free. Since its rank is given by $h^1(\omega_{V_y}) = q(V_y) = 0$, we have that $R^1 a_*\omega_V = 0$. Therefore, by a similar Riemann-Roch computation, one sees that $\chi(a_*\omega_V) \geq (q(V)-1)p_g(V_y)$ and so

$$\chi(\omega_V) = \chi(a_*\omega_V) + q(V) - 1 \leq \chi(a_*\omega_V)(1 + \frac{1}{p_g(V_y)}) \leq p_g(V)(1 + \frac{1}{p_g(V_y)}).$$

□

EXAMPLE 2.2. Let S be a minimal surface of general type admitting a $\mathbb{Z}_2$ action such that $q(S) = 0$, $p_g(S) = 1$ and $p_g(S/\mathbb{Z}_2) = 0$ (cf. (2.6) of [10]). Let C be a curve admitting a fixed point free $\mathbb{Z}_2$ action and let $B = C/\mathbb{Z}_2$. Assume that the genus of B is $b \geq 2$. Let $V = S \times C/\mathbb{Z}_2$ be the quotient by the induced diagonal action. Then V is minimal, Gorenstein of general type such that $p_g(V) = b-1$, $q(V) = b$ and $h^2(\mathcal{O}_V) = 0$. It follows that $\chi(\omega_V) = 2b - 2 = (1 + 1/p_g(V_y))p_g(V)$.

This example shows that the above proposition is close to being optimal.

LEMMA 2.3. *Let X be a minimal 3-fold of general type. Suppose* $\dim \varphi_1(X) = 1$. *Keep the same notations as in 2.1. Replace $\pi : X' \longrightarrow X$, if necessary, by a further birational modification (we still denote it by π). Then*

$$\pi^*(K_X)|_F - \frac{p_g(X) - 1}{p_g(X)}\sigma^*(K_{F_0})$$

is pseudo-effective.

Proof. One has an induced fibration $f : X' \longrightarrow B$.

Case 1. If $b > 0$, we may replace π by a new one as in the proof of Lemma 2.2 of [9] such that $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$. In fact, since the fibers of π are rationally connected[1] and $b > 0$, it follows that $f : X' \to B$ factors through a morphism $f_1 : X \to B$. But since X is minimal and terminal, it follows that a general fiber X_b of f_1 is a smooth minimal surface of general type and hence it can be identified with F_0. It is now clear that $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$.

Thus it suffices to consider the case $b = 0$.

Case 2. If $p_g(X) = 2$, the lemma was verified in section 4 (at page 526 and page 527) in [5]. If $p_g(X) \geq 3$, one may refer to Lemma 3.4 in [9]. □

PROPOSITION 2.4. *Let X be a Gorenstein minimal projective 3-fold of general type. Let $d := \dim \varphi_1(X)$. The following inequalities hold:*

(1) If $d = 2$, then $K_X^3 \geq \ulcorner \frac{2}{3}(g(C) - 1)\urcorner(p_g(X) - 2)$.

(2) If $d = 1$, then $K_X^3 \geq (\frac{p_g(X)-1}{p_g(X)})^2 K_{F_0}^2(p_g(X) - 1)$.

Proof. The inequality (1) is due to Theorem 4.1(ii) in [6].

[1]Shokurov ([18]) proved that if the pair (X, Δ) is klt and the MMP holds, then the fibres of the exceptional locus are always rationally chain connected. Furthermore, the second author and M^{c}Kernan (see [12]) have recently extended Shokurov's result to any dimension and without assuming MMP.

Suppose now that $d = 1$. We may write

$$\pi^*(K_X) \sim S + E_\pi$$

where $S \equiv tF$ with $t \geq p_g(X) - 1$ and E_π is an effective divisor.

Thus we have

$$\begin{aligned} K_X^3 &= \pi^*(K_X)^3 \geq (\pi^*(K_X)^2 \cdot F)(p_g(X) - 1) \\ &\geq (\frac{p_g(X) - 1}{p_g(X)})^2 \sigma^*(K_{F_0})^2 (p_g(X) - 1) \end{aligned}$$

where Lemma 2.3 has been applied to derive the second inequality above. □

2.2. Proof of Theorem 1.1. The Miyaoka-Yau inequality (cf. [15]) says

$$K_X^3 \leq 72\chi(\omega_X).$$

> (**) Denote by V a smooth model of X. Assume that a generic irreducible component in the general fiber of the Albanese morphism is a surface V_y with $q(V_y) = 0$ and $p_g(V_y) = 1$. Because $p_g(V) = p_g(X) \geq 2$, we see that the canonical map of V maps V_y to a point. This means $\dim \varphi_1(X) = 1$, i.e. $|K_X|$ is composed with a pencil. Thus, one sees that in this special situation, the Stein factorization of the Albanese map is the fibration induced by φ_1. So $p_g(F) = p_g(V_y) = 1$.

