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# MIRROR CONGRUENCE FOR RATIONAL POINTS ON CALABI-YAU VARIETIES*

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**0. Introduction.** One of the basic problems in arithmetic mirror symmetry is to compare the number of rational points on a mirror pair of Calabi-Yau varieties. At present, no general algebraic geometric definition is known for a mirror pair. But an important class of mirror pairs comes from certain quotient construction. In this paper, we study the congruence relation for the number of rational points on a quotient mirror pair of varieties over finite fields. Our main result is the following theorem:

THEOREM 0.1. Let $X_0$ be a smooth projective variety over the finite field $\mathbf{F}_q$ with $q$ elements of characteristic $p$. Suppose $X_0$ has a smooth projective lifting $X$ over the Witt ring $W = W(\mathbf{F}_q)$ such that the $W$-modules $H^r(X, \Omega^s_{X/W})$ are free. Let $G$ be a finite group of $W$-automorphisms acting on the right of $X$. Suppose $G$ acts trivially on $H^i(X, \mathcal{O}_X)$ for all $i$. Then for any natural number $k$, we have the congruence

$$\#X_0(\mathbf{F}_{q^k}) \equiv \#(X_0/G)(\mathbf{F}_{q^k}) \pmod{q^k},$$

where $\#X_0(\mathbf{F}_{q^k})$ (resp. $\#(X_0/G)(\mathbf{F}_{q^k})$) denotes the number of $\mathbf{F}_{q^k}$-rational points of $X_0$ (resp. $X_0/G$).

The main application of the above theorem is to Calabi-Yau varieties. This gives the following theorem announced in [W], which was the main motivation of the present paper.

THEOREM 0.2. Let $X_0$ be a geometrically connected smooth projective Calabi-Yau variety of dimension $n$ over the finite field $\mathbf{F}_q$ with $q$ elements of characteristic $p$. Suppose $X_0$ has a smooth projective lifting $X$ over the Witt ring $W = W(\mathbf{F}_q)$ such that the $W$-modules $H^r(X, \Omega^s_{X/W})$ are free. Let $G$ be a finite group of $W$-automorphisms acting on the right of $X$. Suppose $G$ fixes a non-zero $n$-form on $X$. Then for any natural number $k$, we have the congruence

$$\#X_0(\mathbf{F}_{q^k}) \equiv \#(X_0/G)(\mathbf{F}_{q^k}) \pmod{q^k}.$$

*Proof.* If $X$ is a Calabi-Yau scheme over $W$ of dimension $n$, then $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0, n$ and $G$ acts trivially on them. If the generic fiber of $X$ is geometrically connected, then $G$ acts trivially on $H^0(X, \mathcal{O}_X)$. By Serre duality, $H^n(X, \mathcal{O}_X)$ is dual to $H^0(X, \Omega^n_{X/W})$. Since $X$ is Calabi-Yau, $\Omega^n_{X/W}$ is a trivial invertible sheaf. In order

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for $G$ to act trivially on $H^n(X, \mathcal{O}_{X/W})$, it suffices for $G$ to fix a nonzero $n$-form. Theorem 0.2 thus follows from Theorem 0.1.

In particular, we have the following corollary:

COROLLARY 0.3. Let $X_0$ be the smooth $(n-1)$-dimensional hypersurface

$$x_0^{n+1} + \cdots + x_n^{n+1} + \lambda x_0 \cdots x_n = 0$$

in $\mathbf{P}^n_{\mathbf{F}_q}$, where $\lambda \in \mathbf{F}_q$. Let

$$G = \{(\zeta_0, \ldots, \zeta_n) | \zeta_i \in \mathbf{F}_q, \zeta_i^{n+1} = 1, \prod_{i=0}^{n} \zeta_i = 1\}.$$

Consider the action $G \times X_0 \to X_0$ defined by

$$(\zeta_0, \ldots, \zeta_n) \times [x_0 : \ldots : x_n] \mapsto [\zeta_0 x_0 : \ldots : \zeta_n x_n].$$

We have $\#X_0(\mathbf{F}_{q^k}) \equiv \#(X_0/G)(\mathbf{F}_{q^k}) \pmod{q^k}$ for any natural number $k$.

