NONEMPTINESS OF SKEW-SYMMETRIC DEGENERACY LOCI[*]

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Abstract. Let V be a rank N vector bundle on a d-dimensional complex projective scheme X; assume that V is equipped with a skew-symmetric bilinear form with values in a line bundle L and that $\Lambda^2 V^* \otimes L$ is ample. Suppose that the maximum rank of the form at any point of X is r, where $r > 0$ is even. The main result of this paper is that if $d > 2(N-r)$, then the locus of points where the rank of the form is at most $r - 2$ is nonempty. The analogous result for symmetric degeneracy loci was proved in [Gra]; the proof here is similar. If the hypothesis of ampleness is relaxed, we obtain a weaker estimate on the maximum dimension of X (and give a similar result for the symmetric case). We give applications to subschemes of skew-symmetric matrices, and to the stratification of the dual of a Lie algebra by orbit dimension.

Key words. degeneracy loci, vector bundles, matrices

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1. Introduction. This paper proves a nonemptiness result for skew-symmetric degeneracy loci. The analogous result for symmetric degeneracy loci was proved in [Gra]. The methods used in this paper are similar; the proof uses ideas of [IL], which are related to work of Fulton, Lazarsfeld, Tu, Harris, and Sommese ([FL1], [FL2], [Laz], [Tu1], [HT], [Som]). The geometry used in the proof is slightly different than in [Gra]: in place of projective and quadric bundles, we use Grassmannian bundles of 2-planes and isotropic 2-planes. The results of this paper do not require the more refined properties of Gysin maps needed to handle the odd rank case in [Gra].

Before stating the main result we illustrate it with an application. Let $SS_r(N)$ denote the projectivization of the space of skew-symmetric $N \times N$ complex matrices of rank at most r. Because the rank of a skew-symmetric matrix is necessarily even, we will assume r is even. The codimension of $SS_{r-2}(N)$ in $SS_r(N)$ is $2(N - r) + 1$ (cf. [Ful1, Ex. 14.4.11], or [IL, 2.5]), so there exist $2(N - r)$-dimensional closed subschemes of $SS_r(N)$ not meeting $SS_{r-2}(N)$. We prove that this is the largest dimension possible.

THEOREM 1.1. *Assume that $r > 0$ is even. If X is a closed subscheme of $SS_r(N)$ not meeting $SS_{r-2}(N)$, then dim $X \leq 2(N - r)$.*

If U is a vector subspace of the space of $N \times N$ complex matrices, we say that U has constant rank r if every nonzero matrix in U has rank r. As a corollary to the above theorem, we obtain the following linear algebra result.

COROLLARY 1.2. *Assume r is even. If U is a constant rank r vector subspace of the space of skew-symmetric $N \times N$ complex matrices, then dim $U \leq 2(N - r) + 1$.*

If $N = r + 1$ then the corollary says that dim $U \leq 3$. The bound in this case is achieved, as shown by an example of Westwick (see [IL, p. 168]).

If $V \to X$ is a vector bundle with a bilinear form with values in a line bundle L, let X_r denote the subscheme of points in X where the rank of the form is at most r.

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Theorem 1.1 is deduced from the following theorem, which is the main result of the paper.

THEOREM 1.3. *Let X be a d-dimensional complex projective scheme and let $V \to X$ be a rank N vector bundle. Suppose that V is equipped with a skew-symmetric bilinear form with values in a line bundle L such that the restriction of the form to any fiber has rank at most r, where $r > 0$ is even. Assume that $\Lambda^2 V^* \otimes L$ is ample. If the form has constant rank r, then $d \leq 2(N - r)$. Equivalently, if $d > 2(N - r)$, then the locus X_{r-2} is nonempty.*

Theorem 1.1 is an immediate consequence, as the trivial bundle $V = \mathbb{C}^N \times SS_r(N) \to SS_r(N)$ is equipped with a skew-symmetric form with values in $L = \mathcal{O}(1)$, and the bundle $\Lambda^2 V^* \otimes L$ is ample. Observe that as a consequence of this theorem one obtains a stepwise proof of the nonemptiness of skew-symmetric degeneracy loci, a result proved by other methods in [FL2, Prop. 3.5].

If the hypothesis of ampleness in the main theorem is relaxed, we obtain a weaker estimate on the maximum dimension of X (see Theorem 3.3). A similar result holds in the symmetric case; see Remark 3.4.