(1) Assume $\dim \varphi_1(X) = 2$. The above argument implies that $\chi(\omega_X) \leq \frac{3}{2} p_g(X)$ and so by proposition 2.4 and an easy computation, one sees that $g(C) \leq 487$. Furthermore $g(C) \leq 109$ whenever $p_g(X)$ is sufficiently big.

(2) Assume $\dim \varphi_1(X) = 1$. When $b > 0$, we have $p_g(F) \leq 38$ by (both 1.4 and Theorem 1.3 in) [8]. So we only need to study the case $b = 0$.

Suppose $p_g(F) \geq 2$. Then one has $\chi(\omega_X) \leq \frac{3}{2} p_g(X)$ by argument (**) and Proposition 2.1. The Miyaoka-Yau inequality yields $K_X^3 \leq 72\chi(\omega_X) \leq 108 p_g(X)$. Again by Propositions 2.1 and 2.4, we have

$$K_{F_0}^2 \leq 108(\frac{p_g(X)}{p_g(X) - 1})^3 \leq 864.$$

Also $K_{F_0}^2 \leq 108$ whenever $p_g(X)$ is sufficiently big. Taking into account the Noether inequality $K_{F_0}^2 \geq 2p_g(F) - 4$, we get $p_g(F) \leq 434$. This concludes the proof.

3. A Noether type inequality between c_1 and c_2.

3.1. A known inequality. Let X be a nonsingular projective minimal 3-fold of general type. We have a sharp inequality

$$K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3}$$

which was first proved in [7] under the assumption K_X being ample. The general case was recently proved in [2].

3.2. Proof of Theorem 1.3.

Proof. Note that since $K_X^3 > 0$ is an even integer, the Theorem clearly holds for $\chi(\omega_X) \leq 6$. Therefore, we may assume that $\chi(\omega_X) > 0$.

Case 1. $p_g(X) > 0$.

According to Proposition 2.1, we have an inequality $\chi(\omega_X) \leq \frac{3}{2}p_g(X)$ unless a generic irreducible component in the general fiber of the Albanese morphism is a surface V_y with $q(V_y) = 0$ and $p_g(V_y) = 1$.

So in the general case by 3.1 one has the inequality

$$K_X^3 \geq \frac{8}{9}\chi(\omega_X) - \frac{10}{3}$$

or equivalently,

$$c_1^3 \leq \frac{1}{27}c_1c_2 + \frac{10}{3}.$$

In the exceptional case with $p_g(X) > 1$, the argument (**) in 2.2 says that $|K_X|$ is composed with a pencil of surfaces and φ_1 generically factors through the Albanese map. Thus X is canonically fibred by surfaces with $q(V_y) = 0$ and $p_g(V_y) = 1$. According to Theorem 4.1(iii) in [6], one has $K_X^3 \geq 2p_g(X) - 4$. Since by Proposition 2.1 $\chi(\omega_X) \leq 2p_g(X)$, one has the stronger inequality $K_X^3 \geq \chi(\omega_X) - 4$.

In the exceptional case with $p_g(X) = 1$, by Proposition 2.1, one has $\chi(\omega_X) \leq 2$ and so the inequality in Theorem 2.4 holds.

Case 2. $p_g(X) = 0$.

We can not rely on 3.1 in this case. Since $\chi(\omega_X) > 0$, one has $q(X) > 1$. Thus we can study the Albanese map. Let $a : X \longrightarrow Y$ be the Stein factorization of the Albanese morphism. We claim that $\dim(Y) = 1$. In fact, if $\dim(Y) \geq 2$, then the Proof of Theorem 1.1 of [11] shows $p_g(X) \geq \chi(a_*\omega_X) \geq \chi(\omega_X) > 0$, a contradiction.