It is well known that the above hypersurface is Calabi-Yau. A $G$-equivariant nonzero $(n-1)$-form is $\frac{(-1)^i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{1 + \sum_{j \neq i} x_j^{n+1} - \lambda \prod_{j \neq i} x_j}$ on the affine space $x_i = 1$ of $\mathbf{P}^n$.

It is known that for the above hypersurface $X_0$, $X_0/G$ is a strong singular mirror of $X_0$ if $(n+1)|(q-1)$. It is conjectured in [W] that for a strong mirror pair of Calabi-Yau varieties $\{X_0, X_0'\}$ over the finite field $\mathbf{F}_q$, we have $\#X_0(\mathbf{F}_{q^k}) \equiv \#X_0'(\mathbf{F}_{q^k}) \pmod{q^k}$ for any integer $k$. See [W] for a fuller discussion on this and other arithmetic mirror conjectures. In the situation of Theorem 0.2, if $X/G$ is a singular mirror of $X$ and if $Y$ is a smooth crepant resolution of $X/G$, then the pair $(X, Y)$ forms a strong mirror pair of smooth projective Calabi-Yau varieties. The congruence mirror conjecture in this case then reduces to showing the congruence

$$\#(X/G)(\mathbf{F}_{q^k}) \equiv \#Y(\mathbf{F}_{q^k}) \pmod{q^k}.$$

Another application of the theorem is to geometrically connected varieties with the property $H^i(X, \mathcal{O}_X) = 0$ for all $i \neq 0$. Again in this case, $G$ acts trivially on $H^i(X, \mathcal{O}_X)$ for all $i$. Let $\overline{K}$ be the algebraic closure of the fraction field of $W = W(\mathbf{F}_q)$. By [E], if the $l$-adic cohomology group $H^i(X \otimes_W \overline{K}, \mathbf{Q}_l)$ satisfies the coniveau 1 condition for each $i \neq 0$, that is, if any cohomology class in $H^i(X \otimes_W \overline{K}, \mathbf{Q}_l)$ vanishes in $H^i(U, \mathbf{Q}_l)$ when restricted to some nonempty open $U \subset X \otimes_W \overline{K}$, then we have $H^i(X, \mathcal{O}_X) = 0$ for all $i \neq 0$. The converse is true if we assume the generalized Hodge conjecture. It turns out that in this case, we can prove a theorem stronger than Theorem 0.1. We don't need to assume $X_0$ can be lifted to $W$.

THEOREM 0.4. Let $X_0$ be a smooth geometrically connected projective variety over the finite field $\mathbf{F}_q$. Suppose $H^i(X_0, \mathcal{O}_{X_0}) = 0$ for all $i \neq 0$. Then for any natural number $k$, we have

$$\#X_0(\mathbf{F}_{q^k}) \equiv 1 \pmod{q^k}.$$

Let $G$ be a finite group of $\mathbf{F}_q$-automorphisms acting on the right of $X_0$. We have

$$\#(X_0/G)(\mathbf{F}_{q^k}) \equiv \#X_0(\mathbf{F}_{q^k}) \equiv 1 \pmod{q^k}.$$

The liftable condition in Theorem 0.1 cannot be dropped in general. However, if the order of $G$ is prime to $p$, then the liftable condition can be dropped. This is given in the following general result of Berthelot-Bloch-Esnault, proved using their theory of Witt vector cohomology for singular varieties. In contrast, our method is based on crystalline cohomology and the Mazur-Ogus theorem.