One natural example of a skew-symmetric map is the Lie bracket on a Lie algebra $\mathfrak{g}$. From this, one can define a skew-symmetric form ω on the bundle $\mathfrak{g} \times P(\mathfrak{g}^*) \to P(\mathfrak{g}^*)$, with values in $\mathcal{O}_{P(\mathfrak{g}^*)}(1)$. The orbits of the algebraic group G on $\mathfrak{g}^*$ all have even dimension. Let

$$\mathfrak{g}_r^* = \{\lambda \in \mathfrak{g}^* \mid \dim G \cdot \lambda \leq r\};$$

then $\mathfrak{g}_r^*$ is a closed conical subset of $\mathfrak{g}^*$, and the projectivization $P(\mathfrak{g}_r^*)$ coincides with the locus $P(\mathfrak{g}^*)_r$ defined using the form ω. Applying our main theorem yields a result (Proposition 4.3) about the stratification of $\mathfrak{g}^*$ by orbit dimension. As a consequence of this result we obtain the following bound on the size of minimal dimensional orbits in $\mathfrak{g}^*$ (Corollary 4.4): if $N = \dim \mathfrak{g}$, and r is the minimal dimension of a nonzero G-orbit on $\mathfrak{g}^*$, then

$$r \leq 2 \left\lfloor \frac{2N + 1}{6} \right\rfloor.$$

This bound is achieved in the example of the minimal nonzero coadjoint orbits of the group SL_3.

The contents of the paper are as follows. Section 2 contains some preliminary results used in the proof of the main theorem. In particular, we define projection from a subspace in a Grassmannian (or Grassmannian bundle) to a smaller Grassmannian, generalizing the construction for projective space, and we prove that this map is an affine linear bundle map (Proposition 2.5). The proof of the main theorem and its generalized version are given in Section 3, and the application to Lie algebras is given in Section 4.

I would like to thank Robert Varley for encouraging me to consider bundles that are not necessarily ample.

Conventions and notation. Schemes are of finite type over $\mathbb{C}$; all algebraic groups are assumed to be linear. Homology and cohomology are taken with *rational* coefficients, unless otherwise noted. A symplectic form on an r-dimensional vector space is a nondegenerate skew-symmetric form; r must be even. Similarly, we speak of a symplectic form on a rank r vector bundle; such a form may take values in a line bundle. If V and W are vector bundles, $\mathrm{Hom}(V, W)$ denotes the vector bundle $V^* \otimes W$, and $\mathbb{G}_s(V)$ the Grassmann bundle of s-dimensional planes in V.

2. Preliminaries. Let r be even, and let $(\ ,\ )$ be a symplectic form on $\mathbb{C}^r$. We define the conformal symplectic group GSp_r as the set of all g in GL_r such that for all $v, w \in \mathbb{C}^r$,

$$(gv, gw) = \tau(g)(v, w), \tag{2.1}$$

where $\tau(g)$ depends only on g. Equivalently, g must satisfy the condition $g^t M g = \tau(g)M$. This group is, up to isomorphism, independent of the choice of symplectic form. The reason is that all symplectic forms on $\mathbb{C}^r$ are equivalent, in the sense that given a second symplectic form on $\mathbb{C}^r$, there is a linear automorphism of $\mathbb{C}^r$ such that the pullback of the second symplectic form is our original form.

As in the case of the orthogonal groups, (2.1) implies that $(\det g)^2 = \tau(g)^r$. However, in the symplectic case, a stronger result holds. We include a proof for lack of a reference.

PROPOSITION 2.1. *The group GSp_r is connected, and any $g \in GSp_r$ satisfies $\det g = \tau(g)^{r/2}$.*

Proof. For $r = 2$, direct computation shows that $GSp_2 = GL_2$ and $\det g = \tau(g)$. In general, GSp_r acts transitively on P^{r-1}. Let P denote the stabilizer in GSp_r of a point x of P^{r-1}. Then P is a parabolic subgroup of GSp_r, with Levi factor L isomorphic to $\mathbb{C}^* \times GSp_{r-2}$. By induction we may assume that L is connected, and hence so is P. Since $GSp_r/P \simeq P^{r-1}$ is also connected, we see that GSp_r is connected. Since the function

$$g \mapsto \frac{\tau(g)^{r/2}}{\det g}$$

has square 1, its only possible values are ± 1. Hence the function is constant on connected components; since GSp_r is connected, the function is identically 1. $\square$

The (homology) Poincaré polynomial of a space X is by definition $P_t(X) = \sum b_i t^i$, where $b_i = \dim H_i(X)$ is the i-th Betti number of X (recall that we take homology and cohomology with rational coefficients).

PROPOSITION 2.2. *Let W be a symplectic vector space of dimension $r = 2n$, and let $\mathbb{G}_2(W)$ (resp. Z) denote the Grassmanian of (resp. isotropic) 2-planes in W. Then the odd homology groups of $\mathbb{G}_2(W)$ and Z vanish, and, setting $q = t^2$, the Poincaré polynomials of these spaces are given by*

$$
\begin{aligned}
P_t(\mathbb{G}_2(W)) &= 1 + q + 2q^2 + 2q^3 + \cdots + (n-1)q^{r-4} + (n-1)q^{r-3} \\
&\quad + nq^{r-2} + (n-1)q^{r-1} + \cdots + q^{2r-4} \\
P_t(Z) &= 1 + q + 2q^2 + 2q^3 + \cdots + (n-1)q^{r-4} + (n-1)q^{r-3} \\
&\quad + (n-1)q^{r-2} + (n-1)q^{r-1} + \cdots + q^{2r-5}.
\end{aligned}
\tag{2.2}
$$

(Since these spaces are compact orientable manifolds, the polynomials are palindromic.)