So we have a fibration $a : X \longrightarrow Y$ onto a smooth curve Y with $g(Y) = q(X) > 1$. Denote by F a general fiber of a. If $p_g(F) > 0$, then the Proof of Theorem 1.1 of [11] also shows $0 = 2p_g(X) \geq \chi(\omega_X) > 0$, which is also a contradiction. Thus one must have $p_g(F) = 0$. Because F is of general type, one has $q(F) = 0$. Therefore, the sheaves $a_*\omega_X$ and $R^1a_*\omega_X$ have rank $h^0(\omega_F) = p_g(F) = 0$ and $h^1(\omega_F) = q(F) = 0$. Since, by [14], they are torsion free, it follows that they are both zero. So

$$\chi(\omega_X) = \chi(R^2a_*\omega_X) = \chi(\omega_Y) = q(X) - 1.$$

Still looking at the fibration $a : X \longrightarrow Y$, one sees that a is relatively minimal since X is minimal. Therefore $K_{X/Y}$ is nef by Theorem 1.4 of [16]. Thus one has $K_X^3 \geq (2q(X) - 2)K_F^2 \geq 2\chi(\omega_X)$, which is stronger than the required inequality. ◻

4. Examples. In Example 2(e) of [4], one may find a smooth projective 3-fold of general type which is composed with a pencil of surfaces of $p_g(F) = 5$, the biggest value among known examples. Here we present another example which is composed with curves of genus $g = 5$.

Example 4.1. We follow the Example in §4 of [3]. We consider bi-double covers $f_i : C_i \to E_i$ of curves where, $g(E_i) = 0, 0, 2$. We assume that

$$(d_i)_*\mathcal{O}_{C_i} = \mathcal{O}_{E_i} \oplus L_i^\vee \oplus P_i^\vee \oplus L_i^\vee \otimes P_i^\vee$$

where for we have $\deg(L_1) = d_1$, $\deg(L_2) = d_2$, $\deg(P_1) = \deg(P_2) = 1$ and L_3, P_3 are distinct 2-torsion elements in $\mathrm{Pic}^0(E_3)$. In particular $g(C_i) = 2d_i - 1$ for $i \in \{1, 2\}$ and f_3 is étale. It follows that

$$\delta : D_1 \times D_2 \times D_3 \to E_1 \times E_2 \times E_3$$

is a $\mathbb{Z}_2^6$ cover. We denote by $l_i, p_i, l_i p_i$ the elements of $\mathbb{Z}_2^2$ whose eigensheaves with eigenvalues 1 are $L_i^\vee$, $P_i^\vee$ and $(L_i \otimes P_i)^\vee$. Let $X := D_1 \times D_2 \times D_3/G$ where $G \cong \mathbb{Z}_2^4$ is the group generated by

$$\{(1, p_2, l_3), (p_1, l_2, 1), (l_1, 1, p_3), (p_1, p_2, p_3)\}.$$

Then one sees that X is Gorenstein and for the induced morphism $f : X \to E_1 \times E_2 \times E_3$, one has

$$f_*\mathcal{O}_X = (\delta_*\mathcal{O}_{D_1 \times D_2 \times D_3})^G \cong \mathcal{O}_{E_1} \times \mathcal{O}_{E_2} \times \mathcal{O}_{E_3} \oplus$$

$$(L_1^\vee \boxtimes L_2^\vee \otimes P_2^\vee \boxtimes P_3^\vee) \oplus (P_1^\vee \boxtimes L_2^\vee \boxtimes L_3^\vee \otimes P_3^\vee) \oplus (L_1^\vee \otimes P_1^\vee \boxtimes P_2^\vee \boxtimes L_3^\vee).$$

Since $f_*\omega_X = \omega_{E_1 \times E_2 \times E_3} \otimes f_*\mathcal{O}_X$, it follows easily that

$$H^0(\omega_X) \cong H^0(\omega_{E_1} \otimes L_1) \otimes H^0(\omega_{E_2} \otimes L_2 \otimes P_2) \otimes H^0(\omega_{E_3} \otimes P_3).$$

In particular $p_g(X) = (d_1 - 1)(d_2 - 1)$ and φ_1 factors through the map $X \to C_1/\mathbb{Z}_2 \times C_2/\mathbb{Z}_2$. The fibers of φ_1 are then isomorphic to C_3 and hence have genus 5.

QUESTION 4.2. A very natural open problem is to find sharp upper bounds of the invariants of F in Theorem 1.1. It is very interesting to find a new example X which is a Gorenstein minimal 3-fold of general type such that $\Phi_{|K_X|}$ is of fiber type and that the generic irreducible component in a general fiber has larger birational invariants.

We remark that the above question is still open in the surface case. So Question 4.2 is probably quite difficult. A first step should be to construct new examples with bigger fiber invariants.

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