THEOREM 0.5 ([BBE]). Let $X_0$ be a proper scheme over $\mathbf{F}_q$, and $G$ a finite group acting on $X_0$ so that each orbit is contained in an affine open subset of $X_0$. Suppose that the order of $G$ is prime to the characteristic $p$ and suppose that $G$ acts trivially on $H^i(X_0, \mathcal{O}_{X_0})$ for all $i$. Then for any natural number $k$, we have the congruence

$$\#X_0(\mathbf{F}_{q^k}) \equiv \#(X_0/G)(\mathbf{F}_{q^k}) \pmod{q^k}.$$

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**1. Proof of the Theorems.** First we introduce some notations. For any smooth proper scheme $X_0$ over $\mathbf{F}_q$, let $H^i(X_0/W)$ be the crystalline cohomology group of $X_0$. It is a finitely generated module over the Witt ring $W = W(\mathbf{F}_q)$. Denote by $F : X_0 \to X_0$ the Frobenius correspondence, that is, it is the identity map on the underlying topological space of $X_0$, and it maps a section of $\mathcal{O}_{X_0}$ to its $q$-th power.

Let $\kappa$ be a field and let $Z$ be a scheme over $\kappa$. Denote by $|Z|$ the set of Zariski closed points in $Z$. For any $z \in |Z|$, define $\deg(z) = [k(z) : \kappa]$, where $k(z)$ is the residue field at $z$. Let $f : Z \to Z$ be a $\kappa$-endomorphism with isolated fixed points. Set

$$Z^f = \{z \in |Z| | f(z) = z \text{ and } f \text{ induces identity on } k(z)\},$$

and define

$$\Lambda(f) = \sum_{z \in Z^f} \deg(z).$$

Let $\kappa'$ be a field extending $\kappa$ and let $f' : Z \otimes_\kappa \kappa' \to Z \otimes_\kappa \kappa'$ be the base change of $f$. Then we have $\Lambda(f) = \Lambda(f')$.

LEMMA 1.1. Let $X_0$ be a smooth projective variety over the finite field $\mathbf{F}_q$, let $g : X_0 \to X_0$ be an $\mathbf{F}_q$-automorphism of finite order, and let $K = \mathrm{Frac}W$ be the fraction field of $W = W(\mathbf{F}_q)$. Then $\mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K)$ and $\mathrm{Tr}(gF^k, H^i(X_0/W) \otimes_W K)$ are algebraic integers for any positive integer $k$ and any $i$, and

$$\Lambda(F^k) = \sum_{i=0}^{2\dim X_0} (-1)^i \mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K),$$

$$\Lambda(gF^k) = \sum_{i=0}^{2\dim X_0} (-1)^i \mathrm{Tr}(gF^k, H^i(X_0/W) \otimes_W K).$$

*Proof.* Let $l$ be a prime number distinct from $p$. By Deligne's theorem ([D] 3.3.9), $\mathrm{Tr}(F^k, H^i(X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q, \overline{\mathbf{Q}}_l))$ are algebraic integers. By the comparison theorem of Katz-Messing ([KM]), we have

$$\mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K) = \mathrm{Tr}(F^k, H^i(X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q, \overline{\mathbf{Q}}_l)).$$

So $\mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K)$ are algebraic integers. The formula for $\Lambda(F^k)$ follows from the Lefschetz fixed point formula in crystalline cohomology theory ([B] Théorème VII 3.1.9).

We will reduce the statements about $gF^k$ to the corresponding statements for $F^k$. Suppose $g : X_0 \to X_0$ has finite order $m$. Let $X_1 = X_0 \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m}$, and let $\varphi \in \mathrm{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q)$ be the Frobenius substitution. For any $\sigma \in \mathrm{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q)$, we have $\sigma = \varphi^k$ for some integer $k$ uniquely determined modulo $m$. Define

$$f_\sigma : X_1 \to X_1$$

to be the isomorphism of schemes

$$f_\sigma = (\mathrm{id}_{X_0} \times \sigma^*) \circ (g^{-k} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}) : X_0 \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m} \to X_0 \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m}.$$