Proof. If G is a connected reductive group, and $P \supset B$ are a parabolic subgroup and a Borel subgroup, respectively, then there is a fibration $G/B \to G/P$, with fibers P/B. Because the odd cohomology of these spaces vanishes, the cohomology spectral sequence of the fibration degenerates at E_2, so

$$P_t(G/P) = \frac{P_t(G/B)}{P_t(P/B)}.$$

The numerator and denominator of the right hand side can be calculated in terms of Weyl groups. If l is the rank of the semisimple part of $\mathfrak{g} = \text{Lie } G$, then there exist fundamental invariants $d_1, \ldots, d_l$ of the Weyl group W of $\mathfrak{g}$, called exponents, in terms of which we can express the Poincaré polynomial of G/B:

$$P_t(G/B) = \sum_{w \in W} q^{l(w)} = \frac{\prod_i (q^{d_i} - 1)}{(q-1)^l}.$$

Here $l(w)$ is the length of w, the first equality follows from the Bruhat decomposition, and the second is in [Hum]. Because $P/B = L/B_L$, where the Levi factor L of P is reductive, and B_L is a Borel subgroup of L, $P_t(P/B)$ is calculated by a similar formula using the exponents of the Weyl group of $\mathfrak{l} = \text{Lie } L$. The exponents of all Weyl groups of simple complex Lie algebras are known; for $\mathfrak{sl}_r$ (type A_{r-1}) they are $2, 3, \ldots, r$; for $\mathfrak{sp}_r$ (type $C_{r/2}$, with r even), they are $2, 4, 6, \ldots, r$ (see [Hum]). If a group is reductive, then its Lie algebra has a semisimple part which is a product of simple Lie algebras, and its set of exponents is the union (with multiplicities) of the sets of exponents corresponding to those simple Lie algebras.

The Grassmannian $\mathbb{G}_2(W)$ equals G/P, where $G = SL_r$ and $\mathfrak{p} = \text{Lie } P$ has Levi factor with semisimple part isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_{r-2}$. Similarly, the isotropic Grassmannian Z equals G/P, where $G = Sp_r$ and $\mathfrak{p}$ has Levi factor with semisimple part isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sp}_{r-4}$. The desired formulas for $P_t(\mathbb{G}_2(W))$ and $P_t(Z)$ follow (with a little algebra) from this, using the facts in the previous paragraph. ☐

Note that as complex varieties, $\dim \mathbb{G}_2(W) = 2r - 4$ and $\dim Z = 2r - 5$, as follows from the above proposition (or by directly computing the dimensions of the groups G and P used to realize these spaces as homogeneous spaces).

PROPOSITION 2.3. *Let* $W \to X$ *be a rank* r *vector bundle over a* d-*dimensional scheme* X. *Assume that* W *is equipped with a symplectic form with values in a line bundle* L. *Let* $\mathbb{G}_2(W)$ *(resp.* Z*) denote the Grassmannian bundles of (resp. isotropic) 2-planes in* W. *Let* $b_i = \dim H_i(X)$ *for* $i \geq 0$, *and* $b_i = 0$ *for* $i < 0$. *Then*

$$
\begin{aligned}
\dim H_{2d+2r-4}(\mathbb{G}_2(W)) \;=\; & nb_{2d} + (n-1)b_{2d-2} + (n-1)b_{2d-4} + (n-2)b_{2d-6} \\
& + (n-2)b_{2d-8} + \cdots + b_{2d-(4r-6)} + b_{2d-(4r-8)} \\
\dim H_{2d+2r-6}(Z) \;=\; & \dim H_{2d+2r-4}(\mathbb{G}_2(W)) - b_{2d}.
\end{aligned}
$$
$$(2.3)$$

Hence if X *is complete and irreducible,*

$$\dim H_{2d+2r-4}(\mathbb{G}_2(W)) = \dim H_{2d+2r-6}(Z) + 1. \qquad (2.4)$$

Proof. Since both $\mathbb{G}_2(W)$ and Z are partial flag bundles associated to principal bundles for connected groups (which are, respectively, GL_r and GSp_r), the homology of each of these bundles is isomorphic to the tensor product of the homology of X with the homology of the fiber ([Ler]; an argument is also given in [Gra, Prop. 4.4]). Combining this with the previous proposition yields (2.3); (2.4) follows from this, since if X is complete and irreducible then $b_{2d} = 1$ [Ful1, Lemma 19.1.1]. ☐

We now discuss affine linear bundles on schemes. By an affine linear bundle of rank n we mean a fiber bundle with fibers isomorphic to $\mathbb{C}^n$, but with structure group equal to the group $M(n)$ generated by GL_n and translations. We require that such a bundle be locally trivial in the Zariski topology. This definition differs from but is

equivalent to the definition in [Bry]; that paper uses the term affine bundle. However, because this term is used in [Ful1, p. 22] in a weaker sense (there the structure group is not required to be $M(n)$), we have used the term affine linear bundle.

We can identify $M(n)$ modulo translations with GL_n, and so we have a natural surjection $\pi : M(n) \to GL_n$. Given a collection of transition functions for an affine linear bundle $E \to X$, composing those functions with π yields a collection of transition functions for an associated vector bundle $V \to X$ (in [Bry], the term associated vector bundle is used in a different sense). The structure group of E reduces to GL_n if and only if E has a section; in this case, E and its associated vector bundle V are isomorphic schemes.

REMARK 2.4. In the usual (complex) topology, E always has a (continuous) section [Ste, 12.2] and thus E and V are homeomorphic. Therefore E and V are both homotopy equivalent to X. However, in general E and V need not be isomorphic as schemes. For example, if $G = SL_2$, and T (resp. B) is the subgroup of diagonal (resp. upper triangular) matrices, then G/T is an affine variety which is an affine linear bundle over $G/B = P^1$, and the associated vector bundle is the cotangent bundle of P^1 (see [Bry]), which is not an affine variety.

Let

$$0 \to K \to V \xrightarrow{\rho} W \to 0$$

be an exact sequence of vector bundles on a scheme X, of ranks $N - r$, N, and r, respectively. Let $\mathbb{G}_s(V)_K$ denote the open subscheme of $\mathbb{G}_s(V)$ whose fiber over any $x \in X$ consists of those points in $\mathbb{G}_s(V_x)$ corresponding to those subspaces of V_x whose intersection with K_x is $\{0\}$. Assume that $\mathbb{G}_s(V)_K$ is nonempty, which is equivalent to saying that $s \leq r$. Define a map

$$\pi : \mathbb{G}_s(V)_K \to \mathbb{G}_s(W) \tag{2.5}$$

by sending $p \in \mathbb{G}_s(V_x)$ to $\rho(p) \in \mathbb{G}_s(W_x)$. We call π the projection from the subbundle K.

PROPOSITION 2.5. *Let*

$$0 \to K \to V \xrightarrow{\rho} W \to 0 \tag{2.6}$$

be an exact sequence of vector bundles on a scheme X, of ranks $N - r$, N, and r, respectively. Let $s \leq r$. Let $\nu : \mathbb{G}_s(W) \to X$ denote the projection, and let $S \to \mathbb{G}_s(W)$ denote the tautological rank s subbundle of ν^*W. Then $\pi : \mathbb{G}_s(V)_K \to \mathbb{G}_s(W)$ has the structure of an affine linear bundle on $\mathbb{G}_s(W)$ of rank $s(N - r)$, with associated vector bundle $\mathrm{Hom}(S, \nu^*K)$.

Before giving the proof we need some preliminaries. First, note that $\mathbb{G}_r(W) = X$. Also, we can identify $\mathbb{G}_r(V)_K$ with the closed subscheme $E = E(W, K)$ of $\mathrm{Hom}(W, V)$ defined as follows. There is a natural map $\eta : \mathrm{Hom}(W, V) \to \mathrm{Hom}(W, W)$. Let X_1 denote the closed subscheme of $\mathrm{Hom}(W, W)$ corresponding to the identity section, and let

$$E = E(W, K) = \eta^{-1}(X_1). \tag{2.7}$$

Note that the fiber E_x is given by

$$E_x = \{f \in \mathrm{Hom}(W_x, V_x) \mid \rho \circ f = \mathrm{id}\}.$$

The isomorphism

$$E \to \mathbb{G}_r(V)_K$$

takes $f \in E_x$ to the subspace $f(W_x) \in \mathbb{G}_r(V_x)$. The inverse map takes $U_x \in (\mathbb{G}_r(V)_K)_x$ to the map $f = (\rho|_{U_x})^{-1} \in E_x$.

As stated in [Ful2], E is an affine bundle over X. The next lemma (the special case $s = r$ of Proposition 2.5) shows that E is in fact an affine linear bundle. In the case where (2.6) is the tautological exact sequence of bundles on the Grassmannian $\mathbb{G}_{N-r}(\mathbb{C}^N)$, a version of this lemma is [Bry, Ex. 4.3].

LEMMA 2.6. *Let* $0 \to K \to V \xrightarrow{\rho} W \to 0$ *be an exact sequence of vector bundles on a scheme X, of ranks $N - r$, N, and r, respectively, and let E be as defined in (2.7). Then E has a natural structure of an affine linear bundle on X, with associated vector bundle $Hom(W, K)$.*

Proof. We begin by choosing compatible trivializations of the vector bundles K, V, and W. Precisely, cover X by (Zariski) open sets on which we have trivializing isomorphisms

$$\beta_i : V|_{U_i} \xrightarrow{\cong} \mathbb{C}^N \times U_i$$

such that, if α_i is the restriction of β_i to $K|_{U_i}$, then α_i takes $K|_{U_i}$ isomorphically to the subbundle $(\mathbb{C}^{N-r} \times \{0\}) \times U_i$ of $\mathbb{C}^N \times U_i$. There is an induced isomorphism