Note that $f_\sigma$ is independent of the choice of $k$ since $g$ has order $m$. Since $g^{-k} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}$ is an $\mathbf{F}_{q^m}$-morphism of $X_1$, the following diagram commutes:

$$\begin{array}{ccc} X_1 & \stackrel{f_\sigma}{\to} & X_1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathbf{F}_{q^m} & \stackrel{\sigma^*}{\to} & \mathrm{Spec}\mathbf{F}_{q^m}. \end{array}$$

Moreover we have

$$f_\tau f_\sigma = f_{\sigma\tau}$$

for any $\sigma, \tau \in \mathrm{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q)$. By the theory of galois descent, ([S] Chapter V, No. 20, or Corollarie 7.7 in [SGA 1] Exposé VIII), there exists a scheme $X_0'$ over $\mathrm{Spec}\mathbf{F}_q$ such that we have an $\mathbf{F}_{q^m}$-isomorphism

$$X_1 \cong X_0' \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m}$$

and the following diagrams commute:

$$\begin{array}{ccc} X_1 & \stackrel{f_\sigma}{\to} & X_1 \\ \cong\downarrow & & \downarrow\cong \\ X_0' \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m} & \stackrel{\mathrm{id}_{X_0'} \times \sigma^*}{\to} & X_0' \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m}. \end{array}$$

For any scheme $Z$ of characteristic $p$, let $F_Z : Z \to Z$ be the Frobenius correspondence, that is, $F_Z$ is identity on the underlying topological space and the morphism of sheaves $F_Z^\sharp : \mathcal{O}_Z \to F_{Z*}\mathcal{O}_Z$ maps each section to its $q$-th power. On $X_1 = X_0 \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m}$, we have

$$\begin{aligned} F_{X_1} &= (\mathrm{id}_{X_0} \times \varphi^*) \circ (F_{X_0} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}) = f_\varphi \circ (g \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}) \circ (F_{X_0} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}) \\ &= f_\varphi \circ (gF_{X_0} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}). \end{aligned}$$

Through the isomorphism $X_1 \cong X_0' \times_{\mathrm{Spec}\mathbf{F}_q} \mathrm{Spec}\mathbf{F}_{q^m}$, $F_{X_1}$ is identified with $(\mathrm{id}_{X_0'} \times \varphi^*) \circ (F_{X_0'} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}})$. Moreover, the commutative diagram above shows that $f_\varphi$ is identified with $\mathrm{id}_{X_0'} \times \varphi^*$. So the morphism $gF_{X_0} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}$ on $X_0 \times_{\mathbf{F}_q} \mathbf{F}_{q^m}$ is identified with the morphism $F_{X_0'} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}$ on $X_0' \times_{\mathrm{Spec}\mathbf{F}_q} \mathbf{F}_{q^m}$. So we have

$$\begin{aligned}
&\mathrm{Tr}\left(gF_{X_0} \times \mathrm{id}_{\mathbf{F}_{q^m}}, H^i\left(X_0 \times_{\mathbf{F}_q} \mathbf{F}_{q^m}/W(\mathbf{F}_{q^m})\right) \otimes_{W(\mathbf{F}_{q^m})} \mathrm{Frac}(W(\mathbf{F}_{q^m}))\right) \\
&= \mathrm{Tr}\left(F_{X_0'} \times \mathrm{id}_{\mathbf{F}_{q^m}}, H^i\left(X_0' \times_{\mathbf{F}_q} \mathbf{F}_{q^m}/W(\mathbf{F}_{q^m})\right) \otimes_{W(\mathbf{F}_{q^m})} \mathrm{Frac}(W(\mathbf{F}_{q^m}))\right).
\end{aligned}$$