$$\gamma_i : W|_{U_i} \xrightarrow{\cong} \mathbb{C}^r \times U_i.$$

We denote the transition functions for these bundles by α_{ij}, β_{ij}, and γ_{ij}, so, writing $U_{ij} = U_i \cap U_j$ and we have $\beta_{ij} : U_{ij} \to GL_N$, etc. (To be precise, write $V|_{ij} = V|_{U_{ij}}$. Our convention is that if S is a scheme and v and x are S-valued points of $\mathbb{C}^N$ and U_{ij}, respectively, we define $\beta_{ij}(x)$ by the equation

$$\beta_i|_{V_{ij}} \circ (\beta_j|_{V_{ij}})^{-1}(v, x) = (\beta_{ij}(x)v, x),$$

and similarly for α_{ij}, γ_{ij}.) Note that for a point x of U_{ij}, the matrix β_{ij} has the block form

$$\beta_{ij}(x) = \begin{bmatrix} \alpha_{ij}(x) & M_{ij}(x) \\ 0 & \gamma_{ij}(x) \end{bmatrix}$$

for some matrix $M_{ij}(x)$.

Our trivializations induce trivializing isomorphisms

$$\delta_i : \mathrm{Hom}(W, K)|_{U_i} \xrightarrow{\cong} \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r}) \times U_i.$$

Let

$$\delta_{ij} : U_{ij} \to GL(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r}))$$

denote the corresponding transition functions; if $f \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r})$ and $x \in U_{ij}$, then

$$\delta_{ij}(x)(f) = \alpha_{ij}(x) \circ f \circ \gamma_{ij}(x)^{-1}. \tag{2.8}$$

We have additional induced trivializing isomorphisms

$$\zeta_i : \mathrm{Hom}(W, V)|_{U_i} \xrightarrow{\cong} \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r}) \times U_i.$$

Write $\bar{\zeta}_i$ for the restriction of ζ_i to $E|_{U_i}$.

Let $p : \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^N) \to \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r})$ be induced by projection of $\mathbb{C}^N$ onto the first $N - r$ coordinates. Define maps

$$\varepsilon_i : E|_{U_i} \to \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r}) \times U_i$$

by

$$\varepsilon_i = (p \times \mathrm{id}) \circ \bar{\zeta}_i.$$

One can verify that the ε_i are isomorphisms and that the corresponding transition functions ε_{ij} are as follows. If $f \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{N-r})$ and $x \in U_{ij}$, then

$$\varepsilon_{ij}(x)(f) = \alpha_{ij}(x) \circ f \circ \gamma_{ij}(x)^{-1} + M_{ij}(x)\gamma_{ij}(x)^{-1}. \tag{2.9}$$

Therefore the ε_i give E the structure of an affine linear bundle; comparing (2.8) and (2.9), we see that the associated vector bundle is $\mathrm{Hom}(W, K)$. $\square$

Proof of Proposition 2.5. Let $\nu : \mathbb{G}_s(W) \to X$ denote the projection. The pullback by ν of the exact sequence (2.6) is an exact sequence of vector bundles on $\mathbb{G}_s(W)$:

$$0 \to \nu^* K \to \nu^* V \xrightarrow{\nu^* \rho} \nu^* W \to 0.$$

Let $S \subset \nu^* W$ denote the tautological rank s subbundle, and $B = (\nu^* \rho)^{-1}(S)$. The sequence

$$0 \to \nu^* K \to B \to S \to 0$$

is again exact. Let $\tilde{\phi}$ denote the composition

$$\mathbb{G}_s(B) \to \mathbb{G}_s(\nu^* V) \cong \mathbb{G}_s(W) \times_X \mathbb{G}_s(V) \to \mathbb{G}_s(V),$$

where the first map is induced by the bundle inclusion $B \subset \nu^* V$, and the second map is the natural projection. Concretely, if $q \in \mathbb{G}_s(W_x)$ and $p \in \mathbb{G}_s(B_q)$, then the inclusion

$$B_q \subset \nu^* V_q = V_{\nu(q)}$$

means that p defines a point of $\mathbb{G}_s(V_{\nu(q)})$; that point is $\tilde{\phi}(p)$. Note that $\tilde{\phi}^{-1}(\mathbb{G}_s(V)_K) = \mathbb{G}_s(B)_{\nu^* K}$. Let

$$\phi = \tilde{\phi}|_{\mathbb{G}_s(B)_{\nu^* K}} : \mathbb{G}_s(B)_{\nu^* K} \to \mathbb{G}_s(V)_K.$$

We claim that ϕ is an isomorphism of schemes over $\mathbb{G}_s(W)$. By Lemma 2.6, this suffices, since $\mathbb{G}_s(B)_{\nu^* K}$ can be identified with $E(S, \nu^* K)$.