By the base change theorem in crystalline cohomology theory ([B] Corollaire V 3.5.7), we have

$$\begin{aligned}
&\mathrm{Tr}(gF_{X_0}, H^i(X_0/W) \otimes_W K) \\
&= \mathrm{Tr}\left(gF_{X_0} \times \mathrm{id}_{\mathbf{F}_{q^m}}, H^i\left(X_0 \times_{\mathbf{F}_q} \mathbf{F}_{q^m}/W(\mathbf{F}_{q^m})\right) \otimes_{W(\mathbf{F}_{q^m})} \mathrm{Frac}(W(\mathbf{F}_{q^m}))\right), \\
&\mathrm{Tr}(F_{X_0'}, H^i(X_0'/W) \otimes_W K) \\
&= \mathrm{Tr}\left(F_{X_0'} \times \mathrm{id}_{\mathbf{F}_{q^m}}, H^i\left(X_0' \times_{\mathbf{F}_q} \mathbf{F}_{q^m}/W(\mathbf{F}_{q^m})\right) \otimes_{W(\mathbf{F}_{q^m})} \mathrm{Frac}(W(\mathbf{F}_{q^m}))\right).
\end{aligned}$$

So we have

$$\mathrm{Tr}(gF_{X_0}, H^i(X_0/W) \otimes_W K) = \mathrm{Tr}(F_{X_0'}, H^i(X_0'/W) \otimes_W K).$$

In particular, $\mathrm{Tr}(gF_{X_0}, H^i(X_0/W) \otimes_W K)$ are algebraic integers for all $i$. Moreover, we have

$$\begin{aligned}
\Lambda(gF_{X_0}) &= \Lambda(gF_{X_0} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}) \\
&= \Lambda(F_{X_{0'}} \times \mathrm{id}_{\mathrm{Spec}\mathbf{F}_{q^m}}) \\
&= \Lambda(F'_{X_0}) \\
&= \sum_{i=0}^{2\dim X_0} (-1)^i \mathrm{Tr}(F_{X_0'}, H^i(X_0'/W) \otimes_W K) \\
&= \sum_{i=0}^{2\dim X_0} (-1)^i \mathrm{Tr}(gF_{X_0}, H^i(X_0/W) \otimes_W K).
\end{aligned}$$

This proves the statements for $gF$. To prove the statements for $gF^k$, we use the base change from $\mathbf{F}_q$ to $\mathbf{F}_{q^k}$.

Lemma 1.2. Under the condition of Theorem 0.1, we have

$$\mathrm{Tr}(gF^k, H^i(X_0/W) \otimes_W K) \equiv \mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K) \pmod{q^k}$$

for all $i$.

*Proof.* Let $H^i = H^i(X_0/W)$. Recall that $H^i$ can be identified with the de Rham cohomology of the lifting $X$ of $X_0$ to $W = W(\mathbf{F}_q)$. (Confer [B] Théorème V 2.3.2). On $H^i$, we have the Hodge filtration

$$H^i = F^0H^i \supset F^1H^i \supset \cdots$$

and this filtration is $G$ stable. By a result of Mazur (the property (8.2) on page 65 of [M]), we have

$$F(F^1H^i) \subset qH^i.$$

We have

$$H^i/F^1H^i = F^0H^i/F^1H^i \cong H^i(X, \mathcal{O}_X).$$

Choose a basis $\{e_1, \ldots, e_s\}$ of $F^1H^i$ and extend it to a basis $\{e_1, \ldots, e_s, e_{s+1}, \ldots, e_{s+t}\}$ of $H^i$. Since $F^k(F^1H^i) \subset q^kH^i$, the matrix of $F^k$ on $H^i$ with respect to the above basis is of the form

$$\begin{pmatrix} q^kA & q^kB \\ C & D \end{pmatrix},$$

where $A$ is an $s \times s$ matrix, $B$ is an $s \times t$ matrix, $C$ is a $t \times s$ matrix, and $D$ is a $t \times t$ matrix. Since $G$ acts trivially on $H^i/F^1H^i \cong H^i(X, \mathcal{O}_X)$ and $G$ preserves the Hodge filtration, the matrix of $g \in G$ on $H^i$ with respect to the above basis is of the form

$$\begin{pmatrix} P & O \\ Q & I \end{pmatrix},$$

where $P$ is an $s \times s$ matrix, $O$ is the $s \times t$ zero matrix, $Q$ is a $t \times s$ matrix, and $I$ is the $t \times t$ identity matrix. So the matrix of $gF^k$ is

$$\begin{pmatrix} q^kA & q^kB \\ C & D \end{pmatrix} \begin{pmatrix} P & O \\ Q & I \end{pmatrix} = \begin{pmatrix} q^kAP + q^kBQ & q^kB \\ CP + DQ & D \end{pmatrix}.$$