To prove the claim, by working locally on X, we may assume that the bundles K, V and W are trivial, and thereby reduce to the case where X is a point. In this case, K, V, and W are just vector spaces. Then

$$
\begin{aligned}
\mathbb{G}_s(B) &\cong \{(p, q) \mid p \in \mathbb{G}_s(V),\ q \in \mathbb{G}_s(W),\ \tilde{q} := \rho^{-1}(q) \supset p\} \\
&\cong \{(\tilde{q} \supset p) \mid p \in \mathbb{G}_s(V),\ \tilde{q} \in \mathbb{G}_{s+N-r}(V),\ \tilde{q} \supset K\}.
\end{aligned}
$$

The projection $\mathbb{G}_s(B) \to \mathbb{G}_s(W)$ takes $(\tilde{q} \supset p)$ to $\rho(\tilde{q})$. The open subvariety $\mathbb{G}_s(B)_{\nu^* K}$ consists of $(\tilde{q} \supset p)$ as above satisfying the additional condition $p \cap K = \{0\}$, i.e., $p \in \mathbb{G}_s(V)_K$. The map ϕ takes $(\tilde{q} \supset p)$ to p. The map $p \mapsto (p + K, p)$ is inverse to ϕ; hence ϕ is an isomorphism. Finally, ϕ is compatible with the maps to $\mathbb{G}_s(W)$, since given $(\tilde{q} \supset p)$ as above with $p \in \mathbb{G}_s(V)_K$, we have $\rho(\tilde{q}) = \rho(p)$. □

COROLLARY 2.7. *Let $V \to X$ be a rank N vector bundle with a skew-symmetric form with values in a line bundle L. Assume that the form has constant rank r. Let K denote the radical of the form, and let $W = V/K$; then W inherits an L-valued skew-symmetric form V. Let $Z \subset \mathbb{G}_2(W)$ and $\tilde{Z} \subset \mathbb{G}_2(V)$ denote the subbundles of isotropic 2-planes. Then $\mathbb{G}_2(V) - \tilde{Z}$ is an affine linear bundle over $\mathbb{G}_2(W) - Z$, of rank $2(N - r)$.*

Proof. This follows from Proposition 2.5, since under the map

$$\pi : \mathbb{G}_2(V)_K \to \mathbb{G}_2(W),$$

the inverse image of $\mathbb{G}_2(W) - Z$ is $\mathbb{G}_2(V) - \tilde{Z}$. □

3. Proofs of the main results.

3.1. Proof of Theorem 1.3. The proof is parallel to that of the main theorem in [Gra]. We may assume that X is irreducible of dimension d. We assume that the form is of constant rank r, where r is even and positive; we must show that $d \leq 2(N - r)$. Let K denote the radical of the form; K is a vector subbundle of V, of rank $N - r$. Let $W = V/K$; then W is a rank r vector bundle on X, equipped with a nondegenerate skew-symmetric form with values in L. Let $\mathbb{G}_2(V)$ (resp. $\mathbb{G}_2(W)$) denote the Grassmann bundle of 2-planes in V (resp. W), and let $\tilde{Z} \subset \mathbb{G}_2(V)$ (resp. $Z \subset \mathbb{G}_2(W)$) denote the subbundle of isotropic 2-planes. By Corollary 2.7, $\mathbb{G}_2(V) - \tilde{Z}$ is a rank $2(N-r)$ affine linear bundle over $\mathbb{G}_2(W) - Z$, so by Remark 2.4, the homology groups of these spaces are isomorphic.

Let π_1 and π_2 denote the projections from $\mathbb{G}_2(V)$ and $P(\Lambda^2 V)$ to X. We claim that there exists a section of an ample line bundle on $\mathbb{G}_2(V)$ vanishing only on $\tilde{Z}$. Indeed, since $\Lambda^2 V^* \otimes L$ is ample, the line bundle $\mathcal{O}_{P(\Lambda^2 V \otimes L^*)}(1)$ is ample. Under the natural isomorphism between $P(\Lambda^2 V)$ and $P(\Lambda^2 V \otimes L^*)$, this line bundle corresponds to the line bundle $L' = \mathcal{O}_{P(\Lambda^2 V)}(1) \otimes \pi_2^* L$, which is therefore an ample line bundle on $P(\Lambda^2 V)$. Let $S \to \mathbb{G}_2(V)$ be the tautological rank 2 subbundle. Under the Plücker embedding $\mathbb{G}_2(V) \hookrightarrow P(\Lambda^2 V)$, L' pulls back to $L'' = \Lambda^2 S^* \otimes \pi_1^* L$, which is therefore ample. Our skew-symmetric form is a section of the bundle $\Lambda^2 V^* \otimes L \to X$; this pulls back to a section of $\pi_1^*(\Lambda^2 V^* \otimes L) \to \mathbb{G}_2(V)$. Composing this section with the natural map $\pi_1^*(\Lambda^2 V^* \otimes L) \to L''$ yields a section of L''. This section vanishes at $p \in \mathbb{G}_2(V)$ if and only if p is isotropic, proving the claim.