We have

$$\mathrm{Tr}(gF^k, H^i) = \mathrm{Tr}(q^kAP + q^kBQ) + \mathrm{Tr}(D).$$

On the other hand, we have

$$\mathrm{Tr}(F^k, H^i) = \mathrm{Tr}(q^kA) + \mathrm{Tr}(D).$$

So we have

$$\mathrm{Tr}(gF^k, H^i) \equiv \mathrm{Tr}(F^k, H^i) \pmod{q^k}.$$

This finishes the proof of Lemma 1.2.

LEMMA 1.3. Let $X_0$ be a quasi-projective scheme over $\mathbf{F}_q$, let $G$ be a finite group acting on the right of $X_0$. Then for any natural number $k$, we have

$$\#(X_0/G)(\mathbf{F}_{q^k}) = \frac{1}{\#G} \sum_{g \in G} \Lambda(gF^k).$$

*Proof.* This result is well known. We include a proof here for completeness. Let $Y_0 = X_0/G$, and let $|X_0|$ (resp. $|Y_0|$) be the set of Zariski closed point in $X_0$ (resp. $Y_0$). For any $x \in |X_0|$, define the decomposition subgroup at $x$ by

$$G_d(x) = \{g \in G | gx = x\}$$

and the inertia subgroup at $x$ by

$$G_i(x) = \{g \in G_d(x) | g \text{ induces identity on the residue field } k(x) \text{ at } x\}.$$

Let $y$ be the image of $x$ in $Y_0$. By Proposition 1.1 in Exposé V of [SGA 1], we have an isomorphism

$$G_d(x)/G_i(x) \cong \mathrm{Gal}(k(x)/k(y)),$$

and for any $y \in |Y_0|$, there are exactly $\frac{\#G}{\#G_d(x)}$ Zariski closed points in $X_0$ above $y$ and each of these closed points has degree $\deg(y)\frac{\#G_d(x)}{\#G_i(x)}$. We have

$$\begin{aligned}
\#Y_0(\mathbf{F}_{q^k}) &= \sum_{y \in |Y_0|,\ \deg(y)|k} \deg(y) \\
&= \frac{1}{\#G} \sum_{y \in |Y_0|,\ \deg(y)|k} \frac{\#G}{\#G_d(x)} \frac{\#G_d(x)}{\#G_i(x)} \#G_i(x)\deg(y) \\
&= \frac{1}{\#G} \sum_{y \in |Y_0|,\ \deg(y)|k} \ \sum_{x \in |X_0|,\ x \mapsto y} \deg(x)\#G_i(x).
\end{aligned}$$

Let $y \in |Y_0|$ be a Zariski closed point with $\deg(y)|k$, let $x \in |X_0|$ be a point above $y$, and let $\phi_y \in \mathrm{Gal}(k(x)/k(y))$ be the Frobenius substitution. Suppose $g \in G_d(x)$ and $g^{-1} \mapsto \phi_y^{\frac{k}{\deg(y)}}$ under the canonical homomorphism $G_d(x) \to \mathrm{Gal}(k(x)/k(y))$. Then $gF^k(x) = x$ and $gF^k$ induces identity on $k(x)$. Conversely, if $x$ is a Zariski closed point in $X_0$ such that $gF^k(x) = x$ and $gF^k$ induces identity on $k(x)$, then $g \in G_d(x)$, $\deg(y)|k$, and $g^{-1} \mapsto \phi_y^{\frac{k}{\deg(y)}}$, where $y$ is the image of $x$ in $Y_0$. On the other hand, there are exactly $\#G_i(x)$ elements $g$ in $G_d(x)$ such that $g^{-1} \mapsto \phi_y^{\frac{k}{\deg(y)}}$. So we finally get