Because $\mathbb{G}_2(V)$ is projective, the claim implies that $\mathbb{G}_2(V) - \tilde{Z}$ is an affine scheme. Since the dimension of $\mathbb{G}_2(V) - \tilde{Z}$ is $2(N - 2) + d$, and since the homology groups of this space are isomorphic to those of $\mathbb{G}_2(W) - Z$, we conclude that

$$H_j(\mathbb{G}_2(W) - Z) = 0 \quad \text{for } j > 2(N - 2) + d. \tag{3.1}$$

The map $i : Z \hookrightarrow \mathbb{G}_2(W)$ is a regular embedding and there exists a tubular neighborhood of Z in $\mathbb{G}_2(W)$ (cf. [Gra, Prop. 2.5]). Therefore there is a Gysin sequence

$$\cdots \to H_j(\mathbb{G}_2(W) - Z) \to H_j(\mathbb{G}_2(W)) \xrightarrow{i^*} H_{j-2}(Z) \to H_{j-1}(\mathbb{G}_2(W) - Z) \to \cdots \tag{3.2}$$

(see [Gra, Section 3]). This exact sequence and (3.1) imply that $i^* : H_j(\mathbb{G}_2(W)) \to H_{j-2}(Z)$ is injective for $j > 2(N-2) + d$. On the other hand,

$$\dim H_{2d+2r-4}(\mathbb{G}_2(W)) = \dim H_{2d+2r-6}(Z) + 1$$

(Proposition 2.3). Hence $2d + 2r - 4 \le 2(N-2) + d$, so $d \le 2(N-r)$, as desired. This completes the proof.

REMARK 3.1. The preceding proof did not use the more refined properties of Gysin maps needed in [Gra] to handle the case of odd rank symmetric degeneracy loci. In fact, an exact sequence similar to (3.2) can be constructed without reference to Gysin maps, and then the proof will go through. The analogous construction is given in [Gra, Section 4]; we omit details here.

We can relax the hypothesis of ampleness and obtain a weaker bound on the dimension of X, using a result of Goresky and MacPherson. We begin with some definitions. If $\pi : X \to Y$ is a morphism of varieties, where X has pure dimension d, finitely decompose X into subvarieties V_i so that $\pi|_{V_i}$ has constant fiber dimension. Goresky and MacPherson define the defect $D(\pi)$ of π as the supremum over i of the fiber dimension of $\pi|_{V_i}$ minus the codimension of V_i. They prove that if π is proper and Y is affine, then X has the homotopy type of a CW complex of real dimension at most $d + e$, where $e = D(\pi)$. In particular, $H_i(X) = 0$ for $i > d + e$, and $H_{d+e}(X; \mathbb{Z})$ is torsion-free. See [GM, pp. 25, 152].

If L is a line bundle on a complete scheme X, we will say that L has defect $D(L) = e$ if some tensor power $L^{\otimes k}$ is pulled back to X by a morphism $\pi : X \to P^l$ with $D(\pi) = e$. Line bundles with defect 0 are called lef by de Cataldo and Migliorini [dCM], and studied there. Ample line bundles have defect 0.

The following lemma is an immediate consequence of the result of Goresky and MacPherson stated above. For n-ample line bundles, a similar cohomology vanishing theorem appears as [Tu2, Theorem 6.2], with a proof by Harris.

LEMMA 3.2. *Let L be a line bundle on an irreducible projective scheme X of dimension d, and let Z be the zero-scheme of a non-zero section of L. Assume that $D(L) = e$. Then $H_i(X - Z) = 0$ for $i > d + e$, and $H_{d+e}(X - Z; \mathbb{Z})$ is torsion-free.* $\square$

We can now state the generalization of our main theorem.

THEOREM 3.3. *Assume the hypotheses of Theorem 1.3, except replace ampleness of $\Lambda^2 V^* \otimes L$ by the following hypothesis: with notation as in the proof of Theorem 1.3, assume that the bundle $\Lambda^2 S^* \otimes \pi_1^* L$ on $\mathbb{G}_2(V)$ has defect e. If the form has constant rank r, then $d \le 2(N-r) + e$. Equivalently, if $d > 2(N-r) + e$, then the locus X_{r-2} is nonempty.*

Proof. Using Lemma 3.2, the proof of Theorem 3.3 is a simple modification of the proof of Theorem 1.3; it is only necessary to replace the estimate (3.1) by the estimate

$$H_j(\mathbb{G}_2(W) - Z) = 0 \quad \text{for } j > 2(N-2) + d + e. \tag{3.3}$$

$\square$

REMARK 3.4. *In [Gra], we proved* that if $V \to X$ is a rank N vector bundle on a d-dimensional projective scheme X, with an L-valued quadratic form such that $S^2 V^* \otimes L$ is ample, and the quadratic form has constant rank r, then $d \le N - r$.

Let $\pi : P(V) \to X$ denote the projection. If instead of assuming that the bundle $S^2 V^* \otimes L$ on X is ample, we assume that the defect of the bundle $\mathcal{O}_{P(V)}(2) \otimes \pi^* L$ on $P(V)$ is e, then we obtain the estimate $d \le N - r + e$.

4. Example: Duals of Lie algebras. In this section we apply our main theorem to the stratification of the dual of a Lie algebra by orbit dimension.