$$\begin{aligned}
\#Y_0(\mathbf{F}_{q^k}) &= \frac{1}{\#G} \sum_{y \in |Y_0|,\ \deg(y)|k} \ \sum_{x \in |X_0|,\ x \mapsto y} \deg(x)\#G_i(x) \\
&= \frac{1}{\#G} \sum_{g \in G} \Lambda(gF^k).
\end{aligned}$$

This proves Lemma 1.3.

Now we are ready to prove Theorem 0.1. By Lemmas 1.3 and 1.1, we have

$$\begin{aligned}
\#(X_0/G)(\mathbf{F}_{q^k}) &= \frac{1}{\#G} \sum_{g \in G} \Lambda(gF^k) \\
&= \frac{1}{\#G} \sum_{g \in G} \sum_{i=0}^{2\dim X_0} (-1)^i \mathrm{Tr}(gF^k, H^i(X_0/W) \otimes_W K).
\end{aligned}$$

By Lemmas 1.1 and 1.2, $\mathrm{Tr}(gF^k, H^i(X_0/W) \otimes_W K)$ and $\mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K)$ are algebraic integers, and

$$\mathrm{Tr}(gF^k, H^i(X_0/W) \otimes_W K) \equiv \mathrm{Tr}(F^k, H^i(X_0/W) \otimes_W K) \pmod{q^k}.$$

From now on, we work over the integral closure of $p$-adic integers. Let $\mathrm{ord}_q(\#G) = c$, a non-negative rational number. For each $k \geq c$, we have

$$\begin{aligned}
\#(X_0/G)(\mathbf{F}_{q^k}) &= \frac{1}{\#G}\sum_{g\in G}\sum_{i=0}^{2\mathrm{dim}X_0}(-1)^i\mathrm{Tr}(gF^k, H^i(X_0/W)\otimes_W K) \\
&\equiv \frac{1}{\#G}\sum_{g\in G}\sum_{i=0}^{2\mathrm{dim}X_0}(-1)^i\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K) \ (\mathrm{mod}\ q^{k-c}) \\
&\equiv \sum_{i=0}^{2\mathrm{dim}X_0}(-1)^i\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K) \ (\mathrm{mod}\ q^{k-c}) \\
&\equiv \#X_0(\mathbf{F}_{q^k}) \ (\mathrm{mod}\ q^{k-c}).
\end{aligned}$$

Let $Z(X_0, T)$ and $Z(X_0, T)$ be the zeta-functions of $X_0$ and $X_0/G$, respectively. They are rational functions. Recall that we have

$$\begin{aligned}
\frac{d}{dT}\log Z(X_0, T) &= \sum_{k=1}^{\infty}\#X_0(\mathbf{F}_{q^k})T^{k-1}, \\
\frac{d}{dT}\log Z(X_0/G, T) &= \sum_{k=1}^{\infty}\#(X_0/G)(\mathbf{F}_{q^k})T^{k-1}.
\end{aligned}$$

Take a factorization

$$\frac{Z(X_0, T)}{Z(X_0/G, T)} = \prod_{i=1}^{m}(1-\alpha_i T)^{-n_i}, \ \alpha_i \neq 0$$

where the $\alpha_i$'s are distinct and the $n_i$'s are non-zero integers. Taking logarithmic derivative on both sides, we get

$$\sum_{k=1}^{\infty}(\#X_0(\mathbf{F}_{q^k}) - \#(X_0/G)(\mathbf{F}_{q^k}))T^{k-1} = \sum_{i=1}^{m}\frac{n_i\alpha_i}{1-\alpha_i T}.$$