Let G be an algebraic group and $\mathfrak{g}$ its Lie algebra. Let $\langle \ , \ \rangle$ denote the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Define a skew-symmetric form ω on the trivial bundle $\mathfrak{g} \times P(\mathfrak{g}^*) \to P(\mathfrak{g}^*)$, with values in $\mathcal{O}_{P(\mathfrak{g}^*)}(1)$, as follows. If $\lambda \in \mathfrak{g}^* - \{0\}$, let $[\lambda]$ denote the corresponding point in $P(\mathfrak{g}^*)$. Then $[\lambda]$ is a line in $\mathfrak{g}$, with dual space $[\lambda]^*$. If $x, y \in \mathfrak{g}$, then $\omega_{[\lambda]}(x, y)$ is the element of $[\lambda]^*$ satisfying

$$\omega_{[\lambda]}(x, y)(\mu) = \langle \mu, [x, y] \rangle,$$

for $\mu \in [\lambda]$.

REMARK 4.1. The form defined above can be viewed as a special case of the following construction (cf. [Gra]). If $V \to X$ is a vector bundle with a bilinear form with values in a vector bundle W on X, consider $\rho : P(W^*) \to X$. The vector bundle $\rho^* V$ has a bilinear form with values in $\mathcal{O}_{P(W^*)}(1)$. This is defined by composing the natural $\rho^* W$-valued bilinear form on $\rho^* V$ with the projection $\rho^* W \to S^*$, where $S = \mathcal{O}_{P(W^*)}(-1)$ is the tautological subbundle. The form ω on the bundle $\mathfrak{g} \times P(\mathfrak{g}^*) \to P(\mathfrak{g}^*)$ is obtained by applying this construction to $\mathfrak{g}$, viewed as a bundle over a point.

The group G acts on $\mathfrak{g}^*$ via the coadjoint action. Define

$$\mathfrak{g}^*_r = \{\lambda \in \mathfrak{g}^* \mid \dim G \cdot \lambda \le r\}.$$

This is a conical closed subset of $\mathfrak{g}^*$, so we obtain $P(\mathfrak{g}^*_r) \subset P(\mathfrak{g}^*)$.

Any coadjoint orbit has a symplectic form (defined by Kirillov and Kostant) and hence is even-dimensional. The form ω is closely related to the symplectic form on coadjoint orbits; this is reflected in the following lemma (in which we equip closed subschemes with the reduced scheme structures).

LEMMA 4.2. *The subscheme $P(\mathfrak{g}^*_r)$ of $P(\mathfrak{g}^*)$ is equal to the subscheme $P(\mathfrak{g}^*)_r$ of points in $P(\mathfrak{g}^*)$ where the form ω has rank at most r.*

Proof. Let G^λ denote the stabilizer of λ in G, and $\mathfrak{g}^\lambda$ the Lie algebra of G^λ. It is enough to show that the radical of $\omega_{[\lambda]}$ is $\mathfrak{g}^\lambda$. Let ad denote the adjoint action of $\mathfrak{g}$ on itself, and ad^+ the dual action on $\mathfrak{g}^*$. By definition of the dual action, for all $x, y \in \mathfrak{g}$,

$$\langle \mathrm{ad}^+(x)\lambda, y \rangle + \langle \lambda, \mathrm{ad}(x)y \rangle = 0. \tag{4.1}$$

Thus, $x \in \mathfrak{g}$ is in the radical of $\omega_{[\lambda]}$ iff $\langle \lambda, \mathrm{ad}(x)y \rangle = 0$ for all y in $\mathfrak{g}$ iff $\mathrm{ad}^+(x)\lambda = 0$, which is equivalent to $x \in \mathfrak{g}^\lambda$. □

Applying our main theorem to the stratification of $P(\mathfrak{g}^*)$ yields as an immediate consequence a result about conical subvarieties of $\mathfrak{g}^*$.

PROPOSITION 4.3. *Let G be an N-dimensional algebraic group with Lie algebra $\mathfrak{g}$. Let $r > 0$ be even, and let $\mathfrak{g}^*_r$ denote the subscheme of all points in $\mathfrak{g}^*$ whose G-orbits have dimension at most r. If X is a conical closed subscheme of $\mathfrak{g}^*_r$ meeting $\mathfrak{g}^*_{r-2}$ only at 0, then $\dim X \le 2(N - r) + 1$.* □

As a corollary we obtain the following result about minimal orbits.

COROLLARY 4.4. *Let G be an N-dimensional algebraic group with Lie algebra $\mathfrak{g}$. Let r be the minimal dimension of a nonzero G-orbit in $\mathfrak{g}^*$. Then*

$$r \leq 2 \left\lfloor \frac{2N+1}{6} \right\rfloor.$$

Proof. We may assume $r > 0$. Since $\mathfrak{g}_r^*$ is a conical closed subscheme of $\mathfrak{g}^*$ which contains subvarieties of dimension r (namely orbit closures), we have

$$r \leq \dim \mathfrak{g}_r^* \leq 2(N - r) + 1,$$

so $\frac{r}{2} \leq \frac{2N+1}{6}$. Because $\frac{r}{2}$ is an integer, the result follows. $\square$

As noted in the introduction, the bound in the above corollary is achieved for the minimal coadjoint orbits of the group SL_3; in this case $N = 8$ and $r = 4$.

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