Using the congruence

$$\#(X_0/G)(\mathbf{F}_{q^k}) \equiv \#X_0(\mathbf{F}_{q^k}) \ (\mathrm{mod}\ q^{k-c})$$

for all $k \geq c$, one deduces that the above power series is $p$-adic analytic in the open disk $\mathrm{ord}_q(T) > -1$. This implies that each $\alpha_i$ satisfies $\mathrm{ord}_q(\alpha_i) \geq 1$, that is, each $\alpha_i$ is divisible by $q$. We conclude that

$$\#X_0(\mathbf{F}_{q^k}) - \#(X_0/G)(\mathbf{F}_{q^k}) = \sum_{i=1}^{m} n_i\alpha_i^k \equiv 0 \ (\mathrm{mod}\ q^k).$$

This finishes the proof of Theorem 0.1.

Let's prove Theorem 0.4. By Ogus' generalization of Mazur's theorem ([BO] Theorem 8.39), the Newton polygon of the Frobenius correspondence $F$ on $H^i(X_0/W)\otimes_W K$ lies on or above the Hodge polygon of $X_0$. For any $i \neq 0$, we have $H^i(X_0, \mathcal{O}_{X_0}) = 0$. So the slope of each line segment on the Newton polygon is at least 1, that is, all the

eigenvalues of $F^k$ on $H^i(X_0/W)\otimes_W K$ are divisible by $q^k$ (as $p$-adic integers). So we have

$$\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K)\equiv 0 \pmod{q^k}$$

for all $i\neq 0$. Since $X_0$ is geometrically connected, we have

$$\mathrm{Tr}(F^k, H^0(X_0/W)\otimes_W K)=1.$$

So by Lemma 1.1, we have

$$\begin{aligned}\#X_0(\mathbf{F}_{q^k}) &= \sum_{i=0}^{2\dim X_0}(-1)^i\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K)\\ &\equiv 1 \pmod{q^k}.\end{aligned}$$

Now let $G$ be a finite group acting on the right of $X_0$. For any $g\in G$, since $g$ has finite order, the action of $g$ on $H^i(X_0/W)\otimes_W K$ is diagonalizable and all its eigenvalues are roots of unity. Combining with the fact that $F$ commutes with $g$, we see that all the eigenvalues of $gF^k$ on $H^i(X_0/W)\otimes_W K$ are also divisible by $q^k$ for any $i\neq 0$. So we have

$$\mathrm{Tr}(gF^k, H^i(X_0/W)\otimes_W K)\equiv \mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K)\equiv 0 \pmod{q^k}$$

for all $i\neq 0$. Since $X_0$ is geometrically connected, we have

$$\mathrm{Tr}(gF^k, H^0(X_0/W)\otimes_W K)=\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K)=1.$$

Again let $\mathrm{ord}_q(\#G)=c$. For each $k\geq c$, by Lemmas 1.1, 1.3, and the above discussion, we have

$$\begin{aligned}\#(X_0/G)(\mathbf{F}_{q^k}) &= \frac{1}{\#G}\sum_{g\in G}\sum_{i=0}^{2\dim X_0}(-1)^i\mathrm{Tr}(gF^k, H^i(X_0/W)\otimes_W K)\\ &\equiv \frac{1}{\#G}\sum_{g\in G}\sum_{i=0}^{2\dim X_0}(-1)^i\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K) \pmod{q^{k-c}}\\ &\equiv \sum_{i=0}^{2\dim X_0}(-1)^i\mathrm{Tr}(F^k, H^i(X_0/W)\otimes_W K) \pmod{q^{k-c}}\\ &\equiv \#X_0(\mathbf{F}_{q^k}) \pmod{q^{k-c}}.\end{aligned}$$

As in the proof of Theorem 0.1, this implies that

$$\#(X_0/G)(\mathbf{F}_{q^k})\equiv \#X_0(\mathbf{F}_{q^k}) \pmod{q^k}.$$

This finishes the proof of Theorem 0.4.

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